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Ideal-adic Semi-continuity of Minimal Log Discrepancies on Surfaces

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Following Kollár [4], de Fernex, Ein, and Mustaţă [1] proved the ideal-adic semi-continuity of log canonicity effectively, to obtain Shokurov's [6] ACC conjecture for log canonical thresholds on smooth varieties. Mustaţă formulated this semi-continuity for minimal log discrepancies as follows.

Conjecture 1 (Mustaţă; see [3]). *Let (X, Δ) be a pair, Z a closed subset of X, and $\mathcal{I}_Z$ the ideal sheaf of Z. Let $\mathfrak{a} = \prod_{j=1}^{k} \mathfrak{a}_j^{r_j}$ be a formal product of ideal sheaves $\mathfrak{a}_j$ with positive real exponents r_j. Then there exists an integer l such that the following holds: if $\mathfrak{b} = \prod_{j=1}^{k} \mathfrak{b}_j^{r_j}$ satisfies $\mathfrak{a}_j + \mathcal{I}_Z^l = \mathfrak{b}_j + \mathcal{I}_Z^l$ for all j, then*

$$\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) = \mathrm{mld}_Z(X, \Delta, \mathfrak{b}).$$

The case of minimal log discrepancy 0 is the semi-continuity of log canonicity. Conjecture 1 is proved in the Kawamata log terminal (klt) case in [3, Thm. 2.6]. However, log canonical (lc) singularities are inevitably treated in the study of limits of singularities in the ideal-adic topology because the limit of klt singularities is lc in general. For example, the limit of klt pairs $(\mathbb{A}^2_{x,y}, (x, y^n)\mathcal{O}_{\mathbb{A}^2})$ indexed by $n \in \mathbb{N}$ is the lc pair $(\mathbb{A}^2, x\mathcal{O}_{\mathbb{A}^2})$ in the $(x, y)\mathcal{O}_{\mathbb{A}^2}$-adic topology. The purpose of this paper is to settle Mustaţă's conjecture for surfaces.

Theorem 2. *Conjecture 1 holds when X is a surface.*

We must handle a non-klt triple $(X, \Delta, \mathfrak{a})$ that has positive minimal log discrepancy; yet unlike in the klt case, the log canonicity is not retained when the exponent of $\mathfrak{a}$ is increased as $\mathfrak{a}^{1+\varepsilon}$. For surfaces, however, we are reduced to the purely log terminal (plt) case in which $\mathfrak{a}$ has an expression $\mathfrak{a}'\mathcal{O}_X(-C)$; then we can increase only the exponent of the part $\mathfrak{a}'$ to apply the result on log canonicity.

We work over an algebraically closed field of characteristic 0. We use the notation described next for singularities in the minimal model program.

Notation 3. A *pair* (X, Δ) consists of a normal variety X and an effective $\mathbb{R}$-divisor Δ such that $K_X + \Delta$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We treat a *triple* $(X, \Delta, \mathfrak{a})$ by attaching a formal product $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ of finitely many coherent ideal sheaves $\mathfrak{a}_j$ with positive real exponents r_j. A prime divisor E on a normal variety X' with a

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proper birational morphism $\varphi\colon X' \to X$ is called a divisor *over* X, and the image $\varphi(E)$ on X is called the *center* of E on X and denoted by $c_X(E)$. We denote by $\mathcal{D}_X$ the set of divisors over X. The *log discrepancy* $a_E(X, \Delta, \mathfrak{a})$ of E is defined as $1+\mathrm{ord}_E(K_{X'}-\varphi^*(K_X+\Delta))-\mathrm{ord}_E\,\mathfrak{a}$. The triple $(X, \Delta, \mathfrak{a})$ is said to be *log canonical* (resp., *Kawamata log terminal*) if $a_E(X, \Delta, \mathfrak{a})$ is no less (resp., greater) than 0 for all $E \in \mathcal{D}_X$; this triple is said to be *purely log terminal, canonical,* or *terminal* according as whether $a_E(X, \Delta, \mathfrak{a})$ is (respectively) greater than 0, no less than 1, or greater than 1 for all exceptional $E \in \mathcal{D}_X$. A center $c_X(E)$ with $a_E(X, \Delta, \mathfrak{a}) \le 0$ is called a *non-klt center.* Let Z be a closed subset of X. The *minimal log discrepancy* $\mathrm{mld}_Z(X, \Delta, \mathfrak{a})$ over Z is the infimum of $a_E(X, \Delta, \mathfrak{a})$ for all $E \in \mathcal{D}_X$ with center in Z. We say that $E \in \mathcal{D}_X$ *computes* $\mathrm{mld}_Z(X, \Delta, \mathfrak{a})$ if $c_X(E) \subset Z$ and $a_E(X, \Delta, \mathfrak{a}) = \mathrm{mld}_Z(X, \Delta, \mathfrak{a})$ (or is negative when $\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) = -\infty$).

