Michigan Math. J. 62 (2013), 81–115

Torsion in the Cohomology of Desingularized Fiber Products of Elliptic Surfaces

Chad Schoen

0. Introduction

Let W be a smooth projective threefold over an algebraically closed field k. Let l be a prime distinct from $\mathrm{char}(k)$. This paper is concerned with the problem of computing the torsion subgroup $H^3(W,\mathbb{Z}_l)_{\mathrm{tors}}$ of the étale cohomology $H^3(W,\mathbb{Z}_l)$. Before stating the results, we recall five reasons why one is interested in this group.

1. *Relationship to the Brauer group,* $\mathrm{Br}(W)$. The subgroup $H^3(W,\mathbb{Z}_l(1))_{\mathrm{tors}}$ is canonically isomorphic to the l-primary part of the Brauer group modulo its maximal divisible subgroup [Gro, Sec. 8.3].

2. *Birational invariance for smooth projective varieties* [Gro, Thm. 7.4, Thm. 6.1c]. This attribute was used by Artin and Mumford [AMu] to give examples of unirational threefolds that are not rational.

3. *Mirror symmetry.* Take $k=\mathbb{C}$. There is tantalizing empirical evidence (see [BaKr]) that for Calabi–Yau threefolds, $H^3(W(\mathbb{C}),\mathbb{Z})_{\mathrm{tors}}$ should be isomorphic to the first homology of the mirror. No natural isomorphism is currently known.

4. *The integral Tate problem.* Poincaré duality for a smooth projective threefold gives an isomorphism,

$$H^4(W,\mathbb{Z}_l(2))_{\mathrm{tors}} \simeq \mathrm{Hom}(H^3(W,\mathbb{Z}_l)_{\mathrm{tors}},\mathbb{Q}_l/\mathbb{Z}_l(1)).$$

An integral version of the Tate conjecture (or if $k=\mathbb{C}$, of the Hodge conjecture) would imply that $H^4(W,\mathbb{Z}_l(2))_{\mathrm{tors}}$ is generated by classes of codimension-2 algebraic cycles. The integral Hodge conjecture is known to hold for Fano threefolds [V] (see also [Gra]) and for Calabi–Yau threefolds [V]. It is known to fail for certain threefolds of general type [Ko], although in [Ko] $H^4(W,\mathbb{Z}_l(2))_{\mathrm{tors}}=0$. Some of the threefolds studied in this paper are birational to Calabi–Yau varieties but most have Kodaira dimension 1, which is unexplored territory.

5. *The Abel–Jacobi map.* The Abel–Jacobi map applies to algebraic cycles that are integrally homologous to zero. A consequence of Theorem 0.3(ii) is that the much-studied complex multiplication cycles have this property (cf. [ST, 3.2]).

Received September 28, 2011. Revision received April 27, 2012.

Most of this work was done during visits to the Max-Planck-Insitut für Mathematik in Bonn and to the Newton Institute for Mathematical Sciences (special semester on Higher Dimensional Complex Geometry), where a first version of the manuscript was written. Revisions and additions were made at the I.H.E.S. It is a pleasure to thank these institutions for their hospitality and the N.S.F. and N.S.A. for partial support through grants DMS 99-70500 and H98230-08-1-0027.

Despite the interest in $H^3(W,\mathbb{Z}_l)_{\mathrm{tors}}$, little is known about how to compute it. Consider for example the case of hypersurfaces in $\mathbb{P}^4$. If $W \subset \mathbb{P}^4$ is smooth, then the cohomology of W is known by the Lefschetz hyperplane theorem and an Euler characteristic computation. On the other hand, if W is obtained by desingularizing a hypersurface $\bar{W} \subset \mathbb{P}^4$ having only nodes as singularities, then $H^3(W,\mathbb{Z}_l)_{\mathrm{tors}}$ is poorly understood when $\deg(\bar{W}) > 3$ and no instance in which $H^3(W,\mathbb{Z}_l)_{\mathrm{tors}} \neq 0$ seems to be known. A little more is known when W is a desingularization of a double cover $\bar{W} \to \mathbb{P}^3$ branched along a surface $S \subset \mathbb{P}^3$ with ordinary double point singularities. The case $\deg(S) \leq 4$ is treated in [E]. There are a few more known interesting examples with $\deg(S) = 8$ and $H^3(W,\mathbb{Z}_l)_{\mathrm{tors}} \neq 0$ [G]. The most comprehensive results so far are for generic Calabi–Yau hypersurfaces in Gorenstein toric Fano varieties, in which case the Brauer group has a combinatorial description [BaKr]. A case in which the Brauer group is not cyclic is the subject of [GP].

The purpose of this paper is to compute $H^3(W,\mathbb{Z}_l)_{\mathrm{tors}}$ systematically when W is the blowup of a certain type of nodal hypersurface $\bar{W}$ in a smooth projective fourfold. The construction of $\bar{W}$ is as follows. Begin with a smooth, irreducible, projective curve X over k and a pair of elliptic curves E and E' over the function field, $K = k(X)$, whose J-invariants are not contained in k. Write $\pi : Y \to X$ (resp. $\pi' \colon Y' \to X$) for the relatively minimal model of E (resp. E'). The fiber product $\bar{W} := Y \times_X Y'$ is a (nonample) hypersurface in the smooth projective fourfold $Y \times_k Y'$. The following assumption, which will be in force throughout, is equivalent to requiring that the only singularities of $\bar{W}$ be isolated ordinary double points.

0.1. ASSUMPTION. For each $x \in X$ for which both fibers $\pi^{-1}(x)$ and $(\pi')^{-1}(x)$ are singular, both are of multiplicative type.

Now blow up the nodes of $\bar{W}$ to obtain a smooth projective threefold W, which we refer to henceforth as a *desingularized fiber product*. These varieties have found application in many different areas, including algebraic cycles (especially complex multiplication or Heegner cycles) [ST], modularity of Galois representations [M, Chap. 2], phenomena peculiar to positive characteristic [S1], and super string theory [BDo].

The main result of the paper describes $H^3(W,\mathbb{Z}_l)_{\mathrm{tors}}$ in terms of the Galois representations on the torsion in $E(\bar{K})$ and $E'(\bar{K})$ and certain local conditions at places of bad reduction. To be more precise, define $G := \mathrm{Gal}(\bar{K}/K)$-modules

$$E[l^n] := \mathrm{Ker}\, E(\bar{K}) \xrightarrow{l^n} E(\bar{K}) \quad \text{and} \quad E'[l^n] := \mathrm{Ker}\, E'(\bar{K}) \xrightarrow{l^n} E'(\bar{K}).$$

Now consider the group $\mathrm{Hom}_G(E[l^n], E'[l^n])$ or—what amounts to the same thing, thanks to the Weil pairing—the group $(E[l^n] \otimes E'[l^n])^G$. We define a subgroup of this group by imposing local conditions at the places of bad reduction and then relate the subgroup to $H^3(W,\mathbb{Z}_l)[l^n]$.

In order to describe the local conditions, fix for each $x \in X$ a henselization of $\mathcal{O}_{X,x}$, $\mathcal{O}^h_{X,x} \subset \bar{K}$. Denote by $E[l^n]_{0x} \subset E[l^n]$ the largest subgroup whose closure in the Néron model of E at $\mathbf{x} := \mathrm{Spec}(\mathcal{O}^h_{X,x})$ is finite over $\mathbf{x}$ and meets only the identity component of the closed fiber. If E has additive reduction at x, then $E[l^n]_{0x} = 0$ because the identity component of the closed fiber is isomorphic to $\mathbb{G}_a$, whose l^n-torsion subgroup is 0. If E has multiplicative reduction then $E[l^n]_{0x} \simeq \mathbb{Z}/l^n$, since the identity component of the closed fiber is isomorphic to $\mathbb{G}_m$ with l^n-torsion subgroup μ_{l^n}. Define $E'[l^n]_{0x}$ analogously. Let S (resp. S') denote the locus of bad reduction for E (resp. E').

0.2. Definition. $\Theta_{l^n}(W) := \{\theta \in (E[l^n] \otimes E'[l^n])^G : \text{(i), (ii), and (iii) hold}\}$:

(i) for all $s \in S - (S \cap S')$, $\theta \in E[l^n]_{0s} \otimes E'[l^n]$;
(ii) for all $s \in S' - (S \cap S')$, $\theta \in E[l^n] \otimes E'[l^n]_{0s}$.
(iii) for all $s \in S \cap S'$, $\theta \in E[l^n] \otimes E'[l^n]_{0s} + E[l^n]_{0s} \otimes E'[l^n]$.

0.3. Theorem. *Let W be a desingularized fiber product as defined previously. Then the following statements hold.*

(i) $H^i(W, \mathbb{Z}_l)_{\mathrm{tors}} = 0$ *for* $i \notin \{3, 4\}$.
(ii) *If E is isogenous to E' over K, then* $H^\bullet(W, \mathbb{Z}_l)_{\mathrm{tors}} := \bigoplus_{i=0}^6 H^i(W, \mathbb{Z}_l)_{\mathrm{tors}} = 0$.
(iii) *If E is not isogenous to E' over K, then* $\Theta_{l^n}(W) \simeq H^3(W, \mathbb{Z}_l)[l^n] \simeq H^4(W, \mathbb{Z}_l)[l^n]$.

A consequence of the theorem is that the possible isomorphism classes of the group $H^3(W, \mathbb{Z}_l)_{\mathrm{tors}}$ are quite restricted. More precisely, we will prove the following result.

0.4. Theorem. *Let W be as defined before.*

(i) $H^3(W, \mathbb{Z}_l)_{\mathrm{tors}}$ *is a cyclic group.*
(ii) *If E or E' has a place of additive reduction, then* $H^\bullet(W, \mathbb{Z}_l)_{\mathrm{tors}} = 0$.
(iii) *If $S \cap S' = \emptyset$, then* $H^\bullet(W, \mathbb{Z}_l)_{\mathrm{tors}} = 0$.

In concrete situations, the group $\Theta_{l^n}(W)$ (and hence also $H^3(W, \mathbb{Z}_l)_{\mathrm{tors}}$) is frequently quite computable. The next result shows that there are no restrictions on these groups beyond Theorem 0.4(i).

0.5. Theorem. *Given $m \in \mathbb{N}$ prime to $p := \mathrm{char}(k)$, there exists a W as described previously with* $H^3\big(W, \prod_{l \neq p} \mathbb{Z}_l\big)_{\mathrm{tors}} \simeq \mathbb{Z}/m$.

When $H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}} \neq 0$, it is natural to ask whether these cohomology classes come from algebraic cycles. Toward this end, let $Z^2(W)_0$ denote the subgroup of the group of codimension-2 algebraic cycles on W that is generated by cycles in the smooth fibers of $f : W \to X$ (or, equivalently, of $\bar{f} : \bar{W} \to X$) whose intersection with each factor Y and Y' is a degree-0 zero cycle.

0.6. Theorem. *Suppose that E is not isogenous to E'. Then the image of the cycle class map* $Z^2(W)_0 \to H^4(W, \mathbb{Z}_l(2))$ *is* $H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}}$.

The contents of the individual sections of this paper are as follows. For easy reference, the notation is summarized in Section 1. Section 2 recalls tools from étale cohomology that will be used in the sequel. In particular, the groundwork is laid for identifying $H^3(W,\mathbb{Z}_l)_{\text{tors}}$ with the quotient of $H^2(W,\mathbb{Z}/l^n)$ by a maximal free $\mathbb{Z}/l^n$-submodule—provided n is sufficiently large. Thus the focus will be on computing $H^2(W,\mathbb{Z}/l^n)$. The main tool will be the Leray spectral sequence for the canonical morphism $f: W \to X$. To prepare the way, Section 3 treats the Leray spectral sequence for $\pi: Y \to X$ and Section 4 deals with the Leray spectral sequence for the canonical map $\bar{f}: \bar{W} \to X$. In Section 5 we show that $H^3(W,\mathbb{Z}_l)_{\text{tors}}$ is controlled by $H^0(X,R^2 f_*\mathbb{Z}/l^n)$ for large n. The sheaf $R^2 f_*\mathbb{Z}/l^n$ is analyzed in Section 6 with the help of [S2, Sec. 7]. There is a direct factor, $\mathcal{G}_{l^n}$, of $R^2 f_*\mathbb{Z}/l^n$ with the property that if E and E' are not isogenous then $H^0(X,\mathcal{G}_{l^n}) \simeq H^3(W,\mathbb{Z}_l)[l^n]$. Interpreting the global sections of $\mathcal{G}_{l^n}$ in the language of Definition 0.2 proves Theorem 0.3(iii). The proofs of the remaining assertions in Theorems 0.3 and 0.4 are completed in Section 7.

With the explicit formula for $H^3(W,\mathbb{Z}_l)[l^n]$ in Theorem 0.3(iii) in hand, one turns to constructing examples where this group is nonzero. In fact, such examples are rare because the requirements that $\Theta_{l^n}(W) \neq 0$ and E not be isogenous to E' are seldom satisfied simultaneously. Section 8 is devoted to an easy but anomalous example in which the torsion can be made to disappear by extending the base field K. Theorem 0.5 is proved in Section 9 by constructing certain special fiber products with $S \neq S \cap S' \neq S'$. A different construction is presented in Section 10, which gives many examples with $S = S'$. Each construction requires that the genus of X be large if $|H^3(W)_{\text{tors}}|$ is large. In general, it is unknown whether $|H^3(W)_{\text{tors}}|$ can be bounded in terms of the genus of X.

Section 11 concerns the cycle class map to cohomology. Here Theorem 0.6 is proved. The Betti numbers of Y and W are given in Section 12, since this information is useful in other contexts [M, Chap. 2].

The related article [SSha] contains concrete examples of desingularized fiber products W that are birational to Calabi–Yau varieties or Calabi–Yau algebraic spaces and have $H^3(W,\mathbb{Z}_l)_{\text{tors}} \neq 0$ for small primes l.

1. Notation

This section serves as an index for notation that is in force throughout. The notation defined in Section 1.n is first used in Section n.

1.2.

$\mathbb{N} = \mathbb{Z}_{\geq 1}$.
$A[n] = \operatorname{Ker} A \xrightarrow{\cdot n} A$ for an abelian group A and $n \in \mathbb{Z}$.
$M^\vee = \operatorname{Hom}_R(M,R)$ for a module M over a commutative ring R.
k = an algebraically closed field.
l = a prime number distinct from the characteristic of k.
V = a projective k-scheme.
$\mathcal{F}$ is an étale sheaf.
$H^i(V,\mathcal{F})$ denotes étale cohomology.
$h^i(V,\mathbb{Z}/l) := \dim_{\mathbb{Z}/l}(H^i(V,\mathbb{Z}/l))$.
$h^i(V,\mathbb{Q}_l) = \dim_{\mathbb{Q}_l}(H^i(V,\mathbb{Q}_l))$.

X = a smooth irreducible projective curve over k.
$j\colon \eta \to X$ denotes the inclusion of the generic point.
$\mathbf{j}\colon \bar{\eta} \to X$ denotes a geometric generic point algebraic over j.

1.3.

$|A|$ = the cardinality of A, where A is a finite set.
g = the genus of X.
$K = k(X)$.
E is an elliptic curve over K whose J-invariant is not contained in k.
$\pi : Y \to X$ denotes a relatively minimal elliptic surface with generic fiber E.
$S \subset X$, the locus of a bad reduction of E.
$S_a, S_m \subset S$, the locus of an additive (respectively, a multiplicative) reduction of E.
m_s = the number of irreducible components of $\pi^{-1}(s)$.
$\bar{K}$ = a separable closure of K.
$G = \mathrm{Gal}(\bar{K}/K)$.
$E[l^n] = \mathrm{Ker}\, E(\bar{K}) \xrightarrow{l^n} E(\bar{K})$.
$E[l^n]_{0s} \subset E[l^n]$ (depends on the choice of a henselization $\mathcal{O}^h_{X,s} \subset \bar{K}$ of $\mathcal{O}_{X,s}$), the largest subgroup whose closure in the Néron model of E over $\mathcal{O}^h_{X,s}$ is finite over the base and meets only the identity component of the closed fiber. (Equivalently, $E[l^n]_{0s}$ depends on the choice of the inertia group above s, $I_s = \mathrm{Aut}(\bar{K}/\mathcal{O}^h_{X,s}) < G$.)

1.4.

$T := E[l^n]$, $T' := E'[l^n]$, $T_s := E_{0s}[l^n]$, and $T'_s := E'_{0s}[l^n]$ when l and n are understood.
$B_{l^n} = \mathrm{Hom}(E[l^n]^G, \mathbb{Z}/l^n)$.
E' is an elliptic curve over K with J-invariant not contained in k.
$\pi'\colon Y' \to X$, S', S'_a, m'_s, $E'[l^n]_{0s}$, $B'_{l^n}, \ldots$ denote the analogues for E' of $\pi : Y \to X$, S, S_a, m_s, $E[l^n]_{0s}$, $B_{l^n}, \ldots$.
$S'' = S \cap S'$.
$\bar{W}$, $\bar{f}$, W and f are defined only under the additional assumption that $S_a \cap S' = S \cap S'_a = \emptyset$ (cf. Assumption 0.1).
$\bar{W} = Y \times_X Y'$.
$\bar{f}\colon \bar{W} \to X$ denotes the tautological map.

1.5.

$\sigma : W \to \bar{W}$ denotes the blowup of the reduced singular locus of $\bar{W}$.
$f = \bar{f} \circ \sigma : W \to X$.
N_* = the quotient of the finitely generated $\mathbb{Z}/l^n$-module N by a maximal free submodule N_0; the isomorphism class of N_* is independent of the choice of N_0.

