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Quasiconvexity and Relatively Hyperbolic Groups That Split

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The goal of this paper is to examine relative hyperbolicity and quasiconvexity in graphs of relatively hyperbolic vertex groups with almost malnormal quasiconvex edge groups. The paper hinges upon the observation that if G splits as a graph of relatively hyperbolic groups with malnormal relatively quasiconvex edge groups, then a fine hyperbolic graph for G can be built from fine hyperbolic graphs for the vertex groups. This leads to short proofs of the relative hyperbolicity of G as well as to a concise criterion for the relative quasiconvexity of a subgroup H of G.

Bestvina and Feighn [2] proved a combination theorem that characterized the hyperbolicity of groups splitting as graphs of hyperbolic groups. Their geometric characterization is akin to the flat plane theorem characterization of hyperbolicity for actions on CAT(0) spaces and leads to explicit positive results—especially in an "acylindrical" scenario, where some form of malnormality is imposed on the edge groups. The Bestvina–Feighn combination theorem has been revisited multiple times in a hyperbolic setting and more recently, but through diverse methods, in a relatively hyperbolic context.

Dahmani [4] proved a combination theorem for relatively hyperbolic groups using the convergence group approach. Later, Alibegović [1] proved similar results using a method generalizing parts of the Bestvina–Feighn approach. Osin [19] re-proved Dahman's result in the general context of relative Dehn functions. Most recently, Mj and Reeves [17] gave a generalization of the Bestvina–Feighn combination theorem that follows Farb's approach but uses a generalized "partial electrocution". Their result appears to be a far-reaching generalization at the expense of complex geometric language.

Our own results revisit these relatively hyperbolic generalizations, and we offer a very concrete approach employing Bowditch's fine hyperbolic graphs. The most natural formulation of our main combination theorem (which we shall prove as Theorem 1.4) is as follows.

Theorem A (Combining Relatively Hyperbolic Groups along Parabolics). *Let G split as a finite graph of groups. Suppose each vertex group is relatively hyperbolic and each edge group is parabolic in its vertex groups. Then G is hyperbolic relative to $\mathbb{Q} = \{Q_1, \dots, Q_j\}$, where each Q_i is the stabilizer of a "parabolic tree".* (See Definition 1.3.)

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A simplistic example illustrating Theorem A is an amalgamated product $G = G_1 *_C G_2$, where each $G_i = \pi_1 M_i$ and M_i is a cusped hyperbolic manifold with a single boundary torus T_i; here C is an arbitrary common subgroup of $\pi_1 T_1$ and $\pi_1 T_2$. Then G is hyperbolic relative to $\pi_1 T_1 *_C \pi_1 T_2$.

We note that Theorem A is more general than results in the same spirit that were obtained by Dahmani, Alibegović, and Osin. In particular, they require that edge groups be maximal parabolic on at least one side, but we do not. We believe that Theorem A could be deduced from the results of Mj and Reeves.

In Section 4, we employ work of Yang [22] on extended peripheral structures to obtain the following seemingly more natural corollary of Theorem A, which we prove as Corollary 4.6.

COROLLARY B. *Let G split as a finite graph of groups. Suppose*

(a) *each G_v is hyperbolic relative to $\mathbb{P}_v$,*
(b) *each G_e is total and relatively quasiconvex in G_v, and*
(c) *$\{G_e : e$ is attached to $v\}$ is almost malnormal in G_v for each vertex v.*

Then G is hyperbolic relative to $\bigcup_v \mathbb{P}_v - \{repeats\}$.

The “omitted repeats” in the conclusion of Corollary B refer to (some of) the parabolic subgroups of vertex groups that are identified through an edge group.

It is not clear whether Corollary B could be obtained using the method of Dahmani, Alibegović, or Osin. However, we suspect it could be extracted from the result of Mj and Reeves.

DEFINITION 0.1 (Tamely Generated). Let G split as a graph of groups with relatively hyperbolic vertex groups. A subgroup H is *tamely generated* if the induced graph of groups Γ_H has a π_1-isomorphic subgraph of groups Γ'_H that is a finite graph of groups each of whose vertex groups is relatively quasiconvex in the corresponding vertex group of G.

Note that H is tamely generated when H is finitely generated (f.g.) and there are finitely many H-orbits of vertices v in T with H_v nontrivial, and each such H_v is relatively quasiconvex in G_v. However, this condition is not necessary. For instance, let $G = F_2 \times \mathbb{Z}_2$, and consider a splitting where Γ is a bouquet of two circles and where each vertex and edge group is isomorphic to $\mathbb{Z}_2$. Then every f.g. subgroup H of $F_2 \times \mathbb{Z}_2$ is tamely generated, but no subgroup containing $\mathbb{Z}_2$ satisfies the condition that there are finitely many H-orbits of vertices ω with H_ω nontrivial.

The geometric construction proving Theorem A allows us to give a simple criterion for quasiconvexity of a subgroup H relative to $\mathbb{Q}$. Again, coupling this with Yang’s work, we obtain (as Theorem 4.13) the following criterion for quasiconvexity relative to $\mathbb{P}$.

MAIN THEOREM C (Quasiconvexity Criterion). *Let G be hyperbolic relative to $\mathbb{P}$, where each $P \in \mathbb{P}$ is finitely generated. Suppose G splits as a finite graph of groups. Suppose*

(a) *each G_e is total in G,*
(b) *each G_e is relatively quasiconvex in G, and*
(c) *each G_e is almost malnormal in G.*

Let $H \leq G$ be tamely generated. Then H is relatively quasiconvex in G.

Recall that G is *locally relatively quasiconvex* if each finitely generated subgroup H of G is quasiconvex relative to the peripheral structure of G. Kapovich [12] was the first to recognize that hyperbolic limit groups are locally relatively quasiconvex, and subsequently Dahmani [4] proved that all limit groups are locally relatively quasiconvex.

A group P is *small* if there is no embedding $F_2 \hookrightarrow P$, and G has a *small hierarchy* if it can be built from small subgroups by a sequence of amalgamated free products (AFPs) and Higman–Neumann–Neumann (HNN) extensions along small subgroups (see Definition 3.4). When $\mathbb{P}$ is a collection of free abelian groups, the following inductive consequence of Corollary 3.3 generalizes Dahmani's result.

Theorem D. *Let G be hyperbolic relative to a collection of Noetherian subgroups $\mathbb{P}$ and suppose G has a small hierarchy. Then G is locally relatively quasiconvex.*

Although Theorem D is implicit in Dahmani's work, we believe Theorem C is new.

The Main Construction and Its Application. Although we work in somewhat greater generality, let us focus on the simple case of an amalgamated product $G = A *_C B$ where A, B are relatively hyperbolic and C is parabolic on each side. The central theme of this paper is a construction that builds a fine hyperbolic graph $\bar{K}_G$ for G from fine hyperbolic graphs K_A and K_B for A, B. (See Figure 1.) This is done in two steps. Guided by the Bass–Serre tree, we first construct a graph K_G that is a tree of spaces whose vertex spaces are copies of K_A and K_B and whose edge spaces are ordinary edges. Though K_G is fine and hyperbolic, its edges have infinite stabilizers. We remedy this by quotienting these edge

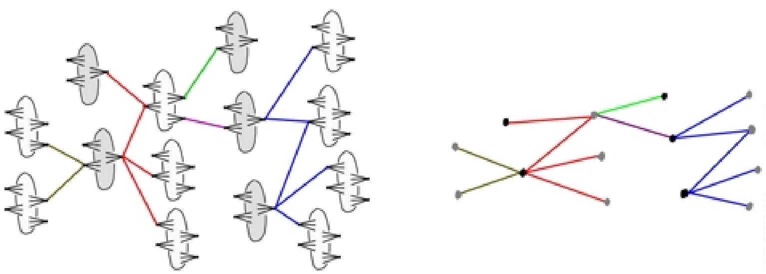

Figure 1 A fine graph K_G for $G = A *_C B$ is built from copies of fine graphs K_A and K_B for A and B by gluing new edges together along vertices stabilized by C, where the parabolic trees of T are images of trees formed from the new edges in K_G; we obtain a fine hyperbolic graph $\bar{K}_G$ with finite edge stabilizers as a quotient $K_G \to \bar{K}_G$

spaces to form the fine hyperbolic graph $\bar{K}_G$. The vertices of $\bar{K}_G$ are quotients of "parabolic trees" in K_G. The fine hyperbolic graph $\bar{K}_G$ quickly proves that G is hyperbolic relative to the collection $\mathbb{Q}$ of subgroups stabilizing parabolic trees. Variations on the construction, hypotheses on the edge groups, and interplay with previous work on peripheral structures lead to a variety of relatively hyperbolic conclusions. The simplest and most immediate in the case described here is that G is hyperbolic relative to $\mathbb{P}_G = \mathbb{P}_A \cup \mathbb{P}_B - \{C\}$ when C is maximal parabolic on each side and A, B are hyperbolic relative to $\mathbb{P}_A$ and $\mathbb{P}_B$.

