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Topological Symmetry Groups and Mapping Class Groups for Spatial Graphs

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Introduction

By a *graph* we shall mean the underlying space of a finite connected simplicial complex of dimension 1. A *spatial graph* is a graph embedded in a 3-manifold. The theory of spatial graphs is a generalization of classical knot theory. For a spatial graph Γ in S^3, the *mapping class group* $\mathrm{MCG}(S^3, \Gamma)$ (resp., $\mathrm{MCG}_+(S^3, \Gamma)$) is defined as the group of isotopy classes of the self-homeomorphisms (resp., orientation-preserving self-homeomorphisms) of S^3 that preserve Γ setwise. The cardinality of the group describes how many symmetries the spatial graph admits. In [16] it is shown that the group $\mathrm{MCG}(S^3, \Gamma)$ is always finitely presented.

Simon [19] (see also [3; 4] for details) introduced a similar concept, called the *topological symmetry group* of a spatial graph Γ in S^3 and denoted by $\mathrm{TSG}(S^3, \Gamma)$, to describe the symmetries of a spatial graph Γ in S^3. This group is defined as the subgroup of the automorphism group of Γ induced by homeomorphisms of the pair (S^3, Γ). When we allow only orientation-preserving homeomorphisms, we obtain the positive topological symmetry group $\mathrm{TSG}_+(S^3, \Gamma)$.

The aim of this paper is to provide complete answers (Theorems 2.5 and 3.2) to the following question.

Question. When is $\mathrm{TSG}(S^3, \Gamma)$ (resp., $\mathrm{TSG}_+(S^3, \Gamma)$) isomorphic to $\mathrm{MCG}(S^3, \Gamma)$ (resp., $\mathrm{MCG}_+(S^3, \Gamma)$)?

We remark that one of the answers to this question (viz., Theorem 2.5) implies that, if the group $\mathrm{MCG}_+(S^3, \Gamma)$ is finite, then by [3] it is a finite subgroup of $\mathrm{SO}(4)$.

Notation. Let X be a subset of a given polyhedral space Y. Throughout the paper, we denote the interior of X by $\mathrm{Int}\, X$. We will use $N(X; Y)$ to denote a closed regular neighborhood of X in Y. If the ambient space Y is clear from the context, we denote it more briefly by $N(X)$. Let M be a 3-manifold, and let $L \subset M$ be a submanifold with or without boundary. When L is of dimension 1 or 2, we write $E(L) = M \setminus \mathrm{Int}\, N(L)$. When L is of dimension 3, we write $E(L) = M \setminus \mathrm{Int}\, L$.

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1. Mapping Class Groups and Topological Symmetry Groups

Throughout this paper, we will work in the piecewise linear category.

1.1. Mapping Class Groups

Let N be a (possibly empty) subspace of a compact orientable manifold M. We will denote by

$$\mathrm{Homeo}(M, N) \quad (\text{resp., } \mathrm{Homeo}(M \text{ rel } N))$$

the space of self-homeomorphisms of M preserving N setwise (resp., pointwise). We call the group

$$\pi_0(\mathrm{Homeo}(M, N)) \quad (\text{resp., } \pi_0(\mathrm{Homeo}(M \text{ rel } N)))$$

—that is, the group of isotopy classes of elements of $\mathrm{Homeo}(M, N)$ (resp., $\mathrm{Homeo}(M \text{ rel } N)$) in which the isotopies are required to preserve N setwise (resp., pointwise)—a *mapping class group* and denote it by

$$\mathrm{MCG}(M, N) \quad (\text{resp., } \mathrm{MCG}(M \text{ rel } N)).$$

The "positive" subscripts, as in $\mathrm{Homeo}_+(M, N)$ and $\mathrm{MCG}_+(M, N)$, mark the subgroups of $\mathrm{Homeo}(M, N)$ and $\mathrm{MCG}(M, N)$ that consist of orientation-preserving homeomorphisms and their isotopy classes, respectively.

Recall that we use *graph* to mean the underlying space of a finite connected simplicial complex of dimension 1. A point x in a graph is called a *vertex* if x does not have an open neighborhood that is homeomorphic to an open interval. We denote by $v(G)$ the set of all vertices of a graph G. We assume throughout that a graph has no valency-1 vertices; that is, no vertex admits any open neighborhood homeomorphic to $[0, 1)$. The closure of each component of $G \setminus v(G)$ is called an *edge*. An edge e of a graph Γ is called a *cut edge* if $\Gamma \setminus \mathrm{Int}\, e$ is disconnected. Observe that our definition of a graph allows multiple edges between two vertices as well as an edge from a vertex to itself (i.e., a *loop*); however, our definition of a vertex excludes vertices of valency 2. These definitions differ from the usual definitions of a graph and a vertex.

