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Quantum Ring of Singularity $X^p + XY^q$

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1. Introduction

Let (X, x) be an isolated complete intersection singularity of dimension $N-1$. This means that X is isomorphic to the fiber $(f^{-1}(0), 0)$ of an analytic map-germ $f: (\mathbb{C}^{N+k-1}, 0) \to (\mathbb{C}^k, 0)$ and that $x \in X$ is an isolated singular point of X. In particular, if $k = 1$ then (X, x) is called a *hypersurface* singularity. The study of the singularity was initiated by H. Whitney and R. Thom and later developed by V. Arnold, K. Saito, and many other mathematicians from the 1960s and 1980s (see [AGV; He; S; ST]). The classification problem is the central topic in singularity theory. Many geometric and topological invariants were introduced to describe the behavior of the singularity—for example, the Milnor ring, intersection matrix, Gauss–Manin system, and periodic map. Singularity theory has tight connections with many fields of mathematics, including differential equations, function theory, and symplectic geometry.

The papers [FJR1; FJR2; FJR3] construct a quantum theory for a hypersurface singularity given by a nondegenerate quasi-homogeneous polynomial W. The starting point of this research is Witten's work [W] that seeks to generalize the Witten–Kontsevich theorem to the moduli problem of r-spin curves. Unlike in the r-spin case, in which the Witten equation has only trivial solution, in the general W case (e.g., the D_n and E_7 cases) the Witten equation may have nontrivial solutions that cannot be ignored in the construction of the virtual cycle $[\mathcal{W}_{g,k}]^{\mathrm{vir}}$. The Witten equation is defined on an orbifold curve and has the following form:

$$\bar{\partial} u_i + \frac{\overline{\partial W}}{\partial u_i} = 0,$$

where the u_i are sections of appropriate orbifold line bundles.

The Witten equation comes from the study of the Landau–Ginzburg (LG) model in supersymmetric quantum field theory, which can be viewed as a geometrical realization of the $N = 2$ superconformal algebra. The other known model is the nonlinear sigma model corresponding to Gromov–Witten theory. In the LG model, the Lagrangian is totally determined by its superpotential—for instance, a quasi-homogeneous polynomial. There are two possible ways of deriving topological

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field theories by twisting the LG model: the so-called LG A model and LG B model. The LG B model has been studied extensively in physics and mathematics. The mathematical theory of the LG A model is just the quantum singularity theory constructed by Fan–Jarvis–Ruan. As pointed out in [IV], the more appropriate model is the orbifold LG model, which should be "identical" to a Calabi–Yau (CY) sigma model by CY–LG correspondence. In fact, the state space of the quantum singularity theory is a space of the dual forms of Lefschetz thimbles orbifolded by an admissible symmetric group G of the polynomial W.

Once we determine the state space and obtain the virtual cycle $[\mathcal{W}_{g,k}]^{\mathrm{vir}}$, we can build up the quantum invariants for the singularity. For instance, we can define the correlators $\langle \tau_{l_1}(\alpha_{i_1}), \ldots, \tau_{l_n}(\alpha_{i_n}) \rangle_g^{W,G}$ for α_{i_j} in the state space $\mathcal{H}_{W,G}$ and the cohomological field theory. All the correlators are assembled into a generating function

$$\mathcal{D}_{W,G} = \exp\Bigl(\sum_{g\geq 0} \hbar^{2g-2} \mathcal{F}_{g,W,G}\Bigr),$$

where $\mathcal{F}_{g,W,G}$ is the genus-g generating function:

$$\mathcal{F}_{g,W,G} = \sum_{k\geq 0} \langle \tau_{l_1}(\alpha_{i_1}), \ldots, \tau_{l_n}(\alpha_{i_n}) \rangle_g^{W,G} \frac{t_{i_1}^{l_1} \cdots t_{i_s}^{l_n}}{n!}.$$

So computing those quantum invariants is critical for understanding the singularity. For an invertible singularity W, Berglund and Hübsch [BH] use Landau–Ginzburg theory to construct the mirror dual singularity W^T (see also [IV; K1; K2; K3]). This has now been generalized by Krawitz [Kr] to a mirror construction containing the symmetry group. The mirror symmetry phenomenon is also described in [C] from the toric point of view. At the Frobenius algebra level, the modern language of mirror symmetry asserts that the quantum ring (also called the FJRW ring) in the A model of the singularity $(W, G_{\max})$ should be isomorphic to the Milnor ring in the B model of the dual singularity W^T. We also have stronger conjectures that relate the generating function $\mathcal{D}_{W,G}$ to Givental's formal generating function. Let us say more about this conjecture.

The genus-g Gromov–Witten (GW) potential function of one point is

$$\begin{aligned}
\mathcal{F}_g^{\mathrm{pt}} &:= \sum_{k\geq 0} \frac{1}{k!} \sum_{d_1,\ldots,d_k} \langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g t_{d_1} \cdots t_{d_k} \\
&= \sum_{k\geq 0} \frac{1}{k!} \sum_{d_1,\ldots,d_k} \int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{d_1} \cdots \psi_k^{d_k} t_{d_1} \cdots t_{d_k}.
\end{aligned}$$

The Witten–Kontsevich generating function is $\mathcal{D}^{\mathrm{pt}} = \exp\bigl(\sum_g \hbar^{g-1} \mathcal{F}_g^{\mathrm{pt}}\bigr)$.

Let A be a finite index set having a distinguished element $\mathbf{1}$. Suppose that the $\mathbb{Q}$-vector space $\mathrm{Vect}(A)$ generated by A is attached with a nondegenerate symmetric bilinear form η. The formal genus-0 GW potential is a power series $\mathcal{F}_0$ in variables $t_{d,l}$ ($d \in \mathbb{N}$, $l \in A$):

$$\mathcal{F}_0 = \sum_{k\geq 0} \frac{1}{k!} \sum_{\substack{d_1,\dots,d_k\\ l_1,\dots,l_k}} \langle \tau_{d_1,l_1}\cdots\tau_{d_k,l_k}\rangle_0 t_{d_1,l_1}\cdots t_{d_k,l_k};$$

this series satisfies the string equation (SE), the dilaton equation (DE), and the topological recursion equation (TRR).

Let $\mathcal{F}_{\mathrm{pr}}$ be the primary potential of $\mathcal{F}$, where $t_{d_k,l_k} = 0$ for $d_k > 0$. Then $\mathcal{F}_{\mathrm{pr}}$ satisfies the WDVV (Witten–Dijkgraaf–Verlinde–Verlinde) equation and forms a Frobenius manifold. We call $\mathcal{F}_0$ *semi-simple of rank* μ if $|A| = \mu$ and the algebra structure on $\mathrm{Vect}(A)$ is semi-simple for generic $t_{0,l}$. Givental [Gi] found that there is a transitive action of the *twisted loop group* on the set of all semi-simple genus-0 GW potentials of rank μ. So given a semi-simple potential $\mathcal{F}_0$ of rank μ, there exists a group element R taking k copies $\mathcal{F}_0^{\mathrm{pt}} \oplus \cdots \oplus \mathcal{F}_0^{\mathrm{pt}}$ to $\mathcal{F}_0$.

Using a method to quantize the quadratic functions (see [Gi]), the group element R can be quantized to obtain an element $\hat{R}(\hbar)$ in *Givental's group*; here $\hat{R}(\hbar)$ acts on the k copies of the tau-functions $\mathcal{D}^{\mathrm{pt}} \oplus \cdots \oplus \mathcal{D}^{\mathrm{pt}}$ to get a power series $\mathcal{D}_{\mathrm{Giv}}$ in $\hbar$, where $\mathcal{D}_{\mathrm{Giv}}$ can be written in the form $\mathcal{D}_{\mathrm{Giv}} = \exp\bigl(\sum_g \hbar^{g-1}\mathcal{F}_g\bigr)$. If $\mathcal{D}_{\mathrm{Giv}}$ is required to satisfy a homogeneity condition, then it is uniquely defined and satisfies the SE, DE, and TRR equations as well as the Virasoro constraints.

Given a genus-0 GW potential of a projective manifold that is semi-simple, Givental conjectured that the total GW potential is the same to $\mathcal{D}_{\mathrm{Giv}}$ constructed from the genus-0 GW potential. We have the same question in the quantum singularity theory. Let $\mathcal{D}_{W,G}$ be the tau-function in our LG A model, and let $\mathcal{D}_{0,W,G}$ be the genus-0 tau-function. If the Frobenius manifold induced by $\mathcal{D}_{0,W,G}$ is semi-simple, then we can get the formal tau-function $\mathcal{D}_{\mathrm{Giv},W,G}$.

CONJECTURE 1.1. $\mathcal{D}_{W,G} = \mathcal{D}_{\mathrm{Giv},W,G}$.

This should be true by Teleman's theorem [Te] if we can show that $\mathcal{D}_{0,W,G}$ is semi-simple. To prove the semi-simple property, it is natural to show that the Frobenius manifold associated to the singularity W/G in the A model is isomorphic to Saito's Frobenius manifold of the dual singularity W^T in the B model (which is easily proved to be semi-simple). If the symmetry group G is chosen suitably, then we should have the following problem.

CONJECTURE 1.2. *$\mathcal{D}_{W,G}$ is identical to $\mathcal{D}_{\mathrm{Giv},W^T}$ under some mirror transformation.*

Conversely, since the construction of the quantum theory depends on choosing the admissible subgroup G such that $\langle J\rangle \leq G \leq G_{\max}$ (see the definitions in Section 2), we should not expect a mirror correspondence from the LG A model of the dual singularity W^T with the trivial symmetry group to the LG B model of the singularity W. However, a mirror correspondence between the orbifold LG A model and the orbifold LG B model has been given in [Kr].

