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Quermaßintegrals and Asymptotic Shape of Random Polytopes in an Isotropic Convex Body

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1. Introduction

The aim of this work is to provide new information on the asymptotic shape of the random polytope

$$K_N = \mathrm{conv}\{\pm x_1, \dots, \pm x_N\} \tag{1.1}$$

spanned by N independent random points $x_1, \dots, x_N$ that are uniformly distributed in an isotropic convex body K in $\mathbb{R}^n$. We fix $N > n$ and further exploit the idea of [9] to compare K_N with the L_q-centroid body $Z_q(K)$ of K for $q \simeq \log N$. Recall that the L_q-centroid body $Z_q(K)$ of K has support function

$$h_{Z_q(K)}(x) = \|\langle \cdot, x\rangle\|_q := \left(\int_K |\langle y, x\rangle|^q \, dy\right)^{1/q}; \tag{1.2}$$

background information on isotropic convex bodies and their L_q-centroid bodies is given in Section 2.

This idea has its roots in previous work [11; 19; 22] on the behavior of symmetric random ± 1-polytopes, the absolute convex hulls of random subsets of the discrete cube $D_2^n = \{-1, 1\}^n$. These articles demonstrated that the absolute convex hull $D_N = \mathrm{conv}(\{\pm x_1, \dots, \pm x_N\})$ of N independent random points $x_1, \dots, x_N$ uniformly distributed over D_2^n has extremal behavior—with respect to several geometric parameters—among all ± 1-polytopes with N vertices at every scale of n, where $n < N \le 2^n$. The main source of this information is the following estimate from [19] (which improves on an analogous result from [11]): for all $N \ge (1+\delta)n$ (where $\delta > 0$ can be as small as $1/\log n$) and for every $0 < \beta < 1$,

$$D_N \supseteq c\left(\sqrt{\beta \log(N/n)}\, B_2^n \cap Q_n\right) \tag{1.3}$$

with probability greater than $1 - \exp(-c_1 n^{\beta} N^{1-\beta}) - \exp(-c_2 N)$. Here B_2^n is the Euclidean unit ball and $Q_n = [-1/2, 1/2]^n$ is the unit cube in $\mathbb{R}^n$.

In a sense, the model of D_N corresponds to the study of the geometry of a random polytope spanned by random points that are uniformly distributed in Q_n. Starting from the observation that $Z_q(Q_n) \simeq \sqrt{q}\, B_2^n \cap Q_n$, whence (1.3) can be equivalently written in the form

$$D_N \supseteq c Z_{\beta \log(N/n)}(Q_n), \tag{1.4}$$

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we proved in [9] that, in full generality, a precise analogue of (1.4) holds for the random polytope K_N spanned by N independent random points $x_1, \dots, x_N$ uniformly distributed in an isotropic convex body K. In particular, for every $N \geq cn$ (where $c > 0$ is an absolute constant) and every isotropic convex body K in $\mathbb{R}^n$,

$$K_N \supseteq c_1 Z_q(K) \quad \text{for all } \; q \leq c_2 \log(N/n) \tag{1.5}$$

with probability tending exponentially fast to 1 as $n, N \to \infty$.

The precise statement is given in Section 3, and it will play a leading role in this paper. The inclusion is sharp; it is proved in [9] that K_N is "weakly sandwiched" between $Z_{q_i}(K)$ $(i = 1, 2)$, where $q_i \simeq \log N$, in the following sense. It can be easily checked that for every $\alpha > 1$ one has

$$\mathbb{E}[\sigma(\{\theta : h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta)\})] \leq N\alpha^{-q}, \tag{1.6}$$

and this implies that if $q \geq c_3 \log(N/n)$ then, for most $\theta \in S^{n-1}$, one has $h_{K_N}(\theta) \leq c_4 h_{Z_q(K)}(\theta)$. It follows that several geometric parameters of K_N are controlled by the corresponding parameters of $Z_{[\log(N/n)]}(K)$. For example, in [9] the volume radius of a random K_N was determined for the full range of values of N as follows. For every $cn \leq N \leq \exp(n)$,

$$\frac{c_5\sqrt{\log(N/n)}}{\sqrt{n}} \leq |K_N|^{1/n} \leq \frac{c_6 L_K \sqrt{\log(N/n)}}{\sqrt{n}} \tag{1.7}$$

with probability greater than $1 - 1/N$, where $c_5, c_6 > 0$ are absolute constants. Actually, combining this argument with a result of Klartag and Milman [15] shows that, in the range $N \in [cn, \exp(\sqrt{n}\,)]$, the isotropic constant L_K of K may be inserted in the lower bound; this leads to the asymptotic formula

$$|K_N|^{1/n} \simeq \frac{L_K\sqrt{\log(N/n)}}{\sqrt{n}}. \tag{1.8}$$

Our first result gives an extension of this formula to the full family of quermaßintegrals $W_{n-k}(K_N)$ of K_N. These are defined through Steiner's formula,

$$|K + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_{n-k}(K) t^{n-k}, \tag{1.9}$$

where $W_{n-k}(K)$ is the mixed volume $V(K, k;\, B_2^n, n-k)$. We work with a normalized variant of $W_{n-k}(K)$: for every $1 \leq k \leq n$, we set

$$Q_k(K) = \left(\frac{W_{n-k}(K)}{\omega_n}\right)^{1/k} = \left(\frac{1}{\omega_k}\int_{G_{n,k}} |P_F(K)|\, d\nu_{n,k}(F)\right)^{1/k}; \tag{1.10}$$

here the last equality follows from Kubota's integral formula (see Section 2 for background information on mixed volumes). In Section 3 we determine the expectation of $Q_k(K_N)$ for all values of k by proving the following theorem.

Theorem 1.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $n^2 \leq N \leq \exp(cn)$ then, for every $1 \leq k \leq n$,*

$$\sqrt{\log N} \lesssim \mathbb{E}[Q_k(K_N)] \lesssim w(Z_{\log N}(K)). \tag{1.11}$$

In the range $n^2 \le N \le \exp(\sqrt{n}\,)$ *we have the following asymptotic formula*: *for every* $1 \le k \le n$,

$$\mathbb{E}[Q_k(K_N)] \simeq L_K\sqrt{\log N}. \tag{1.12}$$

We remark that all our estimates remain valid for $n^{1+\delta} \le N \le n^2$, where $\delta \in (0,1)$ is fixed, if we allow the constants to depend on δ. Working in the range $N \simeq n$ would require more delicate arguments. We chose to simplify the exposition; in fact, Proposition 3.1 is proved for the range $cn \le N \le \exp(cn)$, and it is quite natural that similar extensions can be provided for most statements in this paper (the interested reader may also consult [29] and [3]). We note also that in saying a random K_N *satisfies* a certain asymptotic formula (F) we mean that this holds true with probability greater than $1 - N^{-1}$, where all the constants appearing in (F) are absolute positive constants.

A more careful analysis is carried out in Section 4, where we obtain the equivalence $Q_k(K_N) \simeq L_K\sqrt{\log N}$ with high probability (for a random K_N) in the range $n^2 \le N \le \exp(\sqrt{n}\,)$.

