

Open Gromov–Witten Theory and the Crepanant Resolution Conjecture

RENZO CAVALIERI & DUSTIN ROSS

1. Introduction

1.1. Summary of Results

Gromov–Witten invariants are virtual counts of curves on a fixed target space X , and they are obtained as intersection numbers on moduli spaces of Kontsevich–stable maps to X . Besides being interesting symplectic invariants of X , they exhibit a remarkable amount of algebraic structure: appropriate generating functions of rational Gromov–Witten invariants give a deformation of the intersection ring of X (quantum cohomology), also endowing the cohomology of X with the structure of a Frobenius manifold. There are two typical “families of questions” in Gromov–Witten theory. First, a simple-minded yet difficult question is whether it is possible to compute invariants for a given target. Second, one wishes to draw interesting consequences from the algebraic structure of invariants, such as comparing in some precise way families of invariants of different but related targets or relating different types of curve-counting invariants on the same target.

In this paper we combine these two kinds of questions to investigate two striking (conjectural) features of Gromov–Witten theory.

Crepanant transformation: the equivalence between GW theories of two targets related by a crepanant birational transformation. In particular, when the crepanant transformation is the resolution of singularities of a Gorenstein orbifold, this equivalence is referred to as the *crepanant resolution conjecture* (CRC).

Gluing: the ability to recover GW invariants for a toric variety/orbifold from open invariants of open subspaces covering the target.

We seek to tackle such questions for arbitrary toric spaces (varieties or orbifolds) by reducing them to local questions that are compatible with gluing procedures. We provide an expanded discussion of our motivations in Section 1.2. Here we present the specific results obtained in this paper.

We give a complete and exhaustive description for the specific geometry in Figure 1. The global quotient $\mathfrak{X} = [\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)/\mathbb{Z}_2]$ (with nontrivial

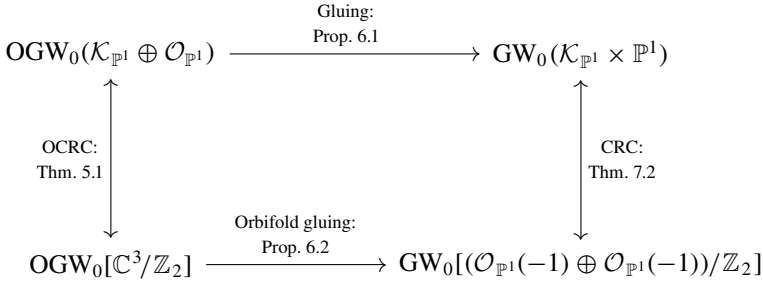


Figure 1 Master diagram for the paper

diagonal action on the fibers and trivial action on the base) is a hard Lefschetz orbifold with $Y = \mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$ (the total space of the canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$) as its crepant resolution. The quotient \mathfrak{X} can be covered by two charts isomorphic to $[\mathbb{C}^3/\mathbb{Z}_2]$ (two copies of the nontrivial representation of \mathbb{Z}_2 and one copy of the trivial one), whose resolutions ($\cong \mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$) cover Y .

The four main results of this paper allow us to “complete the square”.

THEOREM 5.1. *We make and verify a crepant resolution conjecture for the open invariants of $[\mathbb{C}^3/\mathbb{Z}_2]$ and $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.*

This is the first occurrence of a CRC for open invariants. We compute the genus-0 open potential for $[\mathbb{C}^3/\mathbb{Z}_2]$ (Proposition 4.2) using the methods of [BrC]. In order to evaluate invariants for more than one boundary component, we generalize [C2, Thm. 1] to the case of two-part hyperelliptic Hodge integrals with an arbitrary number of descendant insertions (Theorem 2.3). Appearances notwithstanding, Theorem 2.3 is not an instance of the string equation in the orbifold case. The open potential for $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is computed (Proposition 3.3) using the techniques of [KaL]. Some interesting classical combinatorics is required to package the potential in a manageable form.