So-called ADE singularities are simple singularities (according to Arnold's classification) and have special properties. For instance, ADE singularities are self-mirroring; this was shown in [FJR2], where the authors calculated the quantum ring structure of the ADE singularities. Furthermore, by computing the basic four-point correlators (via the WDVV equation and the reconstruction theorems), these authors proved Conjecture 1.2 and thus (via the conclusions in [GiM]) the generalized Witten conjecture for DE cases.

In this paper we prove the Frobenius algebra level of mirror symmetry for chain-type singularities in two variables. This is the first example beyond the ADE case. The Frobenius algebra–level mirror symmetry is also proved for unimodel and bimodel singularities in [Kr+] and in [Ac] for loop-type singularities in two variables. Krawitz [Kr] generalized this symmetry to all invertible singularities.

To prove Conjecture 1.2 for more general singularities, one must compare the Frobenius manifolds. In the B side, it is difficult to calculate the primary potential of Saito's Frobenius manifold associated to a singularity other than ADE singularities. On the other hand, Noumi [N] has considered the following type of singularities:

(i) $x_1^{p_1} + x_2^{p_2} + \cdots + x_N^{p_N}$,
(ii) $x_1^{p_1} + x_1 x_2^{p_2} + x_3^{p_3} + \cdots + x_N^{p_N}$.

Noumi considers the Gauss–Manin system associated to these two singularities. An important fact is that the flat coordinates of the Frobenius manifold and the formula of primary potential is given in [NY]. This gives one possible way to compare the Frobeniu manifolds of both mirror sides.

Since the Frobenius structure of singularities (i) and (ii) in either side is the tensor product of the Frobenius strucures of the A_r singularity and the singularity $x^p + xy^q$, it is natural for us to compute only the primary potential functions of the singularity $x^p + xy^q$ in the A model and then to compare it with the Noumi–Yamada computation in the B model. By the WDVV equation, one may show that the primary potential depends on the metric, three-point correlators, and some basic four-point correlators. We need only compare the metric, the three-point correlators, and some four-point correlators in both sides. Computing the quantum invariants of x^p and $x^p + xy^q$ is important because we can take the direct sum of those singularities to form a Calabi–Yau singularity (whose central charge is a positive integer). Once we know the quantum invariants of the Calabi–Yau singularity, then by the CY–LG correspondence it is hoped we will get the Gromov–Witten invariants of the Calabi–Yau hypersurface defined by the CY singularity. Actually, the genus-0 correspondence for the quintic 3-fold has been verified in [ChR].

In this paper we calculate the quantum ring structure of the chain-type singularity in two variables, $W = x^p + xy^q$ for $p, q \geq 2$, and construct an explicit isomorphism to the Milnor ring of the dual singularity $W^T = x^p y + y^q$. We shall proceed as follows. In Section 2 we give a brief description of the Fan–Jarvis–Ruan–Witten (FJRW) theory and list some useful axioms. In Section 3, we discuss the quantum ring structure for the singularity $x^p + xy^q$ when $\gcd(p-1, q) = 1$; in Section 4, we treat the case $\gcd(p-1, q) > 1$.

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2. The Fan–Jarvis–Ruan–Witten Theory

2.1. The Classical Singularity Theory

DEFINITION 2.1. A polynomial $W: \mathbb{C}^N \to \mathbb{C}$ is *quasi-homogeneous* if there are positive integers $d, n_1, \dots, n_N$ such that, for any $\lambda \in \mathbb{C}^*$,

$$W(\lambda^{n_1}x_1, \dots, \lambda^{n_N}x_N) = \lambda^d W(x_1, \dots, x_N).$$

We also define the *weight* (or *charge*) of x_i to be $q_i := n_i/d$. We say W is *nondegenerate* if the choices of weights q_i are unique; note that W has an isolated critical value only at zero. There are many examples of nondegenerate quasi-homogeneous singularities, including all the nondegenerate homogeneous polynomials such as the famous ADE examples.

A classical invariant of the singularity is the *local algebra,* also known as the *Milnor ring* or *Chiral ring* in physics:

$$\mathcal{Q}_W := \mathbb{C}[x_1, \dots, x_N]/\mathrm{Jac}(W);$$

here $\mathrm{Jac}(W)$ is the Jacobian ideal generated by partial derivatives of W. The degree of the monomial makes the local algebra a graded algebra. There is a unique highest-degree element $\det\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right)$ with degree

$$\hat{c}_W = \sum_{i=1}^{N}(1 - 2q_i). \tag{1}$$

We call $\hat{c}_W$ the *central charge* of W. The dimension of the local algebra is called the *Milnor number* and is given by the formula

$$\mu = \prod_{i=1}^{N}\left(\frac{1}{q_i} - 1\right).$$

Let ϕ_i, $i = 0, 1, \dots, \mu - 1$, be a basis of $\mathcal{Q}_W$. We can consider the miniversal deformation $F_t(x) := W + t_0\phi_0 + \cdots + t_{\mu-1}\phi_{\mu-1}$, $t = (t_0, \dots, t_{\mu-1}) \in \mathbb{C}^\mu$. Let S be a small ball centered at the origin of $\mathbb{C}^\mu$. We have the Milnor fibration $F: \mathbb{C}^N \times S \to \mathbb{C} \times S$ given by $(x, t) \to (F_t(x), t)$. Assume that the critical values of F are in $\mathbb{C}_\delta \times S$, where $\mathbb{C}_\delta := \{z \in \mathbb{C} \mid \|z\| < \delta\}$. Let $z_0 \in \partial\mathbb{C}_\delta$; then $F_t^{-1}(z_0) \to t \in S$ is a fiber bundle. We consider the middle homoly bundle over S with fiber $H_{N-1}(F_t^{-1}(z_0), \mathbb{Z})$ for each $t \in S$. For a generic t, $F_t(x)$ is a holomorphic Morse function. A distinguished basis of $H_{N-1}(F_t^{-1}(z_0), \mathbb{Z})$ can be constructed from a system of paths connecting z_0 to the critical values. A system of paths $l_i: [0, 1] \to \mathbb{C}_\delta$ connecting z_0 to critical values z_i is called *distinguished* if

(i) l_i has no self intersection,
(ii) l_i, l_j has no intersection except $l_i(0) = l_j(0) = z_0$, and

(iii) the paths $l_1, \dots, l_\mu$ are numbered in the same order in which they enter the point z_0 (counterclockwise).

For each l_i, we can associate a homology class $\delta_i \in H_{N-1}(F_t^{-1}(z_0), \mathbb{Z})$ as a vanishing cycle along l_i. More precisely, the neighborhood of the critical point of z_i contains a local vanishing cycle. Then δ_i is obtained by transporting the local vanishing cycle to z_0 using the homotopy lifting property. The cycle δ_i is unique up to the homotopy of l_i provided the homotopy does not pass another critical value. Now $\delta_1, \dots, \delta_\mu$ define a distinguished basis of $H_{N-1}(F_t^{-1}(z_0), \mathbb{Z})$. A different choice of the distinguished system of paths gives a different distinguished basis. The transformation relation between two bases is described by the Picard–Lefschetz transformation. The intersection matrix $(\delta_i \circ \delta_j)$ is an invariant of the singuarity and is used to classify that singularity. Another closely related set of objects are *Lefschetz thimbles,* which are the generators of the relative homology classes $H_N(\mathbb{C}^N, F_t^{-1}(z_0), \mathbb{Z})$. The boundary homomorphism ∂ gives an isomorphism $\partial\colon H_N(\mathbb{C}^N, F_t^{-1}(z_0), \mathbb{Z}) \to H_{N-1}(F_t^{-1}(z_0), \mathbb{Z})$. Geometrically, a Lefschetz thimble Δ_i is the union of the vanishing cycles along the path l_i, and we have $\partial \Delta_i = \delta_i$.

Letting the radius δ of $\mathbb{C}_\delta$ go to ∞, we take $z_0 = -\infty$. Let $\mathrm{Re}(F_t)$ be the real part of F_t. The relative homology class becomes $H_N(\mathbb{C}^N, (\mathrm{Re}(F_t))^{-1}(-\infty, -M), \mathbb{Z})$ for $M \gg 0$. We write $\mathrm{Re}(F_t)^{-1}(-\infty, -M)$ more simply as $F_t^{-\infty}$ and write $\mathrm{Re}(F_t)^{-1}(M, +\infty)$ as $F_t^{+\infty}$. Now the Lefschetz thimble Δ_i in $H_N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{Z})$ is canonically determined by a horizontal path from the critical value to $-\infty$. Although the intersection matrix of the vanishing cycles may be degenerate, the intersection pairing between Lefschetz thimbles is nondegenerate:

$$I\colon H_N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{Z}) \otimes H_N(\mathbb{C}^N, F_t^{+\infty}, \mathbb{Z}) \to \mathbb{Z}.$$

This pairing is given by the intersection of the stable manifold and the unstable manifold of the critical point, and it is preserved by parallel transport via the Gauss–Manin connection. Naturally (see [FJR2]) we have the dual pairing

$$\eta_t\colon H^N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{C}) \otimes H^N(\mathbb{C}^N, F_t^{\infty}, \mathbb{C}) \to \mathbb{C}. \tag{2}$$

2.2. *The Quantum Invariants of the Singularity*

Let $G_{\max} := \mathrm{Aut}(W)$ be the *maximal admissible symmetry group* of W consisting of group element $\gamma = (c_1, \dots, c_N) \in (\mathbb{C}^*)^N$ and such that $W(c_1x_1, \dots, c_Nx_N) = W(x_1, \dots, x_N)$. Observe that $G_{\max}$ always contains the subgroup $\langle J \rangle$, where $J := \mathrm{diag}(e^{2\pi i q_1}, \dots, e^{2\pi i q_N})$ is the *exponential grading element.* We can take any subgroup G such that $\langle J \rangle \le G \le G_{\max}$. Using the group G, we can orbifold the space of Lefschetz thimbles. For any $\gamma \in G$, let $\mathbb{C}_\gamma^{N_\gamma}$ be the fixed locus of γ, let N_γ be the complex dimension of the fixed locus, and let $W_\gamma := W|_{\mathbb{C}_\gamma^{N_\gamma}}$ be the restriction. According to [FJR2, Lemma 3.2.1], 0 is the only critical point of W_γ and G is a subgroup of $\mathrm{Aut}(W_\gamma)$. Let $W_\gamma^\infty = \mathrm{Re}(W_\gamma)^{-1}(-\infty, -M)$ for $M \gg 0$.