THEOREM 1.2. *Let* K *be an isotropic convex body in* $\mathbb{R}^n$. *If* $n^2 \le N \le \exp(\sqrt{n}\,)$ *then, with probability greater than* $1 - N^{-1}$,

$$Q_k(K_N) \simeq L_K\sqrt{\log N} \tag{1.13}$$

for all $1 \le k \le n$.

From Theorem 1.2 one can derive several geometric properties of a random K_N. In Section 4 we describe two such properties that concern the regularity of the covering numbers $N(K_N, \varepsilon B_2^n)$ and the size of random k-dimensional projections of K_N.

THEOREM 1.3. *Let* K *be an isotropic convex body in* $\mathbb{R}^n$ *and let* $n^2 \le N \le \exp(\sqrt{n}\,)$.

(i) *With probability greater than* $1 - N^{-1}$, *a random* K_N *satisfies the entropy estimate*

$$\log N\big(K_N, c_1\varepsilon L_K\sqrt{\log N}B_2^n\big) \le c_2 n \min\left\{\log\left(1 + \frac{c_3}{\varepsilon}\right), \frac{1}{\varepsilon^2}\right\} \tag{1.14}$$

for every $\varepsilon > 0$, *where* $c_1, c_2, c_3 > 0$ *are absolute constants.*

(ii) *Moreover, with probability greater than* $1 - N^{-1}$ *a random* K_N *satisfies the following*: *for every* $1 \le k \le n$,

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \simeq L_K\sqrt{\log N} \tag{1.15}$$

with probability greater than $1 - e^{-ck}$ *with respect to the Haar measure* $\nu_{n,k}$ *on* $G_{n,k}$.

Given $1 \le k \le n$, we can also establish upper bounds for the volume of the projection of a random K_N onto a *fixed* $F \in G_{n,k}$ and onto the *k-dimensional coordinate subspaces* of $\mathbb{R}^n$. These are valid provided that N is not too large, depending on k.

Theorem 1.4. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \le k \le n$.*

(i) *For all $k < N \le e^k$ and for every $F \in G_{n,k}$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \le cL_K\sqrt{\log N} \tag{1.16}$$

with probability greater than $1 - N^{-1}$.

(ii) *For all $k < N \le \exp\big(c_1\sqrt{k/\log k}\big)$, with probability greater than $1 - \exp\big(-c_2\sqrt{k/\log k}\big)$ a random K_N satisfies the following: for every $\sigma \subseteq \{1,\dots,n\}$ with $|\sigma| = k$,*

$$\left(\frac{|P_\sigma(K_N)|}{\omega_k}\right)^{1/k} \le c_3 L_K \log(en/k)\sqrt{\log N}, \tag{1.17}$$

where the $c_i > 0$ are absolute constants.

In Section 5 we generalize a result of Mendelson, Pajor, and Rudelson [22] on the combinatorial dimension of the random polytope D_N. This is defined as follows. For a fixed orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ and for every $\varepsilon > 0$, the Vapnik–Chervonenkis combinatorial dimension $\mathrm{VC}(K,\varepsilon)$ of a symmetric convex body K in $\mathbb{R}^n$ is the largest cardinality of a subset σ of $\{1,\dots,n\}$ for which

$$\varepsilon Q_\sigma \subseteq P_\sigma(K), \tag{1.18}$$

where Q_σ is the unit cube in $\mathbb{R}^\sigma = \mathrm{span}\{e_i : i \in \sigma\}$ and P_σ denotes the orthogonal projection onto $\mathbb{R}^\sigma$. It is proved in [22] that a random D_N satisfies

$$\mathrm{VC}(D_N,\varepsilon) \simeq \min\left\{\frac{c\log(cN\varepsilon^2)}{\varepsilon^2}, n\right\}. \tag{1.19}$$

We extend this estimate to the more general class of random polytopes K_N in which K is an isotropic convex body in $\mathbb{R}^n$ that is unconditional with respect to the basis $\{e_1,\dots,e_n\}$.

Theorem 1.5. *Let K be an unconditional isotropic convex body in $\mathbb{R}^n$. If $c_1 n \le N \le \exp(c_2 n)$, then a random K_N satisfies*

$$\mathrm{VC}(K_N,\varepsilon) \ge \min\left\{\frac{c_3\log(N/n)}{\varepsilon^2}, n\right\} \tag{1.20}$$

for every $\varepsilon \in (0,1)$.

2. Notation and Background Material

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and we write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $1 \le k \le n$ and $F \in G_{n,k}$. We will use P_F to denote the orthogonal projection from $\mathbb{R}^n$ onto F. We also define $B_F := B_2^n \cap F$ and $S_F := S^{n-1} \cap F$.

The letters $c, c', c_1, c_2, \dots$ denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Similarly, if $K, L \subseteq \mathbb{R}^n$ then we will write $K \simeq L$ provided there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$. We also write $\bar{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$; thus, $\bar{A} := A/|A|^{1/n}$.

A *convex body* is a compact convex subset C of $\mathbb{R}^n$ with nonempty interior. We denote the class of convex bodies in $\mathbb{R}^n$ by $\mathcal{K}_n$. We say that C is *symmetric* if $-x \in C$ whenever $x \in C$. We say that C is *centered* if it has center of mass at the origin—that is, if $\int_C \langle x, \theta \rangle \, dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$. For each $-\infty < q < \infty$ $(q \neq 0)$, we define the *q-mean width* of C by

$$w_q(C) := \left(\int_{S^{n-1}} h_C^q(\theta) \sigma(d\theta) \right)^{1/q}. \tag{2.1}$$

The mean width of C is the quantity $w(C) = w_1(C)$. The *radius* of C is defined as $R(C) = \max\{\|x\|_2 : x \in C\}$, and if the origin is an interior point of C then the *polar body* of C is defined as

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in C\}. \tag{2.2}$$

A centered convex body K in $\mathbb{R}^n$ is called *isotropic* if it has volume $|K| = 1$ and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 \, dx = L_K^2 \tag{2.3}$$

for every θ in the Euclidean unit sphere S^{n-1}. For every convex body K in $\mathbb{R}^n$ there exists an affine transformation T of $\mathbb{R}^n$ such that $T(K)$ is isotropic. Moreover, if we ignore orthogonal transformations, then this isotropic image is unique and so the isotropic constant L_K is an invariant of the affine class of K. The reader is referred to [23] and [10] for more information on isotropic convex bodies.

2.1. Quermaßintegrals

The relation between volume and the operations of addition and multiplication of convex bodies by nonnegative reals is described by Minkowski's fundamental theorem, which may be stated as follows. If $K_1, \dots, K_m \in \mathcal{K}_n$ $(m \in \mathbb{N})$ then the volume of $t_1 K_1 + \cdots + t_m K_m$ is a homogeneous polynomial of degree n in $t_i \geq 0$,

$$|t_1 K_1 + \cdots + t_m K_m| = \sum_{1 \leq i_1, \dots, i_n \leq m} V(K_{i_1}, \dots, K_{i_n}) t_{i_1} \cdots t_{i_n}, \tag{2.4}$$

where the coefficients $V(K_{i_1}, \dots, K_{i_n})$ can be chosen to be invariant under permutations of their arguments. The coefficient $V(K_{i_1}, \dots, K_{i_n})$ is called the *mixed volume* of the n-tuple $(K_{i_1}, \dots, K_{i_n})$.