PROPOSITION 6.1. *Closed invariants for an arbitrary toric CY 3-fold can be obtained by gluing open invariants.*

We compare the contributions (to the restriction of the virtual fundamental class of the moduli space of stable maps to a given fixed locus) from the multiple covers of the fixed lines with the contributions of discs that glue to maps in that fixed locus. It is worth pointing out that our definition of the disc function is purely local (i.e., it does not depend on the global geometry of the 3-fold); hence these two contributions are not tautologically equal. It was recently pointed out to us that a similar check of the gluing occurred in [DF, Apx. B].

PROPOSITION 6.2. *Closed invariants for \mathfrak{X} are recovered by gluing open invariants of $[\mathbb{C}^3/\mathbb{Z}_2]$.*

In the orbifold case we content ourselves with proving the gluing for the particular geometry that we are studying. Checking that orbifold invariants glue in general is currently under investigation by the second author.

THEOREM 7.2. *We verify Ruan’s CRC (à la Bryan–Graber) for \mathfrak{X} and Y .*

We mention two interesting aspects of this result. First, although the CRC has been verified in many instances [BGh; BG; BGP], this is one of the first cases (see also [Gi]) in which the Ruan–Bryan–Graber statement is checked for an orbifold that is not simply a representation of a finite group. In a sense we are checking for whether the Ruan–Bryan–Graber CRC does indeed have geometric content and is not just a group-theoretic feature of orbifold invariants. Second, we prove Theorem 7.2 by showing that our open CRC is “compatible with gluing”, thereby gathering some positive evidence that the CRC may be addressed locally in the toric case.

1.2. Context and Motivation

The Atiyah–Bott localization theorem is effectively used in Gromov–Witten theory to reduce the computation of GW invariants for a toric target to a sum of Hodge integrals over loci of fixed maps. Hodge integrals can be evaluated using Grothendieck–Riemann–Roch and Witten’s conjecture; whence the slogan that localization turns toric GW theory into combinatorics. Alas, this slogan is often a camouflaged admission of defeat for us algebraic geometers, as we are typically unable to manage the combinatorial complexity and extract meaningful geometric information from GW invariants. From a physical point of view, open GW invariants (virtual counts of maps from bordered Riemann surfaces) arise naturally from the propagation of open strings. Mathematically, they offer the opportunity to tackle the combinatorial complexity of GW invariants by making their study even more local. The strategy of the topological vertex [AKMV] is first to associate certain combinatorial gadgets to each fixed point of a toric variety and then to give “gluing rules” that reconstruct GW invariants. Philosophically (and physically), these gadgets should correspond to open invariants relative to branes intersecting the fixed lines containing the given vertex. In [Li+], a limiting argument is used to motivate a mathematical theory of the topological vertex in terms of relative GW invariants. Katz and Liu [KaL] take a different approach toward open invariants: when the target admits an antiholomorphic involution σ , they define open invariants by identifying the σ -invariant portion of the obstruction theory in ordinary GW theory.

In [BrC], Katz and Liu’s approach is generalized in two different directions. First, it is noted that the construction can be made local: independent of the global geometry of the target, disc contributions to open invariants are computed by viewing a neighborhood of the fixed (affine) line where the disc is mapping inside a resolved conifold. This gives rise to a local theory that is strongly similar (and possibly identical) to the mathematical topological vertex of Li and colleagues.

However, it is not straightforward that open invariants should glue correctly: this is the significance of Proposition 6.1. The second generalization carries open invariants to the orbifold setting. A formulation of a general theory of the orbifold vertex is given in [Ro], where the author exploits such formalism to compare the GW orbifold vertex with the Donaldson–Thomas orbifold vertex of [BCaY]. Our formulation of open invariants bypasses the technical problem that the foundations of relative stable maps to orbifolds have not yet been laid. One could also argue that the involution-invariant approach is naturally tuned to the study of orbifold geometry (which in essence is “locally G -invariant geometry”).