A *spatial graph* Γ is a graph embedded in a 3-manifold M. Two spatial graphs are said to be *equivalent* if one can be transformed into the other by an ambient isotopy of the 3-manifold. Note that a knot K in M is also a spatial graph. For a spatial graph Γ in S^3, the group $\mathrm{MCG}(S^3, \Gamma)$ describes the symmetries of Γ. For a knot K in S^3, the group $\mathrm{MCG}(S^3, K)$ is called the *symmetry group* of K; see [15].

The following statement is proved in [16].

THEOREM 1.1. *For a spatial graph Γ in S^3, the group $\mathrm{MCG}(S^3, \Gamma)$ is finitely presented.*

1.2. Topological Symmetry Groups of Graphs

Let Γ be a graph, and let X be a 1-dimensional simplicial complex such that $\Gamma = |X|$. We denote by $\mathrm{Aut}(\Gamma)$ the group of all simplicial automorphisms of the simplicial complex X. It is clear that the group $\mathrm{Aut}(\Gamma)$ does not depend on the choice of X. Let Γ be a spatial graph in S^3. The topological symmetry group $\mathrm{TSG}(S^3, \Gamma)$ and the positive topological symmetry group $\mathrm{TSG}_+(S^3, \Gamma)$ of the spatial graph Γ in S^3 are subgroups of $\mathrm{Aut}(\Gamma)$ defined as follows:

$$\mathrm{TSG}(S^3, \Gamma) = \{f \in \mathrm{Aut}(\Gamma) \mid \exists \tilde{f} \in \mathrm{Homeo}(S^3, \Gamma) \text{ s.t. } \tilde{f}|_\Gamma \text{ induces } f\};$$

$$\mathrm{TSG}_+(S^3, \Gamma) = \{f \in \mathrm{Aut}(\Gamma) \mid \exists \tilde{f} \in \mathrm{Homeo}_+(S^3, \Gamma) \text{ s.t. } \tilde{f}|_\Gamma \text{ induces } f\}.$$

These groups were originally defined by Simon [19]; see [3; 4] for details. Obviously, the group $\mathrm{TSG}(S^3, \Gamma)$ is a finite group.

The following proposition is a straightforward consequence of the definitions.

LEMMA 1.2 [16]. *Let Γ be a spatial graph in S^3. Then there is an exact sequence*

$$1 \to \mathrm{MCG}(S^3 \text{ rel } \Gamma) \to \mathrm{MCG}(S^3, \Gamma) \to \mathrm{TSG}(S^3, \Gamma) \to 1$$

$$(\textit{resp.,}\ 1 \to \mathrm{MCG}_+(S^3 \text{ rel } \Gamma) \to \mathrm{MCG}_+(S^3, \Gamma) \to \mathrm{TSG}_+(S^3, \Gamma) \to 1).$$

Hence $\mathrm{MCG}(S^3, \Gamma) \cong \mathrm{TSG}(S^3, \Gamma)$ (*resp.,* $\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma)$) *if and only if* $\mathrm{MCG}(S^3 \text{ rel } \Gamma) \cong 1$ (*resp.,* $\mathrm{MCG}_+(S^3 \text{ rel } \Gamma) \cong 1$).

Given this lemma, we can answer the question posed in the Introduction by determining when $\mathrm{MCG}(S^3 \text{ rel } \Gamma)$ (or $\mathrm{MCG}_+(S^3 \text{ rel } \Gamma)$) is trivial.

1.3. Review of Boundary Patterns

Here we review the notion of a boundary pattern as defined in [14] and [17]. Let M be a compact 3-manifold. A *boundary pattern* for M consists of a set $\underline{\underline{m}}$ of compact connected surfaces in ∂M such that the intersection of any i of them is empty or consists of $(3 - i)$-manifolds. A boundary pattern is said to be *complete* when $\bigcup_{B \in \underline{\underline{m}}} B = \partial M$.

A boundary pattern $\underline{\underline{m}}$ of a 3-manifold M is said to be *useful* if any disk D properly embedded in M, where ∂D intersects ∂B transversely for each $B \in \underline{\underline{m}}$ and where $\#\big(D \cap \big(\bigcup_{B \in \underline{\underline{m}}} \partial B\big)\big) \leq 3$, bounds a disk E in ∂M such that $E \cap \big(\bigcup_{B \in \underline{\underline{m}}} \partial B\big)$ is the cone on $\partial D \cap \big(\bigcup_{B \in \underline{\underline{m}}} \partial B\big)$.

A 3-manifold M with a complete boundary pattern $\underline{\underline{m}}$ is said to be *simple* if it satisfies the following three conditions:

(1) M is irreducible and each element B of $\underline{\underline{m}}$ is incompressible;

(2) every incompressible torus in M is parallel to an element of $\underline{\underline{m}}$ that is a torus; and

(3) every incompressible annulus A in M with $\partial A \cap \left(\bigcup_{B \in \underline{\underline{m}}} \partial B\right) = \emptyset$ is parallel into an element B of $\underline{\underline{m}}$.