Steiner's formula is a special case of Minkowski's theorem; the volume of $K + tB_2^n$ ($t > 0$) can be expanded as a polynomial in t:

$$|K + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_{n-k}(K) t^{n-k}, \tag{2.5}$$

where $W_{n-k}(K) := V(K, k; B_2^n, n-k)$ is the $(n-k)$th quermaßintegral of K. It will be convenient for us to work with a normalized variant of $W_{n-k}(K)$, so for every $1 \le k \le n$ we set

$$Q_k(K) = \left(\frac{1}{\omega_k} \int_{G_{n,k}} |P_F(K)| \, d\nu_{n,k}(F) \right)^{1/k}. \tag{2.6}$$

Note that $Q_1(K) = w(K)$. Kubota's integral formula

$$W_{n-k}(K) = \frac{\omega_n}{\omega_k} \int_{G_{n,k}} |P_F(K)| \, d\nu_{n,k}(F) \tag{2.7}$$

shows that

$$Q_k(K) = \left(\frac{W_{n-k}(K)}{\omega_n} \right)^{1/k}. \tag{2.8}$$

The Aleksandrov–Fenchel inequality states that if $K, L, K_3, \dots, K_n \in \mathcal{K}_n$ then

$$V(K, L, K_3, \dots, K_n)^2 \ge V(K, K, K_3, \dots, K_n) V(L, L, K_3, \dots, K_n). \tag{2.9}$$

This implies that the sequence $(W_0(K), \dots, W_n(K))$ is log-concave: we have

$$W_j^{k-i} \ge W_i^{k-j} W_k^{j-i} \tag{2.10}$$

if $0 \le i < j < k \le n$. Taking into account (2.8), we conclude that $Q_k(K)$ is a decreasing function of k. For the theory of mixed volumes, see [30].

2.2. L_q-Centroid Bodies

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \ge 1$ and every $y \in \mathbb{R}^n$, we set

$$h_{Z_q(K)}(y) := \left(\int_K |\langle x, y \rangle|^q \, dx \right)^{1/q}. \tag{2.11}$$

The L_q-centroid body $Z_q(K)$ of K is the centrally symmetric convex body with support function $h_{Z_q(K)}$. Note that K is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$. Also, if $T \in SL(n)$ then $Z_q(T(K)) = T(Z_q(K))$ for all $q \ge 1$. From Hölder's inequality it follows that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \le p \le q \le \infty$, where $Z_\infty(K) = \mathrm{conv}(K, -K)$. Using Borell's lemma (see [24, [Apx. III]), one can check that

$$Z_q(K) \subseteq c_1 \frac{q}{p} Z_p(K) \tag{2.12}$$

for all $1 \le p < q$. In particular, if K is isotropic then $R(Z_q(K)) \le c_2 q L_K$. One can also check that if K is centered then $Z_q(K) \supseteq c_3 K$ for all $q \ge n$ (see [25] for a proof). We will also use that if K is isotropic then

$$K \subseteq (n+1)L_K B_2^n \tag{2.13}$$

and hence

$$L_K B_2^n = Z_2(K) \subseteq Z_q(K) \subseteq Z_\infty(K) \subseteq (n+1)L_K B_2^n \tag{2.14}$$

for all $q \geq 2$. A proof of the first assertion is given in [14]; the second assertion is clear from Hölder's inequality.

Let C be a symmetric convex body in $\mathbb{R}^n$, and let $\|\cdot\|_C$ denote the norm induced on $\mathbb{R}^n$ by C. The parameter $k_*(C)$ is defined by

$$k_*(C) = n\frac{w(C)^2}{R(C)^2}. \tag{2.15}$$

It is known that, up to an absolute constant, $k_*(C)$ is the largest positive integer $k \leq n$ with the property that $\frac{1}{2}w(C)B_F \subseteq P_F(C) \subseteq 2w(C)B_F$ for most $F \in G_{n,k}$ (to be precise, with probability greater than $n/(n+k)$). The q-mean width $w_q(C)$ is equivalent to $w(C)$ provided $q \leq k_*(C)$: it is proved in [18] that, for every symmetric convex body C in $\mathbb{R}^n$, the following statements hold.

(i) If $1 \leq q \leq k_*(C)$, then $w(C) \leq w_q(C) \leq c_4 w(C)$.
(ii) If $k_*(C) \leq q \leq n$, then $c_5\sqrt{q/n}\,R(C) \leq w_q(C) \leq c_6\sqrt{q/n}\,R(C)$.

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $q \in (-n, \infty)$, $q \neq 0$, we define

$$I_q(K) := \left(\int_K \|x\|_2^q \, dx\right)^{1/q}. \tag{2.16}$$

In [26] and [27] it is proved that, for every $1 \leq q \leq n/2$,

$$I_q(K) \simeq \sqrt{n/q}\, w_q(Z_q(K)) \quad \text{and} \quad I_{-q}(K) \simeq \sqrt{n/q}\, w_{-q}(Z_q(K)). \tag{2.17}$$

Paouris [26] introduced the parameter

$$q_*(K) := \max\{q \leq n : k_*(Z_q(K)) \geq q\}. \tag{2.18}$$

Then the main result of [27] states that, for every centered convex body K of volume 1 in $\mathbb{R}^n$, one has $I_{-q}(K) \simeq I_q(K)$ for every $1 \leq q \leq q_*(K)$; in particular, for all $q \leq q_*(K)$ one has $I_q(K) \leq c_7 I_2(K)$. If K is isotropic then one can check that $q_*(K) \geq c_8\sqrt{n}$, where $c_8 > 0$ is an absolute constant (for a proof, see [26]). Therefore,

$$I_q(K) \leq c_9\sqrt{n}\,L_K \quad \text{for } q \leq \sqrt{n}. \tag{2.19}$$

When $q \simeq q_*(K)$, the result of [18] shows that $w(Z_q(K)) \simeq w_q(Z_q(K))$. Then the following useful estimate is a direct consequence of (2.19) and (2.17).

FACT 2.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $1 \leq q \leq q_*(K)$, then*

$$w(Z_q(K)) \simeq w_q(Z_q(K)) \simeq \sqrt{q}\,L_K. \tag{2.20}$$

In particular, this holds for all $q \leq \sqrt{n}$.

Associated with any centered convex body $K \subset \mathbb{R}^n$ is a family of bodies that was introduced by Ball in [4] (see also [23]); to define these bodies, let us consider a

k-dimensional subspace F of $\mathbb{R}^n$ and its orthogonal subspace E. For every $\phi \in F \setminus \{0\}$ we set $E^+(\phi) = \{x \in \mathrm{span}\{E, \phi\} : \langle x, \phi \rangle \geq 0\}$. Ball proved that, for every $q \geq 0$, the function

$$\phi \mapsto \|\phi\|_2^{1+q/(q+1)} \left(\int_{K \cap E^+(\phi)} \langle x, \phi \rangle^q \, dx \right)^{-1/(q+1)} \tag{2.21}$$

is the gauge function of a convex body $B_q(K, F)$ on F. We shall need some facts about the relation of the bodies $B_q(K, F)$ to the L_q-centroid bodies $Z_q(K)$ and their projections. If K is a centered convex body of volume 1 in $\mathbb{R}^n$ and if $1 \leq k \leq n-1$ then, for every $F \in G_{n,k}$ and every $q \geq 1$, we have

$$P_F(Z_q(K)) = (k+q)^{1/q} |B_{k+q-1}(K, F)|^{1/k+1/q} Z_q(\bar{B}_{k+q-1}(K, F)) \tag{2.22}$$

and

$$|B_{k+q-1}(K, F)|^{1/k+1/q} \leq \frac{e(k+q)}{k} \left(\frac{1}{k+q} \right)^{1/q} \frac{1}{|K \cap F^\perp|^{1/k}}. \tag{2.23}$$