DEFINITION 2.2. For $\gamma \in G$, the *γ-twisted sector* $\mathcal{H}_\gamma$ is defined to be the G-invariant part of the middle-dimensional relative cohomology for W_γ:

$$\mathcal{H}_\gamma := H^{N_\gamma}(\mathbb{C}_\gamma^{N_\gamma}, W_\gamma^\infty, \mathbb{C})^G. \tag{3}$$

For $\alpha \in \mathcal{H}_\gamma$, we say α is *narrow* if $N_\gamma = 0$ or that α is *broad* if $N_\gamma \neq 0$.

DEFINITION 2.3. Suppose $\gamma = (e^{2\pi i \Theta_1^\gamma}, \ldots, e^{2\pi i \Theta_N^\gamma}) \in G$ for rational numbers $0 \leq \Theta_i^\gamma < 1$. Then the *degree-shifting number* of γ is

$$\iota_\gamma := \sum_i (\Theta_i^\gamma - q_i). \tag{4}$$

For a class $\alpha \in \mathcal{H}_\gamma$, we define the complex degree of α to be

$$\deg_{\mathbb{C}}(\alpha) := N_\gamma/2 + \iota_\gamma. \tag{5}$$

The following result was proved as Proposition 3.2.4 in [FJR2].

PROPOSITION 2.4. *Let $\gamma \in G_{\max}$. Then, for any $\alpha \in \mathcal{H}_\gamma$ and $\beta \in \mathcal{H}_{\gamma^{-1}}$,*

$$\begin{aligned} \iota_\gamma + \iota_{\gamma^{-1}} &= \hat{c}_W - N_\gamma, \\ \deg_{\mathbb{C}}(\alpha) + \deg_{\mathbb{C}}(\beta) &= \hat{c}_W. \end{aligned} \tag{6}$$

DEFINITION 2.5. The *state space* of the singularity W/G is defined to be

$$\mathcal{H}_{W,G} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma.$$

The pairing in $\mathcal{H}_{W,G}$ is defined to be the direct sum of the pairings

$$\langle \cdot, \cdot \rangle_\gamma : \mathcal{H}_\gamma \otimes \mathcal{H}_{\gamma^{-1}} \to \mathbb{C},$$

where $\langle \cdot, \cdot \rangle_\gamma$ is just the pairing $\eta(\cdot, \cdot)$ of the singularity W_γ.

The quantum invariants of the singularity W/G are defined via the construction of the virtual fundamental cycle $[\mathcal{W}_{g,k}(\boldsymbol{\gamma})]^{\mathrm{vir}}$ (or $[\mathcal{W}(\Gamma)]^{\mathrm{vir}}$). Let us briefly describe the properties of these virtual fundamental cycles and some axioms related to our computations in this paper. We consider only the case $G = G_{\max}$.

Recall that an orbicurve $\mathcal{C}$ of genus g and k marked points $p_1, \ldots, p_k$ is a Riemann surface with orbifold structure at each marked point. The isotropy group at each marked point p_i is canonically isomorphic to $\mathbb{Z}/m_i$ for some positive integer m_i. Given a nondegenerate quasi-homogeneous polynomial W, we can define a W-structure on $\mathcal{C}$. Roughly speaking, a W-structure on an orbicurve $\mathcal{C}$ is a choice of N orbifold line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_N$ satisfying some relations defined by the polynomial W (see [FJR2] for details).

Any $\gamma_l = (e^{2\pi i \Theta_1^{\gamma_l}}, \ldots, e^{2\pi i \Theta_N^{\gamma_l}}) \in G_{\max}$ gives an action on each orbifold line bundle $\mathcal{L}_j$ at the marked point p_i. We use $|\mathcal{L}_j|$ to denote the *desingularization* of $\mathcal{L}_j$, which is a line bundle on the coarse curve of $\mathcal{C}$. If a W-structure exists on an orbicurve $\mathcal{C}$, then

$$\deg(|\mathcal{L}_j|) = \left(q_j(2g - 2 + k) - \sum_{l=1}^{k} \Theta_j^{\gamma_l} \right) \in \mathbb{Z}. \tag{7}$$

The orbicurve with W-structure is called *W-orbicurve,* and the stack of stable W-orbicurves forms the moduli space $\mathcal{W}_{g,k}$. For any choice $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_k) \in G_{\max}^k$, we define $\mathcal{W}_{g,k}(\boldsymbol{\gamma}) \subseteq \mathcal{W}_{g,k}$ to be the open and closed substack with orbifold decoration $\boldsymbol{\gamma}$, where γ_i is the canonical generator of the isotropy group at p_i. We call $\boldsymbol{\gamma}$ the *type* of any W-orbicurve in $\mathcal{W}_{g,k}(\boldsymbol{\gamma})$. Note that $\mathcal{W}_{g,k}(\boldsymbol{\gamma})$ is not empty if and only if condition (7) holds. Forgetting the W-structure and the orbifold structure gives a morphism $\mathrm{st}\colon \mathcal{W}_{g,k}(\boldsymbol{\gamma}) \to \overline{\mathcal{M}}_{g,k}$. This morphism plays a role similar to that played by the stabilization morphism of stable maps. The following theorem is proved in [FJR2].

THEOREM 2.6. *For any nondegenerate and quasi-homogeneous polynomial W, the stack $\mathcal{W}_{g,k}$ is a smooth compact orbifold* (*Deligne–Mumford stack*) *with projective coarse moduli. In particular, the morphism* $\mathrm{st}\colon \mathcal{W}_{g,k} \to \overline{\mathcal{M}}_{g,k}$ *is flat, proper, and quasi-finite* (*but not representable*).

Moreover, one can consider the decorated dual graph Γ of a stable W-curve and obtain the moduli space $\mathcal{W}_{g,k}(\Gamma)$, which is a closed substack of $\mathcal{W}_{g,k}(\boldsymbol{\gamma})$. Let $T(\Gamma)$ be the set of tails of the decorated graph Γ and attach an element $\gamma_\tau \in G_{\max}$ to each tail τ. The virtual cycle $[\mathcal{W}(\Gamma)]^{\mathrm{vir}}$ was constructed in [FJR2; FJR3] by studying the Witten equation and its moduli problem. It was proved that the virtual cycle $[\mathcal{W}(\Gamma)]^{\mathrm{vir}}$ satisfies a series of axioms analogous to those in Gromov–Witten theory. We will list some of those axioms after introducing the FJRW correlators.

2.3. *Cohomological Field Theory*

For any homogeneous elements $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k)$ with $\alpha_i \in \mathcal{H}_{\gamma_i}$, the map $\Lambda_{g,k}^W \in \mathrm{Hom}(\mathcal{H}_W^{\otimes k}, H^*(\overline{\mathcal{M}}_{g,k}))$ is defined as

$$\Lambda_{g,k}^W(\boldsymbol{\alpha}) := \frac{|G|^g}{\deg(\mathrm{st})} \, \mathrm{PD}\, \mathrm{st}_*\left([\mathcal{W}_{g,k}(W, \boldsymbol{\gamma})]^{\mathrm{vir}} \cap \prod_{i=1}^{k} \alpha_i \right) \tag{8}$$

and then extends linearly to general elements of $\mathcal{H}_W^{\otimes k}$, where PD is the Poincare duality map.

The following statements were proved in [FJR2].

THEOREM 2.7. *The collection $(\mathcal{H}_W, \langle\cdot,\cdot\rangle^W, \{\Lambda_{g,k}^W\}, \mathbf{e}_1)$ is a cohomological field theory with flat identity. The genus-0 theory defines a Frobenius manifold.*

The quantum invariants of the singularity $W/G_{\max}$ consist of the correlators defined as follows.

DEFINITION 2.8. Let ψ_i be the canonical classes in the tautological ring of $\overline{\mathcal{M}}_{g,k}$. Then the FJRW correlators are defined to be

$$\langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_k}(\alpha_k) \rangle_g^W := \int_{[\overline{\mathcal{M}}_{g,k}]} \Lambda_{g,k}^W(\alpha_1, \ldots, \alpha_k) \prod_{i=1}^{k} \psi_i^{l_i}.$$

We mainly use the following axioms about the virtual cycle. For the proofs, see [FJR3].

