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Characteristic Polynomials, η-Complexes, and Freeness of Tame Arrangements

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1. Introduction

We use notation from Section 2 to state the main result of this paper. Let $\mathcal{A}$ be a central ℓ-arrangement over an arbitrary field $\mathbb{K}$. Fix $H_0 \in \mathcal{A}$, and let $(\mathcal{A}'', m)$ be the Ziegler restriction of $\mathcal{A}$ onto H_0. Let $d\mathcal{A}$ be the deconing of $\mathcal{A}$ with respect to H_0. Let

$$\chi_0(\mathcal{A}, t) = \chi(d\mathcal{A}, t) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} b_{\ell-1-i} t^i$$

be a reduced characteristic polynomial of $\mathcal{A}$, and let

$$\chi(\mathcal{A}'', m, t) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \sigma_{\ell-1-i} t^i$$

be a characteristic polynomial of $(\mathcal{A}'', m)$. Note that $\chi_0(\mathcal{A}, t)$ is defined combinatorially and $\chi(\mathcal{A}'', m)$ algebraically. It is well known that $b_0 = \sigma_0 = 1$ and $b_1 = \sigma_1 = |\mathcal{A}| - 1 = |m|$ (use Theorem 2.3, for example). The inequality $b_2 \geq \sigma_2$ has recently been proved, and the equality $b_2 = \sigma_2$ is closely related to the freeness of $\mathcal{A}$ [2, Thm. 5.1]. This is a generalization of Yoshinaga's freeness criterion for 3-arrangements [13, Thm. 3.2]. Also, it is known that $b_i = \sigma_i$ for $i = 0, 1, \dots, \ell-1$ when $\mathcal{A}$ is a free arrangement (see the proof of Corollary 1.2). Hence it is natural to ask whether $b_i \geq \sigma_i$ holds for $i \geq 3$ and whether or not the equality is related to freeness. In fact, we do not know whether σ_i is nonnegative for $i \geq 3$. In this paper, we assume tameness and give the following answer.

Theorem 1.1. *Let $\mathcal{A}$ be a central ℓ-arrangement. Fix $H_0 \in \mathcal{A}$ and let $(\mathcal{A}'', m)$ be the Ziegler restriction of $\mathcal{A}$ with respect to H_0. If $\mathcal{A}$ and $(\mathcal{A}'', m)$ are both tame, then $b_i \geq \sigma_i \geq 0$ $(i = 0, 1, \dots, \ell-1)$.*

Theorem 1.1 gives a lower bound of $|\chi_0(\mathcal{A}, -1)|$ in terms of $\chi(\mathcal{A}'', m, -1)$; in particular, $|\chi_0(\mathcal{A}, -1)| \geq |\chi(\mathcal{A}'', m, -1)|$. Note that $|\chi_0(\mathcal{A}, -1)|$ is the number of chambers when $\mathbb{K} = \mathbb{R}$. In the category of tame arrangements, then, we say that

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$\mathcal{A}$ is a *minimal chamber arrangement* (MCA for short) if $(-1)^{\ell-1}\chi_0(\mathcal{A},-1) = (-1)^{\ell-1}\chi(\mathcal{A}'',m,-1)$. When $\ell = 3$, we can always define an MCA by Yoshinaga's criterion. Also, by that criterion, a 3-arrangement $\mathcal{A}$ is a free arrangement if and only if it is an MCA. As a corollary of Theorem 1.1, we can generalize these characterizations in terms of an MCA as follows.

COROLLARY 1.2. *With the same assumptions and notation as in Theorem 1.1,*

$$(-1)^{\ell-1}\chi_0(\mathcal{A},-1) \geq (-1)^{\ell-1}\chi(\mathcal{A}'',m,-1) \geq 0.$$

Moreover, $\mathcal{A}$ is free if and only if $\mathcal{A}$ is an MCA and $(\mathcal{A}'',m)$ is free.

By Corollary 1.2, if we fix a free multiarrangement $(\mathcal{A}'',m)$ and consider

$$F_t(\mathcal{A}'',m) := \{\mathcal{A} : \text{a tame } \ell\text{-arrangement} \mid \text{the Ziegler restriction of } \mathcal{A} \text{ is } (\mathcal{A}'',m)\},$$

then $\mathcal{A} \in F_t(\mathcal{A}'',m)$ is free if and only if $\mathcal{A}$ is an MCA. Also, for $\mathcal{A} \in F_t(\mathcal{A}'',m)$, the inequalities $(-1)^{\ell-1}\chi_0(\mathcal{A},-1) \geq (-1)^{\ell-1}\chi(\mathcal{A}'',m,-1) \geq 0$ give the lower bound on the number of chambers when $\mathbb{K} = \mathbb{R}$ as well as on the total Betti number of $\mathbb{K}^\ell \setminus \bigcup_{H \in d\mathcal{A}} H$ when $\mathbb{K} = \mathbb{C}$.

For the proof of Theorem 1.1 and Corollary 1.2, we study the multiversion of the η-complex. The η-complex was originally introduced in [10] and then developed in [7] and [12] for simple arrangements. In the proofs, we also study the properties of several η-complexes.

The rest of the paper proceeds as follows. In Section 2 we introduce several definitions and results used in the sequel. In Section 3 we develop several results for the proof, mainly studying properties of η-complexes. Finally, in Section 4 we prove Theorem 1.1 and Corollary 1.2.

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2. Preliminaries

In this section we introduce several definitions and results that will be needed in the rest of the paper. We use [6] as a general reference.