The mapping class group of a manifold M with a boundary pattern $\underline{\underline{m}}$, denoted by $\mathrm{MCG}(M, \underline{\underline{m}})$, is the group $\mathrm{MCG}(M, B_1, B_2, \ldots, B_k)$ if $\underline{\underline{m}} = \{B_1, B_2, \ldots, B_k\}$.

THEOREM 1.3 [14, Prop. 27.1]. *Let $(M, \underline{\underline{m}})$ be a simple 3-manifold with complete and useful boundary pattern. Then $\mathrm{MCG}(M, \underline{\underline{m}})$ is finite.*

2. Positive Topological Symmetry Groups and Positive Mapping Class Groups

Let V be a handlebody and let $D_1, D_2, \ldots, D_n$ be mutually disjoint and mutually nonparallel essential disks in V. Suppose that $V \setminus \left(\bigcup_{i=1}^{n} \mathrm{Int}\, N(D_i; V)\right)$ consists of 3-balls. Then there exists a graph Γ embedded in $\mathrm{Int}\, V$ such that:

(1) Γ is a deformation retract of V;
(2) Γ intersects $\bigcup_{i=1}^{n} D_i$ transversely at the points in the interior of the edges of Γ; and
(3) Γ has exactly n edges $e_1, e_2, \ldots, e_n$ and $\#(e_i \cap D_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

We call such a graph Γ the *dual graph* of $\{D_1, D_2, \ldots, D_n\}$. Note that a dual graph is uniquely determined up to isotopy.

LEMMA 2.1 [16]. *Let V be a handlebody. Let Γ be a graph, embedded in V, that is a deformation retract of V. Then the group $\mathrm{MCG}(V, \Gamma \text{ rel } \partial V)$ is trivial.*

Let Γ be a spatial graph in S^3. Set $V = N(\Gamma)$. Let

$$\Delta = \{D_1, D_2, \ldots, D_{n_1}, D'_1, D'_2, \ldots, D'_{n_2}, D''_1, D''_2, \ldots, D''_{n_3}\}$$

be the family of essential disks in V such that

- Γ is a dual graph of Δ,
- the set $\{D_1, D_2, \ldots, D_{n_1}\}$ corresponds to the set of loops of Γ,
- the set $\{D'_1, D'_2, \ldots, D'_{n_2}\}$ corresponds to the set of cut edges of Γ, and
- the set $\{D''_1, D''_2, \ldots, D''_{n_3}\}$ corresponds to the set of non-loop, non-cut edges of Γ.

For each $1 \leq i \leq n_1$ we take parallel copies $D_{i,1}$, $D_{i,2}$, $D_{i,3}$, and $D_{i,4}$ of D_i, and for each $1 \leq i \leq n_3$ we take parallel copies $D''_{i,1}$ and $D''_{i,2}$ of D''_i; hence all the disks $D_{i,j}$, D', and $D''_{i,j}$ are mutually disjoint.

Denote by Δ' the collection of disks

$$\{D_{i,j} \mid 1 \leq i \leq n_1,\ j = 1,2,3,4\} \cup \{D'_1, D'_2, \ldots, D'_{n_1}\} \cup \{D''_{i,j} \mid 1 \leq i \leq n_3,\ j = 1,2\}.$$

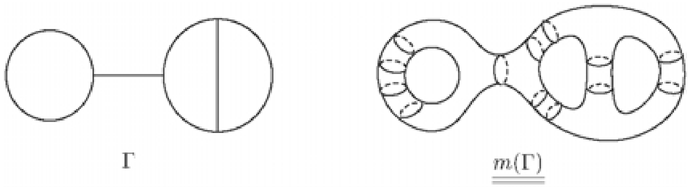

Figure 1 A spatial graph Γ and the corresponding boundary pattern $\underline{\underline{m(\Gamma)}}$

Let $\underline{\underline{m(\Gamma)}}$ be the set consisting of the closure of each component of $\partial N(\Gamma) \setminus \bigcup_{D\in\Delta'} \partial D$. Then $\underline{\underline{m(\Gamma)}}$ is a complete and useful boundary pattern of $E(\Gamma) = E(V)$; see Figure 1.

Lemma 2.2. *Let $\Gamma \subset S^3$ be a spatial graph. Then*

$$\mathrm{MCG}(S^3, \Gamma) \cong \mathrm{MCG}(E(\Gamma), \underline{\underline{m(\Gamma)}}),$$

$$\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{MCG}_+(E(\Gamma), \underline{\underline{m(\Gamma)}}).$$