1.6. $\mathcal{G}_{l^n}$ is a quotient sheaf of $R^2 f_* \mathbb{Z}/l^n$ defined in Section 6.

1.8. $\hat{\mathbb{Z}}' = \prod_{l \neq \mathrm{char}(k)} \mathbb{Z}_l$.

2. Preliminaries on Étale Sheaves and Étale Cohomology

This section collects basic facts about étale cohomology used in the sequel. It may initially be skipped and referred back to as needed. Notation is as in Section 1.2.

2.1. PROPOSITION.

(i) *$H^i(V, \mathbb{Z}/l^n)$ is a finite group.*
(ii) *$H^i(V, \mathbb{Z}_l)$ is a finitely generated $\mathbb{Z}_l$-module.*
(iii) *$H^i(V, \mathbb{Z}/l^n) = 0$ for $i > 2\dim(V)$.*
(iv) *The rank of $H^i(V, \mathbb{Z}_l)$ is independent of l.*

Proof.

(i) [SGA4.5, Théorèmes de finitude, Thm. 1.1].
(ii) [Mi, V.1.11].
(iii) [Mi, VI.1].
(iv) [Mi, VI.12.5b and VI.4.2]. □

The next lemma provides the criteria that will be used to determine whether or not a given cohomology group is torsion free.

2.2. LEMMA. (i) *If $H^i(V, \mathbb{Z}_l)_{\text{tors}} = 0$, then the quotient of $H^i(V, \mathbb{Z}/l^n)$ by a maximal free $\mathbb{Z}/l^n$-submodule is isomorphic to $H^{i+1}(V, \mathbb{Z}_l)[l^n]$ for $n \gg 0$.*

(ii) *If $H^i(V, \mathbb{Z}/l^n)$ is a free $\mathbb{Z}/l^n$-module for $n \gg 0$, then $H^i(V, \mathbb{Z}_l)_{\text{tors}} = 0 = H^{i+1}(V, \mathbb{Z}_l)_{\text{tors}}$.*

(iii) *If $h^i(V, \mathbb{Z}/l) > h^i(V, \mathbb{Z}/l')$ for some prime $l' \neq \text{char}(k)$, then $H^i(V, \mathbb{Z}_l)_{\text{tors}}$ and $H^{i+1}(V, \mathbb{Z}_l)_{\text{tors}}$ are not both zero.*

Proof. The first two assertions follow from Proposition 2.1(ii) and the exact sequence

$$H^i(V, \mathbb{Z}_l) \xrightarrow{l^n} H^i(V, \mathbb{Z}_l) \to H^i(V, \mathbb{Z}/l^n) \to H^{i+1}(V, \mathbb{Z}_l) \xrightarrow{l^n} H^{i+1}(V, \mathbb{Z}_l)$$

[Mi, V.1.11]. When $n = 1$ this sequence implies $h^i(V, \mathbb{Z}/l) = r + t_i(l) + t_{i+1}(l)$, where $r = \text{rank}(H^i(V, \mathbb{Z}_l))$ is independent of l and $t_j(l) = \dim_{\mathbb{Z}/l}(H^j(V, \mathbb{Z}_l)[l])$. Now (iii) follows. □

2.3. PROPOSITION. *Suppose that V is smooth and irreducible and that* $\dim(V) = d$. *Then*:

(i) *the finite groups $H^i(V, \mathbb{Z}_l)_{\text{tors}}$ and $H^{2d-i+1}(V, \mathbb{Z}_l(d))_{\text{tors}}$ are dual and hence (noncanonically) isomorphic; and*
(ii) *$H^i(V, \mathbb{Z}_l)_{\text{tors}} = 0$ for all except finitely many primes l.*

Proof. (i) For each n, Poincaré duality gives an isomorphism

$$H^i(V, \mathbb{Z}/l^n) \simeq \text{Hom}\left(H^{2d-i}\left(V, \frac{1}{l^n}\mathbb{Z}/\mathbb{Z}\right), \mathbb{Q}_l/\mathbb{Z}_l(-d)\right)$$

[Mi, VI.11.1]. Apply $\varprojlim_n$ to get

$$H^i(V,\mathbb{Z}_l) \simeq \mathrm{Hom}(H^{2d-i}(V,\mathbb{Q}_l/\mathbb{Z}_l),\mathbb{Q}_l/\mathbb{Z}_l(-d)).$$

Use the exact sequence

$$H^{2d-i}(V,\mathbb{Q}_l) \xrightarrow{b} H^{2d-i}(V,\mathbb{Q}_l/\mathbb{Z}_l) \to H^{2d-i+1}(V,\mathbb{Z}_l)_{\mathrm{tors}} \to 0$$

[CSaSo, (13)] to conclude

$$H^i(V,\mathbb{Z}_l)_{\mathrm{tors}} \simeq \mathrm{Hom}(H^{2d-i+1}(V,\mathbb{Z}_l)_{\mathrm{tors}},\mathbb{Q}_l/\mathbb{Z}_l(-d)).$$

(ii) [Ga]. □

In the remainder of this section we recall some facts about constructible étale sheaves on curves. Let $\mathbf{S} \subset X$ be a finite subset of closed points on a smooth, projective, connected k-curve. Fix an integer $n > 1$. Denote by $\mathcal{C}$ the category of constructible sheaves of $\mathbb{Z}/n$-modules on X whose restriction to $X - \mathbf{S}$ is locally constant. It is convenient to work in a category $\mathcal{M}$, which is equivalent to $\mathcal{C}$, that we now describe.

Let $j\colon \eta \to X$ be the inclusion of the generic point. Let $\mathbf{j}\colon \bar{\eta} \to X$ denote a geometric point that is algebraic over η. For each $s \in \mathbf{S}$, fix an inertia group $I_s \in \pi_1(X-\mathbf{S},\mathbf{j})$. Now the objects of $\mathcal{M}$ are pairs $(M,\{M_s,\phi_s\}_{s\in\mathbf{S}})$ in which M is a finite $\mathbb{Z}/n$-module with a continuous $\pi_1(X-\mathbf{S},\mathbf{j})$-action, M_s is a finite $\mathbb{Z}/n$-module, and $\phi_s\colon M_s \to M^{I_s}$ is a $\mathbb{Z}/n$-linear map. Morphisms in $\mathcal{M}$ are given by pairs,

$$(\psi,\{\psi_s\}_{s\in\mathbf{S}})\colon (M,\{M_s,\phi_s\}_{s\in\mathbf{S}}) \to (\hat{M},\{\hat{M}_s,\hat{\phi}_s\}_{s\in\mathbf{S}});$$

here $\psi\colon M \to \hat{M}$ is a homomorphism of $\mathbb{Z}/n[\pi_1(X-\mathbf{S},\mathbf{j})]$-modules, $\psi_s\colon M_s \to \hat{M}_s$ is a $\mathbb{Z}/n$-module homomorphism, and $\hat{\phi}_s \circ \psi_s = \psi \circ \phi_s$.

For $\mathcal{F} \in \mathrm{Ob}(\mathcal{C})$, no notational distinction will be made between the sheaf $\mathbf{j}^*\mathcal{F}$ and the associated $\pi_1(X-\mathbf{S},\mathbf{j})$-module (cf. [Mi, II.1.9]).

2.4. Lemma. (i) *The functor $\mathcal{C} \to \mathcal{M}$, $\mathcal{F} \mapsto (\mathbf{j}^*\mathcal{F},\{\mathcal{F}_s,\phi_s\}_{s\in\mathbf{S}})$, where $\phi_s\colon \mathcal{F}_s \to (\mathbf{j}_*\mathbf{j}^*\mathcal{F})_s \simeq (\mathbf{j}^*\mathcal{F})^{I_s}$ is the natural map, is an equivalence of categories.*

(ii) *If $(F,\{F_s,\phi_s\}_{s\in\mathbf{S}})$ corresponds to the étale sheaf $\mathcal{F}$, then*

$$H^0(X,\mathcal{F}) \simeq \left\{(\mathfrak{f},\{f_s\}) \in F \times \prod_{s\in\mathbf{S}} F_s : \mathfrak{f} \in F^{\pi_1(X-\mathbf{S},\bar{\mathbf{j}})},\ f_s \in F_s,\ \phi_s(f_s) = \mathfrak{f}\right\}.$$

(iii) *The kernel $\mathcal{K}$ of the canonical map $\mathcal{F} \xrightarrow{r} j_*j^*\mathcal{F}$ is supported on $\mathbf{S}$. If the stalk $\mathcal{K}_s$ is a free $\mathbb{Z}/l^n$-module for each $s \in \mathbf{S}$, then $\mathcal{K}$ is a direct summand of $\mathcal{F}$.*

Proof. (i) [Mi, II.3.10-16 and V.1.3].

(ii) [Mi, II.3.16(c)].

(iii) The exact sequence in $\mathcal{M}$ corresponding to $0 \to \mathcal{K} \to \mathcal{F} \to j_*j^*\mathcal{F}$ is

$$0 \to (0,\{K_s,0\}_s) \to (M,\{M_s,\phi_s\}_s) \to (M,\{M^{I_s},\psi_s\}_s),$$

where $\psi_s\colon M^{I_s} \to M$ is the inclusion. The hypotheses imply that $M_s \simeq K_s \oplus M'_s$ with $\phi_s\colon M'_s \to M^{I_s}$ injective and $\phi_s(K_s) = 0$. It is apparent that the first morphism has a left inverse. □

Let $\rho\colon \hat{X} \to X$ denote a finite morphism of smooth, projective, irreducible curves, and let $\hat{\mathbf{S}} = \rho^{-1}(\mathbf{S})$. Fix a lifting $\hat{\mathbf{j}}\colon \bar{\eta} \to \hat{X}$ of $\mathbf{j}$. For each $\hat{s} \in \hat{\mathbf{S}}$, choose an inertia group $I_{\hat{s}} \subset \pi_1(\hat{X} - \hat{\mathbf{S}}, \hat{\mathbf{j}})$. Choose $g_{\hat{s}} \in \pi_1(X - S, \mathbf{j})$ such that $I_{\hat{s}} = \rho_*^{-1}(g_{\hat{s}} I_s g_{\hat{s}}^{-1})$, where ρ_* is the induced map on fundamental groups and $\rho(\hat{s}) = s$.

2.5. LEMMA. *If $\mathcal{F} \in \mathrm{Ob}(\mathcal{C})$ corresponds to $(F, \{F_s, \phi_s\}_{s\in\mathbf{S}})$, then $\rho^*\mathcal{F}$ corresponds to $(F, \{F_{\rho(\hat{s})}, g_{\hat{s}} \circ \phi_{\rho(\hat{s})}\}_{\hat{s}\in\hat{\mathbf{S}}})$.*

2.6. REMARK. When $\rho = \mathrm{Id}$, Lemma 2.5 describes the dependence of $\phi_s(F_s) \subset F$ on the choice of the inertia group I_s.

3. Cohomology of Elliptic Surfaces

In preparation for the computation of the cohomology of desingularized fiber products, the cohomology of elliptic surfaces is computed in this section by means of the Leray spectral sequence. The notation of previous sections, and especially that defined in Section 1.3, remains in force. In particular, $\pi : Y \to X$ is a relatively minimal and nonisotrivial elliptic surface with section. Write g for the genus of X and m_s for the number of irreducible components in the fiber $\pi^{-1}(s)$.

3.1. PROPOSITION. (i) *The terms $E_2^{p,q} \simeq H^p(X, R^q\pi_*\mathbb{Z}/l^n)$ in the Leray spectral sequence for π are presented in the usual format in the following table, where numbers indicate ranks of free $\mathbb{Z}/l^n$-modules (e.g., $2g$ stands for $(\mathbb{Z}/l^n)^{2g}$).*

$1 + \sum_{s\in S}(m_s - 1)$	$2g$	1
0	$H^1(X, R^1\pi_*\mathbb{Z}/l^n)$	$\mathrm{Hom}(E[l^n]^G, \mathbb{Z}/l^n)$
1	$2g$	1

(ii) *The differential $d_2^{0,2}\colon E_2^{0,2} \to E_2^{2,1}$ is surjective.*
(iii) *For each i, $H^i(Y, \mathbb{Z}/l^n)$ is a free $\mathbb{Z}/l^n$-module.*

Proof. (i) The form of the bottom row of the E_2-term follows from $\pi_*\mathbb{Z}/l^n \simeq \mathbb{Z}/l^n$. To analyze the E_2^{21}-term, use Poincaré duality:

$$\begin{aligned} H^2(X, R^1\pi_*\mathbb{Z}/l^n) &\simeq H^2(X, j_*j^*R^1\pi_*\mathbb{Z}/l^n) \\ &\simeq H^0(X, j_*j^*R^1\pi_*\mu_{l^n})^\vee \simeq \mathrm{Hom}(E[l^n]^G, \mathbb{Z}/l^n) \end{aligned}$$

[Mi, V.2.2(b)]. The form of the top row follows from the next lemma.

3.2. LEMMA. $R^2\pi_*\mathbb{Z}/l^n \simeq \mathbb{Z}/l^n \oplus \sum_{s\in S} i_{s*}(\mathbb{Z}/l^n)^{m_s-1}$.

Proof. Write $\bar{\pi} : \bar{Y} \to X$ for the Weierstrass model of E. Blowing down the irreducible components of the singular fibers of π that do not meet the identity section gives a morphism, $w\colon Y \to \bar{Y}$. Now $R^2\bar{\pi}_*\mathbb{Z}/l^n \simeq \mathbb{Z}/l^n$ and so the spectral sequence for the composition of functors $\pi_* \simeq \bar{\pi}_* \circ w_*$ gives an exact sequence,

$$0 \to \mathbb{Z}/l^n \to R^2\pi_*\mathbb{Z}/l^n \to \sum_{s\in S} i_{s*}(\mathbb{Z}/l^n)^{m_s-1} \to 0.$$

The map $R^2\pi_*\mathbb{Z}/l^n \to j_*j^*R^2\pi_*\mathbb{Z}/l^n \simeq \mathbb{Z}/l^n$ splits this sequence. □

3.3. Lemma. *In the notation of Section 1.3 and Lemma 2.4, the object of the category $\mathcal{M}$ corresponding to $R^1\pi_*\mathbb{Z}/l^n$ is* $(E[l^n], \{E[l^n]_{0s}, \varphi_s\})$, *where* $\varphi_s\colon E[l^n]_{0s} \to E[l^n]$ *is the inclusion.*

Proof. The espace étalé of the sheaf $R^1\pi_*\mu_{l^n}$ may be identified with the maximal l^n-torsion subgroup scheme of the Néron model of E that meets every fiber in the identity component. This sheaf is constructible, and its restriction to $X - S$ is locally constant [Mi, VI.2 and VI.4.2]. □

3.4. Proposition. $H^0(X, R^1\pi_*\mathbb{Z}/l^n) = 0$.

Proof. Since each φ_s is injective, Lemma 2.4(ii) implies

$$H^0(X, R^1\pi_*\mu_{l^n}) \simeq E[l^n]^G \cap \left(\bigcap_{s\in S} E[l^n]_{0s}\right).$$

If there is a place s of additive reduction, the proposition follows immediately because then $E[l^n]_{0s} = 0$. To treat the general case, it is convenient to base change. Let $K \subset \hat{K}$ be a finite field extension, $\rho\colon \hat{X} \to X$ the normalization of X in $\hat{K}$, and $\hat{\pi}: \hat{Y} \to \hat{X}$ the relatively minimal model of $E_{\hat{K}} := E \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\hat{K})$.

3.5. Lemma. *The canonical map of sheaves,* $\rho^* R^1\pi_*\mathbb{Z}/l^n \to R^1\hat{\pi}_*\mathbb{Z}/l^n$, *is injective. If ρ is unramified over the locus of additive reduction, S_a, it is an isomorphism.*

Proof. Write $w\colon Y \to \bar{Y}$ for the morphism from the relatively minimal model of E to the Weierstrass model. The exceptional fibers of w are trees of $\mathbb{P}^1$s. Thus $R^1 w_*\mathbb{Z}/l^n = 0$ and $R^1\pi_*\mathbb{Z}/l^n \simeq R^1\bar{\pi}_*\mathbb{Z}/l^n$. Hence both Y and $\hat{Y}$ may be replaced by Weierstrass models. If ρ is not ramified over S_a, then the Weierstrass model for $\hat{Y}$ is the pullback of the Weierstrass model for Y and the asserted isomorphism follows from the proper base change theorem. Even if ρ is ramified over S_a, the map is injective since $R^1\pi_*\mathbb{Z}/l^n|_s = 0$ for $s \in S_a$. □

Continuing now with the proof of Proposition 3.4, we choose a field extension $K \subset \hat{K}$ such that all the l-torsion in E becomes rational over $\hat{K}$. Fix an ordered basis e_1, e_2 for the l-torsion. The Weil pairing determines a primitive lth root of unity, $\langle e_1, e_2\rangle = \zeta \in k^*$, and there is an induced morphism from $\mathrm{Spec}(\hat{K})$ to the moduli of generalized elliptic curves with a basis for the l-torsion. This is a fine moduli space unless $l = 2$. When $l = 2$, the argument that follows will need to be modified—say, by choosing $\hat{K}$ such that there is an additional prime $l' \neq \mathrm{char}(k)$ for which the $2l'$-torsion is rational over $\hat{K}$. We leave the necessary modifications to the reader and assume that $l \geq 3$.