Prior to the proof of Theorem 2, we collect standard reductions and known results on Conjecture 1.

LEMMA 4 [3, Rem. 2.5.3, 2.5.4]. *Conjecture 1 can be reduced to the case where X has $\mathbb{Q}$-factorial terminal singularities, $\Delta = 0$, and Z is irreducible. It then suffices to prove the inequality* $\mathrm{mld}_Z(X, \mathfrak{a}) \le \mathrm{mld}_Z(X, \mathfrak{b})$.

THEOREM 5. *Conjecture 1 holds in each of the following cases*:

(i) $\mathrm{mld}_Z(X, \mathfrak{a}) = -\infty$;
(ii) $\mathrm{mld}_Z(X, \mathfrak{a}) = 0$ [1; 4];
(iii) $(X, \mathfrak{a})$ *is klt about* Z [3, Thm. 2.6].

REMARK 6. In Theorem 5(ii), one can take as l any integer greater than the maximum of $\mathrm{ord}_E\,\mathfrak{a}_j/\mathrm{ord}_E\,\mathcal{I}_Z$ by fixing an $E \in \mathcal{D}_X$ that computes $\mathrm{mld}_Z(X, \mathfrak{a})$. The estimate of l in (iii) involves the log canonical threshold of $\mathfrak{a}$.

Conjecture 1 for surfaces is reduced to the plt case.

LEMMA 7. *With respect to Conjecture 1 for surfaces, one may assume the following*:

(i) *X is a smooth surface, $\Delta = 0$, and Z is a closed point*;
(ii) *$(X, \mathfrak{a})$ is plt with unique non-klt center C*;
(iii) *C is a smooth curve.*

Proof. By Lemma 4 we may assume that X is smooth with $\Delta = 0$, and by parts (i) and (ii) of Theorem 5 we may assume that $\mathrm{mld}_Z(X, \mathfrak{a}) > 0$. Let C be the non-klt locus of $(X, \mathfrak{a})$. By Theorem 5(iii), we have only to work about $Z \cap C$. The assumption $\mathrm{mld}_Z(X, \mathfrak{a}) > 0$ means that Z contains no non-klt center, whence $Z \cap C$ consists of finitely many closed points. By replacing Z with $Z \cap C$ and working locally, we may assume that Z is a closed point x and that $(X, \mathfrak{a})$ has the non-klt locus C, which is a curve. The exceptional divisor E of the blow-up of X at x has positive log discrepancy $a_E(X, \mathfrak{a})$, but it is at most $a_E(X, C) = 2 - \mathrm{mult}_x\, C$. Hence C must be smooth at x. □

We work locally about the closed point $x = Z$ under the assumptions given in Lemma 7. We denote by $\mathfrak{m}$ the maximal ideal sheaf at x and use notation similar to that in [3, Def. 2.3].

DEFINITION 8. For $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$ and $l \in \mathbb{N}$, we write $\mathfrak{a} \equiv_l \mathfrak{b}$ if $\mathfrak{a}_j + \mathfrak{m}^l = \mathfrak{b}_j + \mathfrak{m}^l$ for all j.