Our opinion is that the worth of a local theory (especially if defined via localization) should be measured by its success in addressing global questions. One of the most intriguing conjectures in GW theory, the crepant resolution conjecture, predicts a relation between orbifold GW invariants of a Gorenstein orbifold and GW invariants of its crepant resolution—when it exists. (There are various incarnations of the CRC featuring different levels of generality. Here we focus on the most concrete and restrictive version that applies to our geometry. A nice survey of this rich story—and one that contains the most general formulation—is given by Coates and Ruan [CoR].) A natural question is whether the CRC is compatible with gluing and can therefore be addressed locally. In this paper we study this question for a simple yet nontrivial geometry and then give a positive answer.

In [Br], Brini proposes (based on open mirror symmetry) how to relate disc invariants under crepant transformations by comparing B -model quantities that are intrinsically more computable. Verifying that this proposal agrees with Theorem 5.1 is on our immediate agenda because it would not only provide further evidence of our program’s validity but also, and more importantly, validate Brini’s proposal as a conjectural formulation of a vertex CRC.

1.3. Organization of the Paper

In Section 2 we review open Gromov–Witten invariants and describe methods for computing the open invariants of $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $[\mathbb{C}^3/\mathbb{Z}_2]$. We finish the section by computing explicit formulas for certain hyperelliptic Hodge integrals that show up in later computations. Sections 3 and 4 are the computational meat of the paper in which we compute all relevant open invariants. In Section 5 we show that the open invariants satisfy the open crepant resolution conjecture. In Section 6, we show that open invariants can be glued to obtain closed invariants. Finally, in Section 7 we show that the closed CRC for $[(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))/\mathbb{Z}_2]$ can be deduced from the open CRC.

ACKNOWLEDGMENTS. Many thanks to Dagan Karp for suggesting this geometry as a natural first step in our program, to Melissa Liu for helpful comments about the paper and for directing our attention to literature we were unaware of, and to Andrea Brini for constant communication on his parallel work. We are also grateful to Vincent Bouchard, Y. P. Lee, Sara Pasquetti, Yongbin Ruan, and Hsian-Hua Tseng for many interesting discussions related to this project. Finally we would

like to acknowledge the AIM workshop on “Recursion Structures in Topological String Theory and Enumerative Geometry”, at which the idea of a crepant resolution conjecture for open invariants was discussed.

2. Preliminaries

2.1. Open Invariants

In [KaL], Katz and Liu propose a theory for computing open Gromov–Witten invariants that amounts to a generalization of ordinary Gromov–Witten theory computing virtual counts of maps from surfaces whose boundaries satisfy certain boundary conditions. Consider a Calabi–Yau 3-fold X and a special Lagrangian submanifold L . Fix integers g and h and a relative homology class $\beta \in H_2(X, L; \mathbb{Z})$ with $\partial\beta = \sum \gamma_i \in H_1(L, \mathbb{Z})$. Then the open Gromov–Witten invariant $N_{\beta; \gamma_1, \dots, \gamma_h}^{g, h}$ is a virtual count of maps $f: (\Sigma, \partial\Sigma) \rightarrow (X, L)$ satisfying

- $(\Sigma, \partial\Sigma)$ is a Riemann surface of genus g and with h boundary components,
- $f_*[\Sigma] = \beta$, and
- $f_*[\partial\Sigma] = \sum \gamma_i$.

In order to compute open invariants, Katz and Liu propose an obstruction theory for the moduli space of open stable maps $\overline{\mathcal{M}}_{g, h}(X, L \mid \beta; \gamma_i)$ [KaL, Sec. 4.2]. While assuming that the moduli space can be equipped with a well-behaved torus action, they give an explicit formula for how the corresponding virtual cycle restricts to the fixed locus of the torus action. An especially interesting aspect of this theory is that the virtual cycle does depend on the torus action. In other words, different torus actions lead to different invariants. This reflects the framing dependence of open invariants discussed in [AKV].

The computational key to the Katz and Liu setup is the assumption that L is the fixed locus of an antiholomorphic involution. A map from a bordered Riemann surface mapping boundary into L can then be doubled to a map from a closed Riemann surface [KaL, Sec. 3.3]. Open Gromov–Witten invariants are defined/computed from the involution-invariant contributions to the ordinary Gromov–Witten invariants corresponding to the doubled maps.

