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Log-terminal Smoothings of Graded Normal Surface Singularities

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Introduction

Let $(X,0)$ be the germ of an isolated complex normal singularity. Suppose that its canonical module K_X is $\mathbb{Q}$-Cartier; thus, the canonical sheaf $K_{X-\{0\}}$ has some finite order m. The *index 1* (or *canonical*) cover $(T,0) \to (X,0)$ is obtained by normalizing the corresponding cyclic cover. Note that $(T,0)$ has an isolated normal singularity with K_T Cartier. If T is Cohen–Macaulay, then it is Gorenstein; we call such an $(X,0)$ $\mathbb{Q}$-*Gorenstein*. (Warning: some authors require only that X be Cohen–Macaulay with K_X $\mathbb{Q}$-Cartier.) That T is Cohen–Macaulay is automatic if X has dimension 2 but not in general, even if X itself is Cohen–Macaulay. In fact, Singh [16, 6.1] constructs an example of an isolated 3-dimensional *rational* (hence Cohen–Macaulay) singularity X with K_X $\mathbb{Q}$-Cartier whose index 1 cover is not Cohen–Macaulay.

Now suppose $(X,0)$ is a $\mathbb{Q}$-Gorenstein normal surface singularity (e.g., a rational singularity). We will say that a smoothing $f: (\mathcal{X},0) \to (\mathbb{C},0)$ of X is $\mathbb{Q}$-*Gorenstein* if it is the quotient of a smoothing of the index 1 cover of X. The basic examples are certain smoothings of the cyclic quotient singularities of type $rn^2/rnq-1$, first mentioned in [8, (5.9)]. In this particular case, $\mathcal{X}$ is a cyclic quotient of $\mathbb{C}^3$—it is even a *terminal* singularity—and a cyclic quotient by a smaller group is the total space of the smoothing of the index 1 cover of X (which is an A_{rn-1}-singularity). The importance of $\mathbb{Q}$-Gorenstein smoothings was first noticed in the work of Kollár and Shepherd-Barron [7]. But even if the (now 3-dimensional) total space $\mathcal{X}$ of a smoothing of X has $K_{\mathcal{X}}$ $\mathbb{Q}$-Cartier, it does not immediately follow that the smoothing is $\mathbb{Q}$-Gorenstein.

The main point of this work is that, in an important special case, one can deduce that a smoothing is $\mathbb{Q}$-Gorenstein by proving the much stronger result that the total space $\mathcal{X}$ can be chosen to be *log-terminal*. The result is surprising since the original singularities X are generally not even log-canonical (i.e., have discrepancies $-\infty$).

In the early 1980s the author constructed many examples (both published and unpublished) of surface singularities that admit smoothings with Milnor number 0—that is, smoothings whose Milnor fibre is a $\mathbb{Q}$HD (rational homology disk).

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The simplest of these were the aforementioned $n^2/nq-1$ cyclic quotients. These smoothing examples were constructed in one of two ways. The first was an explicit "quotient construction" [19, 5.9]. One starts with (a germ of) an isolated 3-dimensional Gorenstein singularity $(\mathcal{Z},0)$; a finite group G acting on it, freely off the origin; and a G-invariant function f on $\mathcal{Z}$ whose zero locus $(W,0)$ has an isolated singularity (and hence is both normal and Gorenstein). Then $f:(\mathcal{Z},0)\to(\mathbb{C},0)$ is a smoothing of W with a Milnor fibre M, which in these examples is simply connected. One also has $f:(\mathcal{Z}/G,0)\to(\mathbb{C},0)$, which gives a $\mathbb{Q}$-Gorenstein smoothing of $(W/G,0)\equiv(X,0)$ with Milnor fibre M/G (note that G acts freely on M). Here the order of G equals the Euler characteristic of M, so M/G has Euler characteristic 1 and hence is a $\mathbb{Q}$HD Milnor fibre. In all these examples, $\mathcal{Z}$ is actually canonical Gorenstein; therefore, the total space $\mathcal{Z}/G$ of the smoothing is log-terminal.

Yet other examples can be constructed using only Pinkham's [13] method of "smoothing with negative weight" for a weighted homogeneous singularity. For these cases (e.g., the triply infinite family of type $\mathcal{M}$ in [18, (8.3)]), the properties of the total space of the smoothing are much less obvious and it is not at all clear whether the smoothings are $\mathbb{Q}$-Gorenstein.

The problem of finding 3-manifolds that "nicely bound" rational homology disks became of interest in symplectic topology, and it led to a proof that our old list of examples was complete in the weighted homogeneous case (see [18] and [1]). Thus, one knows the resolution dual graph of all weighted homogeneous surface singularities admitting a $\mathbb{Q}$HD smoothing. Nonetheless, this explicit class of singularities is rather mysterious. We show that such smoothings are $\mathbb{Q}$-Gorenstein and can even be assumed to have log-terminal total space. In Theorem 3.4 we prove the following claims.

THEOREM. *Let $(X,0)$ be a weighted homogeneous surface singularity admitting a $\mathbb{Q}$HD smoothing. Then*:

(1) *a $\mathbb{Q}$HD smoothing occurs over a 1-dimensional smoothing component of $(X,0)$; and*
(2) *for $f:(\mathcal{X},0)\to(\mathbb{C},0)$ the induced smoothing over the normalization of that component, $(\mathcal{X},0)$ is log-terminal and the smoothing is $\mathbb{Q}$-Gorenstein.*

The first statement is [21, Cor. 8.2] (one guesses that the smoothing component is smooth, so normalization is unnecessary); the main point is the second assertion. We do *not* claim that log-terminality remains true after base-change (see Remark 3.5(1) to follow).