Proof. Set $V = N(\Gamma)$. Let Δ' be the collection of essential disks in V defined as before. By Lemma 2.1, we see at once that $\mathrm{MCG}(S^3, \Gamma) \cong \mathrm{MCG}\big(S^3, V, \bigcup_{D\in\Delta'} D\big)$ $\big($resp., $\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{MCG}_+\big(S^3, V, \bigcup_{D\in\Delta'} D\big)\big)$ and $\mathrm{MCG}\big(S^3, V, \bigcup_{D\in\Delta'} D\big) \cong \mathrm{MCG}(E(\Gamma), \underline{\underline{m(\Gamma)}})$ $\big($resp., $\mathrm{MCG}_+\big(S^3, V, \bigcup_{D\in\Delta'} D\big) \cong \mathrm{MCG}_+(E(\Gamma), \underline{\underline{m(\Gamma)}})\big)$ by using Alexander's trick. Hence the assertion follows. $\square$

By a *handlebody-knot* we shall mean a handlebody V embedded in the 3-sphere. We note that the exterior $E(V)$ of V is always irreducible. We recall that a 3-manifold is said to be *atoroidal* if it contains no embedded torus that is non-boundary parallel and incompressible.

Lemma 2.3. *Let $V \subset S^3$ be a handlebody-knot of genus at least* 2. *Assume that $E(V)$ is atoroidal. Then* $\mathrm{MCG}_+(E(V) \operatorname{rel} \partial E(V)) = 1$.

Proof. Let Γ be a spatial graph in S^3 such that $N(\Gamma) = V$ and $(E(V), \underline{\underline{m}})$ is simple, where $\underline{\underline{m}} = \underline{\underline{m(\Gamma)}}$. Such a graph can be taken, for instance, as follows. Let g be the genus of V. If $g = 2$, take Γ as a *θ-curve* (i.e., a graph on two vertices with three edges joining them); if $g > 2$, take Γ as a *wheel graph* with $g + 1$ vertices (i.e., the 1-skeleton of a g-gonal pyramid). By Theorem 1.3, $\mathrm{MCG}_+(E(V), \underline{\underline{m}})$ is finite.

Consider the following exact sequence, which corresponds to the fibration $\mathrm{Homeo}_+(E(V)) \to \mathrm{Homeo}_+(\partial E(V))$:

$$\cdots \to \pi_1(\mathrm{Homeo}_+(\partial E(V))) \to \mathrm{MCG}_+(E(V) \operatorname{rel} \partial E(V)) \to \mathrm{MCG}_+(E(V))$$
$$\to \mathrm{MCG}_+(\partial E(V)) \to 0.$$

Since the genus of $\partial E(V)$ is at least 2, we have

$$\pi_1(\mathrm{Homeo}_+(\partial E(V))) = 1$$

by [5; 6; 7]. It follows that the map $\mathrm{MCG}_+(\partial E(V) \text{ rel } \partial E(V)) \to \mathrm{MCG}_+(E(V))$ in the preceding sequence is an injection. On the other hand, the sequence of inclusions

$$\mathrm{Homeo}_+(E(V) \text{ rel } \partial E(V)) \subset \mathrm{Homeo}_+(E(V), \underline{\underline{m}}) \subset \mathrm{Homeo}_+(E(V))$$

induces the following sequence of homomorphisms:

$$\mathrm{MCG}_+(E(V) \text{ rel } \partial E(V)) \to \mathrm{MCG}_+(E(V), \underline{\underline{m}}) \to \mathrm{MCG}_+(E(V)).$$

Since the homomorphism $\mathrm{MCG}_+(E(V) \text{ rel } \partial E(V)) \to \mathrm{MCG}_+(E(V))$ is an injection and since $\mathrm{MCG}_+(E(V), \underline{\underline{m}})$ is finite, the group $\mathrm{MCG}_+(E(V) \text{ rel } \partial E(V))$ is also finite. By [12] (see also [9]), the group $\mathrm{MCG}_+(E(V) \text{ rel } \partial E(V))$ is torsion free; hence $\mathrm{MCG}_+(E(V) \text{ rel } \partial E(V)) = 1$. □

REMARK. In this proof, when working in the piecewise linear category we use results for the homotopy groups of the automorphism groups of 2-manifolds and 3-manifolds in the topological and smooth categories. However, in dimensions not exceeding 3, it has been shown that the information on the homotopy types of automorphism groups of manifolds is the same as in the smooth, topological, and piecewise linear categories (see e.g. [8]).

Because the group $\mathrm{MCG}(V \text{ rel } \partial V)$ is trivial, Lemma 2.3 immediately implies the following result for a handlebody V.

COROLLARY 2.4. *Let V be a handlebody-knot in S^3 whose genus is at least 2. Then $\mathrm{MCG}_+(S^3 \text{ rel } V) \cong 1$ if and only if $E(V)$ is atoroidal.*

This corollary implies that $\mathrm{MCG}_+(S^3, V)$ can be regarded as a subgroup of the handlebody group $\mathrm{MCG}_+(V)$ when $E(V)$ is atoroidal.

DEFINITION. Let Γ be a spatial graph in S^3.

(1) Let P be a 2-sphere that is embedded in S^3 and satisfies:
- the sphere P intersects Γ in a single vertex; and
- each of the two components of $S^3 \setminus P$ contains the nonempty part of Γ.