The value ζ of the Weil pairing determines a connected component and thus a generic point $\varkappa$ of the moduli space. Write $\mathbf{E}$ for the generic fiber of the universal elliptic curve. Write $\mathbf{s}$ for a cusp and define $\mathbf{E}[l]_{0\mathbf{s}} \subset \mathbf{E}[l]$ for the subgroup whose closure in the Néron model meets the identity component of the Néron model over the cusp. We claim that every one-dimensional $\mathbb{Z}/l$-subspace of $\mathbf{E}[l]$ has the form $\mathbf{E}[l]_{0\mathbf{s}}$ for some cusp $\mathbf{s}$. To see this, begin by noting that the fiber of the Néron model at a cusp is isomorphic to $\mathbb{G}_m \times \mathbb{Z}/l$, so its l-torsion may be identified with $\mu_l \times \mathbb{Z}/l$. The Weil pairing extends to the singular fiber by

$$\langle(\zeta^a, b), (\zeta^c, d)\rangle = \zeta^{ad-bc}$$

for $a, b, c, d \in \mathbb{Z}/l$.

An l-gon of rational curves together with an ordered pair of points on the nonsingular locus that identify with l-torsion points (ζ^a, b) and (ζ^c, d) in $\mathbb{G}_m \times \mathbb{Z}/l$ gives a generalized elliptic curve with level structure and thus a cusp s of the moduli space. Assume $ad - bc = 1$ so that s is in the closure of the generic point $\varkappa$. By applying an automorphism of the l-gon as a generalized elliptic curve, we may arrange that the basis e_1, e_2 of $\mathbf{E}[l]$ specializes to $(\zeta^a, b), (\zeta^c, d)$. The automorphism group is dihedral with generators acting on the smooth locus $\mathbb{G}_m \times \mathbb{Z}/l$ by $(t, b) \mapsto (t^{-1}, -b)$ and $(t, b) \mapsto (\zeta^b t, b)$ [DR, II.1.10]. The element $de_1 - be_2 \in \mathbf{E}[l]$ specializes to $(\zeta, 0)$, so $\mathbf{E}[l]_{0s} = \mathrm{Span}\{de_1 - be_2\}$. The only condition on b and d is the existence of a and c with $ad - bc = 1$. Thus every order-l subgroup of $\mathbf{E}[l]$ has the form $\mathbf{E}[l]_{0s}$ for some cusp s in the closure of $\varkappa$. In particular, intersecting over these cusps gives $\bigcap_s \mathbf{E}[l]_{0s} = 0$.

Since $\hat{Y}$ is the pullback of the universal generalized elliptic curve and since the locus of bad reduction of $E_{\hat{K}}$ is the pullback $\hat{S} \subset \hat{X}$ of the cusps, it follows that $\bigcap_{\hat{s}\in\hat{S}} E[l]_{0\hat{s}} = 0$. Thus, for any n, $\bigcap_{\hat{s}\in\hat{S}} E[l^n]_{0\hat{s}} = 0$ and so $H^0(\hat{X}, R^1\hat{\pi}_*\mathbb{Z}/l^n) = 0$. By Lemma 3.5, $H^0(X, R^1\pi_*\mathbb{Z}/l^n) = 0$. □

The proof of Proposition 3.1(i) is now complete. To prove part (ii) it suffices to show $H^3(Y, \mathbb{Z}/l^n) \simeq (\mathbb{Z}/l^n)^{2g}$, which follows from $H^1(Y, \mathbb{Z}/l^n) \simeq (\mathbb{Z}/l^n)^{2g}$ and Poincaré duality. As for part (iii), the fact that $H^i(Y, \mathbb{Z}/l^n)$ is a free $\mathbb{Z}/l^n$-module for all n and $i \neq 2$ implies that $H^i(Y, \mathbb{Z}_l)_{\mathrm{tors}} = 0$ for all i by Lemma 2.2(ii), which implies that $H^2(Y, \mathbb{Z}/l^n)$ is a free $\mathbb{Z}/l^n$-module (cf. the proof of Lemma 2.2(i)). □

4. Cohomology of Fiber Products of Elliptic Surfaces

Recall the terms E, E', $\pi : Y \to X$, $\pi' : Y' \to X$, S, and S' from the Introduction. In this section we consider the cohomology of $\bar{W} := Y \times_X Y'$. Here one can make good progress using the Leray spectral sequence for the canonical map $\bar{f} : \bar{W} \to X$ and the Künneth formula applied to the fiber product. To simplify notation, define

$$T := E[l^n], \quad T' := E'[l^n], \quad T_s := E_{0s}[l^n], \quad T'_s := E'_{0s}[l^n]$$

when l and n are clear from the context. Also define $S'' := S \cap S'$.

4.1. PROPOSITION. $H^i(\bar{W}, \mathbb{Z}_l)_{\mathrm{tors}} = 0$ *for* $i \neq 4$.

Proof. By Lemma 2.2(ii) it suffices to show that $H^i(\bar{W}, \mathbb{Z}/l^n)$ is a free $\mathbb{Z}/l^n$-module for $i \notin \{3, 4\}$ and all n. This will be deduced from Lemmas 4.2–4.5.

4.2. LEMMA.

(i) $\bar{f}_*\mathbb{Z}/l^n \simeq \mathbb{Z}/l^n$.

(ii) $R^1\bar{f}_*\mathbb{Z}/l^n \simeq R^1\pi_*\mathbb{Z}/l^n \oplus R^1\pi'_*\mathbb{Z}/l^n$.

(iii) $R^2\bar{f}_*\mathbb{Z}/l^n \simeq R^2\pi_*\mathbb{Z}/l^n \oplus R^1\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n \oplus R^2\pi'_*\mathbb{Z}/l^n$.

(iv) $R^3\bar{f}_*\mathbb{Z}/l^n \simeq R^2\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n \oplus R^1\pi_*\mathbb{Z}/l^n \otimes R^2\pi'_*\mathbb{Z}/l^n$.

(v) $R^4\bar{f}_*\mathbb{Z}/l^n \simeq R^2\pi_*\mathbb{Z}/l^n \otimes R^2\pi'_*\mathbb{Z}/l^n \simeq \mathbb{Z}/l^n \oplus \sum_{s\in S-S''} i_{s*}(\mathbb{Z}/l^n)^{m_s-1} \oplus \sum_{s\in S'-S''} i_{s*}(\mathbb{Z}/l^n)^{m'_s-1} \oplus \sum_{s\in S''} i_{s*}(\mathbb{Z}/l^n)^{m_s m'_s-1}$.

Proof. By Section 3, $R^i\pi_*\mathbb{Z}/l^n$ and $R^i\pi'_*\mathbb{Z}/l^n$ are flat sheaves of $\mathbb{Z}/l^n$-modules. The assertions now follow from the Künneth formula [Mi, VI.8.6]. The final isomorphism in part (v) is a direct consequence of Lemma 3.2. □

To compactify notation, define:

$$B_{l^n} := \mathrm{Hom}(E[l^n]^G, \mathbb{Z}/l^n) \quad \text{and} \quad B'_{l^n} := \mathrm{Hom}(E'[l^n]^G, \mathbb{Z}/l^n);$$

$$R_{l^n} := \mathrm{Hom}((E[l^n] \otimes E'[l^n])^G, \mathbb{Z}/l^n);$$

$$\mathfrak{R} := R^1\pi_*\mathbb{Z}/l^n \quad \text{and} \quad \mathfrak{R}' := R^1\pi'_*\mathbb{Z}/l^n;$$

$$m'_s := \text{the number of irreducible components in the fiber } (\pi')^{-1}(s).$$

4.3. Lemma. *The E_2-term, $E_2(\bar{f})$, of the Leray spectral sequence for the map $\bar{f}$ and the sheaf $\mathbb{Z}/l^n$ is displayed in the usual format below. The numbers* 0, 1, $2g$, *and* $h^0(X, R^i\bar{f}_*\mathbb{Z}/l) := \dim_{\mathbb{Z}/l}(H^0(X, R^i\bar{f}_*\mathbb{Z}/l))$ *indicate ranks of free $\mathbb{Z}/l^n$-modules.*

$h^0(X, R^4\bar{f}_*\mathbb{Z}/l)$	$2g$	1
$h^0(X, R^3\bar{f}_*\mathbb{Z}/l)$	$H^1(X, R^3\bar{f}_*\mathbb{Z}/l^n)$	$B_{l^n} \oplus B'_{l^n}$
$h^0(X, R^2\bar{f}_*\mathbb{Z}/l)$	$H^1(X, R^2\bar{f}_*\mathbb{Z}/l^n)$	$(\mathbb{Z}/l^n)^2 \oplus R_{l^n}$
0	$H^1(X, R^1\pi_*\mathbb{Z}/l^n \oplus R^1\pi'_*\mathbb{Z}/l^n)$	$B_{l^n} \oplus B'_{l^n}$
1	$2g$	1

Proof. The bottom two rows have the indicated form by Proposition 3.4 and parts (i) and (ii) of Lemma 4.2, and the top row by Lemma 4.2(v). For the second row, compute explicitly the tensor products in Lemma 4.2(iv) to get that $R^3\bar{f}_*\mathbb{Z}/l^n$ is isomorphic to

$$\mathfrak{R} \oplus \mathfrak{R}' \oplus \sum_{s\in S-S''} i_{s*}(\mathbb{Z}/l^n)^{2(m_s-1)} \oplus \sum_{s\in S'-S''} i_{s*}(\mathbb{Z}/l^n)^{2(m'_s-1)} \oplus \sum_{s\in S''} i_{s*}(\mathbb{Z}/l^n)^{m_s+m'_s-2}$$

and then apply the second row in Proposition 3.1(i). To show that E_2^{02} is a free $\mathbb{Z}/l^n$-module, combine Lemma 4.2(iii) with Lemma 3.2 to get that $R^2\bar{f}_*\mathbb{Z}/l^n$ is isomorphic to

$$\mathfrak{R} \otimes \mathfrak{R}' \oplus (\mathbb{Z}/l^n)^2 \oplus \sum_{s\in S-S''} i_{s*}(\mathbb{Z}/l^n)^{m_s-1} \oplus \sum_{s\in S'-S''} i_{s*}(\mathbb{Z}/l^n)^{m'_s-1} \oplus \sum_{s\in S''} i_{s*}(\mathbb{Z}/l^n)^{m_s+m'_s-2}.$$

Now the $E_2^{2,2}$-term has the indicated form by Poincaré duality:

$$H^2(X, \mathfrak{R} \otimes \mathfrak{R}') \simeq H^0(X, j_* j^*(\mathfrak{R} \otimes \mathfrak{R}'))^\vee \simeq \mathrm{Hom}((E[l^n] \otimes E'[l^n])^G, \mathbb{Z}/l^n).$$

[Mi, V.2.2(b)]. The proof of the lemma is now complete—up to the assertion that the E_2^{02}-term is a free $\mathbb{Z}/l^n$-module. This is an immediate consequence of Lemma 3.2, the explicit expression for $R^2 \bar{f}_* \mathbb{Z}/l^n$ in Lemma 4.2(iii), and the following statement. $\square$

4.4. LEMMA. $H^0(X, R^1\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n) = 0.$

Proof. As in the proof of Proposition 3.4, we may reduce to the case where both elliptic surfaces π and π' are relatively minimal models of pullbacks of universal elliptic surfaces with full level l^n structure. The sheaf $R^1\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n$ corresponds (by Lemmas 3.3 and 2.4) to data $(M, \{\mathbf{M}_s, \phi_s\}_{s\in S\cup S'})$, where $M = T \otimes T'$ and, for each $s \in S$, $\phi_s(\mathbf{M}_s) \subset T_s \otimes T'$. By the proof of Proposition 3.4, $\bigcap_{s\in S} T_s = 0$. Thus $\bigcap_{s\in S} T_s \otimes T' = 0$ and the assertion follows from Lemma 2.4(ii). $\square$

4.5. LEMMA. *The differentials $d_2^{02}\colon E_2^{02} \to E_2^{21}$ and $d_2^{04}\colon E_2^{04} \to E_2^{23}$ are each surjective.*

Proof. The projection maps $\bar{W} \to Y$ and $\bar{W} \to Y'$ give rise to morphisms of spectral sequences $E(\pi) \to E(\bar{f})$ and $E(\pi') \to E(\bar{f})$. The surjectivity of d_2^{02} follows from the analogous statements for the spectral sequences $E(\pi)$ and $E(\pi')$ (cf. Proposition 3.1(ii)). The surjectivity of d_2^{04} follows from the surjectivity of d_2^{02} by taking a cup product with the cohomology class of the Cartier divisor $\mathfrak{s} \times_X Y' + Y \times_X \mathfrak{s}'$ in $H^0(X, R^2\pi_*\mathbb{Z}/l^n) \oplus R^2\pi'_*\mathbb{Z}/l^n(1))$, where $\mathfrak{s} \subset Y$ and $\mathfrak{s}' \subset Y'$ denote sections. $\square$

Proof of Proposition 4.1 (continued). It is immediately apparent from Lemma 4.3 that $H^i(\bar{W}, \mathbb{Z}/l^n)$ is a free $\mathbb{Z}/l^n$-module for $i \in \{0, 1, 6\}$. Freeness for $i = 5$ follows from the surjectivity of d_2^{04}. For the case $i = 2$, consider the commutative diagram of exact sequences arising from the morphism of spectral sequences $E(\pi \oplus \pi') := E(\pi) \oplus E(\pi') \to E(\bar{f})$:

$$\begin{array}{ccccccccc}
0 \longrightarrow & E_2^{11}(\pi \oplus \pi') & \longrightarrow & H^2(Y)_\bullet \oplus H^2(Y')_\bullet & \longrightarrow & E_2^{02}(\pi \oplus \pi') & \longrightarrow & B_{l^n} \oplus B'_{l^n} & \longrightarrow 0 \\
& \downarrow\wr & & \alpha\downarrow & & \beta\downarrow & & \downarrow\wr & \\
0 \longrightarrow & E_2^{11}(\bar{f}) & \longrightarrow & H^2(\bar{W}, \mathbb{Z}/l^n)_\bullet & \longrightarrow & E_2^{02}(\bar{f}) & \longrightarrow & B_{l^n} \oplus B'_{l^n} & \longrightarrow 0,
\end{array}$$

where the $\bullet$ means "take the quotient by $H^2(X, \mathbb{Z}/l^n)$". By Lemmas 4.2(iii) and 4.4, β is an isomorphism. Thus α is an isomorphism by the five lemma. Now $H^2(Y, \mathbb{Z}/l^n)$ and $H^2(Y', \mathbb{Z}/l^n)$ are free $\mathbb{Z}/l^n$-modules by Proposition 3.1. Since taking the quotient by the image of $H^2(X, \mathbb{Z}/l^n)$ preserves freeness, $H^2(\bar{W}, \mathbb{Z}/l^n)$ is also free. $\square$

5. Applications to the Cohomology of Desingularized Fiber Products

Write $\sigma : W \to \bar{W}$ for the blowup of the ideal sheaf of the reduced singular locus $\bar{W}_{\mathrm{sing}} \subset \bar{W}$. A point $w = (y, y') \in \pi^{-1}(x) \times (\pi')^{-1}(x) \subset \bar{W}$ lies in $\bar{W}_{\mathrm{sing}}$ precisely when π is not smooth at y and π' is not smooth at y'. Assumption 0.1

implies that, locally analytically, the map π at the first component y of w is given by the standard inclusion of k-algebras: $k[[t]] \to k[[t,u,v]]/(uv-t)$, $t \mapsto t$ (cf. [Si2, IV.8–9]). Since π' has an analogous description, the completion $\widehat{\mathcal{O}_{\bar{W},w}} \simeq k[[u,v,u',v']]/(uv-u'v')$. Consequently W is a smooth projective variety and each connected component of the exceptional divisor $Q \subset W$ is isomorphic to a hypersurface in $\mathbb{P}^3$ defined by $uv - u'v' = 0$.

5.1. Proposition.

(i) *The map* $\sigma^*\colon H^i(\bar{W},\mathbb{Z}/l^n) \to H^i(W,\mathbb{Z}/l^n)$ *is an isomorphism for* $i \in \{0,1,5,6\}$.

(ii) *For* $i \notin \{3,4\}$, $H^i(W,\mathbb{Z}_l)_{\text{tors}} = 0$.

(iii) *The map* $\sigma^*\colon H^4(\bar{W},\mathbb{Z}_l)_{\text{tors}} \to H^4(W,\mathbb{Z}_l)_{\text{tors}}$ *is an isomorphism.*

(iv) $H^3(W,\mathbb{Z}_l)_{\text{tors}}$ *and* $H^4(W,\mathbb{Z}_l)_{\text{tors}}$ *are (noncanonically) isomorphic.*

Proof. (i) This is a straightforward computation using the Leray spectral sequence for the map σ. Note that $\sigma_*\mathbb{Z}/l^n \simeq \mathbb{Z}/l^n$, $R^1\sigma_*\mathbb{Z}/l^n = 0$, and $H^p(\bar{W}, R^q\sigma_*\mathbb{Z}/l^n) = 0$ for $p > 0$ and $q > 0$, since $R^q\sigma_*\mathbb{Z}/l^n$ is supported on the (zero-dimensional) locus $\bar{W}_{\text{sing}}$. Furthermore, the differential

$$d_5^{04}\colon H^0(\bar{W}, R^4\sigma_*\mathbb{Z}/l^n) \to H^5(\bar{W},\sigma_*\mathbb{Z}/l^n)$$

is zero because the map $H^4(W,\mathbb{Z}/l^n) \to H^0(\bar{W}, R^4\sigma_*\mathbb{Z}/l^n)$ is surjective. To see this, take a line L_w in the exceptional divisor $Q_w := \sigma^{-1}(w)$ and compute the intersection product: $L_w \cdot Q_w = c_1(\mathcal{N}_{Q/W}|_{L_w})$, which is minus the class of a point. Thus the cohomology class $[L_w] \in H^4(W,\mathbb{Z}/l^n)(2)$ restricts to a generator of $H^4(Q_w,\mathbb{Z}/l^n)(2)$ for each $w \in \bar{W}_{\text{sing}}$. Since $H^0(\bar{W}, R^4\sigma_*\mathbb{Z}/l^n) \simeq \bigoplus_{w\in\bar{W}_{\text{sing}}} H^4(Q_w,\mathbb{Z}/l^n)$, the surjectivity follows.