Our primary application is to give an easy criterion for recognizing quasiconvexity. A subgroup H is relatively quasiconvex in G if there is an H-cocompact quasiconvex subgraph $\bar{L} \subset \bar{K}_G$ of the fine hyperbolic G-graph. The treelike nature of our graph $\bar{K}_G$ permits us to naturally build the quasiconvex H-graph $\bar{L}$. When H is relatively quasiconvex, there are finitely many H-orbits of nontrivially H-stabilized vertices in the Bass–Serre tree T, and each of these stabilizers is relatively quasiconvex in its vertex group. Choosing finitely many quasiconvex subgraphs in the corresponding copies of K_A and K_B, we are able to combine these together to form L in K_G and then to form a quasiconvex H-subgraph $\bar{L}$ in $\bar{K}_G$.

We conclude by mentioning the following consequence of Corollary 1.5, which is a natural consequence of the viewpoint developed in this paper.

COROLLARY E. *Let M be a compact irreducible 3-manifold, and let $M_1, \dots, M_r$ denote the graph manifolds obtained by removing each (open) hyperbolic piece in the JSJ decomposition of M. Then $\pi_1 M$ is hyperbolic relative to $\{\pi_1 M_1, \dots, \pi_1 M_r\}$.*

As explained to us by the referee, the relative hyperbolicity of $\pi_1(M)$ was previously proved by Druţu and Sapir [5] using work of Kapovich and Leeb [13]. This previous proof is deep in that it uses the structure of the asymptotic cone due to Kapovich and Leeb together with the technical proof of Druţu and Sapir that asymptotically tree graded groups are relatively hyperbolic.

1. Combining Relatively Hyperbolic Groups along Parabolics

The class of relatively hyperbolic groups was introduced by Gromov [8] as a generalization of the class of fundamental groups of complete finite-volume manifolds of pinched negative sectional curvature. Various approaches to relative hyperbolicity were developed by Farb [6], Bowditch [3], and Osin [20]; as surveyed by Hruska [10], these notions are equivalent for finitely generated groups. We follow Bowditch's approach.

DEFINITION 1.1 (Relatively Hyperbolic). A *circuit* in a graph is an embedded cycle. A graph Γ is *fine* if each edge of Γ lies in finitely many circuits of length n for each n.

A group G is *hyperbolic relative to a finite collection of subgroups* $\mathbb{P}$ if G acts cocompactly(without inversions) on a connected, fine, hyperbolic graph Γ with finite edge stabilizers such that each element of $\mathbb{P}$ equals the stabilizer of a vertex of Γ and, moreover, each infinite vertex stabilizer is conjugate to a unique element of $\mathbb{P}$. We refer to a connected, fine, hyperbolic graph Γ equipped with such an action as a $(G; \mathbb{P})$-*graph*. Subgroups of G that are conjugate into subgroups in $\mathbb{P}$ are *parabolic.*

TECHNICAL REMARK 1.2. Given a finite collection of parabolic subgroups $\{A_1, \dots, A_r\}$, we choose $\mathbb{P}$ so that there is a prescribed choice of parabolic subgroup $P_i \in \mathbb{P}$ such that A_i is "declared" to be conjugate into P_i. This is automatic for an infinite parabolic subgroup A but for finite subgroups there could be ambiguity. One way to resolve this is to revise the choice of $\mathbb{P}$ as follows. For any finite collection of parabolic subgroups $\{A_1, \dots, A_r\}$ in G, we also assume that each A_i is conjugate to a subgroup of $\mathbb{P}$ and that no two (finite) subgroups in $\mathbb{P}$ are conjugate. We note that finite subgroups can be freely added to or omitted from the peripheral structure of G (see e.g. [16]).

DEFINITION 1.3 (Parabolic Tree). Let G split as a finite graph of groups, where each vertex group G_ν is hyperbolic relative to $\mathbb{P}_\nu$ and where each edge group G_e embeds as a parabolic subgroup of its two vertex groups. Let T be the Bass–Serre tree. Define the *parabolic forest* F as follows.

(1) A *vertex* in F is a pair (u, P) for $u \in T^0$ and P a G_u-conjugate of an element of $\mathbb{P}_u$.
(2) An *edge* in F is a pair (e, G_e), where e is an edge of T and G_e is its stabilizer.
(3) The edge (e, G_e) is *attached* to $(\iota(e), \iota(P_e))$ and $(\tau(e), \tau(P_e))$, where $\iota(e)$ and $\tau(e)$ are the initial and terminal vertex of e and $\iota(P_e)$ is the $G_{\iota(e)}$-conjugate of an element of $\mathbb{P}$ that is declared to contain G_e. Likewise for $(\tau(e), \tau(P_e))$. We arranged for this unique determination in Technical Remark 1.2.

Each component of F is a *parabolic tree,* and the map $F \to T$ is injective on the set of edges; in particular, each parabolic tree embeds in T. Let $S_1, \dots, S_j$ be representatives of the finitely many orbits of parabolic trees under the G-action on F. Let $Q_i = \text{stab}(S_i)$ for each i.

THEOREM 1.4 (Combining Relatively Hyperbolic Groups along Parabolics). *Let G split as a finite graph Γ of groups. Suppose each vertex group is relatively hyperbolic and each edge group is parabolic in its vertex groups. Then G is hyperbolic relative to $\mathbb{Q} = \{Q_1, \dots, Q_j\}$.*

Proof. For $u \in \Gamma^0$, let G_u be hyperbolic relative to $\mathbb{P}_u$ and let K_u be a $(G_u; \mathbb{P}_u)$-graph. For each $P \in \mathbb{P}_u$, following Technical Remark 1.2, we choose a specific vertex of K_u whose stabilizer equals P. Note that in general there could be more than one possible choice when $|P| < \infty$, but by Technical Remark 1.2 we have a unique choice. Translating determines a "choice" of vertex for conjugates.

We now construct a $(G; \mathbb{Q})$-graph $\bar{K}$. Let K be the tree of spaces whose underlying tree is the Bass–Serre tree T with the following properties.

(1) Vertex spaces of K are copies of appropriate elements in $\{K_u : u \in \Gamma^0\}$. Specifically, K_ν is a copy of K_u for u the image of ν under $T \to \Gamma$.
(2) Each edge space K_e is an ordinary edge, denoted as an ordered pair (e, G_e), that is attached to the vertices in $K_{\iota(e)}$ and $K_{\tau(e)}$ that were chosen to contain G_e.

Note that each G_ν acts on K_ν and there is a G-equivariant map $K \to T$. Let $\bar{K}$ be the quotient of K obtained by contracting each edge space. Observe that G acts on $\bar{K}$ and there is a G-equivariant map $K \to \bar{K}$. Moreover, the preimage of each open edge of $\bar{K}$ is a single open edge of K.

We now show that $\bar{K}$ is a $(G; \mathbb{Q})$-graph. Since any embedded cycle lies in some vertex space, the graph $\bar{K}$ is fine and hyperbolic. There are finitely many orbits of vertices in K and therefore finitely many orbits of vertices in $\bar{K}$. Likewise, there are finitely many orbits of edges in $\bar{K}$. The stabilizer of an (open) edge of $\bar{K}$ equals the stabilizer of the corresponding (open) edge in K and is thus finite. By construction, there is a G-equivariant embedding $F \hookrightarrow K$ where F is the parabolic forest associated to G and T. Finally, the preimage in K of a vertex of $\bar{K}$ is precisely a parabolic tree and thus the stabilizer of a vertex of $\bar{K}$ is a conjugate of some Q_j. □

We now examine some conclusions that arise when the parabolic trees are small. An extreme case arises when the edge groups are isolated from each other as follows.