Set $c := \mathrm{mld}_x(X, \mathfrak{a})$. The nontrivial locus of $\mathfrak{a}$ (i.e., the locus where some $\mathfrak{a}_j$ is nontrivial) is a divisor of the form $C + D$ about x. Since $(X, \mathfrak{a})$ is plt, we can fix $s, t > 0$ and $t' \geq 0$ such that $\mathrm{mld}_x(X, sD, \mathfrak{a}\mathfrak{m}^{t'}) = \mathrm{mld}_x(X, \mathfrak{a}\mathfrak{m}^t) = 0$. We fix a log resolution $\varphi\colon \bar{X} \to X$ of $(X, \mathfrak{a}\mathfrak{m})$; that is, $\prod_j \mathfrak{a}_j \mathfrak{m}\mathcal{O}_{\bar{X}}$ defines a divisor with simple normal crossing support. Let $\bar{C}$ and $\bar{D}$ denote the strict transforms of (respectively) C and D. Since C is smooth, it follows that $\bar{C}$ intersects only one prime divisor F in $\varphi^{-1}(x)$. This will play a crucial role in the proof. If $D \neq 0$ then, by blowing up $\bar{X}$ further, we may assume that every divisor E in $\varphi^{-1}(x)$ intersecting $\bar{D}$ satisfies

$$\mathrm{ord}_E D \geq s^{-1}c - 1. \tag{1}$$

We take an integer l such that

$$l > \mathrm{ord}_E \mathfrak{a}_j / \mathrm{ord}_E \mathfrak{m} \tag{2}$$

for all j and $E \subset \varphi^{-1}(x)$. The following lemma is an application of Theorem 5(ii) and Remark 6 with the inequality (2).

LEMMA 9. $\mathrm{mld}_x(X, sD, \mathfrak{b}\mathfrak{m}^{t'}) = \mathrm{mld}_x(X, \mathfrak{b}\mathfrak{m}^t) = 0$ *for any* $\mathfrak{b} \equiv_l \mathfrak{a}$.

We write

$$\mathfrak{a}_j \mathcal{O}_{\bar{X}} = \mathcal{O}_{\bar{X}}(-H_j - V_j)$$

with divisors H_j and V_j such that $\mathrm{Supp}\, H_j \subset \bar{C} + \bar{D}$ and $\mathrm{Supp}\, V_j \subset \varphi^{-1}(x)$. Let $\mathfrak{b} \equiv_l \mathfrak{a}$. For $E \subset \varphi^{-1}(x)$, we have $\mathrm{ord}_E \mathfrak{a}_j < \mathrm{ord}_E \mathfrak{m}^l$ by (2) and have $\mathrm{ord}_E \mathfrak{a}_j = \mathrm{ord}_E \mathfrak{b}_j$ because $\mathfrak{a}_j + \mathfrak{m}^l = \mathfrak{b}_j + \mathfrak{m}^l$. Hence we can write

$$\mathfrak{b}_j \mathcal{O}_{\bar{X}} = \mathfrak{b}'_j \mathcal{O}_{\bar{X}}(-V_j) \quad \text{and} \quad \mathfrak{m}^l \mathcal{O}_{\bar{X}} = \mathcal{O}_{\bar{X}}(-M_j - V_j)$$

with an ideal sheaf $\mathfrak{b}'_j$ and an effective divisor M_j such that $\mathrm{Supp}\, M_j = \varphi^{-1}(x)$. Then the equality $\mathfrak{a}_j + \mathfrak{m}^l = \mathfrak{b}_j + \mathfrak{m}^l$ induces

$$\mathcal{O}_{\bar{X}}(-H_j) + \mathcal{O}_{\bar{X}}(-M_j) = \mathfrak{b}'_j + \mathcal{O}_{\bar{X}}(-M_j). \tag{3}$$

The next lemma establishes that $\mathrm{mld}_x(X, \mathfrak{b}) \geq c$; when combined with Lemma 4, this completes the proof of Theorem 2.

LEMMA 10. $a_G(X, \mathfrak{b}) \geq c$ *for any* $\mathfrak{b} \equiv_l \mathfrak{a}$ *and* $G \in \mathcal{D}_X$ *with* $c_X(G) = x$.