(ii) By part (i) and the proof of Proposition 4.1, $H^j(W,\mathbb{Z}/l^n)$ is a free $\mathbb{Z}/l^n$-module for $j \in \{0,1,5,6\}$ and each n. The assertion follows from Lemma 2.2.

(iii) The Leray spectral sequence for σ gives an exact sequence

$$0 \to H^4(\bar{W},\mathbb{Z}/l^n(2)) \to H^4(W,\mathbb{Z}/l^n(2)) \to H^4(Q,\mathbb{Z}/l^n(2)) \to 0,$$

which is split by the classes $[L_w]$, $w \in \bar{W}_{\text{sing}}$. Now take the inverse limit over n and use that $H^4(Q,\mathbb{Z}_l(2))_{\text{tors}} = 0$.

(iv) This is a special case of Proposition 2.3(i). □

5.2. Remark. Generally it is not easy to compute $H^4(W,\mathbb{Z}_l)_{\text{tors}} \simeq H^4(\bar{W},\mathbb{Z}_l)_{\text{tors}}$ directly, even with the help of the spectral sequence in Lemma 4.3. The problem is computing the differential d_2^{03}. (See Remark 8.2 for a special situation where this is manageable.)

The next lemma establishes the approach we take when computing $H^4(W,\mathbb{Z}_l)_{\text{tors}} \simeq H^3(W,\mathbb{Z}_l)_{\text{tors}}$: reduce the problem to understanding $H^2(W,\mathbb{Z}/l^n)$.

5.3. Lemma. *For sufficiently large* n, $H^3(W,\mathbb{Z}_l)[l^n]$ *is isomorphic to the quotient of* $H^2(W,\mathbb{Z}/l^n)$ *by a maximal free* $\mathbb{Z}/l^n$*-submodule.*

Proof. Combine Proposition 5.1(ii) with Lemma 2.2(i). □

The main tool for analyzing $H^2(W, \mathbb{Z}/l^n)$ will be the Leray spectral sequence associated to the canonical morphism $f: W \to X$.

5.4. LEMMA. (i) *The E_2-term, $E_2(f)$, of the Leray spectral sequence for f is displayed below in the usual format*; *the numbers* 1, $2g$, *and* $h^0(X, R^4 f_* \mathbb{Z}/l)$ *indicate the ranks of free $\mathbb{Z}/l^n$-modules.*

$$\begin{array}{ccc}
h^0(X, R^4 f_* \mathbb{Z}/l) & 2g & 1 \\[6pt]
H^0(X, R^3 f_* \mathbb{Z}/l^n) & H^1(X, R^3 f_* \mathbb{Z}/l^n) & B_{l^n} \oplus B'_{l^n} \\[6pt]
H^0(X, R^2 f_* \mathbb{Z}/l^n) & H^1(X, R^2 f_* \mathbb{Z}/l^n) & (\mathbb{Z}/l^n)^2 \oplus R_{l^n} \\[6pt]
0 & H^1(X, R^1 f_* \mathbb{Z}/l^n) & B_{l^n} \oplus B'_{l^n} \\[6pt]
1 & 2g & 1
\end{array}$$

(ii) *The differentials $d_2^{02}: E_2^{02}(f) \to E_2^{21}(f)$ and $d_2^{04}: E_2^{04}(f) \to E_2^{23}(f)$ are surjective.*

Proof. (i) Consider the restriction map $\upsilon^i: R^i \bar{f}_* \mathbb{Z}/l^n \to R^i f_* \mathbb{Z}/l^n$. Since $\sigma_* \mathbb{Z}/l^n \simeq \mathbb{Z}/l^n$ and $R^1 \sigma_* \mathbb{Z}/l^n = 0$, we may identify the bottom two rows in $E_2(f)$ with the corresponding rows in $E_2(\bar{f})$ (cf. Lemma 4.3). The top row in $E_2(f)$ may be deduced from the top row in $E_2(\bar{f})$ thanks to the split exact sequence

$$0 \to R^4 \bar{f}_* \mathbb{Z}/l^n \xrightarrow{\upsilon^4} R^4 f_* \mathbb{Z}/l^n \to \bar{f}_* R^4 \sigma_* \mathbb{Z}/l^n \to 0$$

(cf. the proof of Proposition 5.1(iii)). The right-hand columns in $E_2(f)$ and $E_2(\bar{f})$ may be identified. Indeed, $\mathrm{Ker}(\upsilon^i)$ and $\mathrm{Coker}(\upsilon^i)$ are supported on $S \cup S'$ and so $H^2(X, \upsilon^i)$ is an isomorphism for all i.

(ii) This follows from the morphism of spectral sequences, $E(\bar{f}) \to E(f)$, and the analogous fact for the differentials in $E_2(\bar{f})$ (see Lemma 4.5). $\square$

For a finitely generated $\mathbb{Z}/l^n$-module N, let N_* denote the quotient by a maximal free $\mathbb{Z}/l^n$-submodule. The isomorphism class of N_* is independent of the choice of maximal free submodule.

5.5. LEMMA. $H^2(W, \mathbb{Z}/l^n)_* \simeq H^0(X, R^2 f_* \mathbb{Z}/l^n)_*$.

Proof. The map $\sigma: W \to \bar{W}$ gives rise to a morphism of spectral sequences, $E(\bar{f}) \to E(f)$, and to a morphism of exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & E_2^{11}(\bar{f}) \arrow[r] \arrow[d, "\wr"] & H^2(\bar{W}, \mathbb{Z}/l^n)_{\bullet} \arrow[r] \arrow[d, "\alpha"'] & E_2^{02}(\bar{f}) \arrow[r] \arrow[d, "\beta"'] & B_{l^n} \oplus B'_{l^n} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & E_2^{11}(f) \arrow[r] & H^2(W, \mathbb{Z}/l^n)_{\bullet} \arrow[r] & E_2^{02}(f) \arrow[r] & B_{l^n} \oplus B'_{l^n} \arrow[r] & 0;
\end{tikzcd}$$

as before, the $\bullet$ means "take the quotient by the image of $H^2(X, \mathbb{Z}/l^n)$". Splitting each four-term exact sequence into two short exact sequences shows that the map Coker$(\alpha) \to$ Coker(β) is an isomorphism. By Lemma 4.3, $E_2^{02}(\bar{f})$ is a free $\mathbb{Z}/l^n$-module. The same holds for $H^2(\bar{W}, \mathbb{Z}/l^n)_\bullet$ by the proof of Proposition 4.1. If α and β are injective, then they are split injective and the lemma will follow. The injectivity of β follows from the injectivity of the map

$$R^2\bar{f}_*\mathbb{Z}/l^n \xrightarrow{\upsilon^2} R^2 f_*\mathbb{Z}/l^n,$$

which is an easy consequence of the spectral sequence for the composition of functors, $f_* = \bar{f}_*\sigma_*$. The injectivity of α follows from the five lemma. □

5.6. Corollary. $H^0(X, R^2 f_*\mathbb{Z}/l^n)_* \simeq H^3(W, \mathbb{Z}_l)[l^n]$ *for* $n \gg 0$.

Proof. Combine Lemmas 5.3 and 5.5. □

6. The Cohomology Group $H^0(X, R^2 f_*\mathbb{Z}/l^n)$

Let's begin by describing $R^2 f_*\mathbb{Z}/l^n$ as a direct sum of simpler sheaves. Write $j\colon \eta \to X$ for the inclusion of the generic point, and let

$$r\colon R^2 f_*\mathbb{Z}/l^n \to j_*j^*R^2 f_*\mathbb{Z}/l^n$$

denote the canonical map. Define $\mathcal{G}_{l^n}$ to be the image of the composition

$$R^2 f_*\mathbb{Z}/l^n \xrightarrow{r} j_*j^*R^2 f_*\mathbb{Z}/l^n \xrightarrow{j_*\circ \mathrm{pr}} j_*j^*(R^1\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n),$$

where pr denotes the Künneth projector on the generic fiber.

6.1. Proposition.

(i) *The sheaf* Ker(r) *is supported on* $S \cup S'$.
(ii) *For each* $s \in S \cup S'$, *the stalk* Ker$(r)_s$ *is a free* $\mathbb{Z}/l^n$*-module.*
(iii) $R^2 f_*\mathbb{Z}/l^n \simeq \mathrm{Ker}(r) \oplus (\mathbb{Z}/l^n)^2 \oplus \mathcal{G}_{l^n}$.
(iv) $\mathcal{G}_{l^n}|_{X-S''} \simeq (R^1\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n)|_{X-S''}$.
(v) $H^0(X, R^2 f_*\mathbb{Z}/l^n)_* \simeq H^0(X, \mathcal{G}_{l^n})_*$.

Proof. (i) [Mi, II.3.15 and VI.4.2].

(ii) Away from $S \cap S'$ one may identify $R^2 f_*\mathbb{Z}/l^n$ with $R^2\bar{f}_*\mathbb{Z}/l^n$. For $s \notin S \cap S'$, the assertion follows from Lemmas 4.2(iii), 3.2, and 3.3. For $s \in S \cap S'$, this is [S2, 7.3].

(iii) By Lemma 2.4(iii), Ker(r) is a direct summand of $R^2 f_*\mathbb{Z}/l^n$. The decomposition of the image of r as a direct sum follows from the Künneth formula applied to $j_*j^*R^2 f_*\mathbb{Z}/l^n$ [Mi, VI.8.5].

(iv) Apply Lemma 4.2(iii) while using $W - f^{-1}(S'') \simeq \bar{W} - \bar{f}^{-1}(S'')$ and the injectivity of the canonical map

$$R^1\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n \to j_*j^*(R^1\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n)$$

(cf. the proof of Lemma 4.4).

(v) This follows from parts (ii) and (iii). □

6.2. PROPOSITION. *The sheaf $\mathcal{G}_{l^n}$ is a constructible sheaf of $\mathbb{Z}/l^n$-modules that is locally constant on $U := X - (S \cup S')$. Under the equivalence of categories (Lemma 2.4), $\mathcal{G}_{l^n}$ corresponds to $(M, \{M_s, \phi_s\}_{s \in S \cup S'}) \in \mathrm{Ob}(\mathcal{M})$, where $M = T \otimes T'$ and*

$$M_s, \phi_s = \begin{cases} 0, 0 & \text{if } s \in S_a \cup S'_a, \\ T_s \otimes T', \varphi_s \otimes \mathrm{Id} & \text{if } s \in S_m - S'', \\ T \otimes T'_s, \mathrm{Id} \otimes \varphi'_s & \text{if } s \in S'_m - S'', \\ (T \otimes \varphi'_s(T'_s) + \varphi_s(T_s) \otimes T') \cap (T \otimes T')^{I_s}, \phi_s & \text{if } s \in S''. \end{cases}$$

In the last line, ϕ_s is the standard inclusion.

Proof. The only point that is not apparent from Proposition 6.1(iv) and the proof of Lemma 4.4 is the identification of M_s and ϕ_s for $s \in S''$. For this, see [S2, 7.6(iv)]. □

6.3. COROLLARY. *With notation as in Definition 0.2, $H^0(X, \mathcal{G}_{l^n}) \simeq \Theta_{l^n}(W)$.*

Proof. Apply Lemma 2.4(ii). □

6.4. COROLLARY. *If $S_a \cup S'_a \neq \emptyset$, then $H^0(X, \mathcal{G}_{l^n}) = 0$.*

Proof. By Assumption 0.1, $(S_a \cup S'_a) \cap S'' = \emptyset$. For $s \in S_a$ (resp. S'_a), the stalk $R^1\pi_*\mathbb{Z}/l^n|_s$ (resp. $R^1\pi'_*\mathbb{Z}/l^n|_s$) is zero. Now apply Proposition 6.2 and Lemma 2.4(ii). □

6.5. PROPOSITION. *$H^0(X, \mathcal{G}_{l^n})$ is a cyclic group. If $S \cap S' = \emptyset$ then this group is zero.*

Proof. We may and will assume that $n = 1$. We first prove the assertions in the special case when there is an $n' \geq 1$ such that $l^{n'} > 2$ and $E[l^{n'}]$ and $E'[l^{n'}]$ are trivial G-modules. This implies that the elliptic surfaces Y and Y' are relatively minimal models of pullbacks of the universal generalized elliptic curve with level $l^{n'}$ structure over the modular curve $X(l^{n'})$. Assume first that $S \cap S' = \emptyset$. By Lemma 3.3 and Proposition 3.4, $\bigcap_{s \in S} \varphi_s(T_s) = 0$. By Proposition 6.2, $\bigcap_{s \in S} \phi_s(M_s) = 0$ and so $H^0(X, \mathcal{G}_l) \simeq \bigcap_{s \in S \cup S'} \phi_s(M_s) = 0$.

If $S \cap S' \neq \emptyset$, then the hypotheses imply the existence of three distinct $s_1, s_2, s_3 \in S \cup S'$ such that $\varphi_{s_1}(T_{s_1}), \varphi_{s_2}(T_{s_2}) \subset T \simeq (\mathbb{Z}/l)^2$ are distinct one-dimensional subspaces and the same holds for $\varphi'_{s_1}(T'_{s_1}), \varphi'_{s_3}(T'_{s_3}) \subset T'$. The most delicate case to consider is when $\{s_1, s_2, s_3\} \subset S \cap S'$. Set

$$H_i := \varphi_{s_i}(T_{s_i}) \otimes T' + T \otimes \varphi'_{s_i}(T'_{s_i}) \subset T \otimes T'.$$

Now $H^0(X, \mathcal{G}_l) \simeq \bigcap_{s \in S \cup S'} \phi_s(M_s) \subset H_1 \cap H_2 \cap H_3$. We claim

$$\dim(H_1 \cap H_2 \cap H_3) \leq 1.$$

If not, the intersection $\mathbb{P}(H_1) \cap \mathbb{P}(H_2) \cap \mathbb{P}(H_3)$ in the projective space of lines in $T \otimes T'$, $\mathbb{P}(T \otimes T')$, has positive dimension. Equivalently, in the dual projective space the points p_i corresponding to the $\mathbb{P}(H_i)$ are colinear. The nondegenerate bilinear form,

$$(T \otimes T') \otimes (T \otimes T') \to \Lambda^2 T \otimes \Lambda^2 T', \quad (t_1 \otimes t_1') \otimes (t_2 \otimes t_2') \to -t_1 \wedge t_2 \otimes t_1' \wedge t_2'$$

identifies $T \otimes T'$ with its dual and thus $\mathbb{P}(T \otimes T')$ with the dual projective space. Furthermore, under this identification, $p_i = \mathbb{P}(\varphi_{s_i}(T_{s_i}) \otimes \varphi'_{s_i}(T'_{s_i}))$. The three points p_1, p_2, p_3 lie on the Segre quadric of pure tensors. By Bezout's theorem, any line through these three points is necessarily contained in this quadric. But the only lines in the quadric of pure tensors have the form $\mathbb{P}(T_0 \otimes T')$ or $\mathbb{P}(T \otimes T_0')$, where $T_0 \subset T$ and $T_0' \subset T'$ are one-dimensional subspaces. Clearly not all of p_1, p_2, p_3 are contained in a line of this form. The claim follows.

If $s_2 \notin S'$ or $s_3 \notin S$ then $\bigcap_{i=1}^{3} \phi_{s_i}(M_{s_i})$ is contained in an intersection of hypersurfaces of the form just described and thus has dimension at most 1.

It remains only to reduce the general case of the proposition to the special case just proved. This is done in Lemma 6.6. □

Let $\rho\colon \hat{X} \to X$ be a finite morphism of smooth irreducible curves over k. Let $\hat{\pi}: \hat{Y} \to \hat{X}$ and $\hat{\pi}'\colon \hat{Y}' \to \hat{X}$ be the relatively minimal models of the pullbacks of π and π'. Write $\widehat{W}$ for the blowup of $\hat{Y} \times_{\hat{X}} \hat{Y}'$ at the reduced singular locus. Write $\hat{\mathcal{G}}_{l^n}$ for the direct summand of $R^2 \hat{f}_* \mathbb{Z}/l^n$ defined in analogy with $\mathcal{G}_{l^n}$.