Katz and Liu then specialize to compute disk invariants of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, where L is the fixed locus of the antiholomorphic involution $(z, u, v) \rightarrow (1/\bar{z}, \bar{v}\bar{z}, \bar{u}\bar{z})$. Key to the computations are the Riemann–Hilbert bundles $L(2d)$ and $N(d)$ over (D^2, S^1) defined in [KaL, Exm. 3.4.3, Exm. 3.4.4]. The sections of the Riemann–Hilbert bundles are identified torus-equivariantly to the involution-invariant sections of $H^0(\mathbb{P}^1, \mathcal{O}(2d))$ and $H^1(\mathbb{P}^1, \mathcal{O}(-d) \oplus \mathcal{O}(-d))$, respectively.

Our open invariant computations stem from making Katz and Liu’s construction “local”, as we now explain. We represent a toric Calabi–Yau 3-fold via its *web diagram*, a planar trivalent graph in which edges correspond to torus-invariant lines and vertices to torus-invariant points. Once we equip the space with a \mathbb{C}^* -action and then lift that action to the moduli space of open stable maps, the fixed loci consists of maps decomposing as

- compact components of the source curve contracting to the vertices,
- multiple covers of the fixed lines of the 3-fold (fully ramified over fixed points), and
- disks mapping (with appropriate winding) to edges equipped with a Lagrangian.

The contribution from the first two items can be computed using standard Atiyah–Bott localization. The contribution from each disk is computed by applying the Katz–Liu setup to a formal neighborhood of the fixed point where the vertex of the disk is mapped.

2.2. Orientation Convention

A subtlety arises in the computations. Although the sections of $H^0(L(2d))$ are naturally isomorphic to the sections of $H^0(\mathcal{O}_{\mathbb{P}^1}(2d))$, there is no natural choice of isomorphism between the sections of $H^1(N(d))$ and the sections of $H^1(\mathcal{O}_{\mathbb{P}^1}(-d) \oplus \mathcal{O}_{\mathbb{P}^1}(-d))$. The latter correspondence depends instead on a choice of *orientation* for the sections (see [Kal, Sec. 5.2]): a σ -invariant section of $H^1(\mathcal{O}_{\mathbb{P}^1}(-d) \oplus \mathcal{O}_{\mathbb{P}^1}(-d))$ in local coordinates at 0 has the form

$$s = \left(\sum_{i=1}^{d-1} \frac{a_i}{z^i}, \sum_{i=1}^{d-1} \frac{\bar{a}_i}{z^{d-i}} \right) = \left(\sum_{j=1}^{d-1} \frac{\bar{b}_j}{z^{d-j}}, \sum_{j=1}^{d-1} \frac{b_j}{z^j} \right).$$

The space of involution-invariant sections is identified (torus-equivariantly) with a complex vector space by the first (resp. second) projection when using the coordinates a_i (resp. b_j). This choice results in different open invariants: in the first (resp. second) case the weights of the sections are \mathbb{C}^* -weights of sections of the $\mathcal{O}_{\mathbb{P}^1}(-d)$ on the left-hand (resp. right-hand) side. The choice of orientation ultimately yields a global factor of $(-1)^{d+1}$, where d is the winding of the disk.

In order to track the choice of orientation, we establish the following convention.

ORIENTATION CONVENTION 2.1. Throughout the paper, we add an arrow to each edge intersecting a Lagrangian (see Figure 2). The corresponding disk contributions are computed by identifying the involution-invariant sections of $H^1(\mathcal{O}_{\mathbb{P}^1}(-d) \oplus \mathcal{O}_{\mathbb{P}^1}(-d))$ via projection to the sections of the bundle to the *left* of the arrow. The choices of orientations for the geometric objects considered are depicted in Figure 3.