Our main technical tool is to define and study a *graded discrepancy* $\alpha(X)$ for an isolated normal graded singularity X with K_X $\mathbb{Q}$-Cartier. The blow-up of the weight filtration $Z\to X$ has irreducible exceptional fibre E, so that (Z,E) is a log-terminal pair. Then in Definition 1.2 we define $\alpha(X)$ to be the discrepancy of K along E. This invariant depends not only on X but upon the grading as well. A key observation (Proposition 1.4) is that

$$\alpha(X)>-1 \implies X \text{ is log-terminal.}$$

The following basic result relates the graded discrepancy of a singularity to that of a graded hypersurface section (see Theorem 1.5 and Corollary 1.6).

THEOREM. *Let $\mathcal{X} = \operatorname{Spec} A$ be a graded normal isolated singularity, with $K_{\mathcal{X}}$ $\mathbb{Q}$-Cartier, whose minimal set of generators $z_1, z_2, \ldots, z_t$ has weights $m_1, m_2, \ldots, m_t$ with GCD equal to 1. Suppose $X = \operatorname{Spec} A/(f)$ is normal and graded with isolated singularity. Assume further one of the following*:

(1) $f = z_t$ *and* $m_1, \ldots, m_{t-1}$ *have GCD equal to 1*;
(2) $f \in m_A^2$.

Then $\alpha(\mathcal{X}) = \alpha(X) + \text{weight}\, f$.

COROLLARY. *Under the foregoing hypothesis, if in addition X is a $\mathbb{Q}$-Gorenstein surface singularity with $\alpha(X) > -2$, then $\mathcal{X}$ is log-terminal* (*and, in particular, the corresponding smoothing is $\mathbb{Q}$-Gorenstein*).

If X is a weighted homogeneous surface singularity, we show in Section 2 how to compute $\alpha(X)$ in terms of some topological invariants χ and e introduced by W. Neumann. Proposition 2.1 shows that

$$\alpha(X) = -1 - (\chi/e).$$

But [18] limits greatly the possible resolution dual graphs of the singularities with a $\mathbb{Q}$HD smoothing; computing χ and e, we conclude that $\alpha(X) > -2$ in those cases. (Although [1] pinned down the exact list of graphs, that more precise result is not needed in this proof.) Thus $\mathcal{X}$ is the total space of a smoothing of X and, by the theorem just stated, $\alpha(\mathcal{X}) > -1$, so it is log-terminal.

Finally, we apply an old result of Watanabe [23] in Corollary 4.6 to show exactly which weighted homogeneous surface singularities are $\mathbb{Q}$-Gorenstein (an analytic condition). The invariant χ/e shows up naturally.

We have profited from conversations with János Kollár and Sándor Kovács.

1. Seifert Partial Resolution and Graded Discrepancy

Let $z_1, \ldots, z_t$ be coordinate functions on an affine space $\mathbb{C}^t$, where z_i has positive integer weight m_i. Assume that $\text{GCD}(m_1, \ldots, m_t) = 1$. Blowing up the corresponding weight filtration gives the *weighted* or *filtered blow-up* $\pi : \mathcal{Z} \to \mathbb{C}^t$, an isomorphism off $\pi^{-1}(0) \equiv \mathcal{E}$, that is an irreducible Weil divisor isomorphic to the corresponding weighted projective space. Note that $\mathcal{Z}$ is covered by t affine varieties U_i, each of which is a quotient of an affine space V_i by a cyclic group of order m_i. Consider on V_1 coordinates $x, y_2, \ldots, y_t$, related to the z_i via

$$z_1 = x^{m_1},\ z_2 = x^{m_2} y_2,\ \ldots,\ z_t = x^{m_t} y_t.$$

U_1 is the quotient of V_1 by the action of the cyclic group generated by

$$S = [-1/m_1, m_2/m_1, \ldots, m_t/m_1] = (1/m_1)[-1, m_2, \ldots, m_t],$$

where we are using the notation

$$[q_1, \dots, q_t] := (\exp(2\pi i q_1), \dots, \exp(2\pi i q_t)).$$

The hyperplane $\{x = 0\}$ in V_1 maps onto the exceptional fibre $\mathcal{E} \cap U_1$.

The group $\langle S \rangle$ acts on V_1 without pseudo-reflections. We describe the locus where the action is not free. For each prime p dividing m_1, let $J_p = \{j \mid 2 \le j \le t, (p, m_j) = 1\}$; it is nonempty. If $j \notin J_p$, then $S^{m_1/p}$ acts trivially in the jth slot. Thus $S^{m_1/p}$ acts trivially on the linear subspace $V_{1,p}$ defined by the vanishing of x and the y_j with $j \in J_p$. The codimension of $V_{1,p}$ in V_1 is $1 + \#J_p$, and the union of the images in U_1 gives the singular locus there.

Let $X = \operatorname{Spec} A$ be an isolated normal singularity with good $\mathbb{C}^*$-action. Then the graded domain $A = \bigoplus A_k$ can be written as the quotient of a graded polynomial ring $\mathbb{C}[z_1, \dots, z_t]$, where the weights m_i of the z_i are assumed to have GCD 1. Assume further that one has chosen a minimal set of generators—in other words, that the embedding dimension is t. Denote the weight filtration by $I_s = \bigoplus_{k \ge s} A_k$, and consider

$$\bigoplus_{s=0}^{\infty} I_s u^s \subset A[u].$$

The blow-up of the weight filtration of A is

$$\pi : Z = \operatorname{Proj} \bigoplus_{s=0}^{\infty} I_s u^s \to X = \operatorname{Spec} A,$$

which is sometimes called the *Seifert partial resolution* of X (of course, it depends on the choice of grading). It is an isomorphism off $\pi^{-1}(0) \equiv E$, an irreducible Weil divisor isomorphic to $\operatorname{Proj} A$.