THEOREM 2.9. *Let Γ be a decorated W-graph of a genus-g curve with k marked points, and let $\gamma_i \in G$ be the decoration of Γ. Then the virtual cycle $[\mathcal{W}(\Gamma)]^{\mathrm{vir}}$ satisfies the following axioms.*

1. Dimension. *The virtual cycle $[\mathcal{W}(\Gamma)]^{\mathrm{vir}}$ has degree $6g-6+2k-2D$ and lies in $H_r(\mathcal{W}(\Gamma),\mathbb{Q})\otimes\prod_{i=1}^k H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}_{\gamma_i}, W^\infty_{\gamma_i},\mathbb{Q})$. Here $r = 6g-6+2k-2D-\sum_{i=1}^k N_{\gamma_i}$ and*

$$D := -\sum_{i=1}^{N} \operatorname{index}(\mathcal{L}_i) = \hat{c}_W(g-1) + \sum_{j=1}^{k} \iota_{\gamma_j}. \tag{9}$$

Thus $\langle \tau_{l_1}(\alpha_1),\dots,\tau_{l_k}(\alpha_k)\rangle_g^W \neq 0$ implies

$$\sum_{i=1}^{k} \deg_{\mathbb{C}}(\alpha_i) + \sum_{i=1}^{k} l_i = (\hat{c}_W - 3)(1-g) + k. \tag{10}$$

2. Degenerating connected graphs. *Let Γ be a connected, genus-g, stable, decorated W-graph, and let $\tilde{i}\colon \mathcal{W}(\Gamma) \to \mathcal{W}_{g,k}(\boldsymbol{\gamma})$ be the canonical inclusion map. Then*

$$[\mathcal{W}(\Gamma)]^{\mathrm{vir}} = \tilde{i}^*[\mathcal{W}_{g,k}(\boldsymbol{\gamma})]^{\mathrm{vir}}. \tag{11}$$

3. Concavity. *Suppose that all tails of Γ are narrow. If $\pi_*\big(\bigoplus_{i=1}^N \mathcal{L}_i\big) = 0$, then*

$$[\mathcal{W}(\Gamma)]^{\mathrm{vir}} = c_{\mathrm{top}}\Bigg(\bigg(R^1\pi_*\bigoplus_{i=1}^{t}\mathcal{L}_i\bigg)^*\Bigg) \cap [\mathcal{W}(\Gamma)]. \tag{12}$$

4. Index zero. *Suppose $\dim(\mathcal{W}(\Gamma)) = 0$ and let all the decorations on tails be narrow. Let $\pi_*\big(\bigoplus_{i=1}^N \mathcal{L}_i\big)$ and $R^1\pi_*\big(\bigoplus_{i=1}^N \mathcal{L}_i\big)$ be vector bundles of the same rank, and denote the Witten map $\mathcal{D}\colon (x_1,\dots,x_N) \mapsto \big(\overline{\frac{\partial W}{\partial x_1}},\dots,\overline{\frac{\partial W}{\partial x_N}}\big)$. Then*

$$[\mathcal{W}(\Gamma)]^{\mathrm{vir}} = \deg(\mathcal{D})[\mathcal{W}(\Gamma)]. \tag{13}$$

5. Composition. *Given any genus-g decorated stable W-graph Γ with k tails and given any edge e of Γ, let $\hat{\Gamma}$ denote the graph obtained by "cutting" the edge e and replacing it with two unjoined tails τ_+ and τ_- decorated (respectively) with γ_+ and γ_-. Consider the fiber product $F := \mathcal{W}(\hat{\Gamma}) \times_{\overline{\mathcal{M}}(\Gamma)} \mathcal{W}(\Gamma)$ with morphisms $\mathcal{W}(\hat{\Gamma}) \xleftarrow{q} F \xleftarrow{\mathrm{pr}_2} \mathcal{W}(\Gamma)$. We have*

$$\langle[\mathcal{W}(\hat{\Gamma})]^{\mathrm{vir}}\rangle_{\pm} = \frac{1}{\deg(q)} q_*\,\mathrm{pr}_2^*([\mathcal{W}(\Gamma)]^{\mathrm{vir}}), \tag{14}$$

where $\langle\cdot\rangle_\pm$ is the map obtained by contracting via the pairing

$$\langle\cdot,\cdot\rangle\colon H_{N_{\gamma_+}}(\mathbb{C}^N_{\gamma_+}, W^\infty_{\gamma_+},\mathbb{Q}) \otimes H_{N_{\gamma_-}}(\mathbb{C}^N_{\gamma_-}, W^\infty_{\gamma_-},\mathbb{Q}) \to \mathbb{Q}.$$

REMARK 2.10. Let Γ_4 be the graph corresponding to the singular point in $\overline{\mathcal{M}}_{0,4}$. Consider the virtual cycle $\mathcal{W}(\Gamma_4;\alpha_1,\dots,\alpha_4)$ for Γ_4 with decorations $\alpha_1,\dots,\alpha_4$, and let $\langle|\alpha_1,\dots,\alpha_4|\rangle_0^W = \int_{[\Gamma_4]} \Lambda_{0,4}^W(\alpha_1,\dots,\alpha_4)$. Then the composition axiom implies that

$$\langle|\alpha_1,\dots,\alpha_4|\rangle_0^W = \sum_{i,j} \langle\alpha_1,\alpha_2,\alpha_i\rangle_0^W \eta^{i,j} \langle\alpha_j,\alpha_3,\alpha_4\rangle_0^W. \tag{15}$$

2.4. *Quantum Ring* (*Quantum Cohomology Group*) *of the Singularity*

The simplest quantum structure of a singularity is the Frobenius algebra consisting of the state space, the metric, and the quantum multiplication $\bigstar$ given by the genus-0 three-point correlators:

$$\langle\alpha \bigstar \beta,\gamma\rangle = \langle\tau_0(\alpha),\tau_0(\beta),\tau_0(\gamma)\rangle_0^{W,G}. \tag{16}$$

In order to show the mirror symmetry between the LG A model of the quasi-homogenous singularity W and the LG B model of the dual singularity W^T, we must first identify the corresponding Frobenius algebra structures and then compare the Frobenius manifold structure. When these structures are identical, we aim to construct the mirror map between the A model theory (FJRW theory) and the B model theory (Saito–Givental theory).

In the LG B model, the Frobenius algebra is the Milnor ring $\mathcal{Q}_W$ with the residue pairing and the multiplication of the monomials. For $f,g \in \mathcal{Q}_W$, the residue pairing is nondegenerate and defined as

$$\langle f,g\rangle := \operatorname{Res}_{x=0} \frac{fg\, dx_1 \wedge \cdots \wedge dx_N}{\frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_N}} = C\mu. \tag{17}$$

Here $x = (x_1,\dots,x_N)$, μ is the Milnor number, and C is a unique constant that satisfies

$$fg = C \cdot \operatorname{Hessian}(W) \ \text{ modulo } \operatorname{Jac}(W).$$

In the LG A model, the intersection pairing I of the Lefschetz thimbles is dual to η_t in (2). The relative cohomology groups $H^N(\mathbb{C}^N, F_t^{\pm\infty}, \mathbb{C})$ and the pairing η_t can be identified with the space $\Omega^N/dF_t \wedge d\Omega^{N-1}$ via the residue pairing on the deformed Milnor ring $\mathcal{Q}_{F_t}$; see [FJR2, Sec. 5.1] and [Ce]. An explicit isomorphism will be given in [FSh].

3. Quantum Ring of ($W = X^p + XY^q$, $G = G_{\max}$), $\gcd(p-1,q) = 1$

3.1. *Basic Calculation*

Consider the singularity $W = X^p + XY^q$ for $p,q \ge 2$, where $p-1$ and q are coprime. In this case, the group $G = \langle J\rangle \cong \mathbb{Z}/(pq)\mathbb{Z}$. Let $\xi = \exp(\frac{2\pi i}{pq})$; then J acts on $\mathcal{Q}_W\omega$ by (ξ^q, ξ^{p-1}), where $\omega = dX \wedge dY$. We also have

$$q_x = \frac{1}{p},\ q_y = \frac{p-1}{pq}, \quad \hat{c}_W = \frac{2(p-1)(q-1)}{pq}, \quad \Theta_x^J = \frac{1}{p},\ \Theta_y^J = \frac{p-1}{pq}.$$

Let $\Lambda = \{i \mid 1 \le i \le pq - 1,\ p \nmid i\}$. Then the state space is

$$\mathcal{H}_{W,G} = \mathbb{C}\langle y^{q-1}\mathbf{e}_0, \mathbf{e}_k \mid k \in \Lambda\rangle,$$

where $\mathbf{e}_0 := dX \wedge dY \in H^2(\mathbb{C}^2_{J^0}, W^\infty_{J^0}, \mathbb{Q})$ and $\mathbf{e}_k := \mathbf{1} \in H^0(\mathbb{C}^0_{J^k}, W^\infty_{J^k}, \mathbb{Q})$. We remark that only $y^{q-1}\mathbf{e}_0$ is a broad sector. Furthermore,

$$\dim_{\mathbb{C}} \mathcal{H}_{W,G} = pq + 1 - q.$$

We shall use $\{r\}$ to denote the fractional part of the real number r. Then

$$\Theta_x^{J^k} = \left\{\frac{k}{p}\right\} \quad \text{and} \quad \Theta_y^{J^k} = \left\{\frac{k(p-1)}{pq}\right\};$$

by (4) in Definition 2.3, the degree-shifting numbers are

$$\iota_{J^k} = \Theta_x^{J^k} - q_x + \Theta_y^{J^k} - q_y = \left\{\frac{k}{p}\right\} + \left\{\frac{k(p-1)}{pq}\right\} + \frac{1-p-q}{pq}.$$

For any $\alpha \in \mathcal{H}_{J^k}$, we can use the degree formula (7) to obtain:

$$\deg_{\mathbb{C}} \mathbf{e}_k = \left\{\frac{k}{p}\right\} + \left\{\frac{k(p-1)}{pq}\right\} + \frac{1-p-q}{pq}, \quad k \in \Lambda; \tag{18}$$

$$\deg_{\mathbb{C}}(y^{q-1}\mathbf{e}_0) = \frac{(p-1)(q-1)}{pq} = \frac{\hat{c}_W}{2}. \tag{19}$$

3.2. Correlators

For simplicity, we identify $\mathbf{e}_0$ with $y^{q-1}\mathbf{e}_0$ and define the set

$$\hat{\Lambda} := \Lambda \bigcup \{0\}.$$

For any $i, j, k \in \hat{\Lambda}$, the computation of $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\rangle_0^W$ can be classified into four cases.