Let $\mathbb{K}$ be a field and let $V = \mathbb{K}^\ell$. Let $\mathcal{A}$ be an arrangement of affine hyperplanes in V (i.e., a finite set of affine hyperplanes in V). An *ℓ-arrangement* is the arrangement in an ℓ-dimensional vector space. The *intersection lattice* $L(\mathcal{A})$ is a set of affine subspaces consisting of $\bigcap_{H \in \mathcal{B}} H$ for $\mathcal{B} \subset \mathcal{A}$. Let $L(\mathcal{A})$ be a poset with the reverse inclusion order and the unique minimum element V. Define $L_i(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid \operatorname{codim}_V X = i\}$. The *Möbius function* $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is defined by

$$\mu(X) = \begin{cases} 1 & \text{if } X = V, \\ -\sum_{V \supset Y \supsetneq X} \mu(Y) & \text{if } X \neq V. \end{cases}$$

Then a *characteristic polynomial* $\chi(\mathcal{A},t)$ is defined by

$$\chi(\mathcal{A},t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

The arrangement $\mathcal{A}$ is *central* if every $H \in \mathcal{A}$ contains the origin $0 \in V$. Let $\alpha_H \in V^*$ be the defining linear form of $H \in \mathcal{A}$. If $\mathcal{A}$ is central, then the definition of the Möbius function μ implies that $\chi(\mathcal{A},t)$ is divisible by $t-1$. Define a *reduced characteristic polynomial* $\chi_0(\mathcal{A},t)$ by

$$\chi_0(\mathcal{A},t) := \chi(\mathcal{A},t)/(t-1).$$

A central ℓ-arrangement $\mathcal{A}$ is *essential* if $\bigcap_{H\in\mathcal{A}} H = \{0\}$. When $\mathcal{A}$ is a direct product of an essential arrangement $\mathcal{B}$ and an empty arrangement Φ (i.e., $\mathcal{A} \simeq \mathcal{B} \times \Phi$), we say that $\mathcal{B}$ is an *essentialization* of $\mathcal{A}$. For the fixed hyperplane $H_0 \in \mathcal{A}$, the *deconing* $d\mathcal{A}$ of $\mathcal{A}$ with respect to H_0 is defined as $\mathcal{A} \cap \{\alpha_{H_0} = 1\}$, which is an $(\ell-1)$-affine arrangement in H_0. It is well known and easy to check that $\chi_0(\mathcal{A},t) = \chi(d\mathcal{A},t)$. When $V = \mathbb{R}^\ell$, the set of connected components of $V \setminus \bigcup_{H\in\mathcal{A}} H$ is denoted by $C(\mathcal{A})$ and called the set of *chambers.*

Remark 2.1. It is well known that $\pi(\mathcal{A},t) := (-t)^\ell \chi(\mathcal{A},-t^{-1})$ is equal to the topological Poincaré polynomial of $V \setminus \bigcup_{H\in\mathcal{A}} H$ when $V = \mathbb{C}^\ell$. Also, $(-1)^\ell\chi(\mathcal{A},-1) = |C(\mathcal{A})|$ and $|C(d\mathcal{A})| = (-1)^{\ell-1}\chi_0(\mathcal{A},-1)$ when $V = \mathbb{R}^\ell$.

From now on we assume that $\mathcal{A}$ is a central ℓ-arrangement. Let $S := \mathrm{Sym}^*(V^*) = \mathbb{K}[x_1,\dots,x_\ell]$ be a coordinate ring of V. For the module of S-derivations $\mathrm{Der}\, S := \bigoplus_{i=1}^{\ell} S \cdot \partial_{x_i}$, a *module of logarithmic vector fields of* $\mathcal{A}$ is defined by

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}\, S \mid \theta(\alpha_H) \in S\alpha_H \ \forall H \in \mathcal{A}\}.$$

In general, $D(\mathcal{A})$ is a reflexive module. When $D(\mathcal{A})$ is a free S-module with homogeneous basis $\theta_1,\dots,\theta_\ell$ of degrees $d_1,\dots,d_\ell$, we say that $\mathcal{A}$ is *free* with *exponents* $\exp(\mathcal{A}) = (d_1,\dots,d_\ell)$.

A *multiplicity* is a map $m\colon \mathcal{A} \to \mathbb{Z}_{\geq 0}$, and we call a pair $(\mathcal{A},m)$ a *multiarrangement.* When $m \equiv 1$, the multiarrangement $(\mathcal{A},1)$ is equal to the arrangement $\mathcal{A}$, which is said to be a *simple arrangement.* A *module of logarithmic vector fields of* $(\mathcal{A},m)$ is defined by

$$D(\mathcal{A},m) := \{\theta \in \mathrm{Der}\, S \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \ \forall H \in \mathcal{A}\}.$$

The freeness and exponents of a multiarrangement can be defined in the same manner as for a simple arrangement. For $X \in L(\mathcal{A})$, define the *localization* $(\mathcal{A}_X, m_X)$ of $(\mathcal{A},m)$ at X by

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\},$$
$$m_X := m|_{\mathcal{A}_X}.$$

Multiarrangements appear naturally when we restrict a central arrangement onto some hyperplane $H_0 \in \mathcal{A}$. For a central arrangement $\mathcal{A}$ and $H_0 \in \mathcal{A}$, the *Ziegler restriction* $(\mathcal{A}'',m)$ with respect to H_0 is defined by

$$\mathcal{A}'' := \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\},$$
$$m(H \cap H_0) := |\{K \in \mathcal{A} \setminus \{H_0\} \mid K \cap H_0 = H \cap H_0\}|.$$

For the set of Kähler differential p-forms

$$\Omega_S^p := \bigoplus_{1 \le i_1 < \cdots < i_p \le \ell} S\, dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

a *module of logarithmic differential p-forms of* $(\mathcal{A}, m)$ is defined by

$$\Omega^p(\mathcal{A}, m) := \left\{ \omega \in \frac{1}{Q(\mathcal{A}, m)} \Omega_S^p \;\middle|\; \frac{Q(\mathcal{A}, m)}{\alpha_H^{m(H)}}\, d\alpha_H \wedge \omega \in \Omega_S^{p+1} \ \forall H \in \mathcal{A} \right\},$$

where $Q(\mathcal{A}, m) := \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}$. In [1], a *characteristic polynomial of a multiarrangement* is defined algebraically by

$$\chi(\mathcal{A}, m, t) := \lim_{x \to 1} \sum_{p=0}^{\ell} \mathrm{Poin}(\Omega^p(\mathcal{A}, m), x)(t(1-x) - 1)^p,$$

where $\mathrm{Poin}(N, x) := \sum_{k \in \mathbb{Z}} \dim_{\mathbb{K}} N_k x^k$ is the Poincaré series of an S-graded module $N = \bigoplus_{k \in \mathbb{Z}} N_k$.