Also, for every $F \in G_{n,k}$ and every $q \geq 1$,

$$\begin{aligned} \frac{k}{e^2(k+q)} Z_q(\bar{B}_{k+1}(K, F)) &\subseteq Z_q(\bar{B}_{k+q-1}(K, F)) \\ &\subseteq e^2 \frac{k+q}{k} Z_q(\bar{B}_{k+1}(K, F)). \end{aligned} \tag{2.24}$$

If K is isotropic, then

$$L_{\bar{B}_{k+1}(K, F)} \simeq |K \cap F^\perp|^{1/k} L_K. \tag{2.25}$$

For the proofs of these assertions we refer to [26] and [27].

3. Expectation of the Quermaßintegrals

In this section we give the proof of Theorem 1.1, which is a consequence of the following proposition.

Proposition 3.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $cn \leq N \leq \exp(cn)$ then, for every $1 \leq k \leq n$,*

$$c_1 \sqrt{n} |Z_{\log(N/n)}(K)|^{1/n} \leq \mathbb{E}[Q_k(K_N)] \leq c_2 w(Z_{\log N}(K)), \tag{3.1}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. We first recall the precise statements of the main results from [9] on the asymptotic shape of a random polytope with N vertices chosen independently and uniformly from an isotropic convex body.

Fact 3.2. *Let $\beta \in (0, 1/2]$ and $\gamma > 1$. If $N \geq N(\gamma, n) = c\gamma n$, where $c > 0$ is an absolute constant, then for every isotropic convex body K in $\mathbb{R}^n$ we have*

$$K_N \supseteq c_1 Z_q(K) \quad \text{for all } q \leq c_2 \beta \log(N/n) \tag{3.2}$$

with probability greater than $1 - f(\beta, N, n)$, *where* $f(\beta, N, n) \to 0$ *exponentially fast as n and N increase.*

The upper bound obtained in [9] for $f(\beta, N, n)$ is

$$f(\beta, N, n) \leq \exp(-c_3 N^{1-\beta} n^{\beta}) + \mathbb{P}\big(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq \gamma L_K \sqrt{N}\big), \tag{3.3}$$

where $\Gamma : \ell_2^n \to \ell_2^N$ is the random operator $\Gamma(y) = (\langle x_1, y\rangle, \ldots, \langle x_N, y\rangle)$ defined by the vertices $x_1, \ldots, x_N$ of K_N. There are several known bounds for this last probability (see e.g. [13; 21]). The best-known estimate can be extracted from [2, Thm. 3.13]: one has $\mathbb{P}\big(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq \gamma L_K \sqrt{N}\big) \leq \exp\big(-c_0 \gamma \sqrt{N}\big)$ for all $N \geq c\gamma n$. If we assume that $\beta \leq 1/2$, then

$$f(\beta, N, n) \leq \exp(-c_4 \sqrt{n}\,). \tag{3.4}$$

Since $Q_k(\cdot)$ is decreasing in k, we immediately obtain

$$\mathbb{E}[Q_k(K_N)] \geq \mathbb{E}[Q_n(K_N)] = \mathbb{E}\left(\frac{|K_N|}{\omega_n}\right)^{1/n}. \tag{3.5}$$

Then Fact 3.2 shows that

$$\mathbb{E}\left(\frac{|K_N|}{\omega_n}\right)^{1/n} \geq c_5 \left(\frac{|Z_{\log(N/n)}(K)|}{\omega_n}\right)^{1/n}, \tag{3.6}$$

where $c_5 > 0$ is an absolute constant. Combining (3.5) and (3.6) yields the first inequality in (3.1).

We now turn our attention to the opposite direction. Let $N \geq n$. Observe that, for every $\alpha > 0$ and $\theta \in S^{n-1}$, by Markov's inequality we have

$$\mathbb{P}(\alpha, \theta) := \mathbb{P}(\{x \in K : |\langle x, \theta\rangle| \geq \alpha \|\langle \cdot, \theta\rangle\|_q\}) \leq \alpha^{-q}; \tag{3.7}$$

therefore,

$$\begin{aligned} \mathbb{P}(h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta)) &= \mathbb{P}\Big(\max_{j \leq N} |\langle x_j, \theta\rangle| \geq \alpha \|\langle \cdot, \theta\rangle\|_q\Big) \\ &\leq N\,\mathbb{P}(\alpha, \theta) \leq N\alpha^{-q}. \end{aligned} \tag{3.8}$$

Then a standard application of Fubini's theorem shows that, for every $\alpha > 1$,

$$\mathbb{E}[\sigma(\theta : h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta))] \leq N\alpha^{-q}. \tag{3.9}$$

Using that $h_{K_N}(\theta) \leq h_{Z_\infty(K)}(\theta) \leq c_6 n L_K$, which follows from (2.14), we write

$$w(K_N) \leq \int_{A_N} h_{K_N}(\theta)\, d\sigma(\theta) + c_6 \sigma(A_N^c) n L_K, \tag{3.10}$$

where $A_N = \{\theta : h_{K_N}(\theta) \leq \alpha h_{Z_q(K)}(\theta)\}$. Then

$$w(K_N) \leq \alpha \int_{A_N} h_{Z_q(K)}(\theta)\, d\sigma(\theta) + c_6 \sigma(A_N^c) n L_K \tag{3.11}$$

and so, by (3.9),

$$\mathbb{E}[w(K_N)] \leq \alpha w(Z_q(K)) + c_6 N n \alpha^{-q} L_K. \tag{3.12}$$

Since $w(Z_q(K)) \geq w(Z_2(K)) = L_K$, we obtain

$$\mathbb{E}[w(K_N)] \leq (\alpha + c_6 N n \alpha^{-q}) w(Z_q(K)). \tag{3.13}$$

Choosing $\alpha = e$ and $q = 2\log N$, we see that

$$\mathbb{E}[Q_1(K_N)] = \mathbb{E}[w(K_N)] \leq c_7 w(Z_{2\log N}(K)) \leq c_8 w(Z_{\log N}(K)); \tag{3.14}$$

here we have taken into account that $Z_{2\log N}(K) \subseteq c Z_{\log N}(K)$, a consequence of (2.12). Since $Q_k(K)$ is decreasing in k, it follows that

$$\mathbb{E}[Q_k(K_N)] \leq \mathbb{E}[Q_1(K_N)] \leq c_9 w(Z_{\log N}(K)) \tag{3.15}$$

for all $1 \leq k \leq n$, where $c_9 > 0$ is an absolute constant. This completes the proof of the proposition. □

For the proof of Theorem 1.1 we combine Proposition 3.1 with the following known bounds for $|Z_q(K)|$. The first bound, expressed by (3.16), follows from the results of [26] and [15]; the second bound, expressed by (3.17), was obtained in [20].