Case 1. $i = j = k = 0$
By the dimension axiom (10), we have

$$\langle y^{q-1}\mathbf{e}_0, y^{q-1}\mathbf{e}_0, y^{q-1}\mathbf{e}_0\rangle_0^W = 0.$$

Case 2. $i \neq 0$ and $j = k = 0$
By the dimension axiom (10), only $\langle \mathbf{e}_1, y^{q-1}\mathbf{e}_0, y^{q-1}\mathbf{e}_0\rangle_0^W$ is nonzero. From the residue pairing (17) we obtain

$$\eta_{0,0} = \langle \mathbf{e}_1, y^{q-1}\mathbf{e}_0, y^{q-1}\mathbf{e}_0\rangle_0^W = -\frac{1}{q}. \tag{20}$$

Case 3. $ijk \neq 0$

LEMMA 3.1. *If $ijk \neq 0$, then $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\rangle_0^W \neq 0$ if and only if $i + j + k$ is equal to $pq + 1$ or to $2pq + 1$. Furthermore,*

$$\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\rangle_0^W = \begin{cases} 1 & \text{if } \deg|\mathcal{L}_x| = \deg|\mathcal{L}_y| = -1, \\ -q & \text{if } \deg|\mathcal{L}_x| = -2 \text{ and } \deg|\mathcal{L}_y| = 0. \end{cases}$$

Proof. In this case, we can compute the degrees of orbifold line bundles using equation (7):

$$\deg|\mathcal{L}_x| = \frac{1}{p} - \left\{\frac{i}{p}\right\} - \left\{\frac{j}{p}\right\} - \left\{\frac{k}{p}\right\};$$

$$\deg|\mathcal{L}_y| = \frac{p-1}{pq} - \left\{\frac{i(p-1)}{pq}\right\} - \left\{\frac{j(p-1)}{pq}\right\} - \left\{\frac{k(p-1)}{pq}\right\}.$$

We observe that

$$\deg_{\mathbb{C}}(\mathbf{e}_i) + \deg_{\mathbb{C}}(\mathbf{e}_j) + \deg_{\mathbb{C}}(\mathbf{e}_k) = \hat{c}_W - \deg|\mathcal{L}_x| - \deg|\mathcal{L}_y| - 2. \tag{21}$$

Now let us assume $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W$ to be nonzero. Applying the dimension axiom (10) yields

$$\deg|\mathcal{L}_x| + \deg|\mathcal{L}_y| = -2. \tag{22}$$

Furthermore, the degrees of $|\mathcal{L}_x|$ and $|\mathcal{L}_y|$ should be integers. Hence

$$i + j + k \equiv 1 \text{ modulo } (p) \quad \text{and}$$
$$(i + j + k)(p - 1) \equiv p - 1 \text{ modulo } (pq).$$

Since $3 \leq i + j + k \leq 3pq - 3$ and since $p - 1$ and pq are coprime, we must have

$$i + j + k = pq + 1 \text{ or } 2pq + 1.$$

Therefore,

$$-3 < \deg|\mathcal{L}_x| = \frac{1}{p} - \left\{\frac{i}{p}\right\} - \left\{\frac{j}{p}\right\} - \left\{\frac{k}{p}\right\} < 0.$$

Thus (22) implies that $(\deg|\mathcal{L}_x|, \deg|\mathcal{L}_y|)$ is either $(-1, -1)$ or $(-2, 0)$. In the former case, $\pi_*(\mathcal{L}_x \oplus \mathcal{L}_y) = 0$ and $R^1\pi_*(\mathcal{L}_x \oplus \mathcal{L}_y) = 0$. Using the concavity axiom (12), we then obtain

$$\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = 1.$$

In the latter case, for each fiber (isomorphic to $\mathbb{C}P^1$) of the universal curve $\mathcal{C}$ over $\mathcal{W}_{0,3}(W; J^i, J^j, J^k)$, we have

$$H^0(\mathbb{C}P^1, |\mathcal{L}_x| \oplus |\mathcal{L}_y|) = 0 \oplus \mathbb{C} \quad \text{and} \quad H^1(\mathbb{C}P^1, |\mathcal{L}_x| \oplus |\mathcal{L}_y|) = \mathbb{C} \oplus 0.$$

The Witten map from H^0 to H^1 is $(p\bar{x}^{p-1} + \bar{y}^q, q\bar{x}\bar{y}^{q-1})$. This map has degree $-q$ and so, by the index zero axiom (13), we have

$$\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = -q. \qquad \square$$

Combining (20) and Lemma 3.1 yields the following statement.

COROLLARY 3.2. *The inverse matrix $\eta^{\alpha\beta}$ of the metric $\eta_{\alpha\beta}$ has the form*

$$\eta^{\alpha\beta} = \begin{cases} 1 & \textit{if } \alpha + \beta = pq, \\ -q & \textit{if } \alpha = \beta = 0, \\ 0 & \textit{otherwise}. \end{cases}$$

REMARK 3.3. We also have the following conclusions:

- for fixed i and j, there is at most one k such that $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W \neq 0$;
- if $2 \leq (i+j) \leq pq$ then $k = pq + 1 - (i+j)$;
- if $(pq+2) \leq (i+j) \leq (2pq-2)$ then $k = 2pq + 1 - (i+j)$.

Case 4. $ij \neq 0$ and $k = 0$

LEMMA 3.4. *For $i, j \in \Lambda$,*

$$\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W = \begin{cases} \pm 1 & \text{if } i+j = pq+1 \text{ and } \langle |\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j| \rangle_0^W = -q, \\ 0 & \text{otherwise.} \end{cases}$$

Before giving the proof, we discuss the sign in the formula. For a single term—say, $\langle \mathbf{e}_k, \mathbf{e}_M, y^{q-1}\mathbf{e}_0 \rangle_0^W$ as in Lemma 3.5—we know it is a square root of 1. Yet because this term contains a broad sector $y^{q-1}\mathbf{e}_0$, computing the exact value of such a correlator is beyond the reach of current methods. However, we can always fix the sign to be $+1$ by choosing either $y^{q-1}\mathbf{e}_0$ or $-y^{q-1}\mathbf{e}_0$ as an insertion. Once we have made such a choice, the sign will no longer depend on i, j in the formula. See Remark 3.7 for more details.

Proof of Lemma 3.4. We assume that $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W$ is nonzero. On the one hand, since $\deg_{\mathbb{C}}(y^{q-1}\mathbf{e}_0) = \hat{c}_W/2$, dimension axiom (10) implies

$$\deg_{\mathbb{C}}(\mathbf{e}_i) + \deg_{\mathbb{C}}(\mathbf{e}_j) = \frac{(p-1)(q-1)}{pq}. \tag{23}$$

On the other hand, from the composition axiom (2.9) it follows that

$$\begin{aligned} \langle |\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j| \rangle_0^W &= \sum_{\alpha, \beta \in \Lambda} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\beta \rangle_0^W \\ &\quad + (\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W)^2 \eta^{0,0}. \end{aligned} \tag{24}$$

Denote by $\mathcal{L}_{x,i,j,i,j}$ and $\mathcal{L}_{y,i,j,i,j}$ the orbifold line bundles corresponding to the G-decorated graph (J^i, J^j, J^i, J^j). Then using equation (7) yields

$$\begin{aligned} \deg|\mathcal{L}_{x,i,j,i,j}| &= \frac{2}{p} - 2\left\{\frac{i}{p}\right\} - 2\left\{\frac{j}{p}\right\} \quad \text{and} \\ \deg|\mathcal{L}_{y,i,j,i,j}| &= \frac{2p-2}{pq} - 2\left\{\frac{i(p-1)}{pq}\right\} - 2\left\{\frac{j(p-1)}{pq}\right\}. \end{aligned} \tag{25}$$

Because $\gcd(p-1, q) = 1$, there are three cases.