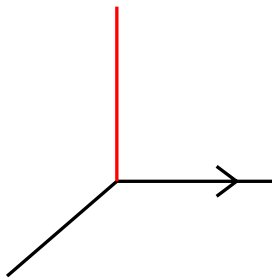


Figure 2 \mathbb{C}^3 with one oriented half-edge indicates that (a) we are computing open invariants with disks lying along the horizontal edge and (b) σ -invariant sections are identified by projecting onto the bundle corresponding to the vertical edge

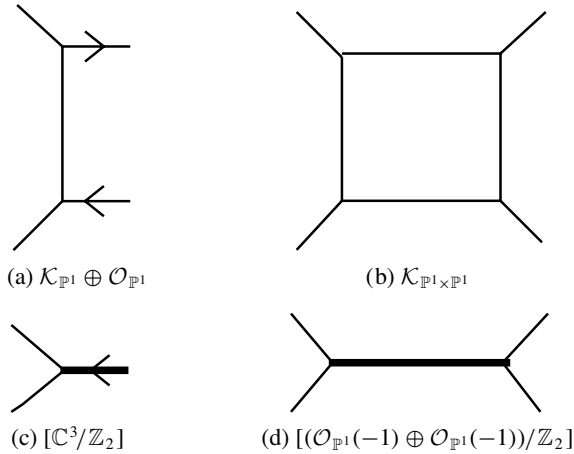


Figure 3 The $\mathcal{B}\mathbb{Z}_2$ -gerbes are marked by heavy lines, and orientations have been chosen via Convention 2.1

In [BrC], the methods of Katz and Liu are extended to the orbifolds $[\mathbb{C}^3/\mathbb{Z}_n]$. Analogously to computing closed orbifold Gromov–Witten invariants, the open orbifold Gromov–Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_n]$ are defined/computed by considering only the contributions to the open invariants that descend to the quotient. In both [KaL] and [BrC], the open invariants defined via the A-model are verified against B-model predictions.

2.3. Hyperelliptic Hodge Integrals

In this section we prove a closed formula for a generating function that packages the hyperelliptic Hodge integrals of the form

$$L(g, i, \bar{m}) := \int_{\bar{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)} \lambda_g \lambda_{g-i} (\bar{\psi})^{\bar{m}}; \tag{1}$$

here \bar{m} is a multi-index (m_1, \dots, m_l) , $|\bar{m}| := m_1 + \dots + m_l = i - 1$, and

$$\bar{\psi}^{\bar{m}} := \psi_1^{m_1} \dots \psi_l^{m_l}.$$

REMARK 2.2. Recall that $\bar{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)$ is the moduli space of maps from genus-0 curves into $\mathcal{B}\mathbb{Z}_2$ with $2g + 2$ twisted marked points. Each such map corresponds to a (possibly nodal) genus g double cover of the source curve ramified over the marked points. We have two natural forgetful maps,

$$\begin{array}{ccc} \bar{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2) & \xrightarrow{F} & \bar{\mathcal{M}}_g \\ \downarrow \pi & & \\ \bar{\mathcal{M}}_{0;2g+2} & & \end{array} \tag{2}$$

by sending a map to the corresponding double cover of its source curve. The λ -classes on $\overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)$ are defined to be

$$\lambda_i := c_i(F^*\mathbb{E}),$$

where \mathbb{E} is the Hodge bundle on $\overline{\mathcal{M}}_g$. The ψ -classes are defined via pull-back from $\overline{\mathcal{M}}_{0;2g+2}$.

For a fixed i and \bar{m} with $|\bar{m}| = i - 1$, define the generating function

$$\mathcal{L}_{i,\bar{m}}(x) := \sum_g L(g, i, \bar{m}) \frac{x^{2g}}{(2g)!}. \tag{3}$$

We know from the $\lambda_g \lambda_{g-1}$ computation [BP; BeCT; FaP] that

$$\mathcal{L}_{1,\emptyset} = \log \sec\left(\frac{x}{2}\right), \tag{4}$$

and we know from [C2] that

$$\mathcal{L}_{i,(i-1)} = \frac{2^{i-1}}{i!} \mathcal{L}_{1,\emptyset}^i. \tag{5}$$

The following theorem generalizes (5).

THEOREM 2.3.

$$\mathcal{L}_{i,\bar{m}} = \binom{m_1 + \dots + m_l}{m_1, \dots, m_l} \frac{2^{i-1}}{i!} \mathcal{L}_{1,\emptyset}^i. \tag{6}$$

REMARK 2.4. This formula was given independently by Gillam [Gi], who verified the result computationally for $l \leq 4$.