Then P is called a *type I sphere* for Γ. See the left part of Figure 2.

(2) Let P be a 2-sphere embedded in S^3 and satisfying:
- the sphere P intersects Γ in exactly two vertices,
- the closure of neither component of $(S^3 \setminus P) \cap \Gamma$ is a single point or a single edge; and
- the annulus $P \cap E(\Gamma)$ is incompressible in $E(\Gamma)$.

Then P is called a *type II sphere* for Γ. See the middle part of Figure 2.

(3) A 2-sphere with two points identified to a single point is called a *pinched sphere,* and the identified point is called its *pinch point*. Let P be a pinched sphere in S^3 with a pinch point p satisfying:

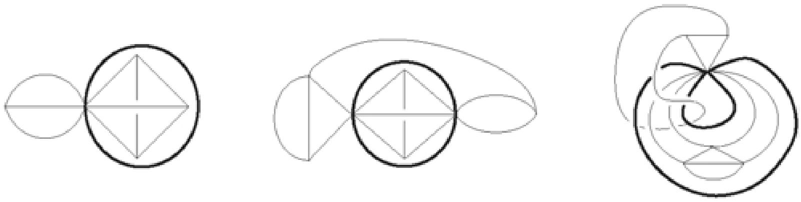

Figure 2 Spheres of type I, II, and III

- the pinch point p is a vertex of Γ such that $P \cap \Gamma = \{p\}$;
- the closure of neither component of $(S^3 \setminus P) \cap \Gamma$ is a single point or a single loop; and
- the annulus $P \cap E(\Gamma)$ is incompressible in $E(\Gamma)$.

Then Γ is called a *type III sphere* for Γ. See the right part of Figure 2.

THEOREM 2.5. *Let Γ be a spatial graph in S^3 that is not a knot. Then*

$$\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma)$$

if and only if $E(\Gamma)$ is atoroidal and Γ does not admit spheres of type I, II, or III.

Proof. The "only if" part is straightforward.

Let Γ be a spatial graph in S^3 such that $E(\Gamma)$ is atoroidal and Γ admits neither type I, II, nor III spheres. By Lemma 1.2, it suffices to show that $\mathrm{MCG}_+(S^3 \text{ rel } \Gamma) = 1$. Set $\underline{\underline{m}} = \underline{\underline{m(\Gamma)}}$. Recall that $\mathrm{MCG}_+(E(\Gamma), \underline{\underline{m}})$ is a finite group by Theorem 1.3.

Let $\mathcal{A} = \{A_1, A_2, \dots, A_{n'}\}$ (resp., $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$) be the set of annulus (resp., non-annulus) components of $\underline{\underline{m}}$. We note that each B_i is a planar surface. Set $\partial B_i = C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_{n_i}}$.

Suppose that $\mathrm{MCG}_+(E(\Gamma), \underline{\underline{m}})$ is not trivial. Then there exists an element $f \in \mathrm{Homeo}_+(E(\Gamma), \underline{\underline{m}})$ that is not isotopic to the identity. The proof is divided into two cases.

Case 1: For each $B_i \in \mathcal{B}$, $f|_{B_i}$ is trivial as an element of $\mathrm{MCG}_+(B_i, C_{i_1}, C_{i_2}, \dots, C_{i_{n_i}})$. We may assume that $f|_{\bigcup_{i=1}^m B_i}$ is the identity. Then there exists a set of pairwise disjoint and pairwise nonparallel essential simple closed curves $\gamma_1, \gamma_2, \dots, \gamma_l$ in $\bigcup_{i=1}^{n'} A_i$ such that $f|_{\partial E(\Gamma)}$ is $\prod_{i=1}^{l} \tau_{\gamma_i}^{\alpha_i}$ as an element of $\mathrm{MCG}_+(\partial E(\Gamma))$, where τ_{γ_i} is a Dehn twist along γ_i and α_i is a nonzero integer. By Lemma 2.3, $\prod_{i=1}^{l} \tau_{\gamma_i}^{\alpha_i} \neq 1 \in \mathrm{MCG}_+(\partial E(\Gamma))$. It follows that the order of $f|_{\partial E(\Gamma)}$ as an element of $\mathrm{MCG}(\partial E(\Gamma))$ is infinite because each γ is a boundary of a meridian disk of $N(\Gamma)$. This contradicts the fact that $\mathrm{MCG}_+(E(\Gamma), \underline{\underline{m}})$ is a finite group.

Case 2: There exists an element $B_i \in \mathcal{B}$ such that $f|_{B_i}$ is not trivial as an element of $\mathrm{MCG}_+(B_i, C_{i_1}, C_{i_2}, \dots, C_{i_{n_i}})$. Since the group $\mathrm{MCG}_+(B_i, C_{i_1}, C_{i_2}, \dots, C_{i_{n_i}})$ is torsion free by [2], this again contradicts the fact that $\mathrm{MCG}_+(E(\Gamma), \underline{\underline{m}})$ is a finite group. □

Let $\Gamma \subset S^3$ be a spatial graph. If Γ admits either type I, II, or III spheres then $\mathrm{MCG}_+(S^3, \Gamma)$ is not a finite group, since $\mathrm{MCG}_+(E(\Gamma), \underline{\underline{m(\Gamma)}})$ admits nontrivial twists along essential disks or annuli corresponding to the spheres. Hence we have our next corollary.