Case 1: p and q are both odd. In this case, for fixed $\alpha, \beta \in \Lambda$ we cannot have both $\alpha = \beta$ and $\alpha + \beta = pq$. Therefore, at least one of $\eta_{\alpha\beta}$, $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W$, and $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\beta \rangle_0^W$ vanishes. By (24) we know that $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W \neq 0$ if and only if $\langle |\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j| \rangle_0^W \neq 0$. So suppose that $\langle |\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j| \rangle_0^W \neq 0$; then, using the dimension axiom (10), again we have

$$\deg|\mathcal{L}_{x,i,j,i,j}| + \deg|\mathcal{L}_{x,i,j,i,j}| = -2. \tag{26}$$

Moreover, $\deg|\mathcal{L}_{x,i,j,i,j}|$ and $\deg|\mathcal{L}_{x,i,j,i,j}|$ are integers; hence

$$\deg|\mathcal{L}_{x,i,j,i,j}| = \frac{2}{p} - 2\left\{\frac{i}{p}\right\} - 2\left\{\frac{j}{p}\right\} < 0 \tag{27}$$

and

$$\deg|\mathcal{L}_{y,i,j,i,j}| = \frac{2p-2}{pq} - 2\left\{\frac{i(p-1)}{pq}\right\} - 2\left\{\frac{j(p-1)}{pq}\right\} < 1. \tag{28}$$

Since p is odd, we must have

$$(\deg|\mathcal{L}_{x,i,j,i,j}|, \deg|\mathcal{L}_{y,i,j,i,j}|) = (-2, 0).$$

Similarly, by the index zero axiom (13) we have

$$\langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W = -q.$$

And since $pq/2$ is not an integer, it follows that $\deg|\mathcal{L}_{y,i,j,i,j}| = 0$ implies $i + j = pq + 1$. Now using equation (24), we obtain $\langle \mathbf{e}_i, \mathbf{e}_{pq+1-i}, y^{q-1}\mathbf{e}_0\rangle_0^W = \pm 1$.

Case 2: p is even and q is odd. In this case, the first term on the right side of (24) is nonzero if and only if $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq/2}\rangle_0^W \neq 0$. Since p is even, we have

$$\deg|\mathcal{L}_{y,i,j,pq/2}| = \frac{p-1}{pq} - \left\{\frac{i(p-1)}{pq}\right\} - \left\{\frac{j(p-1)}{pq}\right\} - \frac{1}{2} < 0.$$

From Lemma 3.1 it follows that $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq/2}\rangle_0^W \neq 0$ implies $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq/2}\rangle_0^W = \pm 1$ for $i + j = pq/2 + 1$ or $3pq/2 + 1$, and

$$\deg|\mathcal{L}_{x,i,j,pq/2}| = \deg|\mathcal{L}_{y,i,j,pq/2}| = -1.$$

Hence the corresponding degree formula is

$$\deg|\mathcal{L}_{x,i,j,i,j}| = \deg|\mathcal{L}_{y,i,j,i,j}| = -1.$$

Thus, by the concavity axiom (12), $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j\rangle_0^W = 1$. Given (24), we know that

$$\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W = 0.$$

Now we need only consider what happens when the first term on the right-hand side of equation (24) vanishes, in which case $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq/2}\rangle_0^W = 0$. We have

$$\langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W = -q(\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W)^2.$$

Using a similar argument as in Case 1, we need only prove

$$(\deg|\mathcal{L}_{x,i,j,i,j}|, \deg|\mathcal{L}_{y,i,j,i,j}|) \neq (-1, -1);$$

otherwise, we can compute $\deg|\mathcal{L}_{x,i,j,pq/2}| = \deg|\mathcal{L}_{y,i,j,pq/2}| = -1$. Using Lemma 3.1 yields $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq/2}\rangle_0^W = 1$, which is a contradiction.

Case 3: q is even. Then p is also even by our assumption $\gcd(p-1, q) = 1$. Hence $pq/2 \notin \Lambda$ and so the first term on the right-hand side of (24) must be zero. We assume that $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W \neq 0$; then, by equation (24), $\langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W \neq 0$. Hence $\deg|\mathcal{L}_{y,i,j,i,j}|$ is an integer and we have

$$2(p-1)(1-i-j) \equiv 0 \text{ modulo } (pq).$$

Now $i + j = pq/2 + 1$ or $pq + 1$ or $3pq/2 + 1$. We claim that $i + j = pq + 1$, and arguing as before shows that $\langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W = -q$.

Otherwise, if $i + j = pq/2 + 1$ or $3pq/2 + 1$ then

$$i + j \equiv 1 \ \text{ modulo } (p)$$

and

$$\frac{(i + j)(p - 1)}{pq} \equiv \frac{1}{2} + \frac{p - 1}{pq} \ \text{ modulo } (1).$$

Using degree formula (18), we obtain

$$\deg_{\mathbb{C}}(\mathbf{e}_i) + \deg_{\mathbb{C}}(\mathbf{e}_j) \equiv \frac{(p - 1)(q - 1)}{pq} + \frac{1}{2} \ \text{ modulo } (1).$$

This contradicts equation (23). □

3.3. Generators and Mirror Symmetry

We will now prove that there exist two generators for the ring structure of $\mathcal{H}_{W,G}$ defined by Fan–Jarvis–Ruan–Witten theory.

LEMMA 3.5. *There exists a unique pair of integers k and m in $\hat{\Lambda}$ that satisfy* $\deg_{\mathbb{C}} \mathbf{e}_k = \frac{q-1}{pq}$, $\deg_{\mathbb{C}} \mathbf{e}_m = \frac{1}{q}$, *and the following congruence equation*:

$$-(k - 1)(p - 1) \equiv (m - 2)(p - 1) \equiv 1 \ \textit{modulo } (pq). \tag{29}$$

Proof. If $p = 2$ then it is easy to derive $M = 1, k = 0, m = 3$, and $k \equiv pq + 1 - M$.

For $p > 2$, since $p - 1$ and q are coprime it follows easily that the congruence equation (29) has a unique solution k such that $1 \le k \le pq - 1$ and $p \nmid k$; this k will satisfy $p \mid (k - 2)$, so

$$\deg_{\mathbb{C}} \mathbf{e}_k = \left\{\frac{k}{p}\right\} + \left\{\frac{k(p - 1)}{pq}\right\} + \frac{1 - p - q}{pq} = \frac{q - 1}{pq}.$$

We can find $\mathbf{e}_m$ similarly. In fact, there exists a unique M with $1 \le M \le (pq - 1)$ and such that

$$(p - 1)M \equiv 1 \ \text{ modulo } (pq).$$

Now we have $k = pq + 1 - M$ and set $m = M + 2$. □

In the rest of this section, m, k, M are the integers in Lemma 3.5. We always have

$$M + 1 = m - 1, \quad p \mid (M + 1), \quad k \equiv pq + 1 - M \ \text{ modulo } (pq). \tag{30}$$

LEMMA 3.6. *There is a canonical bijective map f between the set $\Delta := \{(s, t) \in \mathbb{Z} \oplus \mathbb{Z} \mid 0 \le s \le p - 2, 0 \le t \le q - 1\}$ and the set $\Lambda = \{i \in \mathbb{Z} \mid 1 \le i \le pq - 1,$ $p \nmid i\}$.*

Proof. We define a map $f : \Delta \to \Lambda$ as follows. If there exists an $i \in \Lambda$ such that

$$i \equiv 1 + s(k - 1) + t(m - 1) \ \text{ modulo } (pq),$$

then $f(s, t) = i$. Using (30), we have

$$1 + s(k-1) + t(m-1) \equiv (M+1)(t-s) + (s+1) \ \text{ modulo } (pq).$$

Since $p \mid (M+1)$ and $0 < s+1 \leq p-1$, it follows that the map f is well-defined and so

$$p \nmid (1 + s(k-1) + t(m-1)).$$

Moreover, if $f(s,t) = f(s',t')$ then

$$1 + s(k-1) + t(m-1) \equiv 1 + s'(k-1) + t'(m-1) \ \text{ modulo } (pq).$$

Therefore,

$$s - s' \equiv (M+1)(s' + t' - s - t) \ \text{ modulo } (pq).$$

Since $p \mid (M+1)$ and since $0 \leq s, s' \leq p-2$, we have $s' = s$ and

$$(t' - t)(M+1) \equiv 0 \ \text{ modulo } (pq).$$

Since $0 \leq t, t' \leq q-1$, it follows that $t' = t$ and the map is injective.

Finally, bijectiveness follows because the cardinality of both Δ and Λ is $(p-1)q$. $\square$

Now, for each $i \in \Lambda$ we can identify $\mathbf{e}_i$, $\mathbf{e}_{1+s(k-1)+t(m-1)}$, and (s,t), where $f(s,t) = i$. We define $(p-1,0) = \mp q \cdot y^{q-1}\mathbf{e}_0$ and $(p,0) = (1,0) \bigstar (p-1,0)$.

REMARK 3.7. Consider $\mathbf{e}_k = (1,0)$, $\mathbf{e}_m = (0,1)$, and $\mathbf{e}_M = (p-2,0)$. By (27) and (28), it is not hard to check for nonzero $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W$ in Lemma 3.4. With nonzero $\langle | \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j | \rangle_0^W$, we must have $\mathbf{e}_i = (s,0)$ and $\mathbf{e}_j = (p-1-s,0)$ for some $1 \leq s \leq p-2$. Then the associativity of $\bigstar$ implies

$$\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W = \langle \mathbf{e}_k, \mathbf{e}_M, y^{q-1}\mathbf{e}_0 \rangle_0^W.$$

LEMMA 3.8. *$(s,t) \bigstar (u,v) = (s+u, t+v)$ if $0 \leq s+u \leq p-2$ and $0 \leq t+v \leq q-1$.*

Proof. Let $i = f(s,t)$ and $j = f(u,v)$; then

$$(s,t) \bigstar (u,v) = \mathbf{e}_i \bigstar \mathbf{e}_j = \sum_{\alpha,\beta \in \hat{\Lambda}} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \mathbf{e}_\beta.$$

Using (30) now yields

$$\begin{aligned} i + j &\equiv 1 + s(k-1) + t(m-1) + 1 + u(k-1) + v(m-1) \\ &\equiv (M+1)(t+v-s-u) + (2+s+u) \ \text{ modulo } (pq). \end{aligned}$$

Since $2 \leq 2+s+u \leq q$ and $p \mid (M+1)$, we have

$$i + j \not\equiv 1 \ \text{ modulo } (pq).$$

By Remark 3.3, there exists at most one $\alpha \in \Lambda$ such that $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W$ is nonzero and

$$\alpha \equiv -(M+1)(t+v-s-u) - (1+s+u) \ \text{ modulo } (pq).$$

Therefore,

$$\deg|\mathcal{L}_x| = \frac{1}{p} - \left\{\frac{i}{p}\right\} - \left\{\frac{j}{p}\right\} - \left\{\frac{\alpha}{p}\right\} = -1.$$

Hence $\deg|\mathcal{L}_y| = -1$ also and $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W = 1$.