6.6. Lemma. *There is an injective sheaf homomorphism $\rho^*\mathcal{G}_{l^n} \to \hat{\mathcal{G}}_{l^n}$ that induces an injection $H^0(X, \mathcal{G}_{l^n}) \to H^0(\hat{X}, \hat{\mathcal{G}}_{l^n})$.*

Proof. Since the stalks of $\mathcal{G}_{l^n}$ over points $s \in S_a \cup S_a'$ are zero, we may restrict our attention to $\mathcal{G}_{l^n}|_{X-(S_a \cup S_a')}$. On $\rho^{-1}(X - (S_a \cup S_a'))$, by Lemma 3.5 we have

$$\rho^* R^1\pi_* \mathbb{Z}/l^n \simeq R^1\hat{\pi}_* \mathbb{Z}/l^n, \qquad \rho^* R^1\pi'_* \mathbb{Z}/l^n \simeq R^1\hat{\pi}'_* \mathbb{Z}/l^n.$$

For $s \in S_m$, the description of the subgroup $\varphi_s(T_s) \subset T$ is compatible with base change and passage to the relatively minimal model. It follows from Proposition 6.2 that there is a canonical injection $\rho^*\mathcal{G}_{l^n} \to \hat{\mathcal{G}}_{l^n}$ that is an isomorphism away from $\rho^{-1}(S_a \cup S_a')$. □

7. Proofs of the Main Theorems

7.1. Proposition. (i) *If E and E' are isogenous elliptic curves over $K = k(X)$, then $H^0(X, \mathcal{G}_{l^n}) \simeq \mathbb{Z}/l^n$.*

(ii) *If E and E' are not isogenous over K, then $|H^0(X, \mathcal{G}_{l^n})|$ is bounded as n varies.*

Proof. (i) If E and E' are isogenous, then there is an isogeny whose kernel is a cyclic group. Write $\Gamma \subset W$ for the closure of the graph of this isogeny. Consider the composition

$$\begin{aligned} H^2(W, \mu_{l^n}) &\xrightarrow{\alpha_n} H^0(X, R^2 f_* \mu_{l^n}) \\ &\xrightarrow{\beta_n} H^0(X, \mathcal{G}_{l^n}(1)) \to H^1(E_{\bar{K}}, \mathbb{Z}/l^n) \otimes H^1(E'_{\bar{K}}, \mu_{l^n}), \end{aligned}$$

in which the final map is the restriction of a global section of $\mathcal{G}_{l^n}(1)$ to the stalk at the geometric generic point. Then the image of the cohomology class $[\Gamma]$ under this composition may be identified via the Weil pairing with the element of $\mathrm{Hom}(E[l^n], E'[l^n])$ induced by the isogeny. Since the kernel of the isogeny is cyclic, this element has order l^n. Because $H^0(X, \mathcal{G}_{l^n})$ is a cyclic group (by Proposition 6.5) with at most l^n elements, (i) follows.

(ii) If $H^0(X, \mathcal{G}_{l^n})$ is unbounded as n varies then $\varprojlim_n H^0(X, \mathcal{G}_{l^n}) \neq 0$, which implies

$$\varprojlim_n H^0(X, j_* j^*(R^1\pi_*\mathbb{Z}/l^n \otimes R^1\pi'_*\mathbb{Z}/l^n)) \simeq \varprojlim_n H^0(X, j_* j^*(\mathrm{Hom}(R^1\pi_*\mathbb{Z}/l^n, R^1\pi'_*\mathbb{Z}/l^n)) \neq 0.$$

Thus there is a nontrivial Galois invariant homomorphism of l-adic Tate modules, $T_l(E) \to T_l(E')$. This implies that E is isogenous to E'; when $\mathrm{char}(K) = 0$, use [D, 4.4.13] or [L, Thm. 8]. In general one may reduce to the classical isogeny theorem over a finitely generated base field using [Z, 3.4 and proof of 1.4(i)]. □

7.2. COROLLARY. *If E is not isogenous to E', then $H^0(X, \mathcal{G}_{l^n}) \simeq H^3(W, \mathbb{Z}_l)[l^n]$ for all n.*

Proof. By Propositions 6.1 and 6.5, $H^0(X, R^2 f_*\mathbb{Z}/l^n)_* = 0$ if $H^0(X, \mathcal{G}_{l^n}) \simeq \mathbb{Z}/l^n$ and is isomorphic to $H^0(X, \mathcal{G}_{l^n})$ otherwise. By Proposition 7.1(ii),

$$H^0(X, R^2 f_*\mathbb{Z}/l^n)_* \simeq H^0(X, \mathcal{G}_{l^n})$$

for large n. By Corollary 5.6, the assertion follows for large n. Since there is a natural map for all n, the assertion holds for all n. □

Proof of Theorem 0.3. (i) This is proved in Proposition 5.1(ii).

(ii) By Proposition 7.1(i), $H^0(X, \mathcal{G}_{l^n})_* = 0$ for all n. By Proposition 6.1(v) and Corollary 5.6, $H^3(W, \mathbb{Z}_l)_{\mathrm{tors}} = 0$. The assertion now follows from part (i) and Proposition 2.3(i).

(iii) The assertion follows from Corollaries 7.2 and 6.3. □

Proof of Theorem 0.4. (i) If $H^3(W, \mathbb{Z}_l)_{\mathrm{tors}} \neq 0$, then $H^0(X, \mathcal{G}_{l^n}) \simeq H^3(W, \mathbb{Z}_l)[l^n]$ for all n. This is a cyclic group by Proposition 6.5.

(ii) If $S_a \cup S'_a \neq \emptyset$, then Assumption 0.1 implies that E and E' do not have the same places of bad reduction and thus are not isogenous. Now Corollaries 6.4 and 7.2 imply that $H^3(W, \mathbb{Z}_l)_{\mathrm{tors}} = 0$.

(iii) The hypothesis $J(E) \notin k$ implies that $S \neq \emptyset$. Since $S \neq S'$, E is not isogenous to E'. The assertion then follows from Proposition 6.5 and Corollary 7.2. □

8. A First Example

Define $\hat{\mathbb{Z}}' := \prod_{l \neq \mathrm{char}(k)} \mathbb{Z}_l$. We begin this section with an easily constructed desingularized fiber product W for which $H^3(W, \hat{\mathbb{Z}}')_{\mathrm{tors}} \neq 0$.

Suppose that $\mathrm{char}(k) \neq 2$ and that the genus of X is at least 1. Suppose that $\pi: Y \to X$ is semi-stable. Let $\pi': Y' \to X$ be obtained from π by a quadratic twist with respect to a nontrivial unramified degree-2 cover $\rho: \hat{X} \to X$. Let $\sigma: W \to \bar{W} = Y \times_X Y'$ be the blowup along $\bar{W}_{\mathrm{sing}}$.

8.1. PROPOSITION. $H^3(W, \hat{\mathbb{Z}}')_{\mathrm{tors}} \simeq \mathbb{Z}/2$.

Proof. First we show that E and E' are not isogenous. The Galois representations $E[l]$ and $E'[l]$ are related by tensoring with $\chi\,\mathrm{Id}$, where $\chi: G \to \{\pm 1\}$ is the character of the degree-2 cover of X. For all except finitely many odd primes l, the image of the Galois representation $G \to \mathrm{Aut}(E[l])$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z}/l)$. Indeed, if this were not the case then X would admit morphisms to an infinite tower of modular curves of various level l structures of unbounded genus, which is impossible. For odd primes l, a surjective representation $\kappa: G \to \mathrm{SL}_2(\mathbb{Z}/l)$ is not isomorphic to $\chi\,\mathrm{Id} \otimes \kappa$ because the kernels are not equal. It follows that E and E' are not isogenous.

Since quadratic twisting does not effect the 2-torsion, we may identify the Galois modules $E[2]$ and $E'[2]$. The Weil pairing composed with the isomorphism $\mu_2 \simeq \mathbb{Z}/2$ gives a bilinear form

$$E[2] \otimes E[2] \to \mathbb{Z}/2, \quad a \otimes b \mapsto a \cdot b.$$

Define an isomorphism of Galois modules,

$$E[2] \otimes E[2] \to \mathrm{Hom}(E[2], E[2]), \quad a \otimes b \mapsto [c \mapsto (a \cdot c)b].$$

Write θ for the element of the left-hand side that maps to Id. For any place of bad reduction, $s \in S = S \cap S' = S'$, let $e \in E[2]_{0s}$ be a basis. Extend to a basis e, f of $E[2]$ such that $e \cdot f = 1$. Then

$$\theta = e \otimes f - f \otimes e \in E[2]_{0s} \otimes E[2] + E[2] \otimes E[2]_{0s}.$$

Because Id (and hence θ) is G-invariant, $H^3(W, \mathbb{Z}_2)[2] \neq 0$ by Theorem 0.3(iii). By Theorem 0.4(i), $H^3(W, \mathbb{Z}_2)[2] \simeq \mathbb{Z}/2$.

Define $\widehat{W} := W \times_X \hat{X}$. Write $\upsilon: \widehat{W} \to W$ for the projection. Since the generic elliptic curves E and E' become isogenous over $k(\hat{X})$, by Theorem 0.3(ii) we have $H^3(\widehat{W}, \hat{\mathbb{Z}}')_{\mathrm{tors}} = 0$. The proposition follows from the projection formula applied to any $\vartheta \in H^3(W, \hat{\mathbb{Z}}')_{\mathrm{tors}}$:

$$\begin{aligned} 2\vartheta &= \upsilon_*(1) \cdot \vartheta = \upsilon_*(1 \cdot \upsilon^*(\vartheta)) = \upsilon_*(\upsilon^*(1) \cdot \upsilon^*(\vartheta)) \\ &= 1 \cdot \upsilon_*\upsilon^*(\vartheta) = \upsilon_*\upsilon^*(\vartheta) = 0. \end{aligned}$$

□

8.2. REMARK. The preceding example is exceptional in two respects. First, the torsion disappears after base change ($H^3(W, \hat{\mathbb{Z}}')_{\mathrm{tors}} \simeq \mathbb{Z}/2$ but $H^3(\widehat{W}, \hat{\mathbb{Z}}')_{\mathrm{tors}} = 0$), which by Lemma 6.6 and Corollary 7.2 can occur only when E becomes isogenous to E' after base change. Second, if one admits the additional hypothesis that all singular fibers of π have type I_1, then one can show $H^4(\bar{W}, \mathbb{Z}_2)[2] \simeq \mathbb{Z}/2$ directly from the spectral sequence of Lemma 4.3. The idea is to show $h^4(\bar{W}, \mathbb{Z}/2) = h^4(\bar{W}, \mathbb{Z}/l) + 1$ for odd primes $l \neq \mathrm{char}(k)$ and then to apply Lemma 2.2(iii) and

Proposition 4.1. To verify the equality, consider each term $E_\infty^{pq}(\bar f)$ with $p+q=4$. Since all singular fibers are of type I_1, $E_2^{23}(\bar f)=0$ for all l, which implies that $E_2^{04}(\bar f)=E_\infty^{04}(\bar f)$. By Lemma 4.2(v), $\operatorname{rank}(E_2^{04}(\bar f))$ is independent of l. By Lemmas 4.2(iv) and 3.2 and Proposition 3.4, $E_2^{03}(\bar f)=0$ for all l. Hence the differential d_2^{03} is zero and $E_2^{22}(\bar f)=E_\infty^{22}(\bar f)$ for all l. But $E_2^{22}\simeq(\mathbb{Z}/l)^2$ for odd l. When $l=2$, we have $E_2^{22}\simeq R_2\oplus(\mathbb{Z}/2)^2$ for $R_2\simeq\mathbb{Z}/2$. Finally,

$$\begin{aligned}E_2^{13}&=h^1(X,R^3\bar f_*\mathbb{Z}/l)=h^1(X,R^1\pi_*\mathbb{Z}/l)+h^1(X,R^1\pi'_*\mathbb{Z}/l)\\&=-e(R^1\pi_*\mathbb{Z}/l)-e(R^1\pi'_*\mathbb{Z}/l)\end{aligned}$$

is independent of l by Lemma 3.3 and [Mi, V.2.12].

9. Desingularized Fiber Products with Prescribed Torsion

The purpose of this section is to construct, for any given $m\in\mathbb{N}$ not divisible by $\operatorname{char}(k)$, a desingularized fiber product W with $H^3(W,\hat{\mathbb{Z}}')_{\mathrm{tors}}\simeq\mathbb{Z}/m$. The case $m=1$ follows from Theorem 0.3(ii) or from Theorem 0.4(ii) or (iii); the case $m=2$ follows from Proposition 8.1. Further examples with $m=2$ (and 3) may be found in [SSha, Sec. 7]. These examples have the interesting properties that Y and Y' are rational elliptic surfaces and $S\neq S\cap S'\neq S'$.

Assume henceforth that $m>2$ and that the Galois representations $E[m]$ and $E'[m]$ are both trivial. This assumption is not particularly restrictive because Lemma 6.6 and Corollary 7.2 indicate that torsion is seldom annihilated by base-changing to a finite cover, $\hat X\to X$. It allows us to make use of the theory of moduli of generalized elliptic curves with full level m structure, which we now briefly review (referring to [DR] for details).

A full level m structure on a generalized elliptic curve consists of a basis of the m-torsion sections. There is a fine moduli space for generalized elliptic curves with full level m structure. This is a smooth projective curve over k. The group $\mathrm{GL}_2(\mathbb{Z}/m)$, which operates on the level m structures, operates on this moduli space. The quotient is the J-line, $X(1)$. The subgroup $\mathrm{SL}_2(\mathbb{Z}/m)$ is the stabilizer of each connected component. The $\phi(m)$ connected components are distinguished by the value of the Weil pairing on the chosen basis. The cusps parameterize isomorphism classes of full level m structures on an m-gon of rational curves.

Consider now a smooth, irreducible, projective curve X and two elliptic curves E and E' over the generic point, $\operatorname{Spec}(K)\subset X$, such that the Galois actions on $E[m]$ and $E'[m]$ are trivial. Fixing bases of these $\mathbb{Z}/m$-modules amounts to defining morphisms from $\operatorname{Spec}(K)$ (or, equivalently, from X) to the moduli space of generalized elliptic curves with full level m structure. Write

$$H:=(h,h')\colon X\to X(m)\times X(m)',$$

where $X(m)$ (resp. $X(m)'$) denotes the connected component of the moduli scheme that contains the image of h (resp. h'). The universal elliptic curves over the generic points are denoted E_m and E'_m, so that E_m pulls back via h to E and E'_m via h' to E'. The level m structures give bases $e,f\in E_m[m]\simeq E[m]$ and $e',f'\in E'_m[m]\simeq E'[m]$.

Suppose that an element of order m, $\theta \in E_m[m] \otimes E_m[m]'$, has been fixed. In order to construct a desingularized fiber product W with $H^3(W, \hat{\mathbb{Z}}')_{\text{tors}}$ of order divisible by m, we choose X, h, and h' such that E and E' are not isogenous and θ satisfies all the conditions of Definition 0.2. How this is done depends on whether or not θ is a pure tensor. In this section we assume $\theta = e \otimes e'$.

Write $\Xi \subset X(m)$ and $\Xi' \subset X(m)'$ for the cusps. Fix one cusp $\xi \in \Xi$ (resp. $\xi' \subset \Xi'$) such that $e \in E_m[m]_{0\xi}$ (resp. $e' \in E'_m[m]_{0,\xi'}$). The subgroup $E_m[m]_{0\xi} \subset E_m[m]$ is independent of the choice of inertia group involved in its definition because the Galois action is trivial (cf. Section 1.3 and Remark 2.6). Define the following divisors on $X(m) \times X(m)'$:

$$D_{\xi,\xi'} := \xi \times X(m)' + X(m) \times \xi',$$
$$D'_{\xi,\xi'} := (\Xi - \xi) \times X(m)' + X(m) \times (\Xi' - \xi').$$

Write $\bar{X} \subset X(m) \times X(m)'$ for the image of H.

9.1. PROPOSITION. *Suppose that*

(i) $\bar{X} \cap (\{\xi\} \times X(m)') \not\subset \{\xi\} \times \Xi'$,
(ii) $\bar{X} \cap (X(m) \times \{\xi'\}) \not\subset \Xi \times \{\xi'\}$, *and*
(iii) $\bar{X} \cap D'_{\xi,\xi'} \subset D_{\xi,\xi'} \cap D'_{\xi,\xi'}$.

Then $H^3(W, \hat{\mathbb{Z}}')_{\text{tors}}$ *contains a subgroup isomorphic to* $\mathbb{Z}/m$.

Proof. By (i) we have $S \not\subset S'$, and it follows that E and E' are not isogenous. Hypothesis (iii) implies

$$S = h^{-1}(\Xi) = h^{-1}(\xi) \cup ((h')^{-1}(\xi') \cap S'') \quad \text{and}$$
$$S' = (h')^{-1}(\Xi') = (h')^{-1}(\xi') \cup (h^{-1}(\xi) \cap S'').$$

This implies that $\theta = e \otimes e'$ satisfies the conditions of Definition 0.2 for each $l^n | m$. $\square$

The conditions imposed by Proposition 9.1 on the curve $\bar{X} \subset X(m) \times X(m)'$ are extremely restrictive. The following variant of Bertini's theorem will be helpful in meeting these constraints.