Further, $\pi : Z \to X$ is equal to the proper transform in $\mathcal{Z}$ of the subvariety $\operatorname{Spec} A \subset \mathbb{C}^t$, so Z is covered by closed subvarieties $\bar{U}_i$ of the affine U_i above. More precisely, let $\{g_\alpha(z_1, \dots, z_t)\}$ be homogeneous generators of the graded ideal defining X. Then the proper transform $\bar{V}_1$ of X in V_1 is the affine subvariety with coordinate ring $\mathbb{C}[x, y_2, \dots, y_t]/(g_\alpha(1, y_2, \dots, y_t)) \equiv \mathcal{A}_1$, which is a polynomial ring in x over the subring generated by $y_2, \dots, y_t$. Note that $\bar{V}_1$ is smooth because X has an isolated singularity (for this and other details, consult e.g. Flenner [5] or [20, (2.1)]); thus, (Z, E) is a log-terminal pair that is locally the cyclic quotient of a smooth space plus normal crossing divisors. In particular, the corresponding affine open subset of Z is $\bar{U}_1 = \bar{V}_1/\langle S \rangle$. Since the z_i form a minimal set of generators, all y_i are nonzero in $\mathcal{A}_1$. So the group acts on $\bar{V}_1$ "without pseudo-reflections"; that is, it acts freely off the intersections $\bar{V}_1 \cap V_{1,p} \equiv \bar{V}_{1,p}$, which are necessarily of codimension at least 2 (but possibly less than $1 + \#J_p$).

We wish to compare the Seifert partial resolution $(Z, E) \to (X, 0)$ with that of an appropriate hypersurface section. So let $f \in A_d$ be a homogeneous element with the property that the hypersurface $X' = \operatorname{Spec} A/(f) \subset X$ is also normal with isolated singularity. Then $f = 0$ on Z consists of the proper transform Z' of X' plus the set $E' = E \cap Z'$, which is irreducible and isomorphic to $\operatorname{Proj} A/(f))$. We are interested in achieving the condition

$(*)$ E' is not contained in the singular locus of Z.

When $(*)$ is satisfied, one has

$(**)$ generically E' is a Cartier divisor on both E and Z'.

To verify $(*)$, it suffices to check in some $\bar{U}_i$ for which the pull-back of E' is nonempty. The singular locus is the image of the fixed point locus in $\bar{V}_i$. So in $\bar{V}_1$, say, write $f = x^d f(1, y_2, \ldots, y_t) \equiv x^d \bar{f}$. Assume that $(\bar{f}, x)$ is not the unit ideal. The condition $(*)$ means that the set $x = \bar{f} = 0$ is not contained in any $\bar{V}_{1,p}$.

EXAMPLE. Let $X = \mathbb{C}^3$, with coordinates z_1, z_2, z_3 having weights $2, 2, 1$ (respectively). Then V_1 has coordinates x, y_2, y_3, and the action of S is $(1/2)[-1, 2, 1]$; so U_1 has coordinates $u = x^2$, $v = xy_3$, $w = y_3^2$, and y_2. In other words, U_1 is a line cross the ordinary double point given by $uw = v^2$, with exceptional fibre defined by $u = v = 0$, and singular locus the line cross the point $u = v = w = 0$. Consider now the proper transform of $f = z_3$ in U_1 defined by $v = w = 0$. Its intersection with the exceptional fibre is exactly the singular locus of U_1. Thus, $(*)$ is not satisfied.

The problem in the preceding example was that the weights other than that of z_3 were not relatively prime. However, we have the following result.

PROPOSITION 1.1. *Let $X =$ Spec A be a graded normal isolated singularity with the minimal set of weights $m_1, m_2, \ldots, m_t$ having GCD $= 1$. Suppose $X' =$ Spec $A/(f)$ is normal and graded and with isolated singularity. Then condition $(*)$ is satisfied in both of the following cases*:

(1) *$f = z_t$ and $m_1, \ldots, m_{t-1}$ have GCD $= 1$;*
(2) *$f \in m_A^2$.*

Proof. Consider the first case. As already indicated, it suffices to look in the $\bar{V}_i$. For $i = t$, the corresponding $\bar{f}$ is equal to 1 and so $\bar{U}_t \cap E' = \emptyset$. One may therefore restrict to, say, $\bar{V}_1$. Here $\bar{f} = y_t$.

Now, $A/(z_t)$ is graded normal with isolated singularity *and* its minimal set of generators has relatively prime weights $m_1, \ldots, m_{t-1}$. Thus, the relevant affine V_1' has coordinate ring $\mathcal{A}_1/(y_t)$ with an induced action by $\langle S \rangle$ that is still free off a set of codimension at least 2.

But suppose there exists a prime p such that the locus $x = y_t = 0$ is contained in $\bar{V}_{1,p}$. Then $S^{m/p}$ would act trivially on a divisor of V_1'. This is a contradiction.

The second case is proved similarly. □

Now suppose K_X is $\mathbb{Q}$-Cartier, so that for some positive integer n one has on $U = X - \{0\}$ that $\omega_U^{\otimes n} \cong \mathcal{O}_U$. Then nK_Z has some integral coefficient c along E, and we make the obvious definition.