Fact 3.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $1 \leq q \leq \sqrt{n}$ then*

$$|Z_q(K)|^{1/n} \simeq \sqrt{q/n}\, L_K, \tag{3.16}$$

but if $\sqrt{n} \leq q \leq n$ then

$$c_9\sqrt{q/n} \leq |Z_q(K)|^{1/n} \leq c_{10}\sqrt{q/n}\, L_K. \tag{3.17}$$

Proof of Theorem 1.1. We first assume that $n^2 \leq N \leq \exp(\sqrt{n})$. From (3.16) we have

$$|Z_{\log N}(K)|^{1/n} \geq c_{11}\sqrt{\log N/n}\, L_K, \tag{3.18}$$

and from Fact 2.1 we have

$$w(Z_{\log N}(K)) \leq c_{12}\sqrt{\log N}\, L_K. \tag{3.19}$$

Therefore, (3.1) takes the form

$$\mathbb{E}[Q_k(K_N)] \simeq \sqrt{\log N}\, L_K \tag{3.20}$$

as claimed. If $\exp(\sqrt{n}) \leq N \leq \exp(cn)$ then we use (3.1) and the first inequality in (3.17). It follows that

$$c_{13}\sqrt{\log N} \leq \mathbb{E}[Q_k(K_N)] \leq c_2 w(Z_{\log N}(K)) \tag{3.21}$$

for every $1 \leq k \leq n$, and the proof is complete. □

4. The Range $n^2 \leq N \leq \exp(\sqrt{n})$

Next we prove Theorem 1.2 on the quermaßintegrals of a random K_N in the range $n^2 \leq N \leq \exp(\sqrt{n})$. The precise statement is as follows.

Theorem 4.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $n^2 \leq N \leq \exp(\sqrt{n})$ then, with probability greater than $1 - N^{-1}$, a random K_N satisfies*

$$Q_k(K_N) \le c_1 L_K \sqrt{\log N} \tag{4.1}$$

for all $1 \le k \le n$ *and with probability greater than* $1 - \exp(-\sqrt{n}\,)$ *it satisfies*

$$Q_k(K_N) \ge c_2 L_K \sqrt{\log N} \tag{4.2}$$

for all $1 \le k \le n$, *where* $c_1, c_2 > 0$ *are absolute constants.*

Proof. Let $n^2 \le N \le \exp(\sqrt{n}\,)$. For the proof of (4.2) recall that, with probability greater than $1 - \exp(-\sqrt{n}\,)$, a random K_N contains $c_3 Z_{\log N}(K)$. Then we can use (3.5), (3.6), and the volume estimate from Fact 3.3 to show that any such K_N satisfies

$$Q_k(K_N) \ge Q_n(K_N) \ge c_3\sqrt{n}|Z_{\log N}(K)|^{1/n} \ge c_4 L_K \sqrt{\log N} \tag{4.3}$$

for all $1 \le k \le n$.

For the proof of (4.1) we need two lemmas.

LEMMA 4.2. *Let* K *be an isotropic convex body in* $\mathbb{R}^n$. *For every* $n^2 \le N \le \exp(cn)$ *and for every* $q \ge \log N$ *and* $r \ge 1$, *we have*

$$\int_{S^{n-1}} \frac{h_{K_N}^q(\theta)}{h_{Z_q(K)}^q(\theta)}\, d\sigma(\theta) \le (c_1 r)^q \tag{4.4}$$

with probability greater than $1 - r^{-q}$, *where* $c_1 > 0$ *is an absolute constant.*

Proof. We have assumed that K is isotropic and so, by (2.13) and (2.4), $K_N \subseteq \operatorname{conv}(K, -K) \subseteq (n+1)L_K B_2^n$ and $Z_q(K) \supseteq Z_2(K) = L_K B_2^n$. This implies that $h_{K_N}(\theta) \le (n+1)h_{Z_q(K)}(\theta)$ for all $\theta \in S^{n-1}$. We write

$$\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta) = \int_0^{n+1} q t^{q-1} \sigma(\theta : h_{K_N}(\theta) \ge t h_{Z_q(K)}(\theta))\, dt. \tag{4.5}$$

We fix $\alpha > 1$ (to be chosen) and estimate the expectation over K^N: using (3.9), we obtain

$$\begin{aligned}\mathbb{E}\left(\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta)\right) &\le \alpha^q + \int_\alpha^{n+1} q t^{q-1} N t^{-q}\, dt \\ &\le \alpha^q + qN \log\left(\frac{n+1}{\alpha}\right).\end{aligned} \tag{4.6}$$

We now choose $\alpha = e$. If $q \ge \log N$, then

$$\mathbb{E}\left(\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta)\right) \le c_1^q \tag{4.7}$$

for some absolute constant $c_1 > 0$. Markov's inequality shows that, for every $r \ge 1$,

$$\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta) \le (c_1 r)^q \tag{4.8}$$

with probability greater than $1 - r^{-q}$. □

Lemma 4.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $n^2 \le N \le \exp(cn)$ and for every $q \ge \log N$ and $r \ge 1$, we have*

$$w(K_N) \le c_1 r w_q(Z_q(K)) \tag{4.9}$$

with probability greater than $1 - r^{-q}$.

Proof. Using Hölder's inequality and the Cauchy–Schwarz inequality, we write

$$\begin{aligned}[w(K_N)]^q &\le \left(\int_{S^{n-1}} h_{K_N}(\theta)^{q/2}\, d\sigma(\theta)\right)^2 \\ &\le [w_q(Z_q(K))]^q \int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta).\end{aligned} \tag{4.10}$$

Lemma 4.2 shows that if $q \ge \log N$ and $r \ge 1$, then

$$\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta) \le (c_1 r)^q \tag{4.11}$$

and hence

$$w(K_N) \le c_1 r w_q(Z_q(K)) \tag{4.12}$$

with probability greater than $1 - r^{-q}$. □

We can now prove (4.1). We have assumed that $\log N \lesssim \sqrt{n}$, and we choose $q = \log N$ and $r = e$. Hence, by Lemma 4.3 and Fact 2.1,

$$w(K_N) \le c w_{\log N}(Z_{\log N}(K)) \simeq w(Z_{\log N}(K)) \le c_1 L_K \sqrt{\log N} \tag{4.13}$$

with probability greater than $1 - N^{-1}$. Since $Q_k(K_N) \le w(K_N)$ for all $1 \le k \le n$, the proof of the theorem is now complete. □

Note. Theorem 1.2 and Fact 3.2 show that if $n^2 \le N \le \exp(\sqrt{n})$ then, with probability greater than $1 - N^{-1}$, a random K_N has the two properties

(P1) $K_N \supseteq c_1 Z_{\log N}(K)$ and

(P2) $Q_k(K_N) \simeq L_K \sqrt{\log N}$ for all $1 \le k \le n$.

In Sections 4.1 and 4.2 we derive the two claims of Theorem 1.3 from (P1) and (P2).

4.1. Regularity of the Covering Numbers

Recall that if K and L are nonempty sets in $\mathbb{R}^n$, then the covering number $N(K, L)$ of K by L is defined to be the smallest number of translates of L whose union covers K. If K is a convex body and L is a symmetric convex body in $\mathbb{R}^n$, then a standard volume argument shows that

$$2^{-n} \frac{|K + L|}{|L|} \le N(K, L) \le 2^n \frac{|K + L|}{|L|}. \tag{4.14}$$

The next proposition concerns the covering numbers of a random K_N by multiples of the Euclidean unit ball; in particular, it provides a proof for Theorem 1.3(i).