9.2. LEMMA. *Let $\mathcal{L}$ be an ample invertible sheaf on a smooth projective surface Z. Let $L \subset H^0(Z, \mathcal{L})$ be a linear subspace. For $i \in \{0, 1, 2\}$, let Z_i be the set of closed points $z \in Z$ for which the kernel of the restriction map*

$$L \to \mathcal{L} \otimes \mathcal{O}_{Z,z}/\mathfrak{m}^2_{Z,z}$$

has codimension i. If $\dim(\bar{Z}_i) \leq i - 1$ ($\dim(\emptyset) = -1$ *by convention*), *then the Zariski open subset of nonsingular irreducible curves in the linear system $\mathbb{P}(L)$ is nonempty.*

Proof. Consider the incidence correspondence,

$$I = \{(z, H) \in Z \times \mathbb{P}(L) : z \text{ is a singular point of } H\},$$

and the two projections $p \colon I \to Z$ and $q : I \to \mathbb{P}(L)$. For $z \in Z_i$,

$$\operatorname{codim}_{\mathbb{P}(L)}(p^{-1}(z)) = i.$$

Thus

$$\dim(I) = \max\{\dim(\mathbb{P}(L)) - i + \dim(\bar{Z}_i) : i \in \{0,1,2\}\} \leq \dim(\mathbb{P}(L)) - 1$$

and $q(I) \subset \mathbb{P}(L)$ is a proper closed subset. Any point in $\mathbb{P}(L) - q(I)$ gives a nonsingular curve, which is irreducible because it is ample [H, III.7.9.1]. □

9.3. PROPOSITION. *There exist smooth irreducible curves $\bar{X} \subset X(m) \times X(m)'$ that satisfy the hypotheses of Proposition 9.1.*

Proof. Choose integers $a, b \geq |\Xi| = |\Xi'|$ such that the divisor

$$\mathfrak{d} := a(\xi \times X(m)') + b(X(m) \times \xi') + D_{\xi,\xi'} - D'_{\xi,\xi'}$$

is very ample. Define

$$\mathbf{D}_{\xi,\xi'} := (a+1)(\xi \times X(m)') + (b+1)(X(m) \times \xi'),$$

and let $L \subset H^0(X(m) \times X(m)', \mathcal{O}(\mathbf{D}_{\xi,\xi'}))$ denote the subspace corresponding to the linear system $\mathbb{P}(L)$ generated by $\mathbf{D}_{\xi,\xi'}$ and $D'_{\xi,\xi'} + \mathfrak{d}'$ as $\mathfrak{d}'$ varies in $|\mathfrak{d}|$. Let $\mathbb{P}(L)^\circ$ denote the Zariski open subset of divisors in $\mathbb{P}(L)$ that correspond to reduced, irreducible, nonsingular curves.

9.4. LEMMA. (i) *The reduced base locus of $\mathbb{P}(L)$ is $D_{\xi,\xi'} \cap D'_{\xi,\xi'} = (\Xi - \xi) \times \{\xi'\} \cup \{\xi\} \times (\Xi' - \xi')$.*

(ii) *The locus Z_2 defined in Lemma 9.2 is contained in $D'_{\xi,\xi'}$.*

(iii) *The locus Z_0 defined in Lemma 9.2 is empty.*

(iv) *The locus Z_1 defined in Lemma 9.2 is contained in $\Xi \times \Xi'$.*

(v) *For each $\upsilon \in \Xi - \{\xi\}$, $\upsilon' \in \Xi' - \{\xi'\}$, and $C \in \mathbb{P}(L)^\circ$, we have $C \cdot (\{\upsilon\} \times X(m)') = (b+1)(\upsilon, \xi')$ and $C \cdot (X(m) \times \{\upsilon'\}) = (a+1)(\xi, \upsilon')$.*

(vi) *There exists an $\bar{X} \in \mathbb{P}(L)^\circ$ such that $\bar{X} \cap (X(m) \times \{\xi'\}) \not\subset \Xi \times \{\xi'\}$ and $\bar{X} \cap (\{\xi\} \times X(m)') \not\subset \{\xi\} \times \Xi'$.*

Proof. (i) Since $|\mathfrak{d}|$ is base point free, the reduced base locus is $\mathbf{D}_{\xi,\xi'} \cap D'_{\xi,\xi'} = D_{\xi,\xi'} \cap D'_{\xi,\xi'}$.

(ii) The condition that a divisor in $|\mathfrak{d}|$ must contain a given point $z \in X(m) \times X(m)'$ and be singular there is a codimension-3 condition, since $\mathfrak{d}$ is very ample. Thus $z \notin D'_{\xi,\xi'}$ implies $z \notin Z_2$.

(iii) For each point z in the base locus, there is a divisor of the form $D'_{\xi,\xi'} + \mathfrak{d}'$ that is nonsingular at z.

(iv) Through a nonsingular point $z \in D'_{\xi,\xi'}$ there is a divisor of the form $D'_{\xi,\xi'} + \mathfrak{d}'$ that is nonsingular at z, so $z \notin Z_1$.

(v) Since C is irreducible, it is not in the linear subsystem $D'_{\xi,\xi'} + |\mathfrak{d}|$. Thus the Cartier divisors C and $\mathbf{D}_{\xi,\xi'}$ have equal restrictions to the curve $\{\upsilon\} \times X(m)'$ (resp. $X(m) \times \{\upsilon'\}$)—namely, the point (υ, ξ') (resp. (ξ, υ')) with multiplicity $b+1$ (resp. $a+1$).

(vi) For a general $\mathfrak{d}' \subset |\mathfrak{d}|$, the divisor $D'_{\xi,\xi'} + \mathfrak{d}'$ meets $X(m) \times \{\xi'\}$ transversely at $a+1 > |\Xi|$ distinct points and meets $\{\xi\} \times X(m)'$ transversely at $b+1 > |\Xi'|$

distinct points. Since these conditions are open in $\mathbb{P}(L)$ and hence also in $\mathbb{P}(L)^\circ$, the assertion follows. $\square$

We may now complete the proof of Proposition 9.3. By Lemmas 9.4(ii)–(iv), the linear subspace L satisfies the hypotheses of Lemma 9.2. Thus the Zariski open subset $\mathbb{P}(L)^\circ \subset \mathbb{P}(L)$ is not empty. By Lemma 9.4(v), a general $\bar{X} \in \mathbb{P}(L)^\circ$ satisfies Proposition 9.1(iii); by Lemma 9.4(vi), it satisfies Propositions 9.1(i) and (ii). $\square$

9.5. THEOREM. *Given m not divisible by* $\mathrm{char}(k)$, *there exists a desingularized fiber product W with $H^3(W, \hat{\mathbb{Z}}')_{\mathrm{tors}} \simeq \mathbb{Z}/m$. Furthermore, we may arrange $S \neq S \cap S' \neq S'$.*

Proof. For the case $m = 2$, see [SSha, Sec. 7]. Assume $m > 2$. Choose a and b as in the proof of Proposition 9.3 and satisfying the additional condition that $m, a+1$, and $b+1$ are pairwise coprime. Let $\bar{X}$ be as in Proposition 9.3. By the proof of Lemma 9.4(vi), we may assume that $\bar{X}$ meets $\{\xi\} \times X(m)$ and $X(m) \times \{\xi\}$ transversely. Let $H\colon X \to \bar{X}$ be an isomorphism, and let W be the desingularized fiber product constructed from H at the beginning of this section. By Proposition 9.1, $H^3(W, \hat{\mathbb{Z}}')_{\mathrm{tors}}$ contains a subgroup isomorphic to $\mathbb{Z}/m$.

Fix a prime $l \neq \mathrm{char}(k)$. Let $n \geq 0$ be maximal so that $l^n | m$. It suffices to show that $H^3(W, \mathbb{Z}_l)[l^{n+1}] \subset H^3(W, \mathbb{Z}_l)[l^n]$. This is the case if the Galois representation on $E[l^{n+1}]/E[l^n]$ is irreducible. Indeed, if $H^3(W, \mathbb{Z}_l)[l^{n+1}]$ contains an element of order l^{n+1}, then $\mathrm{Hom}_G(E[l^{n+1}], E'[l^{n+1}])$ contains an element of order l^{n+1} by Theorem 0.3(iii) and so $\mathrm{Hom}_G(E[l^{n+1}]/E[l^n], E'[l^{n+1}]/E'[l^n]) \neq 0$. Because the first representation is irreducible, the second is also and the two are isomorphic. This contradicts the fact that, at points of $s \in S - S''$, the l^{n+1}-torsion in Y' is unramified (since Y' has good reduction) but the l^{n+1}-torsion in Y is ramified (since Y has type-I_m reduction and $l^{n+1} \nmid m$).

It remains to rule out the possibility that both Galois representations $E[l^{n+1}]/E[l^n]$ and $E'[l^{n+1}]/E'[l^n]$ are reducible. If they were, then $E[l^{n+1}]$ and $E'[l^{n+1}]$ would admit G-stable cyclic subgroups of order l^{n+1} because G acts trivially on the l^n-torsion. The theory of coarse moduli schemes would then give factorizations of the J-maps for π and π', $J, J'\colon X \to X(1) \simeq \mathbb{P}^1$, through the modular curve $X_0(l^{n+1})$. Since the canonical map $X_0(l^{n+1}) \to X(1)$ has ramification index l^{n+1} at certain cusps, the tame contribution to the ramification index of J at some cusp would be divisible by l^{n+1} and the same would also hold for J'. To see that both conditions cannot hold simultaneously, note that the J-map associated to the universal elliptic curve E_m is (tamely) ramified with index m at each cusp. By the transversality hypothesis, the ramification index of J at any point in $\bar{X} \cap \{\xi\} \times X(m)$ (resp., of J' at any point in $\bar{X} \cap (X(m) \times \{\xi\})$) is also m. At any point in $\bar{X} \cap (\{\Xi - \xi\} \times X(m))$ (resp., in $\bar{X} \cap (X(m) \times \{\Xi - \xi\})$), the tame contribution to the ramification index of J divides $(b+1)m$ (resp., of J' divides $(a+1)m$) according to Lemma 9.4(v). Since l^n exactly divides m and since m, $a+1$, and $b+1$ are pairwise coprime, it follows that either l^{n+1} does not divide the tame contribution to the ramification index of J over any cusp or that it does not divide the tame contribution to the ramification index of J' over any cusp—a contradiction. $\square$

9.6. REMARK. It is not possible to use the methods of this section to construct desingularized fiber products W with $|H^3(W)_{\text{tors}}|$ that are increasing without bound while the genus of the base curve remains bounded. Indeed, the assumption $\theta = e \otimes e'$ implies that E and E' have cyclic order-m subgroups defined over K. This implies genus$(X) \geq$ genus$(X_0(m)) \geq am + b$ for some $a > 0$ and $b \in \mathbb{R}$ [DiSh, 3.9].

10. A Second Class of Examples with Large Torsion

The goal of this section is to prove the following statement.

10.1. THEOREM. *Given $m \in \mathbb{N}$ such that $\mathrm{char}(k) \nmid m$, there exists a desingularized fiber product W such that $H^3(W, \hat{\mathbb{Z}}')[m] \simeq \mathbb{Z}/m$ and $S = S'$.*

The case $m = 1$ (resp. $m = 2$) was treated in Theorem 0.3(ii) (resp. Proposition 8.1), so we assume $m > 2$. As in Section 9, we fix an irreducible component $X(m)$ of the moduli of generalized elliptic curves with full level m structure; we write E_m for the generic fiber of the universal generalized elliptic curve and $\Xi \subset X(m)$ for the cusps. Let

$$\theta \in E_m[m] \otimes E_m[m] \xrightarrow{\sim} \mathrm{Hom}(E_m[m], E_m[m])$$

map to the identity element. This choice is, in a sense, the opposite of that made in the previous section. Indeed, when m is prime there are two types of nonzero elements up to change of bases: pure tensors (rank-1 homomorphisms) and nonpure tensors (invertible homomorphisms).

To prove the theorem, it suffices to establish the existence of a smooth, irreducible, projective curve X and a morphism $H = (h, h') \colon X \to X(m) \times X(m)$ such that the pullback E of E_m via h is not isogenous to the pullback E' of E_m via h' and such that

$$\theta \in E_m[m] \otimes E_m[m] \xrightarrow{\sim} E[m] \otimes E'[m]$$

satisfies the conditions of Definition 0.2. Indeed, Definitions 0.2(i) and (ii) and the assumption that θ corresponds to an invertible homomorphism together imply $S = S'$.

If $\bar{X} = H(X)$ is a Hecke correspondence then E and E' are isogenous, which implies that the conditions of Definition 0.2 are satisfied (cf. Proposition 7.1(i)). The idea of the proof is to realize $\bar{X}$ by deforming (a multiple of) a Hecke correspondence while keeping the cusps fixed. Thus the isogeny is destroyed while the conditions of Definition 0.2 are preserved.

To prepare for the proof of Theorem 10.1, we recall general facts about Hecke correspondences. Let $n \in \mathbb{N}$ be prime to $\mathrm{char}(k)$. The modular curve $X_0(n)$ is a coarse moduli space for pairs of elliptic curves related by a cyclic degree-n isogeny.

Although there is no pair of universal elliptic curves, there is nonetheless a corresponding pair of J-invariants:

$$(J_n, J_n')\colon X_0(n) \to X(1) \times X(1) \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

Write $\mathbf{T}_n$ for the image of this map.

For $m > 2$ and $n \equiv 1$ module m and prime to $\mathrm{char}(k)$ we construct a Hecke correspondence, $T_n \subset X(m) \times X(m)$, as follows. Begin with the normalization X_n of $X(m) \times_{X(1)} X_0(n)$. Away from the cusps, X_n is a fine moduli space for elliptic curves with a cyclic subgroup of order n and a full level m structure for which the Weil pairing is a prescribed mth root of unity. If A is such an elliptic curve, then taking the quotient by the distinguished cyclic subgroup gives an elliptic curve A' that inherits a level m structure from A. Since $n \equiv 1 \bmod m$, the Weil pairing gives the same mth root of unity [Si1, III.8.2]. Now associating to A the pair (A, A') gives a morphism, $X_n \to X(m) \times X(m)$, whose image is the Hecke correspondence T_n.

Define the divisor

$$D = \Xi \times X(m) + X(m) \times \Xi \subset X(m) \times X(m),$$

and set $I_n = T_n \cap D$.

10.2. Proposition. *Suppose $n \equiv 1 \bmod m$ and $\mathrm{char}(k) \nmid n$. Then*:

(i) *X_n is irreducible*;

(ii) *T_n is invariant under the involution of $X(m) \times X(m)$ that interchanges the factors*; *and*

(iii) $I_n \subset \Xi \times \Xi$.

Assume henceforth that n is prime.

(iv) *If $x \in X(m)$, then $T_n \cdot (x \times X(m)) = n + 1 = T_n \cdot (X(m) \times x)$.*

(v) *Each eigenvalue α of the endomorphism of $H^1(X(m), \mathbb{Q}_l)$ induced by the correspondence T_n is an algebraic integer that satisfies $|\alpha| \leq 2\sqrt{n}$ under any complex embedding.*

Sketch of the proof. (i) The ring $k(X_n)$ of rational functions is a field because the extensions $k(X_0(n))$ and $k(X(m))$ of $k(X(1))$ are linearly disjoint.

(ii) Suppose that the cyclic degree-n isogeny, $\phi\colon A \to A'$, represents a point $(A, A') \in T_n$. Thus the level m structure on A' comes from the level m structure on A. Now the dual isogeny, $\hat{\phi}\colon A' \to A$, maps the level m structure on A' to the original level m structure on A because $\hat{\phi} \circ \phi = n\,\mathrm{Id}$ and $n \equiv 1$ modulo m. Thus $(A', A) \in T_n$.

(iii) $T_n \cap X(m) \times \Xi \subset \Xi \times \Xi$, since the quotient of an elliptic curve by a finite subgroup is again an elliptic curve. Now apply (ii).

(iv) When n is prime there are $n+1$ distinct order-n subgroups of $A[n]$. Generally the quotient curves are pairwise nonisomorphic.

(v) This follows from Weil's theorem on the zeta function of a curve over a finite field and the Eichler–Shimura congruence relation. The latter is a consequence of the degeneration of $X_0(n)$ into two components at the prime n, one of which may be viewed as the graph of Frobenius and the other as the graph of Verschiebung [DR, V; DiSh, 9.5.1]. □

Let X be a smooth, irreducible, projective curve and let $H = (h, h')\colon X \to X(m) \times X(m)$ be a morphism. Define the sheaf $\mathcal{G}_m := \prod \mathcal{G}_{l^n}$ on X, where the product is taken over the maximal prime powers that divide m.

10.3. LEMMA. *Let $n \equiv 1 \bmod m$ be prime to* $\operatorname{char}(k)$. *Suppose that the image of H, $\bar{X} \subset X(m) \times X(m)$, is a curve satisfying*:

(i) *$\bar{X} \cap D \subset I_n$; and*

(ii) *the pullback E of E_m via h is not isogenous to the pullback E' of E_m via h'.*

Then the desingularized fiber product W constructed from E and E' satisfies

(a) *$S = S'$ and*

(b) *$H^3(W, \hat{\mathbb{Z}}')[m] \simeq \mathbb{Z}/m$.*

Proof. (a) $S = h^{-1}(\Xi) = (h')^{-1}(\Xi) = S'$ because $\bar{X} \cap D \subset \Xi \times \Xi$.