DEFINITION 1.2. The graded discrepancy $\alpha(X)$ is c/n.

REMARK 1.3. Note that $\alpha(X)$ depends upon the grading. For instance, it is easy to check that if $\mathbb{C}^2$ has coordinates with weights 1 and m then $\alpha = m$.

However, we also have the following statement.

PROPOSITION 1.4. *If $\alpha(X) > -1$, then X is log-terminal. If $\alpha(X) \geq -1$, then X is log-canonical.*

Proof. The key point is that the pair (Z, E) is log-terminal. So, if $\alpha(X) \geq -1$, resolving singularities will not give any exceptional components whose discrepancy is less than $\alpha(X)$. (See [6, Sec. 5.2] for this and other facts about log-terminal singularities, such as that they are cyclic quotients of rational Gorenstein singularities and hence Cohen–Macaulay.) □

On the other hand, even when $\alpha(X)$ is smaller than -1, it can be a useful invariant despite the usual discrepancy of the singularity being $-\infty$.

EXAMPLE. Let Y be a smooth projective variety and L an ample line bundle. Then the cone $X = \operatorname{Spec} \bigoplus_{i \geq 0} \Gamma(Y, L^{\otimes i})$ is a normal graded ring with isolated singularity. The Seifert partial resolution has total space the geometric line bundle $Z = V(L^{-1}) \to Y$, and the exceptional divisor E is the 0-section. If there exist integers m and $n \neq 0$ such that $K_Y^{\otimes n} \cong L^{\otimes m}$, then K_X is $\mathbb{Q}$-Cartier. Writing $K_Z \equiv cE$ and using the adjunction formula, one easily finds that

$$\alpha(X) = -1 - (m/n).$$

An important use of the invariant $\alpha(X)$ arises when taking graded hypersurface sections—for instance, when one views X as the total space of a graded smoothing of a singularity of dimension one less.

THEOREM 1.5. *Let $\pi : Z \to X = \operatorname{Spec} A$ be the Seifert partial resolution of an isolated normal singularity with good $\mathbb{C}^*$-action for which K_X is $\mathbb{Q}$-Cartier. Suppose $f \in A_d$ is such that the Cartier divisor X' defined by f also has an isolated normal singularity and satisfies condition $(*)$. Then:*

(1) *the Cartier divisor defined by f on Z is generically $dE + Z'$, where $Z' \to X'$ is the Seifert partial resolution of X';*
(2) *the graded discrepancies are related by*

$$\alpha(X) = \alpha(X') + d.$$

Proof. The first assertion follows from the previous discussion relating weighted blow-up of a singularity to the proper transform in the weighted blowing-up of a graded affine space. Condition $(*)$ means that generically along $E' = E \cap Z'$, E and Z are Cartier divisors. For the second assertion, one can write generically $K_Z \equiv \alpha E$. By adjunction, generically along E' one has $K_{Z'} \cong \mathcal{O}_{Z'}(\alpha E + Z')$. But the divisor of f is trivial on Z, so generically $Z' \equiv -dE$. The result follows. □

Corollary 1.6. *Let* $X = \operatorname{Spec} A$ *be an isolated normal singularity with good* $\mathbb{C}^*$*-action and with* K_X $\mathbb{Q}$*-Cartier. Suppose* $f \in A_d$ *also defines an isolated normal singularity* X' *and that* f *satisfies either of the conditions of Proposition 1.1. If* $a(X') > -2$ (*resp.* $a(X') \geq -2$), *then X is log-terminal* (*resp. log-canonical*).

Suppose X' is a singularity with good $\mathbb{C}^*$-action. A deformation of X' is said to have *negative weight* if, roughly speaking, one perturbs the defining equations by terms of smaller degree. Specifically, a smoothing of negative weight consists of an isolated singularity X with good $\mathbb{C}^*$-action, a function $f: X \to \mathbb{C}$ of some positive weight d, and a *graded* isomorphism of the special fibre with X'. Even if $K_{X'}$ is $\mathbb{Q}$-Cartier, K_X need not be (see Remark 3.2). Yet we do have the following result.

Corollary 1.7. *Let* X' *be an isolated normal singularity with good* $\mathbb{C}^*$*-action and with* $K_{X'}$ $\mathbb{Q}$*-Cartier. Suppose* $f: X \to \mathbb{C}$ *gives a smoothing of* X' *of negative weight, so that* K_X *is* $\mathbb{Q}$*-Cartier. If* $a(X') > -2$, *then*:

(1) *X is log-terminal—in particular, rational and* $\mathbb{Q}$*-Gorenstein*;
(2) X' *is Cohen–Macaulay and, in fact,* $\mathbb{Q}$*-Gorenstein*;
(3) *the smoothing is* $\mathbb{Q}$*-Gorenstein—in particular, a quotient of a corresponding smoothing of the index 1 cover of* X'.

Proof. If X and X' have the same embedding dimension, then f satisfies Proposition 1.1(2). Otherwise, f is among a minimal set of generators for the graded ring of X while the other generators have weights that are those of X'; thus, the other weights have GCD $= 1$. The first assertion is therefore just Corollary 1.6 combined with familiar facts about log-terminal singularities. The index 1 cover T of X has an isolated canonical Gorenstein singularity [6, Cor. (5.2.1)], which induces a cover $T' \to X'$ that agrees with the index 1 cover of X' off the singular point. But T' is Cohen–Macaulay (since T is), so it is normal and thus equal to the index 1 cover of X'. □

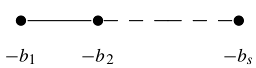
2. *K* for a Normal Graded Surface Singularity

We recall briefly the topological description of normal surface singularities $X = \operatorname{Spec} A$ with good $\mathbb{C}^*$-action; the analytic classification will be described later.