PROPOSITION 4.4. *Let K be an isotropic convex body in $\mathbb{R}^n$, and let $n^2 \leq N \leq \exp(\sqrt{n})$. Then a random K_N satisfies the entropy estimate*

$$\log N(K_N, c_1\varepsilon L_K\sqrt{\log N} B_2^n) \leq c_2 n \min\left\{\log\left(1+\frac{c_3}{\varepsilon}\right), \frac{1}{\varepsilon^2}\right\} \tag{4.15}$$

for every $\varepsilon > 0$, where $c_1, c_2, c_3 > 0$ are absolute constants. Moreover, if $0 < \varepsilon \leq 1$ then

$$c_4 n \log\frac{c_5}{\varepsilon} \leq \log N(K_N, c_6\varepsilon L_K\sqrt{\log N} B_2^n) \leq c_7 n \log\frac{c_8}{\varepsilon} \tag{4.16}$$

for suitable absolute constants c_i, $i = 4, \ldots, 8$.

Proof. We will give estimates for the covering numbers $N(K_N, \varepsilon r_{n,N} B_2^n)$, where K_N satisfies (P1) and (P2) and where

$$r_{n,N} := \left(\frac{|K_N|}{\omega_n}\right)^{1/n} \simeq L_K\sqrt{\log N} \tag{4.17}$$

is the volume radius of K_N. Using the second inequality in (4.14), we write

$$N(K_N, \varepsilon r_{n,N} B_2^n) \leq 2^n \frac{\left|\frac{1}{\varepsilon r_{n,N}} K_N + B_2^n\right|}{\omega_n}. \tag{4.18}$$

By Steiner's formula,

$$\frac{\left|\frac{1}{\varepsilon r_{n,N}} K_N + B_2^n\right|}{\omega_n} = \sum_{k=0}^{n} \binom{n}{k} Q_k^k(K_N) \frac{1}{\varepsilon^k r_{n,N}^k}; \tag{4.19}$$

now, since $Q_k(K_N) \simeq r_{n,N}$ by (P2), we have

$$\frac{\left|\frac{1}{\varepsilon r_{n,N}} K_N + B_2^n\right|}{\omega_n} \leq \sum_{k=0}^{n} \binom{n}{k} \left(\frac{c}{\varepsilon}\right)^k = \left(1+\frac{c}{\varepsilon}\right)^n. \tag{4.20}$$

Returning to (4.18), we see that

$$\log N(K_N, \varepsilon r_{n,N} B_2^n) \leq c_1 n \log\left(1+\frac{c_2}{\varepsilon}\right) \tag{4.21}$$

for suitable absolute constants $c_1, c_2 > 0$. A second upper bound can be given by Sudakov's inequality $\log N(K, tB_2^n) \leq cnw^2(K)/t^2$ (see e.g. [28]). Since $w(K_N) \simeq r_{n,N}$, it follows immediately that

$$\log N(K_N, \varepsilon r_{n,N} B_2^n) \leq \frac{cn}{\varepsilon^2} \tag{4.22}$$

for all $\varepsilon > 0$. This proves (4.15).

A lower bound on the covering numbers can also be obtained for the case where $0 < \varepsilon \leq 1$. For this we can use the lower bound on the volume of K_N from equation (1.7) or (1.8) depending on whether or not (respectively) $\log N \leq \sqrt{n}$. For example, if the inequality holds then

$$N(K_N, \varepsilon r_{n,N} B_2^n)^{1/n} \geq \left(\frac{|K_N|}{|\varepsilon r_{n,N} B_2^n|}\right)^{1/n} = \frac{1}{\varepsilon}. \tag{4.23}$$

Hence $\log N(K_N, \varepsilon r_{n,N} B_2^n) \geq n \log(1/\varepsilon)$. □

4.2. Random Projections of K_N

Next we show that, if K_N has properties (P1) and (P2), then the volume radius of a random projection $P_F(K_N)$ onto $F \in G_{n,k}$ is completely determined by n, k, and N; this is the content of Theorem 1.3(ii).

Proposition 4.5. *Let K be an isotropic convex body in $\mathbb{R}^n$, and let $n^2 \leq N \leq \exp(\sqrt{n})$. Then, with probability greater than $1 - N^{-1}$, a random K_N satisfies the following*: *for every $1 \leq k \leq n$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \simeq L_K \sqrt{\log N} \tag{4.24}$$

with probability greater than $1 - e^{-ck}$ with respect to the Haar measure $\nu_{n,k}$ on $G_{n,k}$.

Proof. The upper bound is a corollary of Theorem 1.2. We know that if $\log N \leq \sqrt{n}$ then K_N satisfies (P2) with probability greater than $1 - N^{-1}$; in particular,

$$Q_k(K_N) = \left(\frac{1}{\omega_k}\int_{G_{n,k}} |P_F(K_N)|\, d\nu_{n,k}(F)\right)^{1/k} \lesssim L_K \sqrt{\log N} \tag{4.25}$$

for all $1 \leq k \leq n$. Applying Markov's inequality then yields the following fact.

Fact 4.6. *If $n^2 \leq N \leq \exp(\sqrt{n})$ then, with probability greater than $1 - N^{-1}$, K_N satisfies the following*: *for every $1 \leq k \leq n$ and every $t \geq 1$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \leq c_1 t \sqrt{\log N} L_K \tag{4.26}$$

with probability greater than $1 - t^{-k}$ with respect to $\nu_{n,k}$.

For the lower bound we use (P1). Integrating in polar coordinates, we have

$$\begin{aligned}
\int_{G_{n,k}} \frac{|P_F^\circ(K_N)|}{\omega_k}\, d\nu_{n,k}(F) &= \int_{G_{n,k}} \int_{S_F} \frac{1}{h^k_{P_F(K_N)}(\theta)}\, d\sigma_F(\theta)\, d\nu_{n,k}(F) \\
&= \int_{G_{n,k}} \int_{S_F} \frac{1}{h^k_{K_N}(\theta)}\, d\sigma_F(\theta)\, d\nu_{n,k}(F) \\
&\leq \left(\int_{G_{n,k}} \int_{S_F} \frac{1}{h^n_{K_N}(\theta)}\, d\sigma_F(\theta)\, d\nu_{n,k}(F)\right)^{k/n} \\
&= \left(\int_{S^{n-1}} \frac{1}{h^n_{K_N}(\theta)}\, d\sigma(\theta)\right)^{k/n} \\
&= \left(\frac{|K_N^\circ|}{\omega_n}\right)^{k/n}.
\end{aligned} \tag{4.27}$$

By the Blaschke–Santaló inequality and the inclusion $K_N \supseteq Z_{c_2 \log N}(K)$, we get

$$\left(\frac{|K_N^\circ|}{\omega_n}\right)^{k/n} \leq \left(\frac{\omega_n}{|K_N|}\right)^{k/n} \leq \left(\frac{\omega_n}{|Z_{c_2 \log N}(K)|}\right)^{k/n}. \tag{4.28}$$

Recall that if $q \leq \sqrt{n}$ then $\left(\frac{|Z_q(K)|}{\omega_n}\right)^{1/n} \geq c_3 \sqrt{q} L_K$, from which we conclude that

$$\int_{G_{n,k}} \frac{|P_F^\circ(K_N)|}{\omega_k} \, d\nu_{n,k}(F) \leq \left(\frac{c_4}{\sqrt{\log N} L_K}\right)^k. \tag{4.29}$$

From Markov's inequality we obtain an upper bound for the volume radius of a random $P_F^\circ(K_N)$, and the reverse Santaló inequality proves the following.