REMARK 2.2. The definition of $\chi(\mathcal{A}, m, t)$ just given is different from the original one in [1]. The original definition was

$$\chi(\mathcal{A}, m, t) := (-1)^{\ell} \lim_{x \to 1} \sum_{p=0}^{\ell} \mathrm{Poin}(D^p(\mathcal{A}, m), x)(t(x-1) - 1)^p,$$

where $D^p(\mathcal{A}, m)$ is an S-dual module of $\Omega^p(\mathcal{A}, m)$. The equivalence of these two definitions was proved in Remark 2.3 of [2]. So we use the definition by differential forms in this article.

Related to these characteristic polynomials, the following local-to-global formula is useful for computing each coefficient.

THEOREM 2.3 [1, Thm. 3.3]. *Put*

$$\chi(\mathcal{A}, m, t) = \sum_{i=0}^{\ell} (-1)^{\ell - i} \sigma_{\ell - i} t^i,$$

$$\chi(\mathcal{A}_X, m_X, t) = t^{\ell - k} \sum_{i=0}^{k} (-1)^{k-i} \sigma_{k-i}^X t^i \quad (X \in L_k(\mathcal{A})).$$

Then $\sigma_k = \sum_{X \in L_k(\mathcal{A})} \sigma_k^X$.

A multiarrangement $(\mathcal{A}, m)$ is *tame* if

$$\mathrm{pd}_S\, \Omega^p(\mathcal{A}, m) \le p \quad (p = 0, 1, \ldots, \ell),$$

where $\mathrm{pd}_S\, \Omega^p(\mathcal{A}, m)$ is the projective dimension of the S-module $\Omega^p(\mathcal{A}, m)$. For example, free arrangements are tame because $\Omega^p(\mathcal{A}, m) \simeq \bigwedge^p \Omega^1(\mathcal{A}, m)$ is free when $\mathcal{A}$ is free. Also, the tameness of generic arrangements was proved in [8]. Tame arrangements were introduced in [7] and named in [12]; they now play an important role in several research areas of arrangements (see, e.g., [3; 4; 9]).

For a central $(\ell-1)$-arrangement $\mathcal{A}''$ and a multiplicity m on $\mathcal{A}''$, assume that $(\mathcal{A}'', m)$ is tame. Let $F_t(\mathcal{A}'', m)$ be the set of central tame ℓ-arrangements whose Ziegler restrictions are equal to $(\mathcal{A}'', m)$. Assume that

$$(-1)^{\ell-1}\chi_0(\mathcal{A}, -1) \geq (-1)^{\ell-1}\chi(\mathcal{A}'', m, -1) \geq 0$$

for all $\mathcal{A} \in F_t(\mathcal{A}'', m)$. Then we say that $\mathcal{A} \in F_t(\mathcal{A}'', m)$ is a *minimal chamber arrangement* (MCA) if

$$(-1)^{\ell-1}\chi_0(\mathcal{A}, -1) = \min_{\mathcal{B} \in F_t(\mathcal{A}'', m)} (-1)^{\ell-1}\chi_0(\mathcal{B}, -1) = (-1)^{\ell-1}\chi(\mathcal{A}'', m, -1).$$

For $D(\mathcal{A}, m) \ni \theta$ and $\Omega^p(\mathcal{A}, m) \ni \omega = \sum g_{i_1,\dots,i_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, define a *contraction*

$$\langle \theta, \omega \rangle := \sum (-1)^{j-1}\theta(x_{i_j}) g_{i_1,\dots,i_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_{j-1}} \wedge dx_{i_{j+1}} \wedge \cdots \wedge dx_{i_p}.$$

If η is a homogeneous p-form, then clearly

$$\langle \theta, \eta \wedge \omega \rangle = \langle \theta, \eta \rangle \wedge \omega + (-1)^p \eta \wedge \langle \theta, \omega \rangle.$$

The following result is a generalized Yoshinaga's freeness criterion.

THEOREM 2.4 [2, Thm. 5.1]. *Let $\mathcal{A}$ be an arrangement and $(\mathcal{A}'', m)$ the Ziegler restriction. Then* (a) $b_2 \geq \sigma_2 \geq 0$ *and* (b) *$\mathcal{A}$ is free if and only if $(\mathcal{A}'', m)$ is free and $b_2 = \sigma_2$ (with notation as in Section 1).*

The *combinatorial restriction map* (introduced in [2, Sec. 4]) is crucial for the proof of Theorem 1.1.

PROPOSITION 2.5. *Let $\mathcal{A}$ be a central ℓ-arrangement, let $H_0 \in \mathcal{A}$, and let $(\mathcal{A}'', m)$ be the Ziegler restriction. Then there is a well-defined map $\rho\colon L(d\mathcal{A}) \to L(\mathcal{A}'')$ that keeps inclusion orders and codimensions of each flat, so ρ induces $\rho_i\colon L_i(d\mathcal{A}) \to L_i(\mathcal{A}'')$. Also, ρ is compatible with a localization.*

3. Several Complexes and Their Properties

Let $\alpha := \alpha_{H_0} \in S = \mathbb{K}[x_1, \dots, x_\ell]$ and $S' = S/\alpha_{H_0}S$. By an appropriate change of coordinates, we may assume that $\alpha = x_\ell$ and $S' = \mathbb{K}[x_1, \dots, x_{\ell-1}]$. In this section we introduce several results needed to prove Theorem 1.1.

REMARK 3.1. In this section we do not use the tameness assumption.

LEMMA 3.2. *The S-morphism*

$$\bigwedge \frac{d\alpha}{\alpha} \colon \Omega^p(\mathcal{A}) \to \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha},$$

$$\omega \mapsto \omega \wedge \frac{d\alpha}{\alpha},$$

is a splitting surjection. In particular, $\operatorname{pd}_S \Omega^p(\mathcal{A}) \geq \operatorname{pd}_S\bigl(\Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}\bigr)$.