COROLLARY 2.6. *Let* Γ *be a spatial graph in* S^3. *Then* $\mathrm{MCG}_+(S^3, \Gamma)$ *is a finite group if and only if* $\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma)$.

It follows from Theorem 2.5 and [3] that the group $\mathrm{MCG}_+(S^3, \Gamma)$ is a finite subgroup of SO(4) when (a) $E(\Gamma)$ is atoroidal and (b) Γ admits neither type I, II, nor III spheres.

Let k be a natural number. Recall that a graph Γ is said to be *k-connected* if there does not exist a set $\{v_1, v_2, \ldots, v_{k-1}\}$ of $k-1$ vertices of Γ such that $\Gamma \setminus \{v_1, v_2, \ldots, v_{k-1}\}$ is not connected as a topological space.

The next corollary follows immediately from Theorem 2.5.

COROLLARY 2.7. *Let* Γ *be a* 3*-connected graph. Then an embedding of* Γ *into* S^3 *satisfies* $\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma)$ *if and only if* Γ *is atoroidal.*

We note that even when Γ is not 3-connected, most embeddings of Γ into S^3 satisfy $\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma)$. See Section 4.

3. Topological Symmetry Groups and Mapping Class Groups

In Section 2 we discussed a topological condition for a spatial graph $\Gamma \subset S^3$ such that the positive mapping class group $\mathrm{MCG}_+(S^3, \Gamma)$ is isomorphic to the positive topological symmetry group $\mathrm{TSG}_+(S^3, \Gamma)$. Of course, even when $\mathrm{MCG}_+(S^3, \Gamma)$ is isomorphic to $\mathrm{TSG}_+(S^3, \Gamma)$, it may be that $\mathrm{MCG}(S^3, \Gamma)$ differs from $\mathrm{TSG}(S^3, \Gamma)$. A trivial example is a spatial 3-connected graph $\Gamma \subset S^3$ contained in an embedded 2-sphere S^2 in S^3. In this case, there exists a reflection f through the 2-sphere; then f fixes the 2-sphere pointwise and, in particular, restricts to an identity on Γ. Obviously, f is orientation reversing and hence $\mathrm{MCG}(S^3 \operatorname{rel} \Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ whereas $\mathrm{MCG}_+(S^3 \operatorname{rel} \Gamma) \cong 1$. By Lemma 1.2, this implies that $\mathrm{MCG}(S^3, \Gamma) \not\cong \mathrm{TSG}(S^3, \Gamma)$ whereas $\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma)$. In this section, we prove that this is the only case.

PROPOSITION 3.1. *Let* Γ *be a spatial graph in* S^3 *such that* Γ *is not a knot. Let* h *be an orientation-reversing homeomorphism of* S^3 *fixing* Γ *pointwise such that* $h^2 \in \mathrm{Homeo}_+(S^3 \operatorname{rel} \Gamma)$ *is isotopic* $(\operatorname{rel} \Gamma)$ *to the identity. Then there exists a homeomorphism* $f \in \mathrm{Homeo}_+(S^3 \operatorname{rel} \Gamma)$ *such that* h *is isotopic* $(\operatorname{rel} \Gamma)$ *to* f *and such that* $f^2 = \mathrm{id}$.

Proof. By an isotopy $(\operatorname{rel} \Gamma)$ we may assume that $h(N(\Gamma)) = N(\Gamma)$. Since $E(\Gamma)$ is not a Seifert fibered 3-manifold, it follows from [10; 11] that h can be isotoped $(\operatorname{rel} \Gamma)$ to a map $h_1\colon (S^3, \Gamma) \to (S^3, \Gamma)$ with $h_1(E(\Gamma)) = E(\Gamma)$ and $(h_1|_{E(\Gamma)})^2 = \mathrm{id}_{E(\Gamma)}$. Let $e_1, e_2, \ldots, e_n$ be the edges of Γ. Using a standard argument of Riemannian geometry, it is easy to see that, for each edge e_i of Γ, there exists a meridian disk D_i of $N(\Gamma)$ such that $h_1(\partial D_i) = \partial D_i$ and D_i intersects Γ once and transversely at a point p_i in $\operatorname{Int} e_i$. Note that $h_1|_{\partial D_i} = \mathrm{id}_{D_i}$ for each i. Since D_i and $h_1(D_i)$ are parallel in $N(\Gamma)$, it follows that h_1 can be isotoped (rel $E(\Gamma) \cup \Gamma$) to a map $h_2\colon (S^3, \Gamma) \to (S^3, \Gamma)$ with $h_2(D_i) = D_i$ for each i. Then, by Alexander's trick, we may isotope h_2 (rel $E(\Gamma) \cup \Gamma$) while preserving each meridian disk D_i as a set to a map $h_3\colon (S^3, \Gamma) \to (S^3, \Gamma)$ with $(h_3|_{D_i})^2 = \mathrm{id}_{D_i}$ for each i. Finally, let B be the closure of a component of $N(\Gamma) \setminus \bigcup_i D_i$. Since $B \cap \Gamma$ is the cone over points in ∂B with vertex at the center of B, we may isotope h_3 (rel $E(B)$) to a map h_4 with $(h_4|_B)^2 = \mathrm{id}_B$. Performing this isotopy for the closure of each component of $N(\Gamma) \setminus \bigcup_i D_i$, we see that h_4 can be isotoped $\big(\text{rel } E(\Gamma) \cup \Gamma \cup \bigcup_i D_i\big)$ to a required orientation-reversing involution $f\colon (S^3, \Gamma) \to (S^3, \Gamma)$. $\square$