FACT 4.7. *If $n^2 \leq N \leq \exp(\sqrt{n})$ then, with probability greater than $1 - N^{-1}$, K_N satisfies the following: for every $1 \leq k \leq n$ and every $t \geq 1$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \geq \frac{c_5 L_K \sqrt{\log N}}{t} \tag{4.30}$$

with probability greater than $1 - t^{-k}$ with respect to $\nu_{n,k}$.

Together, Fact 4.6 and Fact 4.7 prove Proposition 4.5. □

REMARK 4.8. Using [16, Prop. 3.1], one can actually prove that if $k \leq n/4$ (or, more generally, if $k \leq \lambda n$ for some $\lambda \in (0,1)$) then most k-dimensional projections of K_N contain a ball of radius $L_K \sqrt{\log N}$:

$$P_F(K_N) \supseteq \frac{c_6}{t} L_K \sqrt{\log N} B_F \tag{4.31}$$

with probability greater than $1 - t^{-k}$ with respect to $\nu_{n,k}$. This, in turn, shows that (4.30) is satisfied by $P_F(K_N)$. We omit the details.

4.3. Coordinate Projections of K_N

Here we prove Theorem 1.4. Part (i) is proved by our next proposition, which estimates the size of the projection of a random K_N onto a fixed subspace F in $G_{n,k}$.

PROPOSITION 4.9. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leq k \leq n$. For all $k < N \leq e^k$ and for every $F \in G_{n,k}$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \leq c L_K \sqrt{\log N} \tag{4.32}$$

with probability greater than $1 - N^{-1}$.

Proof. Fix $F \in G_{n,k}$. For all $\theta \in S_F$ and all $x \in K$, we have $h_{P_F(Z_q(K))}(\theta) = h_{Z_q(K)}(\theta)$ and $\langle P_F(x), \theta \rangle = \langle x, \theta \rangle$. Hence we can argue as in Lemma 4.2 to show that if $q \geq \log N$ then a random K_N satisfies

$$\int_{S_F} \frac{h^q_{P_F(K_N)}(\theta)}{h^q_{P_F(Z_q(K))}(\theta)}\, d\sigma_F(\theta) \le c_1^q. \tag{4.33}$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\big[w_{-q/2}\big(P_F(Z_q(K))\big)\big]^{-q} &= \left(\int_{S_F} \frac{1}{h^{q/2}_{P_F(Z_q(K))}(\theta)}\, d\sigma_F(\theta)\right)^2 \\ &\le \left(\int_{S_F} \frac{1}{h^q_{P_F(K_N)}(\theta)}\, d\sigma_F(\theta)\right)\left(\int_{S_F} \frac{h^q_{P_F(K_N)}(\theta)}{h^q_{P_F(Z_q(K))}}\, d\sigma_F(\theta)\right) \\ &\le w_{-q}(P_F(K_N))^{-q} c_1^q;\end{aligned}$$

therefore, if $q \ge \log N$ then

$$w_{-q}(P_F(K_N)) \le c_1 s w_{-q/2}\big(P_F(Z_q(K))\big) \tag{4.34}$$

with probability greater than $1 - s^{-q}$.

Assume that $q \le k$. Using Hölder's inequality and taking polars in the subspace F yields

$$\begin{aligned}\left(\frac{|(P_F(K_N))^\circ|}{|B_2^k|}\right)^{1/k} &= \left(\int_{S_F} \frac{1}{h^k_{P_F(K_N)}(\theta)}\, d\sigma_F(\theta)\right)^{1/k} \\ &\ge \left(\int_{S_F} \frac{1}{h^q_{P_F(K_N)}(\theta)}\, d\sigma_F(\theta)\right)^{1/q} \\ &= w_{-q}(P_F(K_N))^{-1}.\end{aligned} \tag{4.35}$$

Applying the Blaschke–Santaló inequality on F, we see that

$$|P_F(K_N)|^{1/k} \le \frac{c_2}{\sqrt{k}} w_{-q}(P_F(K_N)) \tag{4.36}$$

for a suitable absolute constant $c_2 > 0$. Then (4.34) shows that

$$|P_F(K_N)|^{1/k} \le \frac{c_3 s}{\sqrt{k}} w_{-q/2}\big(P_F(Z_q(K))\big) \tag{4.37}$$

with probability greater than $1 - s^{-q}$ for $\log N \le q \le k$. From (2.22) we know that

$$P_F(Z_q(K)) = (k+q)^{1/q} |B_{k+q-1}(K,F)|^{1/k+1/q} Z_q(\bar{B}_{k+q-1}(K,F)), \tag{4.38}$$

and from (2.24) we obtain $Z_q(\bar{B}_{k+q-1}(K,F)) \subseteq c_4 Z_{q/2}(\bar{B}_{k+1}(K,F))$ for a new absolute constant $c_4 > 0$. Hence, with probability greater than $1 - s^{-q}$, if $\log N \le q \le k$ then

$$\begin{aligned}&|P_F(K_N)|^{1/k} \\ &\quad \le \frac{c_5 s}{\sqrt{k}} (k+q)^{1/q} |B_{k+q-1}(K,F)|^{1/k+1/q} w_{-q/2}\big(Z_{q/2}(\bar{B}_{k+1}(K,F))\big).\end{aligned} \tag{4.39}$$

But $\bar{B}_{k+1}(K,F)$ is easily checked to be isotropic, and from (2.17) and (2.19) it then follows that

$$w_{-q/2}\big(Z_{q/2}(\bar{B}_{k+1}(K,F))\big) \leq c_6 \frac{\sqrt{q}}{\sqrt{k}} I_{-q/2}(\bar{B}_{k+1}(K,F)) \\ \leq c_7 \sqrt{q}\, L_{\bar{B}_{k+1}(K,F)}. \tag{4.40}$$

From (2.23) and (2.25) we have

$$L_{\bar{B}_{k+1}(K,F)} \leq c_8 |K \cap F^{\perp}|^{1/k} L_K \tag{4.41}$$

and

$$(k+q)^{1/q} |B_{k+q-1}(K,F)|^{1/k+1/k} |K \cap F^{\perp}| \leq e \frac{k+q}{k} \leq 2e \tag{4.42}$$

for $q \leq k$. We can now use (4.39) to conclude that

$$|P_F(K_N)|^{1/k} \leq c L_K \frac{\sqrt{q}}{\sqrt{k}} \tag{4.43}$$

with probability greater than $1 - s^{-q}$ for all q satisfying $\log N \leq q \leq k$. The proposition follows if we choose $q = \log N$ for $N \leq e^k$. □

In Proposition 4.9, F may be one of the k-dimensional coordinate subspaces of $\mathbb{R}^n$. Using a result from [1] gives us a uniform estimate of the same order on the size of all projections of a random K_N onto k-dimensional coordinate subspaces of $\mathbb{R}^n$. This is part (ii) of Theorem 1.4.