Proof. It suffices to show that the morphism $\bigwedge \frac{d\alpha}{\alpha}$ has a section. By [6, Prop. 4.86],

$$\omega \wedge \frac{d\alpha}{\alpha} \mapsto (-1)^p \left\langle \theta_E, \omega \wedge \frac{d\alpha}{\alpha} \right\rangle$$

is the section of $\bigwedge \frac{d\alpha}{\alpha}$, where $\theta_E = \sum_{i=1}^{\ell} x_i \partial_{x_i}$ is the Euler derivation. Hence $\Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}$ is a direct summand of $\Omega^p(\mathcal{A})$. From this the inequality $\mathrm{pd}_S\, \Omega^p(\mathcal{A}) \geq \mathrm{pd}_S\big(\Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}\big)$ follows immediately. □

REMARK 3.3. By [6, Prop. 4.79],

$$\omega \wedge \frac{d\alpha}{\alpha} \in \Omega^{p+1}(\mathcal{A})$$

for $\omega \in \Omega^p(\mathcal{A})$. Also, it is known that the complex $\big(\Omega^*(\mathcal{A}), \bigwedge \frac{d\alpha}{\alpha}\big)$ is exact (see e.g. [6, Prop. 4.86]). Hence $\Omega^p(\mathcal{A})$ splits, via the section described in Lemma 3.2, into

$$\Omega^p(\mathcal{A}) \simeq \left(\Omega^{p-1}(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}\right) \oplus \left(\Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}\right).$$

The isomorphism is given by

$$\left(\Omega^{p-1}(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}\right) \oplus \left(\Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}\right) \ni (\omega, \omega') \mapsto \omega + (-1)^p \langle \theta_E, \omega \rangle \in \Omega^p(\mathcal{A}).$$

Let $\mathrm{res}^p \colon \Omega^p(\mathcal{A}) \to \Omega^p(\mathcal{A}'', m)$ be the residue map defined by

$$\Omega^p(\mathcal{A}) \ni \sigma \wedge \frac{d\alpha}{\alpha} + \delta \mapsto \delta|_{H_0} \in \Omega^p(\mathcal{A}'', m),$$

where σ and δ are (respectively) the $(p-1)$-forms and p-forms generated by $dx_1, \ldots, dx_{\ell-1}$ (see [14] or [13, Thm. 2.5]). Note that the residue map factors through

$$\Omega^p(\mathcal{A}) \xrightarrow{\bigwedge (d\alpha/\alpha)} \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha} \xrightarrow{r^p} \Omega^p(\mathcal{A}'', m).$$

The second arrow, $r^p \colon \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha} \to \Omega^p(\mathcal{A}'', m)$, is also denoted by res^p. Let $M^p := \mathrm{Im}(\mathrm{res})^p \subset \Omega^p(\mathcal{A}'', m)$ denote the image of the residue map and let $C^p := \mathrm{coker}(\mathrm{res})^p$ denote its cokernel; then

$$0 \to M^p \to \Omega^p(\mathcal{A}'', m) \to C^p \to 0.$$

We shall use res instead of res^p when the index p is obvious.

LEMMA 3.4. *Let*

$$\Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha} \xrightarrow{\cdot\alpha} \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}$$

denote the S-morphism defined by

$$\Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha} \ni \omega \mapsto \alpha\omega \in \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}.$$

Then the sequence

$$0 \to \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha} \xrightarrow{\cdot\alpha} \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha} \xrightarrow{\mathrm{res}^p} M^p \to 0$$

is exact. In particular, $\mathrm{pd}_{S'} M^p \leq \mathrm{pd}_S \Omega^p(\mathcal{A})$.

Proof. Let $\omega = \delta \wedge \frac{d\alpha}{\alpha} + \sigma \in \Omega^p(\mathcal{A})$ such that σ and δ are generated by $dx_1, \dots, dx_{\ell-1}$. If $\omega \wedge \frac{d\alpha}{\alpha} = \alpha\omega' \wedge \frac{d\alpha}{\alpha}$ for $\omega' \in \Omega^p(\mathcal{A})$, then it is clear that $\mathrm{res}\big(\omega \wedge \frac{d\alpha}{\alpha}\big) = \sigma|_{H_0} = 0$. Assume that $\mathrm{res}\big(\omega \wedge \frac{d\alpha}{\alpha}\big) = \sigma|_{H_0} = 0$. Because σ has no poles along $\alpha = 0$, we can write $\sigma \wedge \frac{d\alpha}{\alpha} = \alpha\sigma' \wedge \frac{d\alpha}{\alpha} \in \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}$ for some p-form σ' that is generated over $dx_1, \dots, dx_{\ell-1}$ and has no poles along $\alpha = 0$. Now apply the S-linear section in Lemma 3.2 to both sides and obtain $\omega' := \alpha\langle\theta_E, \sigma'\rangle \wedge \frac{d\alpha}{\alpha} + (-1)^p\alpha\sigma' \in \Omega^p(\mathcal{A})$. By our choice of σ', it follows that $\frac{\omega'}{\alpha} \in \Omega^p(\mathcal{A})$. So $\frac{\omega'}{\alpha} \wedge \frac{d\alpha}{\alpha} \in \Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}$ and $\alpha\big(\frac{\omega'}{\alpha} \wedge \frac{d\alpha}{\alpha}\big) = (-1)^p\omega \wedge \frac{d\alpha}{\alpha}$, which shows the exactness. Let us now prove the inequality. Since the action of S to M^p factors through $S' = S/\alpha S$, it follows that $\mathrm{depth}_S M^p = \mathrm{depth}_{S'} M^p$. Hence the Auslander–Buchsbaum formula shows that $\mathrm{pd}_{S'} M^p + 1 = \mathrm{pd}_S M^p$. Also, the long exact sequence of Ext functions shows that $\mathrm{pd}_S\big(\Omega^p(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}\big) + 1 \geq \mathrm{pd}_S M^p$. Combining this with Lemma 3.2, we have $\mathrm{pd}_{S'} M^p \leq \mathrm{pd}_S \Omega^p(\mathcal{A})$. □

Next consider the η-complex (for details, see [6, Def. 4.87]). It is the complex $\big(\Omega^*(\mathcal{A}), \bigwedge \eta\big)$, where $\eta \in \Omega^1_S$ is some generic 1-form in the sense of Proposition 3.6 (to follow) and where the map $\Omega^p(\mathcal{A}) \to \Omega^{p+1}(\mathcal{A})$ is given by $\omega \mapsto \omega \wedge \eta$. We can define the cohomology group $H^p(\Omega^*(\mathcal{A}))$ of this complex. Let $\bar{\eta} := \eta|_{H_0}$. We have the following lemma.