Proof. We treat the three different cases corresponding to the possible positions of $c_{\bar{X}}(G)$:

(i) $c_{\bar{X}}(G) \not\subset \bar{C} + \bar{D}$;
(ii) $c_{\bar{X}}(G) \subset \bar{D}$;
(iii) $c_{\bar{X}}(G) \subset \bar{C}$.

(i) By equation (3) we have $\operatorname{Supp} H_j \cap \operatorname{Supp} M_j = \operatorname{Supp} \mathcal{O}_{\bar{X}}/\mathfrak{b}'_j \cap \operatorname{Supp} M_j$, whence $\operatorname{Supp} \mathcal{O}_{\bar{X}}/\mathfrak{b}'_j \cap \varphi^{-1}(x) \subset \bar{C} + \bar{D}$. In particular, $c_{\bar{X}}(G) \not\subset \operatorname{Supp} \mathcal{O}_{\bar{X}}/\mathfrak{b}'_j$. This implies that $\operatorname{ord}_G \mathfrak{b}_j = \operatorname{ord}_G V_j = \operatorname{ord}_G \mathfrak{a}_j$, so $a_G(X, \mathfrak{b}) = a_G(X, \mathfrak{a}) \geq c$.

(ii) Take a prime divisor E in $\varphi^{-1}(x)$ such that $c_{\bar{X}}(G) \subset E$. By inequality (1), we have $\operatorname{ord}_G D = \operatorname{ord}_E D \cdot \operatorname{ord}_G E + \operatorname{ord}_G \bar{D} \geq \operatorname{ord}_E D + 1 \geq s^{-1}c$. Lemma 9 for $(X, sD, \mathfrak{b}\mathfrak{m}^{t'})$ implies that $a_G(X, \mathfrak{b}) \geq s \operatorname{ord}_G D$, and these two inequalities yield $a_G(X, \mathfrak{b}) \geq c$.

(iii) We know that $c_{\bar{X}}(G)$ is in the unique divisor $F \subset \varphi^{-1}(x)$ intersecting $\bar{C}$. There exists a divisor E in $\varphi^{-1}(x)$ with $a_E(X, \mathfrak{a}\mathfrak{m}^t) = 0$. Let L be the union of all such E. Then $L \cup \bar{C}$ is connected by the connectedness lemma [5, Thm. 17.4]. Hence $F \subset L$—that is, $a_F(X, \mathfrak{a}\mathfrak{m}^t) = 0$—and so $\operatorname{ord}_F \mathfrak{m}^t = a_F(X, \mathfrak{a}) \geq c$ (actually the equality holds by precise inversion of adjunction [2]). Lemma 9 for $(X, \mathfrak{b}\mathfrak{m}^t)$ now implies that $a_G(X, \mathfrak{b}) \geq \operatorname{ord}_G \mathfrak{m}^t$. Given $c_{\bar{X}}(G) \subset F$, we obtain $a_G(X, \mathfrak{b}) \geq \operatorname{ord}_G \mathfrak{m}^t \geq \operatorname{ord}_F \mathfrak{m}^t \geq c$. □

REMARK 11. The case division in the proof of Lemma 10 is in terms of the union H of divisors E, with $\operatorname{ord}_E \mathfrak{a} > 0$ and with $c_X(E) \not\subset Z$, on a suitable log resolution $\bar{X}$. We write $H = H' + H''$ so that H' is the union of those E with $a_E(X, \mathfrak{a}) = 0$. Then the cases (i), (ii), (iii) in the proof of Lemma 10 correspond to these respective conditions: $c_{\bar{X}}(G) \not\subset H$; $c_{\bar{X}}(G) \subset H''$ and $c_{\bar{X}}(G) \not\subset H'$; and $c_{\bar{X}}(G) \subset H'$. The proof of (i) works in any dimension, and that of (ii) works provided $(X, \mathfrak{a})$ is plt (or, more generally, dlt). However, the proof of (iii) does not work unless H' intersects only one divisor in $\varphi^{-1}(Z)$.

REMARK 12. In [3], Conjecture 1 is formulated for $(X, \Delta, \mathfrak{a})$ with $\mathfrak{a}$ an $\mathbb{R}$-ideal sheaf as an equivalence class of formal products of ideal sheaves. Our proof is valid also for this formulation.

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