COROLLARY 1.5. *Let G split as a finite directed graph of groups where each vertex group G_ν is hyperbolic relative to $\mathbb{P}_\nu$. Suppose that:*

(1) *each edge group is parabolic in its vertex groups;*
(2) *each outgoing infinite edge group $G_{\vec{e}}$ is maximal parabolic in its initial vertex group G_ν and, for each other incoming and outgoing infinite edge group $G_{\overleftarrow{e}}$ or $G_{\vec{d}}$ or $G_{\overleftarrow{d}}$, none of its conjugates lie in $G_{\vec{e}}$.*

Then G is hyperbolic relative to $\mathbb{P} = \bigcup_\nu \mathbb{P}_\nu - \{$outgoing edge groups$\}$.

Proof. We can arrange for finitely stabilized edges of F to be attached to distinct chosen vertices when they correspond to distinct edges of T. Thus, parabolic trees are singletons and/or i-pods consisting of edges that all terminate at the same vertex $\{(\nu, P^g)\}$ for $P \in \mathbb{P}_\nu$ and $g \in G_\nu$. Recall that an *i-pod* is a tree consisting of i edges glued to a central vertex. □

COROLLARY 1.6. *Let G split as a finite graph of groups. Suppose each vertex group G_ν is hyperbolic relative to $\mathbb{P}_\nu$. For each G_ν, assume that the collection $\{G_e : e$ is attached to $\nu\}$ is a collection of maximal parabolic subgroups of G_ν. Then G is hyperbolic relative to $\mathbb{P} = \bigcup_\nu \mathbb{P}_\nu - \{$repeats$\}$. Specifically, we remove an element of $\bigcup_\nu \mathbb{P}_\nu$ if it is conjugate to another one.*

Parts (1) and (2) of our next corollary were treated by Dahmani [4], Alibegović [1], and Osin [19].

COROLLARY 1.7. (1) *Let G_1 and G_2 be hyperbolic relative to $\mathbb{P}_1$ and $\mathbb{P}_2$. Let $G = G_1 *_{P_1 = P_2'} G_2$, where each $P_i \in \mathbb{P}_i$ and where P_1 is identified with the subgroup P_2' of P_2. Then G is hyperbolic relative to $\mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\}$.*

(2) *Let G_1 be hyperbolic relative to $\mathbb{P}$. Let $P_1 \in \mathbb{P}$ be isomorphic to a subgroup P_2' of a maximal parabolic subgroup P_2 not conjugate to P_1. Let $G = G_1 *_{P_1^t = P_2'}$, where $P_1^t = t^{-1} P_1 t$. Then G is hyperbolic relative to $\mathbb{P} - \{P_1\}$.*

(3) *Let G_1 be hyperbolic relative to $\mathbb{P}$. Let $P \in \mathbb{P}$ be isomorphic to $P' \leq P$. Let $G = G_1 *_{P^t = P'}$. Then G is hyperbolic relative to $\mathbb{P} \cup \langle P, t \rangle - \{P\}$.*

Remark 1.8. Note that in Corollary 1.7 (and similarly elsewhere), a collection $\mathbb{P}$ of peripheral subgroups contains only representatives of conjugacy classes. Thus notation such as $\mathbb{P} - \{P_1\}$ actually means: remove the element of $\mathbb{P}$ that is conjugate to P_1.

Proof of Corollary 1.7. (1) In this case, the parabolic trees are either singletons stabilized by a conjugate of an element of $\mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\}$, or parabolic trees are i-pods stabilized by conjugates of P_2.

(2) The proof is similar.

(3) All parabolic trees are singletons except for those that are translates of a copy of the Bass–Serre tree for $P *_{P^t = P'}$. Following the proof of Theorem 1.4, let $\nu \in \bar{K}$. If the preimage of ν in K is not attached to an edge space, then G_ν is conjugate to an element of $\mathbb{P} - \{P\}$; otherwise, G_ν is conjugate to $\langle P, t \rangle$. □

Example 1.9. We encourage the reader to consider Theorem 1.4 and Corollaries 1.6 and 1.7 in the scenario where G splits as a graph of free groups with cyclic edge groups. A very simple case is to let $G = \langle a, b, t \mid (W^n)^t = W^m \rangle$, where $W \in \langle a, b \rangle$ and $m, n \geq 1$. Then G is hyperbolic relative to $\langle W, t \rangle$.

2. Relative Quasiconvexity

Dahmani introduced the notion of a relatively quasiconvex subgroup in [4]. This notion was further developed by Osin in [20], and later Hruska investigated several equivalent definitions of relatively quasiconvex subgroups in [10]. Martínez-Pedroza and Wise [16] introduced a definition of relative quasiconvexity in the context of fine hyperbolic graphs and showed that this definition is equivalent to Osin's definition. We will study relative quasiconvexity using this fine hyperbolic viewpoint. Our aim is to examine the relative quasiconvexity of certain subgroups that are themselves amalgams, and we note that powerful results in this direction are given in [15].

Definition 2.1 (Relative Quasiconvexity). Let G be hyperbolic relative to $\mathbb{P}$. A subgroup H of G is *quasiconvex relative to* $\mathbb{P}$ if, for some (and hence any) $(G; \mathbb{P})$-graph K, there is a nonempty connected and quasi-isometrically embedded H-cocompact subgraph L of K. In the sequel, we sometimes refer to L as a *quasiconvex H-cocompact subgraph* of K.

Remark 2.2. It is immediate from the Definition 2.1 that, in a relatively hyperbolic group, any parabolic subgroup is relatively quasiconvex and any relatively quasiconvex subgroup is relatively hyperbolic. In particular, the relatively quasiconvex subgroup H is hyperbolic relative to the collection $\mathbb{P}_H$ consisting of representatives of H-stabilizers of vertices of $L \subseteq K$. Note that a conjugate of a relatively quasiconvex subgroup is also relatively quasiconvex and that the intersection of two relatively quasiconvex subgroups is relatively quasiconvex. Specifically, this last statement was proved in [15] when G is f.g. and in [10] when G is countable.

Relative quasiconvexity has the following transitive property proved by Hruska [10] for countable relatively hyperbolic groups.

LEMMA 2.3. *Let G be hyperbolic relative to $\mathbb{P}_G$. Suppose that B is relatively quasiconvex in G, and note that B is then hyperbolic relative to $\mathbb{P}_B$ as in Remark 2.2. Then $A \leq B$ is quasiconvex relative to $\mathbb{P}_B$ if and only if A is quasiconvex relative to $\mathbb{P}_G$.*

Proof. Let K be a $(G; \mathbb{P}_G)$-graph. Since B is quasiconvex relative to $\mathbb{P}_G$, there is a B-cocompact and quasiconvex subgraph $L \subset K$. Note that L is a $(B; \mathbb{P}_B)$-graph. Let $A \leq B$.

If A is quasiconvex in B relative to $\mathbb{P}_B$, there is an A-cocompact quasiconvex subgraph $M \subset L$. Since the composition $L_A \to L_B \to K$ is a quasi-isometric embedding, A is quasiconvex relative to $\mathbb{P}_G$. Conversely, if A is quasiconvex in G relative to $\mathbb{P}_G$ then there is an A-cocompact quasiconvex subgraph $M \subset K$. Let $L' = L \cup BM$. Note that L' is B-cocompact and hence also quasiconvex; thus L' also serves as a fine hyperbolic graph for B. Now $M \subset L'$ is quasiconvex because $M \subset L$ is quasiconvex, so A is relatively quasiconvex in B. □

REMARK 2.4. One consequence of Theorem 1.4 and its corollaries is that, when G splits as a graph of relatively hyperbolic groups with parabolic subgroups, each of the vertex groups is quasiconvex relative to the peripheral structure of G. (For Theorem 1.4 this is $\mathbb{Q}$, and for Corollary 1.6 this is $\mathbb{P} - \{\text{repeats}\}$.) Indeed, K_v is a G_v-cocompact quasiconvex subgraph in the fine graph K constructed in the proof.