LEMMA 3.5. *Let $\bar{f}$ denote the image of $f \in S$ by the canonical surjection $S \to S'$. Then the following statements hold.*

(i) *Let $\eta = \sum_{i=1}^{\ell} f_i\, dx_i \in \Omega^1_S$ and $\omega = \delta \wedge dx_\ell/x_\ell + \sigma \in \Omega^p(\mathcal{A})$ such that σ and δ are generated over $dx_1, \dots, dx_{\ell-1}$. Then*

$$\mathrm{res}(\omega \wedge \eta) = \mathrm{res}(\omega) \wedge \mathrm{res}(\eta);$$

that is, the wedge product with $\bigwedge \eta$ and the residue map are commutative.

(ii) $\mathrm{res}(\eta) = \bar{\eta} \in \Omega^1_{S'}$.

Proof. (i) Note that

$$\mathrm{res}(\eta) = \sum_{i=1}^{\ell-1} \bar{f}_i\, dx_i \quad \text{and} \quad \mathrm{res}(\omega) = \bar{\sigma}.$$

Hence

$$\begin{aligned} \mathrm{res}(\omega \wedge \eta) &= \mathrm{res}\left(\delta \wedge \frac{dx_\ell}{x_\ell} \wedge \left(\sum_{i=1}^{\ell-1} f_i\, dx_i\right) + \sigma \wedge \left(\sum_{i=1}^{\ell} f_i\, dx_i\right)\right) \\ &= \bar{\sigma} \wedge \left(\sum_{i=1}^{\ell-1} \bar{f}_i\, dx_i\right) \\ &= \mathrm{res}(\omega) \wedge \mathrm{res}(\eta), \end{aligned}$$

which completes the proof. Part (ii) is a direct consequence of part (i). □

Since $\bar{\eta} \in \Omega^1_{S'}$, it follows that

$$\bar{\omega} \wedge \bar{\eta} \in \Omega^{p+1}(\mathcal{A}'', m) \quad (\bar{\omega} \in \Omega^p(\mathcal{A}'', m)).$$

In other words, for $p = 0, 1, \ldots, \ell - 1$, we have the maps

$$\begin{aligned} \bigwedge \bar{\eta} \colon \Omega^p(\mathcal{A}'', m) &\to \Omega^{p+1}(\mathcal{A}'', m), \\ \bar{\omega} &\mapsto \bar{\omega} \wedge \bar{\eta}. \end{aligned}$$

Then, by Remark 3.3 and Lemma 3.4, we have the complexes

$$\big(M^*, \bigwedge \bar{\eta}\big) \colon 0 \to M^0 \xrightarrow{\bigwedge \bar{\eta}} M^1 \xrightarrow{\bigwedge \bar{\eta}} \cdots \xrightarrow{\bigwedge \bar{\eta}} M^\ell \to 0,$$

$$\begin{aligned} \big(\Omega^*(\mathcal{A}'', m), \bigwedge \bar{\eta}\big) \colon 0 \to \Omega^0(\mathcal{A}'', m) &\xrightarrow{\bigwedge \bar{\eta}} \Omega^1(\mathcal{A}'', m) \\ &\xrightarrow{\bigwedge \bar{\eta}} \cdots \xrightarrow{\bigwedge \bar{\eta}} \Omega^{\ell-1}(\mathcal{A}'', m) \to 0. \end{aligned}$$

Moreover, for the inclusion $i_p \colon M^p \to \Omega^p(\mathcal{A}'', m)$, we have $i_p(\mathrm{res}^p(\omega)) \wedge \bar{\eta} = i_{p+1}(\mathrm{res}^p(\omega) \wedge \bar{\eta})$ for $\omega \in \Omega^p(\mathcal{A})$. Hence the complexes $\big(M^*, \bigwedge \bar{\eta}\big)$ and $\big(\Omega^*(\mathcal{A}'', m), \bigwedge \bar{\eta}\big)$ induce the complex

$$\big(C^*, \bigwedge \bar{\eta}\big) \colon 0 \to C^0 \xrightarrow{\bigwedge \bar{\eta}} C^1 \xrightarrow{\bigwedge \bar{\eta}} \cdots \xrightarrow{\bigwedge \bar{\eta}} C^{\ell-1} \to 0.$$

Let $H^p(M^*)$, $H^p(\Omega^*(\mathcal{A}'', m))$, and $H^p(C^*)$ denote the respective cohomology groups of these complexes.

PROPOSITION 3.6. *For an integer $d \geq 0$, there exists a generic 1-form $\eta \in \Omega^1_S$ of homogeneous degree d such that all cohomology groups of both complexes, $\big(\Omega^*(\mathcal{A}), \bigwedge \eta\big)$ and $\big(\Omega^*(\mathcal{A}'', m), \bigwedge \bar{\eta}\big)$, are finite-dimensional over $\mathbb{K}$.*

Proof. Apply the same proof as that for [6, Prop. 4.91]. Observe that we may assume $\mathbb{K}$ to be algebraically closed if necessary because $L(\mathcal{A})$ and $\chi(\mathcal{A}'', m, t)$ are stable under field extension (see [10, Sec. 2] and the proof of [1, Thm. 2.5]). □

DEFINITION 3.7. Let $\eta_d \in \Omega^1_S$ denote a generic 1-form with properties as in Proposition 3.6 of homogeneous degree d. Also, the complexes $\big(\Omega^*(\mathcal{A}), \bigwedge \eta_d\big)$, $\big(M^*, \bigwedge \bar{\eta}_d\big)$, $\big(\Omega^*(\mathcal{A}'', m), \bigwedge \bar{\eta}_d\big)$, and $\big(C^*, \bigwedge \bar{\eta}_d\big)$ are said to be *generic η_d- and $\bar{\eta}_d$-complexes.*