Now $(s,t) \bigstar (u,v) = \mathbf{e}_i \bigstar \mathbf{e}_j = \mathbf{e}_{pq-\alpha} = \mathbf{e}_{1+(s+u)(k-1)+(t+v)(m-1)} = (s+u, t+v)$. □

LEMMA 3.9. *We have the following identities characterizing the multiplication* $\bigstar$:

(i) $(p-2,0) \bigstar (1,0) = \mp q \cdot y^{q-1}\mathbf{e}_0 = (p-1,0)$;
(ii) $(p-1,0) \bigstar (0,1) = 0$;
(iii) $(p,0) + q(0,q-1) = 0$;
(iv) $(s,t) \bigstar (u,v) = 0$ *if* $t+v \geq q$;
(v) $(s,t) \bigstar (u,v) = 0$ *if* $s+u \geq p-1$ *and* $t+v \neq 0$;
(vi) $(s,0) \bigstar (u,0) = -q(s+u-p, q-1)$ *for* $0 \leq s,u \leq p-1$ *and* $p \leq s+u \leq 2p-2$.

Proof. We can check directly as follows.

(i) Since $\mathbf{e}_{1+(p-2)(k-1)} = \mathbf{e}_M$ and $M+k = pq+1$, we know that $\langle \mathbf{e}_{1+(p-2)(k-1)}, \mathbf{e}_k, \mathbf{e}_\alpha \rangle_0^W \neq 0$ if and only if $\alpha = 0$. Therefore,

$$\begin{aligned}(p-2,0) \bigstar (1,0) &= \mathbf{e}_{1+(p-2)(k-1)} \bigstar \mathbf{e}_k \\ &= \langle \mathbf{e}_M, \mathbf{e}_k, y^{q-1}\mathbf{e}_0 \rangle_0^W \eta^{0,0} y^{q-1}\mathbf{e}_0 = \mp q \cdot y^{q-1}\mathbf{e}_0.\end{aligned}$$

By associativity, for any $0 \leq s \leq p-2$ we have $(p-1-s,0) \bigstar (s,0) = (p-1,0)$.

(ii) $\langle y^{q-1}\mathbf{e}_0, \mathbf{e}_m, \mathbf{e}_\alpha \rangle_0^W \neq 0$ implies $m+\alpha = pq+1$. Then $p \mid \alpha$ and $\alpha \notin \hat{\Lambda}$. Thus

$$(p-1,0) \bigstar (0,1) = \mp q \cdot y^{q-1}\mathbf{e}_0 \bigstar \mathbf{e}_m = \mp q \sum_{\alpha,\beta \in \hat{\Lambda}} \langle y^{q-1}\mathbf{e}_0, \mathbf{e}_m, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \mathbf{e}_\beta = 0.$$

(iii) Recall that $\langle \mathbf{e}_k, y^{q-1}\mathbf{e}_0, \mathbf{e}_M \rangle_0^W = \pm 1$, $(p-1,0) = \mp q \cdot y^{q-1}\mathbf{e}_0$, and $\mathbf{e}_{pq-M} = (0, q-1)$. Hence

$$\begin{aligned}(p,0) &= \mathbf{e}_k \bigstar (\mp q \cdot y^{q-1}\mathbf{e}_0) \\ &= \mp q \langle \mathbf{e}_k, y^{q-1}\mathbf{e}_0, \mathbf{e}_M \rangle_0^W \eta^{M,pq-M} \mathbf{e}_{pq-M} = -q(0,q-1).\end{aligned}$$

(iv) For $t+v \geq q$ we have $(0,q-1) \bigstar (0,1) = -\frac{1}{q}(p,0) \bigstar (0,1) = 0$. Therefore,

$$(s,t) \bigstar (u,v) = (s,0) \bigstar [(0,q-1) \bigstar (0,1)] \bigstar (0, v+t-q) \bigstar (u,0) = 0.$$

(v) For $s+u \geq p-1$ and $t+v \neq 0$,

$$(s,t) \bigstar (u,v) = [(0,t) \bigstar (p-1,0)] \bigstar (s+u+1-p, v) = 0.$$

(vi) For this last case, we have

$$\begin{aligned}(s,0) \bigstar (u,0) &= (p,0) \bigstar (s+u-p, 0) \\ &= -q(0,q-1) \bigstar (s+u-p,0) \\ &= -q(s+u-p, q-1).\end{aligned}$$

□

THEOREM 3.10. *If* $\gcd(p-1,q)=1$ *then* $\mathbf{e}_k$ *and* $\mathbf{e}_m$ *in Lemma 3.5 generate the quantum ring of* $(W = X^p + XY^q, G_{\max})$. *The multiplication is given by*

- $\mp q \cdot y^{q-1}\mathbf{e}_0 = \mathbf{e}_k^{p-1}$,
- $\mathbf{e}_i = \mathbf{e}_k^s \bigstar \mathbf{e}_m^t$, *if* $i \in \Lambda$ *such that* $f(s,t) = i$ *for* $(s,t) \in \Delta$.

Here $\mathbf{e}_k^{p-1}$ *denotes the* $(p-1)$th *power of* $\mathbf{e}_k$ *under the multiplication* $\bigstar$. *Moreover, we have*:

- $\mathbf{e}_k^{p-1} \bigstar \mathbf{e}_m = (p-1,0) \bigstar (0,1) = 0$;
- $\mathbf{e}_k^p + q\mathbf{e}_m^{q-1} = (p,0) + q(0,q-1) = 0$.

The mirror symmetry phenomenon between two dual singularities is formalized in our next corollary.

COROLLARY 3.11. *If* $\gcd(p-1,q)=1$, *then for the pair* $(W = X^p + XY^q, G = G_{\max})$ *we have a* $\mathbb{C}$*-algebra isomorphism* $\mathcal{H}_{W,G} \cong \mathcal{Q}_{W^T}$, *where* $W^T = X^pY + Y^q$ *is the dual singularity.*

Proof. Define a $\mathbb{C}$-algebra epimorphism $F\colon \mathbb{C}[X,Y] \to \mathcal{H}_{W,G}$ such that $F(X) = \mathbf{e}_k$ and $F(Y) = \mathbf{e}_m$. Theorem 3.10 implies that both $X^{p-1}Y$ and $X^p + qY^{q-1}$ are in $\mathrm{Ker}(F)$. Observe that the dimension of the vector space $\mathbb{C}[X,Y]/(pX^{p-1}Y, X^p + qY^{q-1})$ and of $\mathcal{H}_{W,G}$ are both $pq - q + 1$. Hence we have the $\mathbb{C}$-algebra isomorphism

$$\bar{F}\colon \mathbb{C}[X,Y]/(pX^{p-1}Y, X^p + qY^{q-1}) \to \mathcal{H}_{W,G}.$$

The corollary now follows because $\mathbb{C}[X,Y]/(pX^{p-1}Y, X^p + qY^{q-1}) = \mathcal{Q}_{W^T}$. $\square$

4. Quantum Ring of $(W = X^p + XY^q, G = G_{\max})$, $\gcd(p-1,q) = d \neq 1$

4.1. Basic Calculation

We have the same fractional degrees and the central charge:

$$q_x = \frac{1}{p}, \quad q_y = \frac{p-1}{pq}, \quad \hat{c}_W = \frac{2(p-1)(q-1)}{pq}.$$

Let $\xi = \exp\left(\frac{2\pi i}{pq}\right)$ and let λ act on $\mathcal{Q}_W\omega$ by (ξ^{-q}, ξ). Then λ generates the maximal admissible abelian group $G = \mathbb{Z}/(pq)\mathbb{Z}$. Now $\Theta_x^J = \frac{p-1}{p}$ and $\Theta_y^J = \frac{1}{pq}$. The G-invariant state space of the polynomial W is

$$\mathcal{H}_{W,G} = \mathbb{C}\langle y^{q-1}\mathbf{e}_0, \mathbf{e}_k \mid k \in \Lambda\rangle.$$

Here $\mathbf{e}_0$, $\mathbf{e}_k$, and Λ are defined as before. We also have

$$\dim_{\mathbb{C}} \mathcal{H}_{W,G} = pq + 1 - q.$$

Moreover,

$$\Theta_x^{J^k} = \left\{\frac{p-k}{p}\right\}, \qquad \Theta_y^{J^k} = \left\{\frac{k}{pq}\right\},$$

$$\iota_{J^k} = \left\{\frac{p-k}{p}\right\} + \left\{\frac{k}{pq}\right\} - \frac{p+q-1}{pq},$$

$$\deg_{\mathbb{C}}(y^{q-1}\mathbf{e}_0) = \frac{(p-1)(q-1)}{pq} = \frac{\hat{c}_W}{2},$$

$$\deg_{\mathbb{C}} \mathbf{e}_k = \left\{\frac{p-k}{p}\right\} + \left\{\frac{k}{pq}\right\} + \frac{1-p-q}{pq}.$$

REMARK 4.1. In this case, $\mathbf{e}_{p-1}$ will be the unit in the quantum ring.