A spatial graph $\Gamma \subset S^3$ is called a *plane graph* if Γ is embedded in a sphere S^2 in S^3.

THEOREM 3.2. *Let Γ be a spatial graph in S^3 such that Γ is not a knot and $\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma)$. Then $\mathrm{MCG}(S^3, \Gamma) \cong \mathrm{TSG}(S^3, \Gamma)$ if and only if Γ is not a plane graph.*

Proof. The "only if" part is clear.

Assume that $\mathrm{MCG}(S^3, \Gamma) \ncong \mathrm{TSG}(S^3, \Gamma)$. This is equivalent to saying that $\mathrm{MCG}(S^3 \text{ rel } \Gamma) \neq 1$ and hence that $\mathrm{MCG}(S^3 \text{ rel } \Gamma) \cong \mathbb{Z}/2\mathbb{Z}$. Then there exists an orientation-reversing homeomorphism $h \in \mathrm{Homeo}(S^3 \text{ rel } \Gamma)$ such that $h^2 \in \mathrm{Homeo}_+(S^3 \text{ rel } \Gamma)$ is isotopic (rel Γ) to the identity.

By Proposition 3.1, h can be isotoped (rel Γ) to an orientation-reversing involution $f\colon (S^3, \Gamma) \to (S^3, \Gamma)$ with $\Gamma \subset \mathrm{Fix}(f)$, where $\mathrm{Fix}(f)$ for a map $f\colon S^3 \to S^3$ is the set of fixed points of f. Since the fixed point set of an orientation-reversing involution of S^3 is either a 2-sphere or two points by Smith theory ([20]; see also [13]), $\mathrm{Fix}(f)$ is the 2-sphere. This implies that Γ is a plane graph. $\square$

REMARK. Theorem 3.2 implies that $\mathrm{MCG}(S^3 \text{ rel } \Gamma) \neq \mathrm{MCG}_+(S^3 \text{ rel } \Gamma)$ if and only if Γ is a plane graph under the condition $\mathrm{MCG}_+(S^3 \text{ rel } \Gamma) \cong 1$. Finding a general condition capable of detecting when $\mathrm{MCG}(S^3, \Gamma)$ differs from $\mathrm{MCG}_+(S^3, \Gamma)$ is another interesting problem (see e.g. [18]).

Let Γ be the spatial graph depicted in Figure 3. It is easily seen that Γ is not a plane graph and that $\mathrm{MCG}(S^3, \Gamma)$ differs from $\mathrm{MCG}_+(S^3, \Gamma)$.

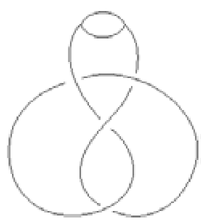

Figure 3

4. Easy Examples

One of the simple but important examples of spatial graphs in S^3 is a tunnel number 1 knot or link together with a specific tunnel attached. A *tunnel number 1 knot* (or link) in S^3 is a knot (or link) K with an arc τ, called a *tunnel* for K, such that $K \cap \tau = K \cap \partial\tau$ and $E(K \cup \tau)$ is a genus-2 handlebody. Allowing that $\partial\tau$ could be a single point (i.e., a tunnel could be a circle rather than an arc), the spatial graph $\Gamma = K \cup \tau$ is either a θ-curve or a bouquet of two circles if K is a knot, or a handcuff if K is a link (a tunnel number 1 link necessarily consists of at most two components).

In this section, we give a complete list of the groups $\mathrm{MCG}_+(S^3, \Gamma)$, $\mathrm{MCG}(S^3, \Gamma)$, $\mathrm{TSG}_+(S^3, \Gamma)$, and $\mathrm{TSG}(S^3, \Gamma)$ for every spatial graph Γ that is the union of a tunnel number 1 knot or link K and its tunnel τ. The following lemma is just a direct translation of [1, Prop. 17.2] to our cases.