COROLLARY 3.8. *With notation as before, the cohomology group $H^p(M^*)$ of the complex $\big(M^*, \bigwedge \bar{\eta}_d\big)$ is also finite-dimensional over $\mathbb{K}$.*

Proof. By Proposition 3.6 and Remark 3.3, $H^p\big(\Omega^*(\mathcal{A}) \wedge \frac{d\alpha}{\alpha}\big)$ is of finite dimension. Therefore, the exact sequence in Lemma 3.4 shows that $H^p(M^*)$ is also of finite dimension. □

COROLLARY 3.9. *The cohomology group $H^p(C^*)$ of the complex $\big(C^*, \bigwedge \bar{\eta}_d\big)$ is also finite-dimensional over $\mathbb{K}$.*

Proof. Apply Lemma 3.5, Proposition 3.6, and Corollary 3.8 to the cohomology long exact sequence of $0 \to M^p \to \Omega^p(\mathcal{A}'', m) \to C^p \to 0$. □

Before starting the next proposition, we recall that $C^0 = C^{\ell-1} = 0$. We follow the proof by Schulze in [9, Sec. 2]. Since $\Omega^0(\mathcal{A}) = S$ and $\Omega^0(\mathcal{A}'', m) = S'$, it follows that $C^0 = 0$. Also, since the complex $\left(\Omega^*(\mathcal{A}), \bigwedge \frac{d\alpha}{\alpha}\right)$ is exact, it follows that

$$\Omega^{\ell-1}(\mathcal{A}) \wedge \frac{d\alpha}{\alpha} = \Omega^{\ell}(\mathcal{A}) = \frac{S}{Q(\mathcal{A})}\, dx_1 \wedge \cdots \wedge dx_\ell.$$

Then

$$\Omega^{\ell-1}(\mathcal{A}'', m) = \frac{S'}{Q(\mathcal{A}'', m)}\, dx_1 \wedge \cdots \wedge dx_{\ell-1}$$

implies that $C^{\ell-1} = 0$.

PROPOSITION 3.10.

$$\sum_{p=0}^{\ell-1} \operatorname{Poin}(C^p, x)(t(1-x)-1)^p \in \mathbb{Q}[x, x^{-1}, t],$$

$$\sum_{p=0}^{\ell-1} \operatorname{Poin}(M^p, x)(t(1-x)-1)^p \in \mathbb{Q}[x, x^{-1}, t];$$

that is, there are no poles along $x = 1$.

Proof. Apply the same proof as that of [6, Prop. 4.133] combined with Proposition 3.6 and Corollaries 3.8 and 3.9. □

The following result will be useful in proving Theorem 1.1.

THEOREM 3.11 [7, Thm. 5.8]. *Let $S = \mathbb{K}[x_1, \ldots, x_\ell]$ and $F^* = (0 \to F^0 \to F^1 \to \cdots \to F^\ell \to 0)$ be a complex of finite S-modules such that every morphism is S-linear and every cohomology group is finite-dimensional. If a nonnegative integer q satisfies*

$$\operatorname{pd}_S F^p < \ell + p - q$$

for all p, then $H^q(F^*) = 0$.

4. Proofs of Theorem 1.1 and Corollary 1.2

In this section we prove Theorem 1.1 and Corollary 1.2. Recall that we have not yet used the tameness assumption, although it is used in this section.

Proof of Theorem 1.1

The proof proceeds by induction on the dimensions ℓ and i (in the setup of the theorem). For $i = \ell - 1$, we have $b_0 = \sigma_0 = 1$; for $i = \ell - 2$, we have $b_1 = \sigma_1 = |\mathcal{A}| - 1 = |m|$. Also, $b_2 \geq \sigma_2$ follows from [2, Thm. 5.1]. Hence Theorem 1.1 holds when $\ell \leq 3$ or when $i = \ell - 3, \ell - 2$, or $\ell - 1$. Assume that $\ell \geq 4$ and $0 \leq i < \ell - 3$. Put $j = \ell - 1 - i$; then $3 \leq j \leq \ell - 1$. Recall the combinatorial restriction map $\rho\colon L(d\mathcal{A}) \to L(\mathcal{A}'')$ from Proposition 2.5. First, by the definition of a Möbius function we have

$$b_j = \sum_{X \in L_j(\mathcal{A}'')} b_j^X,$$

where

$$b_j^X = \sum_{Y \in L_j(d\mathcal{A}),\, \rho(Y)=X} |\mu(Y)|.$$

Let $\sigma_j^X := |\chi_{\mathrm{red}}(\mathcal{A}''_X, m_X, 0)|$, where $\chi_{\mathrm{red}}(\mathcal{A}''_X, m_X, t) := \chi(\mathcal{A}''_X, m_X, t)/t^{\ell-1-j}$ for $X \in L_j(\mathcal{A}'')$ (and "red" denotes "reduced"). Then Theorem 2.3 implies that

$$\sigma_j = \sum_{X \in L_j(\mathcal{A}'')} \sigma_j^X = \sum_{X \in L_j(\mathcal{A}'')} |\chi_{\mathrm{red}}(\mathcal{A}''_X, m_X, 0)|.$$

Now recall the tameness condition on $\mathcal{A}$ and $(\mathcal{A}'', m)$. By the definition of tameness and the localization's exactness, $\mathcal{A}_X$ and $(\mathcal{A}''_X, m_X)$ are also tame. However, $\mathcal{A}_X$ and $(\mathcal{A}''_X, m_X)$ are not essential. Hence to apply the induction hypothesis, we need the following lemma.