LEMMA 2.5. *Let G be a f.g. group that splits as a finite graph of groups Γ. If each edge group is f.g. then each vertex group is f.g.*

Proof. Let $G = \langle g_1, \ldots, g_n \rangle$. We regard G as π_1 of a 2-complex corresponding to Γ. We show that each vertex group G_v equals $\langle \{G_e\}_{e \text{ attached to } v} \cup \{g \in G_v : g \text{ in normal form of some } g_i\} \rangle$. Let $a \in G_v$ and consider an expression of a as a product of normal forms of the $g_i^{\pm 1}$. Then a equals some product $a_1 t_1^{\varepsilon_1} b_1 t_2^{\varepsilon_2} a_2 \cdots a_n t_m^{\varepsilon_m} b_k$. There is a disc diagram D whose boundary path is $a^{-1} a_1 t_1^{\varepsilon_1} b_1 t_2^{\varepsilon_2} a_2 \cdots a_n t_m^{\varepsilon_m} b_k$. See Figure 2. The region of D that lies along a shows that a equals the product of elements in edge groups adjacent to G_v, together with elements of G_v that lie in the normal forms of $g_1, \ldots, g_n$. □

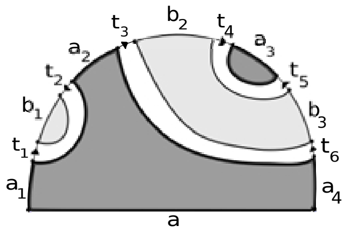

Figure 2

THEOREM 2.6 (Quasiconvexity of a Subgroup in Parabolic Splitting). *Let G split as a finite graph Γ of relatively hyperbolic groups such that each edge group is parabolic in its vertex groups.* (*Note that G is hyperbolic relative to $\mathbb{Q} = \{Q_1, \dots, Q_j\}$ by Theorem 1.4.*) *Let $H \leq G$ be tamely generated. Then H is quasiconvex relative to $\mathbb{Q}$. Moreover, if each H_ν in the Bass–Serre tree T is finitely generated then H is finitely generated.*

Proof. Since there are finitely many orbits of vertices whose stabilizers are finitely generated, H is finitely generated. For each $u \in \Gamma^0$, let G_u be hyperbolic relative to $\mathbb{P}_u$ and let K_u be a $(G_u; \mathbb{P}_u)$-graph. Let K be the $(G; \mathbb{Q})$-graph constructed in the proof of Theorem 1.4 and let $\bar{K}$ be its quotient. We will construct an H-cocompact quasiconvex, connected subgraph $\bar{L}$ of $\bar{K}$.

Let T_H be the minimal H-invariant subgraph of T. Recall that each edge of T (and hence of T_H) corresponds to an edge of K. Let F_H denote the subgraph of K that is the union of all edges corresponding to edges of T_H. Let $\{\nu_1, \dots, \nu_n\}$ be representatives of H-orbits of vertices of T_H. For each i, let $L_i \hookrightarrow K_{\nu_i}$ be an $(H \cap G_{\nu_i}^{g_i})$-cocompact quasiconvex subgraph such that L_i contains $F_H \cap K_{\nu_i}$. (There are finitely many $(H \cap G_{\nu_i}^{g_i})$-orbits of such endpoints of edges in K_{ν_i}.) Let $L = F_H \cup \bigcup_{i=1}^n HL_i$ and let $\bar{L}$ be the image of L under $K \to \bar{K}$. Observe that L is quasiconvex in K, since K is a "tree union" and since each such L_i of L is quasiconvex in K_{ν_i}. Likewise, $\bar{L}$ is quasiconvex in $\bar{K}$. □

COROLLARY 2.7 (Characterizing Quasiconvexity in Maximal Parabolic Splitting). *Let G split as a finite graph of countable groups. For each ν, let G_ν be hyperbolic relative to $\mathbb{P}_\nu$ and let the collection $\{G_e : e$ is attached to $\nu\}$ be a collection of maximal parabolic subgroups of G_ν.* (*Note that G is hyperbolic relative to $\mathbb{P} = \bigcup_\nu \mathbb{P}_\nu - \{$repeats$\}$ by Corollary 1.6.*) *Let T be the Bass–Serre tree and let H be a subgroup of G. Then the following statements are equivalent.*

(1) *H is tamely generated and each H_ν in the Bass–Serre tree T is f.g.*
(2) *H is f.g. and quasiconvex relative to $\mathbb{P}$.*

Proof. (1) ⇒ (2) This implication follows from Theorem 1.4 and Theorem 2.6.

(2) ⇒ (1) Since H is f.g., the minimal H-subtree T_H is H-cocompact and so H splits as a finite graph of groups Γ_H. Since H is quasiconvex in $\mathbb{P}$, it is hyperbolic relative to intersections with conjugates of $\mathbb{P}$. In particular, the infinite edge groups in the induced splitting of H are maximal parabolic; hence they are f.g. because the maximal parabolic subgroups of a f.g. relatively hyperbolic group are f.g. [20]. Each vertex group of Γ_H is f.g. by Lemma 2.5.

By Remark 2.4, each vertex group of G is quasiconvex relative to $\mathbb{P}$. Therefore, each G_ν is relatively quasiconvex (by Remark 2.2) since it is a conjugate of a vertex group. Thus $H_\nu = H \cap G_\nu$ is quasiconvex relative to $\mathbb{P}$ by Remark 2.2. Finally, H_ν is quasiconvex in G_ν by Lemma 2.3. □

3. Local Relative Quasiconvexity

A relatively hyperbolic group G is *locally relatively quasiconvex* if each f.g. subgroup of G is relatively quasiconvex. The focus of this section is the following criterion for showing that the combination of locally relatively quasiconvex groups is again locally relatively quasiconvex.

Recall that N is *Noetherian* if each subgroup of N is f.g. We now give a criterion for local quasiconvexity of a group that splits along parabolic subgroups.

THEOREM 3.1 (A Criterion for Locally Relative Quasiconvexity). (1) *Let G_1 and G_2 be locally relatively quasiconvex relative to $\mathbb{P}_1$ and $\mathbb{P}_2$. Let $G = G_1 *_{P_1=P_2'} G_2$, where each $P_i \in \mathbb{P}_i$ and P_1 is identified with the subgroup P_2' of P_2. Suppose P_1 is Noetherian. Then G is locally quasiconvex relative to $\mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\}$.*

(2) *Let G_1 be locally relatively quasiconvex relative to $\mathbb{P}$. Let $P_1 \in \mathbb{P}$ be isomorphic to a subgroup P_2' of a maximal parabolic subgroup P_2 not conjugate to P_1. Let $G = G_1 *_{P_1^t=P_2'}$. Suppose P_1 is Noetherian. Then G is locally quasiconvex relative to $\mathbb{P} - \{P_1\}$.*

(3) *Let G_1 be locally quasiconvex relative to $\mathbb{P}$. Let P be a maximal parabolic subgroup of G_1 that is isomorphic to $P' \leq P$. Let $G = G_1 *_{P^t=P'}$ and suppose P is Noetherian. Then G is also locally quasiconvex relative to $\mathbb{P} \cup \langle P, t\rangle - \{P\}$.*

Proof. (1) By Corollary 1.7, G is hyperbolic relative to $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\}$. Let H be a finitely generated subgroup of G. We show that H is quasiconvex relative to $\mathbb{P}$. Let T be the Bass–Serre tree of G. Since H is f.g., the minimal H-subtree T_H is H-cocompact and so H splits as a finite graph of groups Γ_H. Moreover, the edge groups of this splitting are f.g. because the edge groups of G are Noetherian by hypothesis. Thus each vertex group of Γ_H is f.g. by Lemma 2.5. Since G_1 and G_2 are locally relatively quasiconvex, each vertex group of T_H is relatively quasiconvex in its "image vertex group" under the map $T_H \to T$. Now, by Theorem 2.6, H is quasiconvex relative to $\mathbb{P}$. The proofs of (2) and (3) are similar. □

DEFINITION 3.2 (Almost Malnormal). A subgroup H is *malnormal* in G if $H \cap H^g = \{1\}$ for $g \notin H$, and similarly H is *almost malnormal* if the intersection $H \cap H^g$ is always finite. Likewise, a collection of subgroups $\{H_i\}$ is almost malnormal if $H_i^g \cap H_j^h$ is finite *unless* $i = j$ and $gh^{-1} \in H_i$.

COROLLARY 3.3. *Let G split as a finite graph of groups. Suppose*

(a) *each G_v is locally relatively quasiconvex,*
(b) *each G_e is Noetherian and maximal parabolic in its vertex groups, and*
(c) *$\{G_e : e$ is attached to $v\}$ is almost malnormal in G_v for any vertex v.*

Then G is locally relatively quasiconvex relative to $\mathbb{P}$ (see Corollary 1.6).