PROPOSITION 4.10. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leq k \leq n$. For all $k < N \leq \exp\big(c_1\sqrt{k/\log k}\big)$, with probability greater than $1 - \exp\big(-c_2\sqrt{k/\log k}\big)$ a random K_N satisfies the following: for every $\sigma \subseteq \{1,\dots,n\}$ with $|\sigma| = k$,*

$$\left(\frac{|P_\sigma(K_N)|}{\omega_k}\right)^{1/k} \leq c_3 L_K \log(en/k)\sqrt{\log N}, \tag{4.44}$$

where $c_i > 0$ are absolute constants.

Proof. Let $1 \leq k \leq n$. It is proved in [1, Thm. 1.1] that, for every $t \geq 1$,

$$\mathbb{P}\left(\max_{|\sigma|=k} \|P_\sigma(x)\|_2 \geq c_1 t L_K \sqrt{k} \log\left(\frac{en}{k}\right)\right) \leq \exp\left(-\frac{t\sqrt{k}\log\left(\frac{en}{k}\right)}{\sqrt{\log(ek)}}\right). \tag{4.45}$$

Assume that $N \leq \exp\big(c_2\sqrt{k/\log k}\big)$. Then, with probability greater than $1 - \exp\big(-c_3\sqrt{k/\log k}\big)$, we have that N random points $x_1,\dots,x_N$ from K satisfy the following: for every $\sigma \subseteq \{1,\dots,n\}$ and for every $1 \leq i \leq N$,

$$\|P_\sigma(x_i)\|_2 \leq c_4 L_K \sqrt{k} \log\left(\frac{en}{k}\right). \tag{4.46}$$

Now we recall a well-known volume bound that was obtained independently in [5], [8], and [12]: if $z_1,\dots,z_N \in \mathbb{R}^k$ and $\max\|z_i\|_2 \leq \alpha$, then

$$|\mathrm{conv}(\{z_1,\dots,z_N\})|^{1/k} \le \frac{c_5\alpha\sqrt{\log N}}{k}. \tag{4.47}$$

In our case inequality this implies that, for every σ with $|\sigma| = k$,

$$\left(\frac{|P_\sigma(K_N)|}{\omega_k}\right)^{1/k} \le c_6 L_K \log(en/k)\sqrt{\log N}, \tag{4.48}$$

as claimed. □

5. Combinatorial Dimension in the Unconditional Case

In this section we assume that K is an unconditional isotropic convex body in $\mathbb{R}^n$. Thus K is symmetric, and the standard orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ is a 1-unconditional basis for $\|\cdot\|_K$: for every choice of real numbers $t_1,\dots,t_n$ and every choice of signs $\varepsilon_j = \pm 1$,

$$\|\varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n\|_K = \|t_1 e_1 + \cdots + t_n e_n\|_K. \tag{5.1}$$

It is known that the isotropic constant of K satisfies $L_K \simeq 1$. Moreover, Bobkov and Nazarov [7] have proved that $K \supseteq c_2 Q_n$ for $Q_n = \left[-\frac{1}{2},\frac{1}{2}\right]^n$.

We will use that, for this class of convex bodies, the family of L_q-centroid bodies of the cube Q_n is extremal (the argument is due to R. Łatała).

LEMMA 5.1. *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then*

$$Z_q(K) \supseteq c Z_q(Q_n) \tag{5.2}$$

for all $q \ge 1$, where $c > 0$ is an absolute constant.

Proof. Let $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$ be independent and identically distributed ± 1 random variables defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ and with distribution $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. For every $\theta \in S^{n-1}$, the following expressions are a consequence of the unconditionality of K, Jensen's inequality, and the contraction principle:

$$\begin{aligned}
\|\langle\cdot,\theta\rangle\|_{L^q(K)} &= \left(\int_K \left|\sum_{i=1}^n \theta_i x_i\right|^q dx\right)^{1/q} \\
&= \left(\int_\Omega \int_K \left|\sum_{i=1}^n \theta_i \varepsilon_i |x_i|\right|^q dx\, d\mathbb{P}(\varepsilon)\right)^{1/q} \\
&\ge \left(\int_\Omega \left|\sum_{i=1}^n \theta_i \varepsilon_i \int_K |x_i|\, dx\right|^q d\mathbb{P}(\varepsilon)\right)^{1/q} \\
&\ge \left(\int_\Omega \left|\sum_{i=1}^n t_i \theta_i \varepsilon_i\right|^q d\mathbb{P}(\varepsilon)\right)^{1/q} \\
&\ge \left(\int_{Q_n} \left|\sum_{i=1}^n t_i \theta_i y_i\right|^q dy\right)^{1/q} = \|\langle\cdot,(t\theta)\rangle\|_{L^q(Q_n)};
\end{aligned} \tag{5.3}$$

here $t_i = \int_K |x_i|\,dx \simeq L_K \simeq 1$ and $t\theta = (t_1\theta_1, \dots, t_n\theta_n)$. Recall that

$$\|\langle\cdot,\theta\rangle\|_{L^q(Q_n)} \simeq \sum_{j\le q} \theta_j^* + \sqrt{q}\Big(\sum_{q<j\le n} (\theta_j^*)^2\Big)^{1/2} \tag{5.4}$$

(see [6]). Since $t_i \simeq 1$ for all $i = 1, \dots, n$, it follows that

$$\|\langle\cdot,\theta\rangle\|_{L^q(K)} \ge \|\langle\cdot,(t\theta)\rangle\|_{L^q(Q_n)} \ge c\|\langle\cdot,\theta\rangle\|_{L^q(Q_n)}, \tag{5.5}$$

and this proves the lemma. □

Since $Z_q(Q_n) \simeq \sqrt{q}B_2^n \cap Q_n$, from Fact 3.1 we immediately get the following.

PROPOSITION 5.2. *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. If $c_1 n \le N \le \exp(c_2 n)$ and if $K_N = \mathrm{conv}\{x_1, \dots, x_N\}$ is a random polytope spanned by N independent random points $x_1, \dots, x_N$ uniformly distributed in K, then for every $\sigma \subseteq \{1, \dots, n\}$ we have*

$$P_\sigma(K_N) \supseteq c_1\big(\sqrt{\log(N/n)}\,B_\sigma \cap Q_\sigma\big) \tag{5.6}$$

with probability $1 - o_n(1)$.

Proof of Theorem 1.5. Let $\varepsilon \in (0,1)$. For every $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| = k$, we have $Q_\sigma \subseteq \sqrt{k}\,B_\sigma$ and hence

$$P_\sigma(K_N) \supseteq c_1 \min\left\{\frac{\sqrt{\log(N/n)}}{\sqrt{k}}, 1\right\} Q_\sigma \supseteq \varepsilon Q_\sigma \tag{5.7}$$

provided that

$$\varepsilon \le \frac{c_2\sqrt{\log(N/n)}}{\sqrt{k}}. \tag{5.8}$$

This shows that

$$\mathrm{VC}(K_N, \varepsilon) \ge \min\left\{\frac{c_3\log(N/n)}{\varepsilon^2}, n\right\}, \tag{5.9}$$

which is the lower bound in Theorem 1.5. □

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