LEMMA 4.1. *Let $V = V_1 \oplus V_2$ and $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a central ℓ-arrangement in V that decomposes into the product of a d-arrangement $\mathcal{A}_1$ in V_1 and an $(\ell - d)$-arrangement $\mathcal{A}_2$ in V_2. For $m \colon \mathcal{A} \to \mathbb{Z}_{\geq 0}$, let m_i $(i = 1, 2)$ denote the restriction of m onto $\mathcal{A}_i$. Let S_i $(i = 1, 2)$ denote the coordinate ring of V_i. Then* $\mathrm{pd}_S\, \Omega^p(\mathcal{A}, m) \geq \mathrm{pd}_{S_i}\, \Omega^p(\mathcal{A}_i, m_i)$ *for $i = 1, 2$.*

Proof. It suffices to show the claim when $i = 1$. Note that $S_1 \otimes_{\mathbb{K}} S_2 = S$. Recall the decomposition

$$\Omega^p(\mathcal{A}, m) = \bigoplus_{q+r=p} \Omega^q(\mathcal{A}_1, m_1) \otimes_{\mathbb{K}} \Omega^r(\mathcal{A}_2, m_2),$$

which follows by dualizing [1, Lemma 1.4]. Then $\Omega^p(\mathcal{A}, m)$ contains

$$\Omega^p(\mathcal{A}_1, m_1) \otimes_{\mathbb{K}} \Omega^0(\mathcal{A}_2, m_2) \simeq S \cdot \Omega^p(\mathcal{A}_1, m_1)$$

as a direct summand, so $\mathrm{pd}_S\, \Omega^p(\mathcal{A}, m) \geq \mathrm{pd}_S\, S \cdot \Omega^p(\mathcal{A}_1, m_1)$. Let us prove

$$\mathrm{pd}_S\, S \cdot \Omega^p(\mathcal{A}_1, m_1) \geq \mathrm{pd}_{S_1}\, \Omega^p(\mathcal{A}_1, m_1).$$

Observe that $S \cdot \Omega^p(\mathcal{A}_1, m_1) \simeq \Omega^p(\mathcal{A}_1, m_1) \otimes_{\mathbb{K}} S_2$ and that S_2 is flat over $\mathbb{K}$. Thus S is flat over S_1 because $\bigotimes_{\mathbb{K}} S_2 = \bigotimes_{S_1} (S_1 \otimes_{\mathbb{K}} S_2) = \bigotimes_{S_1} S$. Also, it is known that

$$\mathrm{Ext}^i_{S_1}(N, S_1) \otimes_{\mathbb{K}} S_2 \simeq \mathrm{Ext}^i_S(S \cdot N, S)$$

for any finitely generated S_1-module N (see e.g. [5, (3.E)]). Hence

$$\mathrm{Ext}^q_{S_1}(\Omega^p(\mathcal{A}_1, m_1), S_1) \otimes_{\mathbb{K}} S_2 \simeq \mathrm{Ext}^q_S(S \cdot \Omega^p(\mathcal{A}_1, m_1), S).$$

Therefore, $\mathrm{pd}_S\, S \cdot \Omega^p(\mathcal{A}_1, m_1) \geq \mathrm{pd}_{S_1}\, \Omega^p(\mathcal{A}_1, m_1)$. □

We continue with the proof of Theorem 1.1. Since the essentializations of $\mathcal{A}_X$ and $(\mathcal{A}''_X, m_X)$ are tame arrangements of dimension $\leq \ell - 2$ (by Lemma 4.1), we can apply the induction hypothesis on dimensions. Note that $(-1)^j b_j^X$ is the constant term of the characteristic polynomial of the essentialization of $\mathcal{A}_X$ for $X \in L_j(d\mathcal{A})$

as well as that $\chi_{\mathrm{red}}(\mathcal{A}''_X, m_X, t)$ is equal to the characteristic polynomial of the essentialization of $(\mathcal{A}''_X, m_X)$. Hence the induction hypothesis shows that

$$b_j^X \geq |\chi_{\mathrm{red}}(\mathcal{A}''_X, m_X, 0)| = \sigma_j^X \quad (j = 0, 1, \ldots, \ell - 2).$$

Combining these observations yields

$$b_j = \sum_{X \in L_j(\mathcal{A}'')} b_j^X \geq \sum_{X \in L_j(\mathcal{A}'')} \sigma_j^X = \sigma_j \quad (j = 0, 1, \ldots, \ell - 2).$$

Next we show that $b_{\ell-1} \geq \sigma_{\ell-1}$. By assumption and Lemma 3.4, $\mathrm{pd}_{S'} M^p \leq p$ $(p = 0, 1, \ldots, \ell - 1)$. Hence Theorem 3.11 and Proposition 3.6 show that $H^p(M^*) = 0$ $(p \leq \ell - 2)$ for every generic η_d-complex. And for every generic $\bar{\eta}_d$-complex, we have $H^p(\Omega^*(\mathcal{A}'', m)) = 0$ $(p \leq \ell - 2)$ by Theorem 3.11, Proposition 3.6, and the tameness of $(\mathcal{A}'', m)$. Hence the long exact sequence of cohomology of

$$0 \to M^p \to \Omega^p(\mathcal{A}'', m) \to C^p \to 0$$

shows that $H^p(C^*) = 0$ $(0 \leq p \leq \ell - 3)$ for every generic η_d-complex. By the argument in the proof of [9, Prop. 4],

$$\chi_0(\mathcal{A}, t) = \sum_{p=0}^{\ell-1} \mathrm{Poin}(M^p, x)(t(1-x) - 1)^p \Big|_{x=1},$$

$$\chi(\mathcal{A}'', m, t) = \sum_{p=0}^{\ell-1} \mathrm{Poin}(\Omega^p(\mathcal{A}'', m), x)(t(1-x) - 1)^p \Big|_{x=1}.$$

Therefore,

$$\chi_0(\mathcal{A}, t) - \chi(\mathcal{A}'', m, t) = -\sum_{p=0}^{\ell-1} \mathrm{Poin}(C^p, x)(t(1-x) - 1)^p \Big|_{x=1}.$$

Now apply the vanishing of cohomologies just described to a generic η_0-complex. Then

$$\begin{aligned}
\chi_0(\mathcal{A}, 0) - \chi(\mathcal{A}'', m, 0) &= -\sum_{p=0}^{\ell-1} \mathrm{Poin}(C^p, x)(-1)^p \Big|_{x=1} \\
&= -\sum_{p=1}^{\ell-2} \mathrm{Poin}\, H^p(C^p)(-1)^p \\
&= (-1)^{\ell-1} \dim_{\mathbb{K}} H^{\ell-2}(C^*).
\end{aligned}$$

Hence

$$b_{\ell-1} - \sigma_{\ell-1} = (-1)^{\ell-1}(\chi_0(\mathcal{A}, 0) - \chi(\mathcal{A}'', m, 0)) = \dim_{\mathbb{K}} H^{\ell-2}(C^*) \geq 0.$$

Therefore, Theorem 1.1 is proved (contingent on the following proposition).