One has the cyclic quotient singularities of type n/q, where $0 < q < n$, $(q, n) = 1$, defined as the quotient of $\mathbb{C}^2$ by the cyclic group action generated by

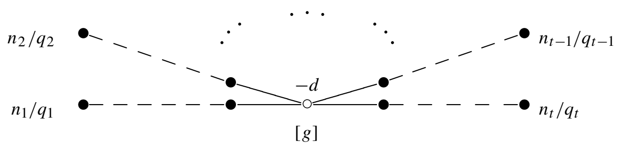
$$[1/n, q/n].$$

Writing the continued fraction expansion $n/q = b_1 - 1/b_2 - \cdots - 1/b_s$, each $b_i \geq 2$, the minimal resolution of the singularity consists of a string of rational curves with dual graph



This singularity is isomorphic to n/q', where $qq' \equiv 1 \bmod n$, and the continued fraction expansion n/q' gives the b_i in the opposite order.

Every other normal surface singularity has a unique $\mathbb{C}^*$-action with minimal good resolution graph Γ that is star-shaped:



The strings of Γ are described uniquely by the continued fractions shown, starting from the node. The central curve C has genus g. If $t = 0$ then one has a cone. If $g = 0$, minimality and the exclusion of cyclic quotients require $t \geq 3$. The Seifert partial resolution of the singularity contains the one exceptional curve C with t cyclic quotient singularities on it.

The graph Γ completely determines the topology of the singularity (i.e., gives the Seifert invariants). One has two topological invariants introduced by Neumann (e.g., [11, p. 250]):

$$e = d - \sum_{i=1}^{t} \frac{q_i}{n_i};$$

$$\chi = 2g - 2 + t - \sum_{i=1}^{t} \frac{1}{n_i} = 2g - 2 + \sum_{i=1}^{t} \left(1 - \frac{1}{n_i}\right).$$

While $e > 0$, one can easily show that $\chi < 0$ exactly for quotient singularities (which are log-terminal) and that $\chi = 0$ exactly for the familiar list of graded log-canonical singularities.

Let $(\tilde{X}, E) \to (X, 0)$ be the minimal good resolution with resolution graph as described previously. The line bundle $K_{\tilde{X}}$ is numerically equivalent to a $\mathbb{Q}$-combination of exceptional curves whose coefficients we examine. For a cyclic quotient, this is an exercise using Cramer's rule (cf. [8, 5.9.(iii)]); but we shall not use this. Exclude the cyclic quotients and consider the $\mathbb{Q}$-cycle $Z \equiv -(K + E)$, writing in terms of cycles supported on the t strings plus a multiple of the central curve C:

$$Z = \sum_{k=1}^{t} Y_k + \beta C.$$

We solve for each Y_k in terms of n_k, q_k, and β, and then we solve for the appropriate β. For any irreducible exceptional curve F, set $d_F = -F \cdot F$ and denote by u_F the number of neighbors of F. Thus $Z \cdot F = -(d_F + 2g_F - 2 + u_F - d_F) = 2 - 2g_F - u_F$, so that $Z \cdot F$ equals 1 on the t ends and $2 - 2g - t$ at $F = C$ (and is 0 elsewhere).

Consider first one string of type n/q. Suppose the exceptional curves counting out from C are $E_1, E_2, \ldots, E_s$. Let e_i be the $\mathbb{Q}$-cycle supported on the string defined by $e_i(E_j) = -\delta_{ij}$. Then the cycle $Y = \beta e_1 - e_s$ dots to $-\beta$ with E_1, to 1 with E_s, and to 0 with the intermediate E_j. It is easy to see that the $\mathbb{Q}$-cycle e_1 has coefficient $1/n$ at E_s and that its coefficients increase strictly as one approaches E_1, where it is q/n. By the same argument, e_s has coefficient $1/n$ at E_1 that is increasing to q'/n at E_s, where as usual $qq' \equiv 1 \bmod n$. In particular, the coefficient of Y at E_1 is $(\beta q - 1)/n$, and the coefficient at the end E_s is $(\beta - q')/n$. Note that if $s = 1$ then $q = q' = 1$, so the formulas are still correct.

Choosing the corresponding Y_k for each string, we show that for appropriate β, the cycle $Z = \sum_{k=1}^{t} Y_k + \beta C$ represents $-(K+E)$. Given the coefficient of each Y_k at the curve intersecting C, the condition $Z \cdot C = 2 - 2g - t$ becomes

$$-\beta d + \sum_{k=1}^{t} \frac{\beta q_k - 1}{n_k} = 2 - 2g - t.$$

We therefore must set

$$\beta = \chi/e.$$

PROPOSITION 2.1. *Writing $K_{\tilde{X}}$ numerically as a rational combination of exceptional curves, the coefficient of the central curve C is $-1 - (\chi/e)$.*

COROLLARY 2.2. *Suppose X is a $\mathbb{Q}$-Gorenstein weighted homogeneous surface singularity* (*e.g.*, $g = 0$) *and not a cyclic quotient. Then the graded discrepancy is*

$$\alpha(X) = -1 - (\chi/e).$$

Proof. The Seifert partial resolution is obtained from the minimal good resolution by blowing-down the t strings. The parenthetical assertion in the corollary (that a weighted homogeneous singularity with rational central curve is $\mathbb{Q}$-Gorenstein) follows, for instance, from the main theorem of [10]; this implies the stronger result that such an X is a quotient of a complete intersection. □