LEMMA 4.1. *Let Γ be the union of a nontrivial tunnel number 1 knot or link K and its tunnel τ in S^3, which is either a θ-curve or a handcuff.*

(1) *If K is a Hopf link and τ is its upper or lower tunnel when K is considered as a* 2*-bridge link, then* $\mathrm{MCG}(S^3, \Gamma)$ *is isomorphic to the dihedral group D_4 of order 8.*
(2) *If K is a nontrivial* 2*-bridge knot or link that is not a Hopf link and if τ is its upper or lower tunnel, then* $\mathrm{MCG}(S^3, \Gamma)$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
(3) *Otherwise,* $\mathrm{MCG}(S^3, \Gamma)$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.

From Lemma 4.1 and Theorems 2.5 and 3.2 with some computations, the following proposition is immediate. We note that the plane handcuff and the plane bouquet admit the spheres of type I whereas the other embeddings of these graphs considered in this proposition do not.

PROPOSITION 4.2. *Let Γ be the union of a tunnel number* 1 *knot or link and its tunnel in S^3.*

(1) *Let Γ be a θ-curve.*
   (a) *If Γ is a plane graph, then*
   $$\mathrm{MCG}_+(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma) = \mathrm{TSG}(S^3, \Gamma) = \mathrm{Aut}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \times D_3,$$
   $$\mathrm{MCG}(S^3, \Gamma) \cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{MCG}_+(S^3, \Gamma),$$
   where D_3 is the dihedral group of order 6.
   (b) *If Γ is the union of a nontrivial* 2*-bridge knot and its upper or lower tunnel, then*
   $$\begin{aligned}\mathrm{MCG}_+(S^3, \Gamma) = \mathrm{MCG}(S^3, \Gamma) &\cong \mathrm{TSG}_+(S^3, \Gamma) = \mathrm{TSG}(S^3, \Gamma) \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$
   (c) *Otherwise,*
   $$\mathrm{MCG}_+(S^3, \Gamma) = \mathrm{MCG}(S^3, \Gamma) \cong \mathrm{TSG}_+(S^3, \Gamma) = \mathrm{TSG}(S^3, \Gamma) \cong \mathbb{Z}/2\mathbb{Z}.$$

(2) *Let Γ be a handcuff.*

(a) *If Γ is a plane graph, then*

$$\mathrm{MCG}_+(S^3,\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}),$$
$$\mathrm{MCG}(S^3,\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{MCG}_+(S^3,\Gamma);$$
$$\mathrm{TSG}_+(S^3,\Gamma) = \mathrm{TSG}(S^3,\Gamma) = \mathrm{Aut}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

In this case, $\mathrm{MCG}_+(S^3,\Gamma)$ *and* $\mathrm{MCG}(S^3,\Gamma)$ *are not finite whereas* $\mathrm{TSG}_+(S^3,\Gamma)$ *and* $\mathrm{TSG}(S^3,\Gamma)$ *are finite and of order* 8.

(b) *If Γ is the union of a Hopf link and its upper or lower tunnel, then*

$$\mathrm{MCG}_+(S^3,\Gamma) \cong \mathrm{TSG}_+(S^3,\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$
$$\mathrm{MCG}(S^3,\Gamma) \cong \mathrm{TSG}(S^3,\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{MCG}_+(S^3,\Gamma) \cong D_4.$$

(c) *If Γ is the union of any nontrivial* 2*-bridge link* (*except the Hopf link*) *and its upper or lower tunnel, then*

$$\begin{aligned}\mathrm{MCG}_+(S^3,\Gamma) = \mathrm{MCG}(S^3,\Gamma) \cong \mathrm{TSG}_+(S^3,\Gamma) &= \mathrm{TSG}(S^3,\Gamma)\\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

(d) *Otherwise,*

$$\mathrm{MCG}_+(S^3,\Gamma) = \mathrm{MCG}(S^3,\Gamma) \cong \mathrm{TSG}_+(S^3,\Gamma) = \mathrm{TSG}(S^3,\Gamma) \cong \mathbb{Z}/2\mathbb{Z}.$$

(3) *Finally, let Γ be a spatial bouquet of two circles. Then the following statements hold.*

(a) *If Γ is a plane graph, then*

$$\mathrm{MCG}_+(S^3,\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}),$$
$$\mathrm{MCG}(S^3,\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{MCG}_+(S^3,\Gamma);$$
$$\mathrm{TSG}_+(S^3,\Gamma) = \mathrm{TSG}(S^3,\Gamma) = \mathrm{Aut}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

(b) *Otherwise,*

$$\mathrm{MCG}_+(S^3,\Gamma) = \mathrm{MCG}(S^3,\Gamma) \cong \mathrm{TSG}_+(S^3,\Gamma) = \mathrm{TSG}(S^3,\Gamma) \cong \mathbb{Z}/2\mathbb{Z}.$$

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