(b) The identifications $E_m[m] \simeq E[m]$ and $E_m[m] \simeq E'[m]$ induced by h and h' give, for $s \in S''$, the identifications $E_m[m]_{0h(s)} \simeq E[m]_{0s}$ and $E_m[m]_{0h'(s)} \simeq E'[m]_{0s}$. Since $S = S' = S''$, it follows (from Theorem 0.3, Lemma 2.4(ii), and Corollary 6.3) that there is an isomorphism

$$H^3(W, \hat{\mathbb{Z}}')[m] \simeq \vartheta := \bigcap_{s \in S''} E_m[m]_{0h(s)} \otimes E_m[m] + E_m[m] \otimes E_m[m]_{0h'(s)}.$$

By (i), this group contains the subgroup

$$\Theta := \bigcap_{(\xi,\xi') \in I_n} E_m[m]_{0\xi} \otimes E_m[m] + E_m[m] \otimes E_m[m]_{0\xi'} \subset E_m[m] \otimes E_m[m],$$

which is isomorphic to $H^0(X_n, \tilde{\mathcal{G}}_m)$. Here $\tilde{\mathcal{G}}_m$ is constructed from the product of the pair of universal elliptic curves over the generic point of X_n exactly as $\mathcal{G}_m$ is constructed from the $E \times E'$. By Proposition 7.1(i), $\Theta \simeq \mathbb{Z}/m$. Since ϑ is cyclic by Theorem 0.4(i), we have $\Theta = \vartheta$. □

The next result is a tool for constructing curves $\bar{X} \subset X(m) \times X(m)$ that satisfy the hypotheses of Lemma 10.3.

10.4. LEMMA. *Let $B, C \subset Z$ be reduced curves on a smooth, irreducible, projective surface. Assume that B and C intersect properly. If $C \cdot C > 0$ then, for all sufficiently large j, the subset of divisors $\mathbf{d} \in |jC|$ that satisfy $\mathbf{d} \cap B \subset C \cap B$ has positive dimension.*

Proof. Write $\nu\colon \tilde{B} \to B$ for the normalization and consider the restriction map

$$r\colon H^0(Z, \mathcal{O}_Z(jC)) \to H^0(B, \mathcal{O}_Z(jC)|_B) \to H^0(\tilde{B}, \nu^*(\mathcal{O}_Z(jC)|_B)).$$

By the Riemann–Roch theorem, the dimension of the term on the right grows linearly with j while the dimension of the term on the left grows quadratically.

Let $s_j \in H^0(Z, \mathcal{O}_Z(jC))$ have divisor jC and define $L_j := \mathrm{Span}\{r^{-1}(r(s_j))\} \subset H^0(Z, \mathcal{O}_Z(jC))$. The subset $|L_j|^\circ \subset |L_j|$ of divisors that do not contain a component of B is open. Since $s_j \in |L_j|^\circ$ this subset is nonempty. For large j, $\dim(L_j) > 1$ and so $\dim(|L_j|^\circ) > 0$. For $\mathbf{d} \in |L_j|^\circ$, $\mathbf{d} \cap B = C \cap B$. □

In order to apply Lemma 10.4 when $C = T_n$ for certain n, we prove the following result.

10.5. LEMMA. *For sufficiently large primes n, $T_n \cdot T_n > 0$.*

Proof. The Lefschetz fixed point formula [K, 1.3.6(c)] applied to a correspondence $C \subset X(m) \times X(m)$ gives

$$C^T \cdot C = \sum_{i \geq 0} (-1)^i \operatorname{Tr}((C \circ C)|_{H^i}),$$

where C^T is the image of C under switching factors in $X(m) \times X(m)$. Take $C = T_n$. By Proposition 10.2(ii), $T_n^T = T_n$. According to Proposition 10.2(iv), T_n acts on H^0 and on H^2 via multiplication by $n+1$. Thus

$$T_n \cdot T_n = 2(n+1)^2 - \operatorname{Tr}((T_n \circ T_n)|_{H^1}).$$

By Proposition 10.2(v), $|\operatorname{Tr}((T_n \circ T_n)|_{H^1})| \leq 4nh^1(X(m), \mathbb{Q}_l)$. By Dirichlet's theorem on primes in arithmetic progression, there are infinitely many primes n with $n \equiv 1 \bmod m$. The lemma follows. □

10.6. LEMMA. *There are only finitely many irreducible curves $C \subset X(m) \times X(m)$ of bounded degree with respect to the product polarization such that the pullbacks of E_m to the generic point of C via the two projections are isogenous.*

Proof. Suppose that the two pullbacks are isogenous over the generic point of C. Write $m'n'$ for the minimal degree of an isogeny with cyclic kernel, where $m'|m$ and $\gcd(m, n') = 1$. Now the pullback of E_m to C acquires a cyclic subgroup of order n' defined over C. But the image of the Galois representation $\mathrm{Gal}(\bar{K}_m/K_m) \to \mathrm{Aut}(E_m[n'])/\mathrm{Aut}(E_m)$, where $\mathrm{Spec}(K_m) \in X(m)$ is the generic point, is isomorphic to $\mathrm{SL}_2(\mathbb{Z}/n')/(\pm \mathrm{Id})$. Thus the degree of C over $X(m)$ via the first projection is at least as large as the index of the stabilizer of a subgroup $\mathbb{Z}/n' \subset (\mathbb{Z}/n')^2$. Since there are only finitely many n' for which this index is below a fixed value, the lemma follows. □

Proof of Theorem 10.1. Recall that we may assume $m > 2$. Choose a sufficiently large prime n such that $n \equiv 1 \bmod m$, $n \neq \mathrm{char}(k)$, and $T_n \cdot T_n > 0$. Apply Lemma 10.4 with $B = D, C = T_n$, and j sufficiently large such that $\dim(|L_j|) > 0$. Since the base field is algebraically closed, $|L_j|^\circ$ is an infinite set. For each $\mathbf{d} \in |L_j|^\circ$ we have $\mathbf{d} \cap D \subset T_n \cap D = I_n$. Choose $\mathbf{d}$ so that it contains an irreducible component $\bar{X}$ that satisfies Lemma 10.3(ii); this is possible by Lemma 10.6. Thus $\bar{X}$ satisfies the hypotheses of Lemma 10.3, so Theorem 10.1 follows. □

11. A Cycle Class Map to the Torsion Subgroup of Cohomology

The purpose of this section is to show that degree-4 torsion cohomology classes on desingularized fiber products are cohomology classes of algebraic cycles. Toward this end we introduce the following notation pertaining to a closed point $x \in X - (S \cup S')$:

$$Y_x := \pi^{-1}(x), \qquad Y'_x := (\pi')^{-1}(x),$$
$$W_x := f^{-1}(x) \simeq Y_x \times Y'_x, \qquad \iota_x : W_x \to W.$$

Write $e_x \in Y_x$ and $e'_x \in Y'_x$ for the identity elements in the elliptic curves. Let $Z^2(W)$ denote the group of codimension-2 algebraic cycles on W. Define

$$\mathrm{Div}(W_x)_0 := \{z_x \in \mathrm{Div}(W_x) : z_x \cdot (Y_x \times e'_x) = 0 = z_x \cdot (e_x \times Y'_x)\},$$

$$Z^2(W)_0 := \mathrm{Image}\Bigg[\bigoplus_{x \in X-(S\cup S')} \mathrm{Div}(W_x)_0 \xrightarrow{\bigoplus_x \iota_{x*}} Z^2(W)\Bigg].$$

By Theorem 0.4(ii), we may (and will) assume that both Y and Y' are semi-stable.

11.1. THEOREM. *Assume that E is not isogenous to E'. Then the image of the cycle class map* $\mathrm{cl}\colon Z^2(W)_0 \to H^4(W, \hat{\mathbb{Z}}'(2))$ *is* $H^4(W, \hat{\mathbb{Z}}'(2))_{\mathrm{tors}}$.

Proof. It suffices to show for each prime $l \neq \mathrm{char}(k)$ that the image of the cycle class map $\mathrm{cl}\colon Z^2(W)_0 \to H^4(W, \mathbb{Z}_l(2))$ is $H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}}$.

The image of the cohomology class of $z_x \in \mathrm{Div}(W_x)_0$ under the composition.

$$\begin{aligned}\mathrm{Div}(W_x)_0 &\to H^2(W_x, \mathbb{Z}_l(1)) \xrightarrow{\mathrm{pr}} H^1(Y_x, \mathbb{Z}_l) \otimes H^1(Y'_x, \mathbb{Z}_l)(1)\\ &\simeq H^2_x(X, j_* j^*(R^1\pi_*\mathbb{Z}_l \otimes R^1\pi'_*\mathbb{Z}_l)(2))\\ &\qquad \to H^2(X, j_* j^*(R^1\pi_*\mathbb{Z}_l \otimes R^1\pi'_*\mathbb{Z}_l)(2)) \xrightarrow{b} H^4(W, \mathbb{Z}_l(2)) \qquad (11.2)\end{aligned}$$

is torsion, because the second-to-last group in the sequence is torsion. Indeed, Poincaré duality [Mi, V.2.2c] gives

$$H^2(X, j_* j^*(R^1\pi_*\mathbb{Q}_l \otimes R^1\pi'_*\mathbb{Q}_l(2)) \simeq H^0(X, j_* j^*(R^1\pi_*\mathbb{Q}_l \otimes R^1\pi'_*\mathbb{Q}_l(1))^\vee,$$

and the vector space on the right is zero by the isogeny theorem argument in the proof of Proposition 7.1(ii).

Consider the cup product pairing

$$H^2(W, \mathbb{Z}/l(1)) \otimes H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}} \xrightarrow{\cup} H^6(W, \mathbb{Z}/l(3)) \simeq \mathbb{Z}/l. \qquad (11.3)$$

The left-hand factor may be replaced by $H^0(X, \mathcal{G}_l)(1)$, since this is isomorphic to the quotient of $H^2(W, \mathbb{Z}/l(1))$ by the subgroup $H^2(W, \mathbb{Z}_l(1)) \otimes \mathbb{Z}/l$, which is the annihilator of $H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}}$ (cf. Lemma 2.2(i) and Corollary 7.2). Fix a generator $\tau \in H^0(X, \mathcal{G}_l)(1)$.

11.4. LEMMA. *The image of z_x in $H^4(W,\mathbb{Z}_l(2))_{\text{tors}}$ under the map* (11.2) *is a generator if* $\tau \cup [z_x] \neq 0$.

Proof. Since $H^4(W,\mathbb{Z}_l(2))_{\text{tors}}$ is cyclic, it suffices to show that z_x is not divisible by l. □

For computational purposes it is convenient to replace (11.3) with

$$H^0(X,\mathcal{G}_l)(1) \otimes H^2(X, j_*j^*(R^1\pi_*\mathbb{Z}_l \otimes R^1\pi'_*\mathbb{Z}_l)(2)) \xrightarrow{\cup} H^2(X, R^4f_*\mathbb{Z}/l(3)) \simeq \mathbb{Z}/l.$$

With the help of the commutative diagram

$$\begin{array}{ccccc}
H^0(X,\mathcal{G}_l(1)) & \otimes & H^2_x(X, j_*j^*(R^1\pi_*\mathbb{Z}_l \otimes R^1\pi'_*\mathbb{Z}_l)(2)) & \longrightarrow & H^2_x(X, R^4f_*\mathbb{Z}/l(3)) \\
\downarrow & & \wr\downarrow & & \wr\downarrow \\
H^1(Y_x,\mathbb{Z}/l) \otimes H^1(Y'_x,\mathbb{Z}/l)(1) & \otimes & H^1(Y_x,\mathbb{Z}_l) \otimes H^1(Y'_x,\mathbb{Z}_l)(1) & \longrightarrow & H^4(f^{-1}(x),\mathbb{Z}/l(2)) \\
 & & & & \\
\tau_x & \otimes & [z_x] & \longrightarrow & \tau_x \cup [z_x],
\end{array}$$

in which τ_x denotes the stalk of the section τ at x, the computation is reduced to an ordinary cup product in the cohomology of the fiber at x.

Observe that if the elliptic curves Y_x and Y'_x are not isogenous then the cycle class map composed with projection on the middle Künneth component,

$$\mathrm{Div}(Y_x \times Y'_x) \to H^2(Y_x \times Y'_x,\mathbb{Z}_l(1)) \xrightarrow{\mathrm{pr}} H^1(Y_x,\mathbb{Z}_l) \otimes H^1(Y'_x,\mathbb{Z}_l)(1), \tag{11.5}$$

is zero. If the two curves are isogenous, then there is an isogeny with cyclic kernel of some order n. This situation may be expressed in terms of the J-invariants of the elliptic curves E and E': $J, J'\colon X \to X(1) \simeq \mathbb{P}^1$. Identify $\mathbb{A}^1$ with the complement of the cusp in $X(1)$. Recall that the nth modular polynomial defines a curve $\dot{\mathbf{T}}_n \subset \mathbb{A}^1 \times \mathbb{A}^1$ that contains the point (j, j') precisely when there is a cyclic degree-n isogeny between an elliptic curve with J-invariant j and one with J-invariant j' [L, Sec. 5.2]. Thus the map (11.5) has nonzero image exactly when $(J(x), J'(x)) \in \dot{\mathbf{T}}_n$ for some n.

Choose a finite separable field extension $K \subset \hat{K}$ such that $\mathrm{Gal}(\bar{K}/\hat{K})$ acts trivially on $E[l]$ and $E'[l]$. Corresponding to the field extension is a morphism of smooth projective curves, $\rho\colon \hat{X} \to X$. Fix full level l structures on $E[l]$ and $E'[l]$ for which the Weil pairing gives the same value, $\zeta \in \mu_l \subset k^*$. The level structures give nonconstant morphisms $h, h'\colon \hat{X} \to X(l)$, where $X(l)$ is the irreducible component of the moduli of elliptic curves with full level l structure for which the Weil pairing takes the value ζ. Write E_l for the generic fiber of the universal generalized elliptic curve over $X(l)$. For primes n with $n \equiv 1$ modulo l and $n \neq \mathrm{char}(k)$, we have as in Section 10 the Hecke correspondence $T_n \subset X(l) \times X(l)$. The pullback of E_l to the generic point $\mathrm{Spec}(K_n) \in T_n$ by projection on the first (resp. second) factor will be denoted $E_{l\ K_n}$ (resp. $E'_{l\ K_n}$). There is a canonical isogeny $E_{l\ K_n} \to E'_{l\ K_n}$. Because the isogeny has cyclic kernel, the middle Künneth component of the cohomology class of its graph,

$$[z_n] \in H^1(E_{l\ \bar{K}_n}, \mathbb{Z}/l) \otimes H^1(E'_{l\ \bar{K}_n}, \mathbb{Z}/l)(1) \simeq \mathrm{Hom}(E_l[l], E'_l[l]),$$

is nonzero. By the triviality of the Galois action, $E_l[l]$ (resp. $E'_l[l]$) may be thought of as a constant sheaf over $X(l) \times X(l)$ pulled back from the projection to the first (resp. second) factor. Via h (resp. h'), $E_l[l]$ and $E[l]$ (resp. $E'_l[l]$ and $E'[l]$) are identified. Thus $\tau \in H^0(X, \mathcal{G}_l)(1) \subset H^0(\hat{X}, \mathcal{G}_l)(1)$ and $[z_n]$ may be viewed as elements in the same group, $E[l] \otimes E'[l]$. The cup product in the preceding diagram corresponds to the nondegenerate bilinear form on $E[l] \otimes E'[l]$ given by the Weil pairing on each factor. Since $n \equiv 1 \bmod l$, the element $[z_n]$ is identified with an isomorphism under the identification $E[l] \otimes E'[l] \simeq \mathrm{Hom}(E[l], E'[l])$. Consequently, $[z_n] \in E[l] \otimes E'[l]$ is not a pure tensor. Thus the orthogonal complement of the orbit,

$$\{(1 \otimes g)[z_n] : g \in \mathrm{SL}_2(\mathbb{Z}/l)\} \subset E[l] \otimes E'[l],$$

is zero. In particular, there exists a $g \in \mathrm{SL}_2(\mathbb{Z}/l)$ such that $\tau \cup (1 \otimes g)[z_n] \neq 0$.

Write $D \subset X(l) \times X(l)$ for the locus where at least one coordinate is a cusp.

11.6. PROPOSITION. *Let $C \subset X(l) \times X(l)$ be an irreducible curve whose support is not contained in D. Then, for all sufficiently large primes n satisfying $n \equiv 1 \bmod l$ and for any $g \in \mathrm{SL}_2(\mathbb{Z}/l)$, the intersection $C \cap (1 \times g)T_n$ is not contained in D.*

Proof of Theorem 11.1 (*assuming Proposition 11.6*). Write $\bar{X} \subset X(l) \times X(l)$ for the image of $\hat{X}$ under $H = (h, h')$. Choose n prime ($n \equiv 1 \bmod l$, $n \neq \mathrm{char}(k)$) such that, for any $g \in \mathrm{SL}_2(\mathbb{Z}/l)$, there is some $\bar{x} \in \bar{X} \cap (1 \times g)T_n$ that is not contained in D. Choose g so that $\tau \cup (1 \otimes g)[z_n] \neq 0$. The universal isogeny over T_n gives an isogeny on the fibers, $Y_{\bar{x}} \to Y'_{\bar{x}}$. We may identify the middle Künneth component of the cycle class of this isogeny, $[z_{\bar{x}}]$, with $(1 \otimes g)[z_n]$. It follows that, for any $x \in \rho(H^{-1}(\bar{x})) \subset X$, there is an element $z_x \in \mathrm{Div}_0(W_x)$ such that

$$\tau_x \cup [z_x] \neq 0 \in H^2_x(X, R^4 f_* \mathbb{Z}/l(3)) \simeq H^2(X, R^4 f_* \mathbb{Z}/l(3)) \simeq H^6(W, \mathbb{Z}/l(3)).$$

By Lemma 11.4, $[z_x]$ generates the cyclic group $H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}}$. □

To prove the proposition, we exploit fundamental properties of the $\mathbb{Q}$-subalgebra $\mathcal{H} \subset \mathrm{End}(\mathrm{Jac}(X(l)) \otimes \mathbb{Q}$ generated by the action of the correspondences T_n for n prime with $n \equiv 1 \bmod l$ and $n \neq \mathrm{char}(k)$.