Small-hierarchies and Local Quasiconvexity

The main result in this section is a consequence of Theorem 3.1 that employs results of Yang [22] (stated in Theorems 4.7 and 4.2) and also depends on Lemma 4.9, which is independent of the other results in Section 4. The reader may choose to read this section and refer ahead to those results or may return to this section after reading Section 4.

DEFINITION 3.4 (Small-hierarchy). A group is *small* if it has no rank-2 free subgroup. Any small group has a *length-0 small-hierarchy.* G has a *length-n small-hierarchy* if $G \cong A *_C B$ or $G \cong A *_{C'=C'}$, where A and B have length-$(n-1)$ small-hierarchies and where C is small and f.g. We say G has a *small-hierarchy* if it has a length-n small-hierarchy for some n.

We can define $\mathcal{F}$*-hierarchy* by replacing "small" by a class of groups $\mathcal{F}$ closed under subgroups and isomorphisms. For instance, when $\mathcal{F}$ is the class of finite groups, the class of groups with an $\mathcal{F}$-hierarchy is precisely the class of virtually free groups.

REMARK 3.5. The *Tits alternative* for relatively hyperbolic groups states that every f.g. subgroup is either elementary or parabolic or contains a subgroup isomorphic to F_2. The Tits alternative was proved for countable relatively hyperbolic groups in [8, Thm. 8.2.F]. A proof is given for convergence groups in [21]. It is shown in [20] that every cyclic subgroup H of a f.g. relatively hyperbolic group G is relatively quasiconvex.

THEOREM 3.6. *Let G be f.g. and hyperbolic relative to $\mathbb{P}$, where each element of $\mathbb{P}$ is Noetherian. Suppose G has a small-hierarchy. Then G is locally relatively quasiconvex.*

Proof. The proof is by induction on the length of the hierarchy. Since edge groups are f.g., the Tits alternative shows that there are three cases according to whether the edge group is finite, virtually cyclic, or infinite parabolic; we note that the edge group is relatively quasiconvex in each case. These three cases are each divided into two subcases according to whether $G = A *_{C_1} B$ or $G = A *_{C_1^t = C_2}$.

Since C_1 and G are f.g., the vertex groups are f.g. by Lemma 2.5. Thus, since C_1 is relatively quasiconvex, the vertex groups are relatively quasiconvex by Lemma 4.9.

When C_1 is finite, the conclusion follows in each subcase from Theorem 3.1.

When C_1 is virtually cyclic but not parabolic, C_1 lies in a unique maximal virtually cyclic subgroup Z that is almost malnormal and relatively quasiconvex by [18]. Thus G is hyperbolic relative to $\mathbb{P}' = \mathbb{P} \cup \{Z\}$ by Theorem 4.2.

Observe that C_1 is maximal infinite cyclic on at least one side, since otherwise there would be a nontrivial splitting of Z as an amalgamated free product over C_1. We equip the (relatively quasiconvex) vertex groups with their induced peripheral structures. Note that C_1 is maximal parabolic on at least one side and so G is locally relatively quasiconvex relative to $\mathbb{P}'$ by Theorem 3.1. Finally, by Theorem 4.7, any subgroup H is quasiconvex relative to the original peripheral structure $\mathbb{P}$ because intersections between H and conjugates of Z are quasiconvex relative to $\mathbb{P}$.

When C_1 is infinite parabolic, we will first produce a new splitting before verifying local relative quasiconvexity.

Suppose $G = A *_{C_1} B$. Let D_a, D_b be the maximal parabolic subgroups of A, B containing C_1, and refine the splitting to

$$A *_{D_a} (D_a *_{C_1} D_b) *_{D_b} B.$$

The two outer splittings are along a parabolic that is maximal on the outside vertex group. The inner vertex group $D_a *_{C_1} D_b$ is a single parabolic subgroup of G.

Indeed, since C_1 is infinite, $D_a \supset C_1 \subset D_b$ must all lie in the same parabolic subgroup of G. It is obvious that $D_a *_{C_1} D_b$ is locally relatively quasiconvex with respect to its induced peripheral structure since this subgroup is itself parabolic in G. Hence $(D_a *_{C_1} D_b) *_{D_b} B$ is locally relatively quasiconvex by Theorem 3.1, so $G = A *_{D_a} ((D_a *_{C_1} D_b) *_{D_b} B)$ is locally relatively quasiconvex by Theorem 3.1.

When $G \cong A *_{C_1^t = C_2}$, let M_i be the maximal parabolic subgroup of G containing C_i. There are two subsubcases as follows.

[$t \in M_1$] Then $C_2 \leq M_1$ and we revise the splitting to $G \cong A *_{D_1} M_1$, where $D_1 = M_1 \cap A$. In this splitting, the edge group is maximal parabolic at $D_1 \subset A$ and M_1 is parabolic.

[$t \notin M_1$] Let D_i denote the maximal parabolic subgroup of A containing C_i. Observe that $\{D_1, D_2\}$ is almost malnormal since $D_i = M_i \cap A$. We revise the HNN extension to

$$(D_1^t *_{C_1^t = C_2} A) *_{D_1^t = D_1},$$

where the conjugated copies of D_1 in the HNN extension embed in the first and second factor of the AFP.

In both cases, the local relative quasiconvexity of G now holds by Theorem 3.1 as before. □

4. Relative Quasiconvexity in Graphs of Groups

Gersten [7] and then Bowditch [3] showed that a hyperbolic group G is hyperbolic relative to an almost malnormal quasiconvex subgroup. Generalizing work of Martínez-Pedroza [14], Yang [22] introduced and characterized a class of parabolically extended structures for countable relatively hyperbolic groups. We use his results to generalize our previous results. The following structure was defined in [22] for countable groups.

DEFINITION 4.1 (Extended Peripheral Structure). A *peripheral structure* consists of a finite collection $\mathbb{P}$ of subgroups of a group G. Each element $P \in \mathbb{P}$ is a *peripheral subgroup* of G. The peripheral structure $\mathbb{E} = \{E_j\}_{j \in J}$ is said to *extend* $\mathbb{P} = \{P_i\}_{i \in I}$ if for each $i \in I$ there exists a $j \in J$ such that $P_i \subseteq E_j$. For $E \in \mathbb{E}$, we let $\mathbb{P}_E = \{P_i : P_i \subseteq E, P_i \in \mathbb{P}, i \in I\}$.

We will use the following result of Yang [22].

THEOREM 4.2 (Hyperbolicity of Extended Peripheral Structure). *Let G be hyperbolic relative to $\mathbb{P}$ and let the peripheral structure $\mathbb{E}$ extend $\mathbb{P}$. Then G is hyperbolic relative to $\mathbb{E}$ if and only if the following hold*:

(1) *$\mathbb{E}$ is almost malnormal*;
(2) *each $E \in \mathbb{E}$ is quasiconvex in G relative to $\mathbb{P}$.*

DEFINITION 4.3 (Total). Let G be hyperbolic relative to $\mathbb{P}$. The subgroup H of G is *total relative to* $\mathbb{P}$ either if $H \cap P^g = P^g$ or if $H \cap P^g$ is finite for each $P \in \mathbb{P}$ and $g \in G$.

The following result is proved in [5].

LEMMA 4.4. *If G is f.g. and hyperbolic relative to $\mathbb{P} = \{P_1, \ldots, P_n\}$ and if each P_i is hyperbolic relative to $\mathbb{H}_i = \{H_{i1}, \ldots, H_{im_i}\}$, then G is hyperbolic relative to $\bigcup_{1 \le i \le n} \mathbb{H}_i$.*

As an application of Theorem 4.2, we now generalize Corollary 1.7 to handle the case where edge groups are quasiconvex and not merely parabolic.