Next, it is natural to find the numerical (i.e., topological) order of the class of K_X (or equivalently $K_{\tilde{X}} + E$) in the discriminant group $\Delta = \mathbb{E}^*/\mathbb{E}$ of the singularity. A multiple rZ is integral if and only if its image in Δ dots to 0 with every element in the discriminant group or with any set of generators. Yet the duals of any $t-1$ of the end curves are generators, and Z dotted with an end curve dual is equal to the (negative of the) coefficient of the corresponding end curve in Z. At the end of the kth string, this is

$$(\beta - q_k')/n_k.$$

We summarize as follows.

PROPOSITION 2.3. *Consider the graph of a weighted homogeneous surface singularity as described before. Then*:

(1) *the coefficient of C in $-K$ is* $1 + (\chi/e)$;
(2) *all other coefficients of $-K$ are strictly smaller than* $1 + (\chi/e)$; *and*
(3) *the order of the class of K in the discriminant group is the least common denominator for any $t-1$ of the terms*

$$(\chi/e - q_k')/n_k.$$

The second assertion must be checked separately for quotient singularities.

Combining Corollary 2.2 with Corollary 1.7, one sees there is a special role played by singularities with $\chi/e < 1$. Observe that $q_i = 1$ means the corresponding string has length 1. The following result is easily verified from the definitions of χ and e.

LEMMA 2.4. *Consider a weighted homogeneous surface singularity with rational central curve of self-intersection $-d$ and the Seifert invariants (n_i, q_i), $1 \leq i \leq t$. Then $\chi/e < 1$ in each of the following cases*:

(1) $t = 3$ *and either*
   (a) $d \geq 4$,
   (b) $d = 3$ *and* $q_1 = 1$, *or*
   (c) $d = 2$ *and* $q_1 = q_2 = 1$;
(2) $t = 4$, $d \geq 3$, *and* $q_1 = q_2 = q_3 = 1$.

3. The Main Theorem

Let $(X, 0)$ be the germ of a normal surface singularity, topologically the cone over its neighborhood boundary L, its link. This L is a compact 3-manifold and can be viewed as the boundary of a tubular neighborhood of the exceptional fibre in a resolution of X. If now $f : (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ is a smoothing, then the general fibre M of f, the *Milnor fibre,* is a compact 4-manifold with boundary L. Note that M has the homotopy type of a complex of dimension 2, and its first Betti number is 0. If N denotes the link of $\mathcal{X}$, then N is a compact 5-manifold containing L whose complement fibres over the circle via $f/|f|$, with general fibre M; this is called an *open-book decomposition of* N. In particular, N can be constructed from M by adjoining cells of dimension > 2, so that they have the same fundamental group and $H_2(M) \to H_2(N)$ is surjective. For further details, see [8, Lemma 5.1].

Before stating and proving the main theorem, we start with a general proposition.

PROPOSITION 3.1. *Let $f : (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ be a smoothing of a normal surface singularity $(X, 0)$ such that the Milnor fibre M is a rational homology disk* (*i.e,* $b_2(M) = 0$). *Then*:

(1) $(X, 0)$ *is a rational surface singularity*;
(2) $(\mathcal{X}, 0)$ *is a rational 3-fold singularity*;
(3) *the link N of $\mathcal{X}$ is a rational homology sphere*;
(4) $\mathrm{Pic}(\mathcal{X} - \{0\})$ *is finite and so $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier.*

Proof. That $(X, 0)$ is rational follows by the Durfee–Steenbrink formula $\mu_0 + \mu_+ = 2p_g(X)$ [17]. Elkik's theorem [4] then implies that $(\mathcal{X}, 0)$ is rational. As mentioned previously, $H_1(M; \mathbb{Q}) \cong H_1(N; \mathbb{Q}) = 0$ in general and $H_2(N; \mathbb{Q}) = 0$ because M has no rational homology. Thus, the compact 5-manifold is a rational homology sphere.

Following Mumford's original arguments in [9], let $U = \mathcal{X} - \{0\}$ and consider the cohomology sequence

$$H^1(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U^*) \to H^2(U, \mathbb{Z}).$$

The first term is 0 because the depth of $\mathcal{X}$ is 3. The 5-manifold N is homotopic to U, so $H^2(U, \mathbb{Z}) \cong H^2(N, \mathbb{Z})$ is finite; this is then isomorphic to the torsion in $H_1(N; \mathbb{Z})$, which is in turn isomorphic to H_1 of the Milnor fibre. In particular, $\mathrm{Pic}(U)$ is finite. □

REMARK 3.2. Artin component smoothings of a rational surface singularity are definitely not $\mathbb{Q}$-Gorenstein (except for the rational double points), for the Milnor fibre—being diffeomorphic to the minimal resolution—is simply connected and so is not the quotient of the Milnor fibre of the index 1 cover. In fact, the total space $\mathcal{X}$ of such a smoothing does not have $K_{\mathcal{X}}$ $\mathbb{Q}$-Cartier.

REMARK 3.3. As mentioned in the Introduction, it does not follow immediately from Proposition 3.1 that the total space $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein, since it is not clear that its index 1 cover $\mathcal{Y} \to \mathcal{X}$ is Cohen–Macaulay. In particular, $\mathcal{Y}$ could conceivably give a smoothing of a nonnormal model of the index 1 cover of X.