4.2. Correlators

The computation of $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W$ is also classified into four cases.

Case 1. $i = j = k = 0$
By dimension axiom (10), all correlators of this type vanish.

Case 2. $i \neq 0$ *and* $j = k = 0$
The only nonzero correlator is

$$\langle \mathbf{e}_{p-1}, y^{q-1}\mathbf{e}_0, y^{q-1}\mathbf{e}_0 \rangle_0^W = -\frac{1}{q}.$$

Case 3. $ijk \neq 0$

LEMMA 4.2. *If* $i, j, k \in \Lambda$, *then*

$$\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = \begin{cases} -q & \text{if } i+j+k = p-1, \\ 1 & \text{if } i+j+k = pq+p-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\deg|\mathcal{L}_x| = \frac{1}{p} - \left\{\frac{p-i}{p}\right\} - \left\{\frac{p-j}{p}\right\} - \left\{\frac{p-k}{p}\right\}, \tag{31}$$

$$\deg|\mathcal{L}_y| = \frac{p-1}{pq} - \left\{\frac{i}{pq}\right\} - \left\{\frac{j}{pq}\right\} - \left\{\frac{k}{pq}\right\}. \tag{32}$$

If $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W \neq 0$, then $\deg|\mathcal{L}_x|$ and $\deg|\mathcal{L}_y|$ are integers. Using (31) and (32) together with dimension axiom (10), we have

$$\deg|\mathcal{L}_x| + \deg|\mathcal{L}_y| = -2. \tag{33}$$

Since $\deg|\mathcal{L}_y|$ is an integer, (32) now implies

$$i + j + k \equiv p - 1 \text{ modulo } (pq). \tag{34}$$

Thus $i+j+k=p-1$, $pq+p-1$, or $2pq+p-1$. The last case is impossible because if $i+j+k=2pq+p-1$ then $\deg|\mathcal{L}_y|=-2$. By (31) and (33), this contradicts the fact that

$$\deg|\mathcal{L}_x| = \frac{1}{p} - \left\{\frac{p-i}{p}\right\} - \left\{\frac{p-j}{p}\right\} - \left\{\frac{p-k}{p}\right\} < 0.$$

If $i+j+k=p-1$ then $(\deg|\mathcal{L}_x|, \deg|\mathcal{L}_y|) = (-2,0)$. In this case, index zero axiom (13) implies

$$\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\rangle_0^W = -q.$$

If $i+j+k=pq+p-1$ then $(\deg|\mathcal{L}_x|, \deg|\mathcal{L}_y|) = (-1,-1)$. Using the concavity axiom (12) now yields $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\rangle_0^W = 1$. □

REMARK 4.3. The metric $\eta_{\alpha\beta}$ has the same form as in Corollary 3.2.

Case 4. $ij \neq 0$ and $k=0$

LEMMA 4.4. *For $ij \neq 0$,*

$$\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W = \begin{cases} \pm 1 & \text{if } i+j=p-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\deg|\mathcal{L}_{x,i,j,i,j}| = \frac{2}{p} - 2\left\{\frac{p-i}{p}\right\} - 2\left\{\frac{p-j}{p}\right\}, \tag{35}$$

$$\deg|\mathcal{L}_{y,i,j,i,j}| = \frac{2(p-1)}{pq} - 2\left\{\frac{i}{pq}\right\} - 2\left\{\frac{j}{pq}\right\}. \tag{36}$$

If $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W \neq 0$ then, by (35), (36), and dimension axiom (10),

$$\deg|\mathcal{L}_{x,i,j,i,j}| + \deg|\mathcal{L}_{y,i,j,i,j}| = -2. \tag{37}$$

From (35) it follows that $-3 < \deg|\mathcal{L}_{x,i,j,i,j}| < 0$. Then $\deg|\mathcal{L}_{y,i,j,i,j}| = 0$ or -1. Using (36), we have

$$i+j = p-1 \text{ or } p-1+\frac{pq}{2}. \tag{38}$$

We see that the second case holds only if $2 \mid (pq)$. On the one hand, composition axiom (2.9) yields

$$\begin{aligned}\langle |\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W &= \sum_{l\in\Lambda} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_l\rangle_0^W \eta^{l,pq-l} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq-l}\rangle_0^W \\ &\quad + (\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W)^2 \eta^{0,0}.\end{aligned}$$

On the other hand, $\langle |\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W \neq 0$ if and only if

$$(\deg|\mathcal{L}_{x,i,j,i,j}|, \deg|\mathcal{L}_{y,i,j,i,j}|) = (-2,0) \text{ or } (-1,-1). \tag{39}$$

Hence we have three cases.

Case 1: q is even and $pq/2 \notin \Lambda$. Then

$$\langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W = (\langle\mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W)^2\eta^{0,0}.$$

If $i + j = p - 1$ then (35) shows that $\deg|\mathcal{L}_{x,i,j,i,j}| = -2$. Thus (36) implies $\deg|\mathcal{L}_{y,i,j,i,j}| = 0$ and so $\langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W = -q$.

If $i + j = pq/2 + p - 1$, then

$$i + j \equiv -1 \ \text{modulo}\ (p).$$

So (35) and (36) imply $\deg|\mathcal{L}_{x,i,j,i,j}| = -2$ and $\deg|\mathcal{L}_{y,i,j,i,j}| = -1$, which contradict with (37). Therefore $\langle\mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W \neq 0$ if and only if $i + j = p - 1$.

Case 2: p is even, q is odd, and $pq/2 \in \Lambda$. In this case, if $i + j = pq/2 + p - 1$ then (35) and (36) imply that $\deg|\mathcal{L}_{x,i,j,i,j}| = \deg|\mathcal{L}_{y,i,j,i,j}| = -1$. Thus $\langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W = 1$ and we have

$$\sum_{l\in\Lambda}\langle\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_l\rangle_0^W\eta^{l,pq-l}\langle\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq-l}\rangle_0^W$$
$$= \langle\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq/2}\rangle_0^W\eta^{pq/2,pq/2}\langle\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq/2}\rangle_0^W = 1.$$

Thus $\langle\mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W = 0$.

If $i + j = p - 1$, then $\deg|\mathcal{L}_{x,i,j,i,j}| = -2$ and $\deg|\mathcal{L}_{y,i,j,i,j}| = 0$. Therefore,

$$(\langle\mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W)^2\eta^{0,0} = \langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W = -q$$

and so $\langle\mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W = \pm 1$.

Case 3: p and q are both odd, and $pq/2$ is not an integer. Hence

$$(\langle\mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W)^2\eta^{0,0} = \langle|\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j|\rangle_0^W.$$

Once again, $\langle\mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0\rangle_0^W = \pm 1$ if and only if $i + j = p - 1$. □

4.3. *Generators and Mirror Symmetry*

LEMMA 4.5. *There is a canonical bijective map g between the set $\Delta = \{(s,t) \in \mathbb{Z} \oplus \mathbb{Z} \mid 0 \le s \le p - 2, 0 \le t \le q - 1\}$ and the set $\Lambda = \{i \in \mathbb{Z} \mid 1 \le i \le pq - 1, p \nmid i\}$.*

Proof. We define a map $g\colon \Delta \to \Lambda$ by $g(s,t) = i$ for $i \in \Lambda$ such that

$$i \equiv p - 1 - s + tp \ \text{modulo}\ (pq).$$ □

For any $i \in \Lambda$, we identify $\mathbf{e}_i$ and (s,t) as the same element if $g(s,t) = i$. Then $(1,0) = \mathbf{e}_{p-2}$ and $(0,1) = \mathbf{e}_{2p-1}$. If we replace $\mathbf{e}_k$ and $\mathbf{e}_m$ in Theorem 3.10 by $\mathbf{e}_{p-2}$ and $\mathbf{e}_{2p-1}$, respectively, then it is straightforward (as in Section 3) to prove the following theorem.

THEOREM 4.6. *For $\gcd(p-1,q) = d \neq 1$, we have that $\mathbf{e}_{p-2}$ and $\mathbf{e}_{2p-1}$ generate the quantum ring of $(W = X^p + XY^q, G = G_{\max})$. The multiplication is determined as follows:*

- $\mathbf{e}_{p-1}$ *is the unit of this ring*;
- $\mp q \cdot y^{q-1}\mathbf{e}_0 = \mathbf{e}_{p-2}^{p-1}$;
- $\mathbf{e}_i = \mathbf{e}_{p-2}^{s} \bigstar \mathbf{e}_{2p-1}^{t}$ *for each* $i \in \Lambda$, *where* $(s,t) \in \Delta$ *such that* $g(s,t) = i$.

We also have the following equalities:

- $\mathbf{e}_{p-2}^{p-1} \bigstar \mathbf{e}_{2p-1} = (p-1,0) \bigstar (0,1) = 0$;
- $\mathbf{e}_{p-2}^{p} + q\mathbf{e}_{2p-1}^{q-1} = (p,0) + q(0,q-1) = 0$.

COROLLARY 4.7. *Let* $\gcd(p-1,q) = d \neq 1$, $W = X^p + XY^q$, *and* $G = G_{\max}$. *Then we have a* $\mathbb{C}$*-algebra isomorphism* $\mathcal{H}_{W,G} \cong \mathcal{Q}_{W^T}$, *where* $W^T = X^pY + Y^q$ *is the dual singularity.*

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