THEOREM 4.5 (Combination along Total, Malnormal, and Quasiconvex Subgroups). (1) *Let G_i be hyperbolic relative to $\mathbb{P}_i$ for $i = 1, 2$. Let $C_i \le G_i$ be almost malnormal, total, and relatively quasiconvex. Let $C_1' \le C_1$. Then $G = G_1 *_{C_1' = C_2} G_2$ is hyperbolic relative to $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 - \{P_2 \in \mathbb{P}_2 : P_2^g \subseteq C_2$ for some $g \in G_2\}$.*

(2) *Let G_1 be hyperbolic relative to $\mathbb{P}$. Let $\{C_1, C_2\}$ be almost malnormal and assume that each C_i is total and relatively quasiconvex. Let $C_1' \le C_1$. Then $G = G_1 *_{C_1' = C_2^t}$ is hyperbolic relative to $\mathbb{P} = \mathbb{P} - \{P_2 \in \mathbb{P}_2 : P_2^g \subseteq C_2$ for some $g \in G_2\}$.*

Proof. (1) For each i, let

$$\mathbb{E}_i = \mathbb{P}_i - \{P \in \mathbb{P}_i : P^g \le C_i \text{ for some } g \in G_i\} \cup \{C_i\}.$$

Without loss of generality, we can assume that $\mathbb{E}_i$ extends $\mathbb{P}_i$, since we can replace an element of $\mathbb{P}_i$ by its conjugate. We now show that G_i is hyperbolic relative to $\mathbb{E}_i$ by verifying the two conditions of Theorem 4.2. First, $\mathbb{E}_i$ is malnormal in G_i because $\mathbb{P}_i$ is almost malnormal and C_i is total and almost malnormal. Second, each element of $\mathbb{E}_i$ is relatively quasiconvex since C_i is relatively quasiconvex (by hypothesis) and since each element of $\mathbb{P}_i$ is relatively quasiconvex (by Remark 2.2).

We now regard each G_i as hyperbolic relative to $\mathbb{E}_i$. Therefore, since the edge group $C_2 = C_1'$ is maximal on one side, it follows from Corollary 1.7 that G is hyperbolic relative to $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2 - \{C_2\}$.

We now apply Lemma 4.4 to show that G is hyperbolic relative to $\mathbb{P}$. We have already shown that G is hyperbolic relative to $\mathbb{E}$. But each element of $\mathbb{E}$ is hyperbolic relative to $\mathbb{P}$ that it contains. Thus, by Lemma 4.4, we obtain the result.

(2) The proof is analogous to the proof of (1). □

The next result can be obtained by induction using Theorem 4.5 or can be proved directly using the same mode of proof.

COROLLARY 4.6. *Let G split as a finite graph of groups. Suppose*

(a) *each G_ν is hyperbolic relative to $\mathbb{P}_\nu$,*
(b) *each G_e is total and relatively quasiconvex in G_ν, and*
(c) *$\{G_e : e$ is attached to $\nu\}$ is almost malnormal in G_ν for each vertex ν.*

Then G is hyperbolic relative to $\bigcup_\nu \mathbb{P}_\nu - \{repeats\}$.

Yang characterized relative quasiconvexity with respect to extensions in [22] as follows.

THEOREM 4.7 (Quasiconvexity in Extended Peripheral Structure). *Let G be hyperbolic relative to $\mathbb{P}$ and relative to $\mathbb{E}$. Suppose that $\mathbb{E}$ extends $\mathbb{P}$. Then the following statements hold.*

(1) *If $H \leq G$ is quasiconvex relative to $\mathbb{P}$, then H is quasiconvex relative to $\mathbb{E}$.*
(2) *Conversely, if $H \leq G$ is quasiconvex relative to $\mathbb{E}$, then H is quasiconvex relative to $\mathbb{P}$ if and only if $H \cap E^g$ is quasiconvex relative to $\mathbb{P}$ for all $g \in G$ and $E \in \mathbb{E}$.*

We recall the following observation of Bowditch (see [16, Lemmas 2.7 and 2.9]).

LEMMA 4.8 (G-attachment). *Let G act on a graph K. Let $p, q \in K^0$ and let e be a new edge whose endpoints are p and q. The G-*attachment *of e is the new graph $K' = K \cup Ge$ that consists of the union of K and copies ge of e attached at gp and gq for any $g \in G$. Note that K' is G-cocompact/fine/hyperbolic if K is.*

In the following lemma we prove that, when a relatively hyperbolic group G splits, relative quasiconvexity of vertex groups is equivalent to relative quasiconvexity of the edge groups.

LEMMA 4.9 (Quasiconvex Edges $\Leftrightarrow$ Quasiconvex Vertices). *Let G be hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups whose vertex groups and edge groups are finitely generated. Then the edge groups are quasiconvex relative to $\mathbb{P}$ if and only if the vertex groups are quasiconvex relative to $\mathbb{P}$.*

Proof. If the vertex groups are quasiconvex relative to $\mathbb{P}$ then so are the edge groups, since relative quasiconvexity is preserved by intersection (see [10; 15]) in the f.g. group G. Assume the edge groups are quasiconvex relative to $\mathbb{P}$. Let K be a $(G;\mathbb{P})$-graph and let T be the Bass–Serre tree for G. Let $f: K \to T$ be a G-equivariant map that sends vertices to vertices and edges to geodesics. Subdivide K and T, so that each edge is the union of two length-$\frac{1}{2}$ *half-edges*. Let ν be a vertex in T. It suffices to find a G_ν-cocompact quasiconvex subgraph L of K.

Let $\{e_1, \dots, e_m\}$ be representatives of the G_ν-orbits of half-edges attached to ν. Let ω_i be the other vertex of e_i for $1 \leq i \leq m$. Since each $G_{\omega_i} = G_{e_i}$ is f.g. by hypothesis, we can perform finitely many G_{ω_i}-attachments of arcs so that the preimage of ω_i is connected for each i. This leads to finitely many G-attachments to K to obtain a new fine hyperbolic graph K'. By mapping the newly attached edges to their associated vertices in T, we thus obtain a G-equivariant map $f': K' \to T$ such that $M_i' = f'^{-1}(\omega_i)$ is connected and G_{ω_i}-cocompact for each i.

Consider $L' = f'^{-1}(N_{1/2}(\nu))$, where $N_{1/2}(\nu)$ is the closed $\frac{1}{2}$-neighborhood of ν. To see that L' is connected, consider a path σ in K' between distinct components of L'. Moreover, choose σ so that its image in T is minimal among all such choices. Then σ must leave and enter L' through the same $g_\nu M_i'$, which is connected by construction.

We now show that L' is quasiconvex. Consider a geodesic γ that intersects L' exactly at its endpoints. As before, the endpoints of γ lie in the same $g_\nu M_i'$. Since

$g_\nu M_i'$ is κ_i-quasiconvex for some κ_i, we see that γ lies in κ-neighborhood of $g_\nu M_i'$ and hence in the κ-neighborhood of L'. $\square$

LEMMA 4.10 (Total Edges $\Leftrightarrow$ Total Vertices). *Let G be hyperbolic relative to $\mathbb{P}$. Let G act on a tree T. For each $P \in \mathbb{P}$, let T_P be a minimal P-subtree. Assume that no T_P has a finite edge stabilizer in the P-action. Then edge groups of T are total in G if and only if vertex groups are total in G.*

Proof. The intersection of two total subgroups is total. Therefore, if the vertex groups are total then the edge groups are also total. We now assume that the edge groups are total. Let G_ν be a vertex group and $P \in \mathbb{P}$ such that $P^g \cap G_\nu$ is infinite for some $g \in G$. If $|P^g \cap G_e| = \infty$ for some edge e attached to ν, then $P \subseteq G_e$; thus $P \subseteq G_e \subseteq G_\nu$. Now suppose that $|P^g \cap G_e| < \infty$ for each e attached to ν. If $P^g \nsubseteq G_\nu$ then the action of P^g on gT violates our hypothesis. $\square$

REMARK 4.11. Suppose G is f.g. and hyperbolic relative to $\mathbb{P}$. Let $P \in \mathbb{P}$ such that $P = A *_C B$ $[P = A *_{C=C'^t}]$ where C is a finite group. Since P is hyperbolic relative to $\{A, B\}$ $[\{A\}]$, it follows from Lemma 4.4 that G is hyperbolic relative to $\mathbb{P}' = \mathbb{P} - \{P\} \cup \{A, B\}$ $[\mathbb{P}' = \mathbb{P} - \{P\} \cup \{A\}]$.

We now describe a more general criterion for relative quasiconvexity which is proven by combining Corollary 2.7 with Theorem 4.7.