We now state and prove the main theorem of this paper.

THEOREM 3.4. *Let $(X, 0)$ be a weighted homogeneous surface singularity that possesses a rational homology disk smoothing. Then the following statements hold.*

(1) *The smoothing is induced by the graded smoothing $f\colon (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ coming from a 1-dimensional smoothing component in the base space of the semi-universal deformation of $(X, 0)$.*
(2) *$(\mathcal{X}, 0)$ is log-terminal, and the smoothing is $\mathbb{Q}$-Gorenstein.*

Proof. The result is known for the relevant cyclic quotients by explicit construction, so we skip this case. The grading on X extends to a grading on the base space and total space of the semi-universal deformation. According to [21, Cor. 8.2], a rational homology disk smoothing occurs over a 1-dimensional smoothing component; it must also be graded. We consider the induced smoothing $f\colon (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ over the normalization of this component. By Proposition 3.1, $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. By [18, Cor. 2.5], the resolution dual graph of X satisfies one of (1) or (2) in Lemma 2.4; therefore, $\chi/e < 1$. By Corollary 2.2, it follows that $\alpha(X) > -2$. Corollary 1.7 now yields the desired result. □

REMARK 3.5. (1) As J. Kollár has pointed out, the log-terminality result need not hold after base-change of the deformation, given that the condition on the weights in Proposition 1.1 need not hold. For instance, one would not want to consider the deformation of the elliptic cone given by $x^3 + y^3 + z^3 + t^{3m} = 0$; although the total space is a graded ring, its weights $(m, m, m, 1)$ do not induce the original weights $(1, 1, 1)$ when one sets $t = 0$.

(2) It is important to note that, among all the X that admit rational homology disk smoothings, the only log-canonical examples are the cyclic quotients plus three more (which are quotients of elliptic cones).

(3) The proof did not use the precise list of resolution graphs found in [1] but only the restrictions on the graph found earlier in [18].

4. When Is X $\mathbb{Q}$-Gorenstein?

The topological data of a weighted homogeneous surface singularity X is given by its graph—that is, the data $(g, d, \{n_i/q_i\})$. (We exclude cyclic quotients.) The additional information of the analytic type is: the central curve C, if $g > 0$; the isomorphism class of a conormal divisor D of C of the resolution, which has degree d; and the t points $P_1, \ldots, P_t$ on C. A theorem of Pinkham [12] (also due to Dolgachev) shows that the graph together with this analytic data allows one to write down the graded pieces of the ring of the singularity. We describe this result following the approach of Demazure [2], who proved a more general result describing graded normal domains of any dimension.

Consider a $\mathbb{Q}$-divisor on C, say $F = \Sigma r_j Q_j$, where $r_j \in \mathbb{Q}$ and $Q_j \in C$. Two such divisors are equivalent if their difference is an integral divisor linearly equivalent to 0. Define the integral divisor

$$\lfloor F \rfloor = \Sigma \lfloor r_j \rfloor Q_j$$

and the invertible sheaf

$$\mathcal{O}(F) \equiv \mathcal{O}(\lfloor F \rfloor) \subset k(C);$$

as usual, $\lfloor r \rfloor$ means greatest integer $\leq r$.

Now let $X = \operatorname{Spec} A$ be a weighted homogeneous surface singularity with graph Γ and analytic data $C, D, \{P_1, \ldots, P_t\}$. Define the $\mathbb{Q}$-divisor

$$E = D - \Sigma (q_i/n_i) P_i.$$

The degree of E is the topological invariant $e > 0$.

THEOREM 4.1 [12]. *Let $X = \operatorname{Spec} A$ be a graded normal surface singularity with resolution graph Γ and analytic invariants as before. Then*

$$A = \bigoplus_{k=0}^{\infty} A_k = \bigoplus_{k=0}^{\infty} H^0(\mathcal{O}(kE))T^k \subset k(C)[T].$$

(In the notation of [12] and [20], one has $\lfloor kE \rfloor = D^{(k)}$.)

In [22], Watanabe computed the graded local cohomology of A (in all dimensions) and hence the canonical sheaf K_X of X. Consider the $\mathbb{Q}$-divisor

$$\Xi = K + \Sigma (1 - 1/n_i) P_i,$$

where K is a canonical divisor on C. The degree of Ξ is the topological invariant χ.

THEOREM 4.2 [22, (2.8)]. *With notation as before, the dualizing module of A is*

$$\omega_A = \bigoplus_{k=-\infty}^{\infty} H^0(C, \Xi + kE)T^k.$$

Corollary 4.3 (see also [20, (2.1)]). *A is Gorenstein if and only if there is a $t \in \mathbb{Z}$ with*

$$\Xi \equiv tE;$$

that is, $tq_i \equiv 1(n_i)$ for all i and $\lfloor tE \rfloor \equiv K_C$. Necessarily, $t = \chi/e$.

Watanabe's method yields a formula as well for sK_X—namely, the double dual $\omega_A^{(s)}$ of the sth tensor power of ω_A (this is first stated in [23, Lemma 3.2]). By [22, Thm. 1.6], there is an isomorphism from the group of equivalence classes of certain $\mathbb{Q}$-divisors whose fractional part involves only the P_i onto the divisor class group of A; this map sends

$$F \mapsto \bigoplus_{k=-\infty}^{\infty} H^0(C, F + kE)T^k.$$

Theorem 4.4 (Watanabe). *With notation as before,*

$$\omega_A^{(s)} = \bigoplus_{k=-\infty}^{\infty} H^0(C, s\Xi + kE)T^k.$$

Corollary 4.5. *sK_X is Cartier if and only if there is a $t \in \mathbb{Z}$ with*

$$s\Xi \equiv tE.$$

Taking degrees, the last analytic condition implies $s\chi = te$. From this equality, one can easily prove the following statement.