Proposition 4.2. *If $(\mathcal{A}'', m)$ is tame, then $\sigma_j \geq 0$ $(j = 0, 1, \ldots, \ell - 1)$.*

Proof. We use induction on ℓ. By the same argument as in the proof of Theorem 1.1, the statement is true when $\ell \leq 3$. So assume that $\ell \geq 4$. Since (by Lemma 4.1) the essentialization of $(\mathcal{A}''_X, m_X)$ is also tame, we may apply the induction hypothesis. Again the same argument as in the proof of Theorem 1.1 shows that $\sigma_j^X \geq 0$ for $X \in L_j(\mathcal{A}'')$ and $0 \leq j \leq \ell - 2$. Hence the equality

$$\sigma_j = \sum_{X \in L_j(\mathcal{A}'')} \sigma_j^X$$

from Theorem 2.3 establishes that $\sigma_j \geq 0$ $(j = 0, 1, \ldots, \ell - 2)$. It therefore suffices to show that $\sigma_{\ell-1} \geq 0$.

As we saw in the proof of Theorem 1.1,

$$\chi(\mathcal{A}'', m, t) = \sum_{p=0}^{\ell-1} \operatorname{Poin}(\Omega^p(\mathcal{A}'', m), x)(t(1-x) - 1)^p \Big|_{x=1}.$$

Hence $\sigma_{\ell-1} = (-1)^{\ell-1} \sum_{p=0}^{\ell-1} (-1)^p \operatorname{Poin}(\Omega^p(\mathcal{A}'', m), x)\big|_{x=1}$. Let us show that $\sigma_{\ell-1}$ is not negative. By the tameness condition, we have

$$\begin{aligned}
\sigma_{\ell-1} &= (-1)^{\ell-1} \sum_{p=0}^{\ell-1} (-1)^p \operatorname{Poin}(\Omega^p(\mathcal{A}'', m), x)\Big|_{x=1} \\
&= (-1)^{\ell-1}(-1)^{\ell-1} H^{\ell-1}(\Omega^*(\mathcal{A}'', m)) \geq 0;
\end{aligned}$$

this completes the proof of the proposition. □

COROLLARY 4.3. *In the notation of Theorem 1.1,* $(-1)^{\ell-1}\chi(\mathcal{A}'', m, -1) \geq 0$.

Proof. The statement follows from Proposition 4.2 and the equality

$$(-1)^{\ell-1}\chi_0(\mathcal{A}'', m, -1) = \sum_{i=0}^{\ell-1} \sigma_i. \qquad \square$$

Proof of Corollary 1.2

The first statement follows immediately from Theorem 1.1 and Corollary 4.3. Let us prove the second statement. Assume that $\mathcal{A}$ is free; then, by [14, Thm. 11], its Ziegler restriction $(\mathcal{A}'', m)$ is also free. Now Terao's factorization theorem [11, Main Theorem] and its multiversion [1, Thm. 4.1] show that the characteristic polynomials split into the same form:

$$\chi_0(\mathcal{A}, t) = \prod_{i=1}^{\ell-1} (t - d_i) = \chi(\mathcal{A}'', m, t).$$

Hence $\chi_0(\mathcal{A}, -1) = \chi(\mathcal{A}'', m, -1)$, which shows that $\mathcal{A}$ is a minimal chamber arrangement. Assume that $\mathcal{A}$ is an MCA and that $(\mathcal{A}'', m)$ is free. Then, by Theorem 1.1, $b_i \geq \sigma_i$ $(i = 0, 1, \ldots, \ell - 1)$. Therefore, by the minimality of chambers we have $b_i = \sigma_i$ $(i = 0, 1, \ldots, \ell - 1)$. Thus $\chi_0(\mathcal{A}, t) = \chi(\mathcal{A}'', m, t)$, and an application of Theorem 2.4 completes the proof of Corollary 1.2.

COROLLARY 4.4. *Assume that $\mathcal{A}$ is a 4-arrangement. Then Theorem 1.1 holds if $\mathcal{A}$ is tame.*

Proof. Since $\Omega^p(\mathcal{A}'', m)$ is reflexive, the statement follows from the Auslander–Buchsbaum formula combined with the inequality $\operatorname{depth}_{S'} \Omega^p(\mathcal{A}'', m) \geq 2$. □

The Auslander–Buchsbaum formula also shows that 2- and 3-(multi)arrangements are tame. So as a corollary to Theorem 1.1 and Corollary 1.2, we can give another proof of Yoshinaga's criterion [13].

COROLLARY 4.5 [13, Thm. 3.2]. *A 3-arrangement $\mathcal{A}$ is free if and only if it is an MCA.*

Proof. Since 2- and 3-multiarrangements are tame, we can use Theorem 1.1 and Corollary 1.2. The "only if" part follows from Corollary 1.2. Assume that $\mathcal{A}$ is an MCA. Since $C^0 = C^2 = 0$, the complex of cokernels is $0 \to C^1 \to 0$. The minimality of chambers implies that $|b_1 - \sigma_1| = H^1(C^*) = 0$. Hence $C^1 = 0$, which is equal to the freeness of $\mathcal{A}$ (see e.g. [13, Cor. 2.6]). □

Theorem 1.1 and Corollary 1.2 give rise to the following problem, which involves relating tameness to geometry and combinatorics of arrangements.

PROBLEM 4.6. Do Theorem 1.1 and Corollary 1.2 hold true without the assumption of tameness? Moreover, is a free arrangement an MCA in general?

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