11.7. PROPOSITION.

(i) *$\mathcal{H}$ is commutative.*

(ii) *$\mathcal{H}$ is semi-simple.*

Proof. (i) [DiSh, Sec. 5.2].

(ii) This follows from Proposition 10.2(ii) and the positive definiteness of the form

$$\mathrm{End}(\mathrm{Jac}(X(l)) \times \mathrm{End}(\mathrm{Jac}(X(l)) \to \mathbb{Z}, \quad (T, U) \mapsto \mathrm{Tr}(T \circ U^T),$$

where $U \to U^T$ corresponds to the Rosati involution [DiSh, 5.5.4; Mu, Sec. 21]. □

Write $\Xi \subset X(l)$ for the set of cusps, and define the sum of the local intersection multiplicities at the cusps as

$$\kappa_n(g) := \sum_{(\xi,\xi')\in\Xi\times\Xi} (C \cdot (1 \times g)T_n)_{(\xi,\xi')}.$$

11.8. PROPOSITION. *Let n be a prime number with $n \equiv 1 \bmod l$ and $n \neq \mathrm{char}(k)$.*

(i) *The sequence $\{\kappa_n(g)\}_n$ is bounded independently of $g \in \mathrm{SL}_2(\mathbb{Z}/l)$.*
(ii) *For large n, $(C \cdot (1 \times g)T_n) > n/2$.*

Observe that Proposition 11.6 follows from Proposition 11.8 because

$$(1 \times g)T_n \cap D \subset (1 \times g)T_n \cap \Xi \times \Xi.$$

Thus, for large n, the local contribution to the global intersection number $(C \cdot (1 \times g)T_n)$ from points contained in D is constant; hence the two curves must also meet outside of D.

Proof of Proposition 11.8. (i) Locally near the cusp $(\infty,\infty) \in \mathbb{P}^1\times\mathbb{P}^1$, the curve $\mathbf{T}_n$ consists of two smooth branches meeting transversely. One branch is tangent to $\{\infty\}\times\mathbb{P}^1$ with multiplicity n while the other is tangent to $\mathbb{P}^1\times\{\infty\}$ with multiplicity n. This is because $\mathbf{T}_n$ is the image of the map $X_0(n) \xrightarrow{(J,\, J\circ w_n)} X(1) \times X(1)$, where w_n is the Atkin–Lehner involution. At one cusp of $X_0(n)$, J is étale; at the other cusp, J has ramification index n. The Atkin–Lehner involution interchanges these two cusps. Thus a local analytic equation for $\mathbf{T}_n$ at the cusp is $(t - u^n)(t^n - u) = 0$.

Locally at $(\xi,\xi') \in \Xi \times \Xi$, the map $X(l) \times X(l) \to \mathbb{P}^1 \times \mathbb{P}^1$ given by the J-function on each factor is a quotient map for the action of $\mathbb{Z}/l \times \mathbb{Z}/l$. This gives the local analytic equation for the inverse image of $\mathbf{T}_n = \bigcup_{g\in\mathrm{SL}_2(\mathbb{Z}/l)}(1 \times g)T_n$,

$$(s^l - v^{nl})(s^{ln} - v^l) = \prod_{i=0}^{l-1}(s - \zeta_l^i v^n)(s^n - \zeta_l^i v),$$

which consists of $2l$ nonsingular branches. So at a cusp that lies on $(1 \times g)T_n$, the curve is either locally smooth and tangent to a component of D to order n or it consists of two smooth components meeting transversely, one tangent to one order-n component of D and the other tangent to the other order-n component of D. Therefore, to compute the local intersection multiplicity $(C \cdot (1 \times g)T_n)_{(\xi,\xi')}$ for n sufficiently large, one need only compute the local intersection multiplicities of C with the components of D at (ξ,ξ'). This is independent of n and so the result follows.

(ii) To estimate the global intersection number, $C \cdot (1 \times g)T_n = (1 \times g)^{-1}C \cdot T_n$, set $C' = (1 \times g)^{-1}C$; then use Proposition 10.2(ii) and the Lefschetz fixed point formula [K, 1.3.6c] to obtain

$$C' \cdot T_n = \sum_{i=0}^{2}(-1)^i \,\mathrm{Tr}(C' \circ T_n|_{H^i(X(l))}).$$

By Proposition 10.2(iv), T_n acts on both $H^0(X(l))$ and $H^2(X(l))$ via multiplication by $n+1$. The correspondence C' acts on $H^0(X(l))$ via multiplication by the scalar $C'\cdot(X(l)\times x)$ and on $H^2(X(l))$ via multiplication by $C'\cdot(x\times X(l))$. Thus

$$C'\cdot T_n \geq n+1-\mathrm{Tr}(C'\circ T_n|_{H^1(X(l))}).$$

Since $\mathcal{H}$ is a commutative semi-simple algebra, we may choose a basis for $H^1(X(l))$ that simultaneously diagonalizes the action of all T_n. By Proposition 10.2(v), the absolute value of each diagonal entry of T_n is bounded by $2\sqrt{n}$. Thus the trace is bounded by $2h^1(X(l))c\sqrt{n}$, where c is the maximal absolute value of a diagonal element in the matrix representing the action of C'. By Dirichlet's theorem, there are infinitely many primes n with $n\equiv 1$ modulo l. Thus, for sufficiently large n, $C'\cdot T_n\geq n/2$. □

11.9. REMARK. By Theorem 11.1, $H^4(W,\hat{\mathbb{Z}}'(2))_{\mathrm{tors}}$ is contained in the image of the cycle class map. However, the images of the cycle class maps to $H^4(W,\mathbb{Q}_l(2))$ and $H^2(W,\mathbb{Q}_l(1))$ are not generally known because the Tate conjecture for elliptic surfaces is an open problem.

12. Betti Numbers

In the process of computing the torsion in the cohomology of W, much of the work needed to compute the Betti numbers has been done. Since this information is often useful (e.g., for investigating modularity of Galois representations; cf. [M, Chap. 2]), the results are recorded here. Because of its relevance, the cohomology of Y is given as well.

Let $\pi: Y\to X$ be an elliptic surface as in Section 3. Let α_x denote the wild conductor of $R^1\pi_*\mathbb{Z}/l$ at $x\in X$ [Mi, V.2.9], which is independent of $l\neq \mathrm{char}(k)$. We say that π is *tame* if $\alpha_x=0$ for all $x\in |X|_0$. Let e denote the l-adic Euler characteristic.

12.1. PROPOSITION.

(i) *$H^\bullet(Y,\mathbb{Z}_l)$ is torsion free.*
(ii) *$h^0(Y,\mathbb{Q}_l)=1=h^4(Y,\mathbb{Q}_l)$ and $h^1(Y,\mathbb{Q}_l)=2g=h^3(Y,\mathbb{Q}_l)$.*
(iii) *$e(Y)=|S_a|+\sum_{s\in S}m_s+\sum_{x\in X}\alpha_x$.*
(iv) *$e(Y)\geq\sum_{x\in X}e(\pi^{-1}(x))$, with equality if and only if π is tame.*

Proof. Part (i) was proved in the last paragraph of Section 3, and part (ii) follows from Proposition 3.1. For (iii), recall that $e(Y)$ is independent of l by the Lefschetz fixed point theorem or Proposition 2.1(iv). Choose l such that $E[l]^G=0$. Now $H^1(X,R^1\pi_*\mathbb{Z}/l^n)$ is a free $\mathbb{Z}/l^n$-module by Propositions 3.1(i) and (iii). Its rank is

$$h^1(X,R^1\pi_*\mathbb{Z}/l)=-e(X,R^1\pi_*\mathbb{Z}/l)=4g-4+|S_m|+2|S_a|+\sum_{x\in X}\alpha_x$$

by [Mi, V.2.9-12] and the known monodromy at a place of bad reduction [Si2, 10.2a]. Now use Proposition 3.1 to compute the last term in the equation

$$e(Y, \mathbb{Q}_l) = e(Y, \mathbb{Z}/l) = \sum_{i=0}^{2} (-1)^i e(X, R^i \pi_* \mathbb{Z}/l).$$

Finally, part (iv) follows from (iii) and the elementary computation of $e(\pi^{-1}(x))$. □

Let α'_x be the counterpart of α_x for π'. Set $\boldsymbol{\alpha} = \sum_{x \in X} \alpha_x + \alpha'_x$. Define

$$\varepsilon = \begin{cases} 1 & \text{if } E \text{ and } E' \text{ are isogenous over } K, \\ 0 & \text{otherwise,} \end{cases}$$

12.2. Proposition. *Let W be a desingularized fiber product as in Section 5.*

(i) $h^0(W, \mathbb{Q}_l) = 1 = h^6(W, \mathbb{Q}_l)$ *and* $h^1(W, \mathbb{Q}_l) = 2g = h^5(W, \mathbb{Q}_l)$.
(ii) $h^2(W) = h^4(W) = \varepsilon - 5 + 8g + |S_a| + |S'_a| + \sum_{s \in S - S''} m_s + \sum_{s \in S' - S''} m'_s + \sum_{s \in S''} (2 m_s m'_s + 1) + \boldsymbol{\alpha}$.
(iii) $h^3(W) = 2\big(6g - 4 + |S''| + \varepsilon + |S_a \cup S'_a| + \boldsymbol{\alpha} + \sum_{s \in S - S''} m_s + \sum_{s \in S' - S''} m'_s\big)$.
(iv) $e(W) = 4 \sum_{s \in S''} m_s m'_s = \sum_{x \in X} e(f^{-1}(x))$.

Proof. Part (i) follows from Lemma 5.4. For (ii), use Lemma 4.2(ii) and the proof of Lemma 5.4(i) to get

$$H^1(X, R^1 f_* \mathbb{Z}/l^n) \simeq H^1(X, R^1 \pi_* \mathbb{Z}/l^n) \oplus H^1(X, R^1 \pi'_* \mathbb{Z}/l^n).$$

By the proof of Proposition 12.1, this is a free $\mathbb{Z}/l^n$-module of rank

$$8g - 8 + |S_m| + |S'_m| + 2|S_a| + 2|S'_a| + \boldsymbol{\alpha}$$

provided l is chosen such that $B_{l^n} = 0 = B'_{l^n}$ in Lemma 5.4. The choice of l does not affect the Betti number (see Proposition 2.1(iv)). By [S2, 7.3], the stalks of the subsheaf of $R^2 f_* \mathbb{Z}/l^n$ supported on S'' are free $\mathbb{Z}/l^n$-modules and the contribution of this subsheaf, which is a direct summand, to $H^0(X, R^2 f_* \mathbb{Z}/l^n)$ has rank $\sum_{s \in S''} (2 m_s m'_s - 1)$. The stalks of the subsheaf supported at $S - S''$ (resp. $S' - S''$) are also free $\mathbb{Z}/l^n$-modules, and the rank of the contribution to $H^0(X, R^2 f_* \mathbb{Z}/l^n)$ is $\sum_{s \in S - S''} (m_s - 1)$ $\big($resp. $\sum_{s \in S' - S''} (m'_s - 1)\big)$ by Lemmas 3.2 and 4.2(iii) and the proof of Proposition 6.1(ii). For large n, the contribution to the rank of the free part of $H^0(X, R^2 f_* \mathbb{Z}/l^n)$ from the subsheaf complementary to the subsheaf with finite support is $2 + \varepsilon$ by Propositions 6.1(iii) and 7.1. There are no nontrivial differentials in the spectral sequence of Lemma 5.4 that enter in the computation of $H^2(W, \mathbb{Z}/l^n)$. Taking the rank-1 E_2^{20}-term into account and passing to the inverse limit gives the answer.

(iii) When both π and π' are semi-stable, this is [S1, 3.1]. That argument may be modified to allow for additive reduction by subtracting from the right-hand side of [S1, 3.6] the sum over closed points $x \in X$ of the wild conductors. By Assumption 0.1, this sum is $2\boldsymbol{\alpha}$. For each $s \in S_a \cup S'_a$, the expression in square brackets in [S1, 3.6] is 4 (again by Assumption 0.1). The remainder of the proof of [S1, 3.1] goes through unchanged. In particular: $h^2(X, R^1 f_* \mathbb{Q}_l) = 0$; $h^0(X, R^3 f_* \mathbb{Q}_l)$ is computed as in [S1, 3.5]; and (iii) follows.

(iv) This follows from (i)–(iii) and the easy computation of the Euler characteristics of the fibers. In contrast to Proposition 12.1(iii), the wild conductor does not appear in the formula. □

References

[AMu] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational,* Proc. London Math. Soc. (3) 25 (1972), 75–95.

[BaKr] V. Batyrev and M. Kreuzer, *Integral cohomology and mirror symmetry for Calabi–Yau 3-folds,* AMS/IP Stud. Adv. Math., 38, pp. 255–270, Amer. Math. Soc., Providence, RI, 2006.

[BDo] V. Bouchard and R. Donagi, *An $SU(5)$ heterotic standard model,* Phys. Lett. B 633 (2006), 783–791.

[CSaSo] J.-L. Colliot-Thélène, J.-J. Sansuc, and C. Soulé, *Torsion dans le groupe de Chow de codimension deux,* Duke Math. J. 50 (1983), 763–801.

[D] P. Deligne, *Théorie de Hodge. II,* Inst. Hautes Études Sci. Publ. Math. 40 (1971), 5–57.

[SGA4.5] P. Deligne, *Cohomologie étale,* Lecture Notes in Math., 569, Springer-Verlag, Berlin, 1977.

[DR] P. Deligne and M. Rapoport, *Les schémas de modules des courbes elliptiques,* Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, 1972), Lecture Notes in Math., 349, pp. 143–316, Springer-Verlag, Berlin, 1973.

[DiSh] F. Diamond and J. Shurman, *A first course in modular forms,* Grad. Texts in Math., 228, Springer-Verlag, New York, 2005.

[E] S. Endrass, *On the divisor class group of double solids,* Manuscripta Math. 99 (1999), 341–358.

[Ga] O. Gabber, *Sur la torsion dans la cohomologie l-adique d'une variété,* C. R. Acad. Sci. Paris Sér. I Math. 297 (1983), 179–182.

[Gra] C. Grabowski, *On the integral Hodge conjecture for 3-folds,* Ph.D. thesis, Duke Univ., 2004.

[G] M. Gross, *Some calculations of Brauer groups,* Appendix to *Stable singularities in string theory* by P. Aspinwall and D. Morrison, Comm. Math. Phys. 178 (1996), 115–134.

[GP] M. Gross and S. Pavanelli, *A Calabi–Yau threefold with Brauer group* $(\mathbb{Z}/8)^2$, Proc. Amer. Math. Soc. 136 (2008), 1–9.

[Gro] A. Grothendieck, *Le Groupe de Brauer, III,* Dix exposes sur la cohomologie des schemas, pp. 88–188, North-Holland, Amsterdam, 1968.

[H] R. Hartshorne, *Algebraic geometry,* Grad. Texts in Math., 52, Springer-Verlag, New York, 1977.

[K] S. Kleiman, *Algebraic cycles and Weil conjectures,* Dix exposes sur la cohomologie des schemas, pp. 359–386, North-Holland, Amsterdam, 1968.

[Ko] J. Kollar, *Trento examples,* Classification of irregular varieties (Trento, 1990), Lecture Notes in Math., 1515, pp. 134–139, Springer-Verlag, Berlin, 1992.

[L] S. Lang, *Elliptic functions,* Addison-Wesley, Reading, MA, 1973.

[M] C. Meyer, *Modular Calabi–Yau threefolds,* Fields Inst. Monogr., 22, Amer. Math. Soc., Providence, RI, 2005.

[Mi] J. Milne, *Étale cohomology,* Princeton Math. Ser., 33, Princeton Univ. Press, Princeton, NJ, 1980.

[Mu] D. Mumford, *Abelian varieties,* Tata Inst. Fund. Res. Stud. Math., 5, Bombay, 1985.

[S1] C. Schoen, *Desingularized fiber products of semi-stable elliptic surfaces with vanishing third Betti number,* Compositio Math. 145 (2009), 89–111.

[S2] ———, *Invariants of regular models of the product of two elliptic curves at a place of multiplicative reduction,* Arithmetic and geometry of K3 surfaces and Calabi–Yau threefolds, Springer-Verlag, Berlin (to appear).

[SSha] C. Schoen and S. Sharif, *Geometric Tate–Shafarevich groups and certain elliptic threefolds,* in preparation.

[ST] C. Schoen and J. Top, *Drinfeld modules and torsion in the Chow groups of certain threefolds,* Proc. London Math. Soc. (3) 95 (2007), 545–566.

[Si1] J. Silverman, *The arithmetic of elliptic curves,* Grad. Texts in Math., 106, Springer-Verlag, New York, 1986.

[Si2] ———, *Advanced topics in the arithmetic of elliptic curves,* Grad. Texts in Math., 151, Springer-Verlag, New York, 1994.

[V] C. Voisin, *On integral Hodge classes on uniruled or Calabi–Yau threefolds,* Moduli spaces and arithmetic geometry, Adv. Stud. Pure Math., 45, pp. 43–73, Math. Soc. Japan, Tokyo, 2006.

[Z] Y. Zarhin, *Homomorphisms of abelian varieties over geometric fields of finite characteristic,* preprint, 2011, arXiv:1012.5083v2.

Department of Mathematics
Duke University
Box 90320
Durham, NC 27708

schoen@math.duke.edu