Corollary 4.6. *X is $\mathbb{Q}$-Gorenstein if and only if the class of the degree 0 $\mathbb{Q}$-divisor*

$$\Xi - (\chi/e)E$$

is torsion.

Remark 4.7. (1) If X is $\mathbb{Q}$-Gorenstein, then the order of K_X in the (analytic) divisor class group is divisible by the denominator of χ/e written as a reduced fraction, since $t/s = \chi/e$ (cf. Proposition 2.3(3)).

(2) Corollary 4.6 gives an explicit condition on a conormal divisor D such that a weighted homogeneous singularity is $\mathbb{Q}$-Gorenstein once one fixes the topological data as well as the isomorphism type of the curve C and the points P_i. Recall that Popescu-Pampu [14] proved that, for the resolution graph of *any* normal surface with fixed analytic type of the reduced exceptional divisor, one can choose normal bundles so that there exists a $\mathbb{Q}$-Gorenstein singularity with the given topological and analytic data.

(3) An alternate description of a weighted homogeneous surface singularity A, due to I. Dolgachev, is

$$A = \bigoplus_{k=0}^{\infty} H^0(U, L^{-k})^{\Gamma},$$

where L is a line bundle on the universal covering space U of Proj A and $\Gamma \subset \operatorname{Aut} U$ is a discrete subgroup. From this point of view, the Gorenstein condition is found by Dolgachev in [3] and the $\mathbb{Q}$-Gorenstein condition is due to Pratoussevitch [15].

References

[1] M. Bhupal and A. Stipsicz, *Weighted homogeneous singularities and rational homology disk smoothings,* Amer. J. Math. 133 (2011), 1259–1297.

[2] M. Demazure, *Anneaux gradués normaux,* Introduction à la théorie des singularités II, Travaux en Cours, 37, pp. 35–68, Hermann, Paris, 1988.

[3] I. Dolgachev, *On the link space of a Gorenstein quasihomogeneous surface singularity,* Math. Ann. 265 (1983), 529–540.

[4] R. Elkik, *Singularités rationnelles et déformations,* Invent. Math. 47 (1978), 139–147.

[5] H. Flenner, *Rationale quasihomogene Singularitäten,* Arch. Math. 36 (1981), 35–44.

[6] J. Kollár and S. Mori, *Birational geometry of algebraic varieties,* Cambridge Tracts in Math., 134, Cambridge Univ. Press, Cambridge, 1998.

[7] J. Kollár and N. Shepherd-Barron, *Threefolds and deformations of surface singularities,* Invent. Math. 91 (1988), 299–338.

[8] E. Looijenga and J. Wahl, *Quadratic functions and smoothing surface singularities,* Topology 25 (1986), 261–291.

[9] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity,* Inst. Hautes Études Sci. Publ. Math. 9 (1961), 5–22.

[10] W. D. Neumann, *Abelian covers of quasihomogeneous surface singularities,* Singularities (Arcata, 1981), Proc. Sympos. Pure Math., 40, pp. 233–243, Amer. Math. Soc., Providence, RI, 1983.

[11] ———, *Geometry of quasihomogeneous surface singularities,* Singularities (Arcata, 1981), Proc. Sympos. Pure Math., 40, pp. 245–258, Amer. Math. Soc., Providence, RI, 1983.

[12] H. Pinkham, *Normal surface singularities with $\mathbb{C}^*$ action,* Math. Ann. 227 (1977), 183–193.

[13] ———, *Deformations of normal surface singularities with $\mathbb{C}^*$ action,* Math. Ann. 232 (1978), 65–84.

[14] P. Popescu-Pampu, *Numerically Gorenstein surface singularities are homeomorphic to Gorenstein ones,* Duke Math. J. 159 (2011), 539–559.

[15] A. Pratoussevitch, *On the link space of a $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularity,* Real and complex singularities, Trends Math., pp. 311–325, Birkhäuser, Basel, 2007.

[16] A. Singh, *Cyclic covers of rings with rational singularities,* Trans. Amer. Math. Soc. 355 (2002), 1009–1024.

[17] J. H. M. Steenbrink, *Mixed Hodge structure associated with isolated singularities,* Singularities (Arcata, 1981), Proc. Sympos. Pure Math., 40, pp. 513–536, Amer. Math. Soc., Providence, RI, 1983.

[18] A. Stipsicz, Z. Szabó, and J. Wahl, *Rational blowdowns and smoothings of surface singularities,* J. Topol. 1 (2008), 477–517.

[19] J. Wahl, *Smoothings of normal surface singularities,* Topology 20 (1981), 219–246.

[20] ———, *Deformations of quasi-homogeneous surface singularities,* Math. Ann. 280 (1988), 105–128.

[21] ———, *On rational homology disk smoothings of valency 4 surface singularities,* Geom. Topol. 15 (2011), 1125–1156.
[22] K. Watanabe, *Some remarks concerning Demazure's construction of normal graded rings,* Nagoya Math. J. 83 (1981), 203–211.
[23] ———, *Rational singularities with k^*-action,* Commutative algebra (Trento, 1981), Lecture Notes in Pure and Appl. Math., 84, pp. 339–351, Dekker, New York, 1983.

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