THEOREM 4.12. *Let G be f.g. and hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups. Suppose*

(a) *each G_e is total in G,*
(b) *each G_e is relatively quasiconvex in G, and*
(c) *$\{G_e : e$ is attached to $\nu\}$ is almost malnormal in G_ν for each vertex ν.*

Let $H \le G$ be a tamely generated subgroup of G. Then H is relatively quasiconvex in G.

TECHNICAL REMARK. By splitting certain elements of $\mathbb{P}$ to obtain $\mathbb{P}'$ as in Remark 4.11, we can assume that (i) G is hyperbolic relative to $\mathbb{P}'$ and (ii) each G_ν is hyperbolic relative to the conjugates of elements of $\mathbb{P}'$ that it contains.

Proof of Theorem 4.12. For any $P \in \mathbb{P}$, if the action of P on a minimal subtree T_P of the Bass–Serre tree T yields a finite graph Γ of groups some of whose edge groups are finite, then by Remark 4.11 we can replace $\mathbb{P}$ by the groups that complement these finite edge groups (i.e., the fundamental groups of the subgraphs obtained by deleting these edges from Γ). Therefore, G is hyperbolic relative to $\mathbb{P}'$.

No $P \in \mathbb{P}'$ has a nontrivial induced splitting as a graph of groups with a finite edge group. The edge groups are total relative to $\mathbb{P}'$ since they are total relative to $\mathbb{P}$. Hence by Lemma 4.10 the vertex groups are total in G relative to $\mathbb{P}'$. By Lemma 4.9, each vertex group G_ν is relatively quasiconvex in G relative to $\mathbb{P}$; therefore, by Theorem 4.7, each G_ν is quasiconvex in G relative to $\mathbb{P}'$. Thus G_ν

has an induced relatively hyperbolic structure $\mathbb{P}'_\nu$ as in Remark 2.2. By totality of G_ν, we can assume each element of $\mathbb{P}_n u'$ is a conjugate of an element of $\mathbb{P}'$. As usual, we may omit the finite subgroups in $\mathbb{P}'_\nu$.

Step 1. We now extend the peripheral structure of each G_ν from $\mathbb{P}_n u'$ to $\mathbb{E}_\nu$, where

$$\mathbb{E}_\nu = \{G_e : e \text{ is attached to } \nu\} \cup \{P \in \mathbb{P}'_\nu : P^g \nleq G_e \text{ for any } g \in G_\nu\}.$$

Almost malnormality of $\mathbb{E}_\nu$ follows from condition (c) and the totality of the edge groups in their vertex groups, which in turn follows from the totality of the edge groups in G. The relative quasiconvexity of new elements G_e is condition (b). Hence G_ν is hyperbolic relative to $\mathbb{E}_\nu$ by Theorem 4.7.

Step 2. For each $\tilde{\nu}$ in the Bass–Serre tree, its H-stabilizer $H_{\tilde{\nu}}$ lies in $G_{\tilde{\nu}}$, which we identify (by a conjugacy isomorphism) with the chosen vertex stabilizer G_ν in the graph of group decomposition. Then $H_{\tilde{\nu}}$ is quasiconvex in G_ν relative to $\mathbb{E}_\nu$ for each ν by Theorem 4.7, since $\mathbb{E}_\nu$ extends $\mathbb{P}_n u'$ and each $H_{\tilde{\nu}}$ is quasiconvex in G_ν relative to $\mathbb{P}'_\nu$. Therefore, H is quasiconvex relative to $\bigcup \mathbb{E}_\nu$ by Corollary 2.7.

Step 3. H is quasiconvex relative to $\mathbb{P}' = \bigcup \mathbb{P}'_\nu$. Since $\bigcup \mathbb{E}_\nu$ extends $\mathbb{P} = \bigcup \mathbb{P}'_\nu$ by Theorem 4.7, it suffices to show that $H \cap K^g$ is quasiconvex relative to $\mathbb{P}'$ for all $K \in \bigcup \mathbb{E}_\nu$ and $g \in G$. There are two cases.

Case 1: $K \in \mathbb{P}'_\nu$ for some ν. Now $H \cap K^g$ is a parabolic subgroup of G relative to $\mathbb{P}'$ and is thus quasiconvex relative to $\mathbb{P}'$.

Case 2: $K = G_e$ for some e attached to some ν. The group K is relatively quasiconvex in G_ν; therefore, by Remark 2.2, K^g is also relatively quasiconvex but in $G_{g\nu}$. Now, since $K^g \cap H = K^g \cap H_{g\nu}$ and K^g and $H_{g\nu}$ are both relatively quasiconvex in $G_{g\nu}$, the group $K^g \cap H$ is relatively quasiconvex in $G_{g\nu}$. Since (by Lemma 4.9) $G_{g\nu}$ is quasiconvex relative to $\mathbb{P}'$, Lemma 2.3 implies that $K^g \cap H$ is quasiconvex relative to $\mathbb{P}'$.

Now H is quasiconvex relative to $\mathbb{P}$ by Theorem 4.7, since $\mathbb{P}$ extends $\mathbb{P}'$. $\square$

The following result strengthens Theorem 4.12 by relaxing condition (c).

THEOREM 4.13 (Quasiconvexity Criterion for Relatively Hyperbolic Groups That Split). *Let G be f.g. and hyperbolic relative to $\mathbb{P}$ such that G splits as a finite graph of groups. Suppose*

(a) *each G_e is total in G,*
(b) *each G_e is relatively quasiconvex in G, and*
(c) *each G_e is almost malnormal in G.*

Let $H \leq G$ be tamely generated. Then H is relatively quasiconvex in G.

REMARK 4.14. By Lemma 2.3 and Remark 2.4, condition (b) is equivalent to requiring that each G_e be quasiconvex in G_ν. Also, we can replace condition (a) by requiring G_e to be total in G_ν.

Proof of Theorem 4.13. We prove the result by induction on the number of edges of the graph of groups Γ. The base case where Γ has no edge is contained in the hypothesis. Suppose that Γ has at least one edge e (regarded as an open edge). If e is nonseparating, then $G = A *_{C^t=D}$, where A is the graph of groups over $\Gamma - e$ and where C, D are the two images of G_e. Condition (c) ensures that $\{C, D\}$ is almost malnormal in A; therefore, by induction, the various nontrivial intersections $H \cap A^g$ are relatively quasiconvex in A^g and so H is relatively quasiconvex in G by Theorem 4.12. A similar argument concludes the separating case. $\square$

COROLLARY 4.15. *Let G be f.g. and hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups. Assume that*:

(a) *each G_v is locally relatively quasiconvex*;
(b) *each G_e is Noetherian, total, and relatively quasiconvex in G*; *and*
(c) *each G_e is almost malnormal in G.*

Then G is locally relatively quasiconvex relative to $\mathbb{P}$.

THEOREM 4.16. *Let G be hyperbolic relative to $\mathbb{P}$. Suppose G splits as a graph Γ of groups with relatively quasiconvex edge groups. Suppose Γ is bipartite with $\Gamma^0 = V \sqcup U$, where each edge joins vertices of V and U. Suppose that each G_v is maximal parabolic for $v \in V$ and that for each $P \in \mathbb{P}$, there is at most one v with P conjugate to G_v. Let $H \leq G$ be tamely generated. Then H is quasiconvex relative to $\mathbb{P}$.*

The scenario of Theorem 4.16 arises from the JSJ decomposition of a compact aspherical manifold M. The manifold M decomposes as a bipartite graph Γ of spaces with $\Gamma^0 = U \sqcup V$. The submanifold M_v is hyperbolic for each $v \in V$, and M_u is a graph manifold for each $u \in U$. The edges of Γ correspond to the "transitional tori" between these hyperbolic and complementary graph manifold parts. Some of the graph manifolds are complex but others are simpler Seifert fibered spaces—in the simplest cases, thickened tori between adjacent hyperbolic parts or I-bundles over Klein bottles where a hyperbolic part terminates. Hence $\pi_1 M$ decomposes accordingly as a graph Γ of groups, and $\pi_1 M$ is hyperbolic relative to $\{\pi_1 M_u : u \in U\}$ by Theorem 1.4 or, indeed, Corollary 1.5.

Proof of Theorem 4.16. Let K_o be a fine hyperbolic graph for G. Each vertex group is quasiconvex in G by Lemma 4.9. So for each $u \in U$ let K_u be a G_u-quasiconvex subgraph, and in this way we obtain finite hyperbolic G_u-graphs; for $v \in V$, we let K_v be a singleton. We apply the construction in the proof of Theorem 1.4 to obtain a fine hyperbolic G-graph K and quotient $\bar{K}$. Note that the parabolic trees are i-pods. We form the H-cocompact quasiconvex subgraph L by combining H_ω-cocompact quasiconvex subgraphs K_ω as in the proof of Theorem 2.6. $\square$

THEOREM 4.17. *Let G be f.g. and hyperbolic relative to $\mathbb{P}$. Suppose G splits as a graph Γ of groups with relatively quasiconvex edge groups. Suppose that Γ is bipartite with $\Gamma^0 = V \sqcup U$ and that each edge joins vertices of V and U. Suppose each G_v is almost malnormal and total in G for $v \in V$. Let $H \leq G$ be tamely generated. Then H is quasiconvex relative to $\mathbb{P}$.*

Theorem 4.17 covers the case where edge groups are almost malnormal on both sides, since we can subdivide to put barycenters of edges in V.

Another special case where Theorem 4.17 applies is when $G = G_1 *_{C_1'=C_2} G_2$ is hyperbolic relative to $\mathbb{P}$ and $C_2 \leq G_2$ is total and relatively quasiconvex in G as well as almost malnormal in G_2.

Proof of Theorem 4.17. Following the Technical Remark preceding the proof of Theorem 4.12, by splitting certain elements of $\mathbb{P}$ to obtain $\mathbb{P}'$ (as in Remark 4.11) we can assume that G is hyperbolic relative to $\mathbb{P}'$, where each $P' \in \mathbb{P}'$ is elliptic with respect to the action of G on the Bass–Serre tree T. Since $\mathbb{P}$ extends $\mathbb{P}'$ and since each $G_v \cap P^g$ is conjugate to an element of $\mathbb{P}'$, we see that each G_v is quasiconvex in G relative to $\mathbb{P}'$ by Theorem 4.7; moreover, since elements of $\mathbb{P}'$ are vertex groups of elements of $\mathbb{P}$, each G_v is total relative to $\mathbb{P}'$. Therefore, each G_v is hyperbolic relative to a collection $\mathbb{P}_v'$ of conjugates of elements of $\mathbb{P}'$.

We argue by induction on the number of edges of Γ. If Γ has no edge, the result is contained in the hypothesis. Suppose Γ has at least one edge e. If e is separating and $\Gamma = \Gamma_1 \sqcup e \sqcup \Gamma_2$, where e attaches $v \in \Gamma_1^0$ to $u \in \Gamma_2^0$, then $G = G_1 *_{G_e} G_2$ (here $G_i = \pi_1(\Gamma_i)$). Each G_e is the intersection of vertex groups and hence is quasiconvex relative to $\mathbb{P}'$. By Lemma 4.9, the groups G_1 and G_2 are quasiconvex in G relative to $\mathbb{P}'$. Thus G_i is hyperbolic relative to $\mathbb{P}_i'$ by Remark 2.2.

Observe that T contains subtrees T_1 and T_2, which are the Bass–Serre trees of Γ_1 and Γ_2, and that $T - G\tilde{e} = \{gT_1 \cup gT_2 : g \in G\}$. The Bass–Serre tree $\bar{T}$ of $G_1 *_{G_e} G_2$ is the quotient of T obtained by identifying each gT_i to a vertex.

Since H is relatively finitely generated, there is a finite graph of groups Γ_H for H and a map $\Gamma_H \to \Gamma$. Removing the edges mapping to e from Γ_H, we obtain a collection of finitely many graphs of groups—some over Γ_1 and some over Γ_2. Each component of Γ_H corresponds to the stabilizer of some gT_i and is denoted by H_{gT_i}; since that component is a finite graph with relatively quasiconvex vertex stabilizers, we see that each H_{gT_i} is relatively quasiconvex in G_i relative to $\mathbb{P}_i'$ by induction on the number of edges of Γ_H.

We extend the peripheral structure $\mathbb{P}_1'$ of G_1 to $\mathbb{E}_1 = \{G_1\}$. Note that now each H_{gT_1} is quasiconvex in G_1 relative to $\mathbb{E}_1$ by Theorem 4.7. Let

$$\mathbb{E} = \mathbb{E}_1 \cup \mathbb{P}_2' - \{P \in \mathbb{P}_2' : P^g \leq G_e \text{ for some } g \in G_2\}.$$

Observe that $\mathbb{E}$ extends $\mathbb{P}'$. Since G_v is total and quasiconvex in G relative to $\mathbb{P}'$ and since $\mathbb{E}$ extends $\mathbb{P}'$, it follows from Theorem 4.7 that the group G_1 is total and quasiconvex in G relative to $\mathbb{E}$. Therefore, G is hyperbolic relative to $\mathbb{E}$ by Theorem 4.2.

Since G_1 is maximal parabolic in G, by Theorem 4.16 H is quasiconvex in G relative to $\mathbb{E}$. The graph Γ_H shows that H is generated by finitely many hyperbolic elements and vertex stabilizers $H_{g\bar{T}_i}$, and each $H_{g\bar{T}_i} = H_{gT_i}$—which, as we explained previously, is relatively quasiconvex in G_i.

We now show that H is quasiconvex relative to $\mathbb{P}'$ and therefore relative to $\mathbb{P}$ by Theorem 4.7. Since $\mathbb{E}$ extends $\mathbb{P}'$, by Theorem 4.7 it suffices to show that $H \cap E^g$ is quasiconvex relative to $\mathbb{P}'$ for all $E \in \mathbb{E}$ and $g \in G$. There are two cases.

Case 1: $E \in \mathbb{P}_2'$. Then $H \cap E^g$ is a parabolic subgroup of G relative to $\mathbb{P}'$ and is thus quasiconvex relative to $\mathbb{P}'$.

Case 2: $E = G_1$. Then $H \cap E^g$ is quasiconvex relative to $\mathbb{P}_1'$ because $(H \cap E^g) = H_{gT_1}$ is quasiconvex in G_1^g relative to $\mathbb{E}_1^g = \{G_1^g\}$. Since $E^g = G_1^g$ is quasiconvex relative to $\mathbb{P}'$, Lemma 2.3 implies that $H \cap E^g$ is quasiconvex relative to $\mathbb{P}'$.

Now assume that e is nonseparating. Let $u \in U$ and $v \in V$ be the endpoints of e. Then $G = G_1 *_{C^t = D}$, where G_1 is the graph of groups over $\Gamma - e$, and C and D are the images of G_e in G_v and G_u, respectively. We first reduce the peripheral structure of G from $\mathbb{P}$ to $\mathbb{P}'$, and then we extend from $\mathbb{P}'$ to $\mathbb{E}$ with

$$\mathbb{E} = \{G_v\} \cup \mathbb{P}' - \{P \in \mathbb{P}' : P^g \leq G_v \text{ for some } g \in G\}.$$

By Theorem 4.2, G is hyperbolic relative to $\mathbb{E}$ since G_v is almost malnormal, total, and quasiconvex relative to $\mathbb{P}$. The argument follows, as in the separating case, by induction and Theorem 4.16. □

Theorem 4.13 suggests the following criterion for relative quasiconvexity.

Conjecture 4.18. *Let G be hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups with f.g. relatively quasiconvex edge groups. Suppose $H \leq G$ is tamely generated such that each H_v is f.g. for each v in the Bass–Serre tree. Then H is relatively quasiconvex in G.*

When the edge groups are separable in G, there is a finite index subgroup G' whose splitting has relatively malnormal edge groups (see e.g. [11; 9]). Consequently, if moreover the edge groups of G are total, then the induced splitting of G' satisfies the criterion of Theorem 4.13 and we see that Conjecture 4.18 holds in this case. In particular, Conjecture 4.18 holds when G is virtually special and hyperbolic relative to virtually abelian subgroups—provided that edge groups are also total. We suspect the totalness assumption can be dropped altogether.

As a closing thought, consider a hyperbolic 3-manifold M virtually having a malnormal quasiconvex hierarchy (conjecturally all closed M). Theorem 4.13 suggests an alternate approach to the tameness theorem, which could be re-proved by verifying the following statement.

> *If the intersection of a f.g. H with a malnormal quasiconvex edge group is infinitely generated, then H is a virtual fiber.*

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