Proof of Theorem 2.3. We use induction on the multi-index \bar{m} . Given $\bar{m} = (m_1, \dots, m_k)$ with $|\bar{m}| = j - 1$, we know that the statement is true if either $j = 1$ or $k = 1$. Suppose (6) holds when $j < i$ and also when $j = i$ and $k \leq l$. For this case, we show that (6) holds when $j = i$ and $k = l + 1$.

NOTATION. Write $\bar{m} = (m_1, \dots, m_l, m_{l+1})$ and set $\bar{m}' = (m_1, \dots, m_{l-1}, m'_l)$, where $m'_l := m_l + m_{l+1}$. For a subset $A \subseteq \{1, \dots, l + 1\}$, we write $\bar{m}(A)$ for the multi-index that is equal to \bar{m} in the entries indexed by numbers in A and equal to 0 in the other entries. As usual, A^c denotes the complement of A . We use $\bar{m}[k]$ to denote the multi-index \bar{m} with the first entry replaced by k .

We shall prove the recursion by evaluating via localization auxiliary integrals on $\overline{\mathcal{M}}_{0;2g+2,0}(\mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2, 1)$. This moduli space parameterizes double covers of the source curve with a special rational component picked out. By postcomposing the usual evaluation maps with projection onto the first factor, we obtain evaluation maps to \mathbb{P}^1 that we denote by e_i . The auxiliary integrals are

$$\text{A1: } \int \lambda_g \lambda_{g-i} \bar{\psi}^{\bar{m}(\{1\}^c)} e_i^*(0) e_{l+1}^*(0) e_{2g+2}^*(\infty);$$

$$\text{A2: } \int \lambda_g \lambda_{g-i} \bar{\psi}^{\bar{m}'(\{1\}^c)} e_i^*(0) e_{l+1}^*(0) e_{2g+2}^*(\infty).$$

REMARK 2.5. (i) In each integrand, we do not include the ψ_1 -part of the Hodge integral. The ψ_1 -classes in the result make an appearance through node smoothing. The other ψ -classes correspond to the marked points with the matching index.

(ii) We have abused notation in order to make the expression legible. By λ_{g-i} we intend $c_{g-i}^{\text{eq}}(R^1\pi_* f^*\mathcal{O})$, where the trivial bundle is linearized with 0 weights: the λ -classes are how these classes restrict to the fixed loci. By $e_i^*(0)$ (resp. $e_i^*(\infty)$) we denote $c_1^{\text{eq}}(e_i^*\mathcal{O}(1))$ linearized with weight 1 over 0 and weight 0 over ∞ (resp. 0 over 0 and -1 over ∞). These classes essentially localize to require the corresponding mark point to map over 0 (resp. ∞).

(iii) The difference in the two auxiliary integrals is that we have “spread” the ψ -classes on the two points fixed over 0 in two different ways.

(iv) Both integrals vanish for dimensional reasons. In both integrals, the degree of the class we integrate is $m_2 + \dots + m_{l+1} + 3 + 2g - i$, which is strictly less than $2g + 2$ (because $m_1 + \dots + m_{l+1} = i - 1$ and $m_1 > 0$).

(v) Localizing A1 yields relation (8) among Hodge integrals, where all terms are already known by induction. Localizing A2 yields relation (9) by computing one unknown Hodge integral in terms of inductively known ones. Then, since (8) and (9) are proportional to each other, we can determine the desired integral.

Analyzing the obstruction theory via the normalization sequence of the source curve, one sees that the maps in the contributing fixed loci satisfy the following properties (see [CI] for more details).

- The preimages of 0 and ∞ in the corresponding double cover must be connected.
- One distinguished projective line in the source curve maps to the main component of the target with degree 1. The corresponding double cover has a rational component over the distinguished projective line.
- The l th and $(l + 1)$ th marked points must map to 0, and the $(2g + 2)$ th marked point must map to ∞ .

The contributing fixed loci are as follows.

F_g : All marked points—except for the $(2g + 2)$ th—map to 0. The corresponding double cover contracts a genus- g component over 0 and does not have a positive-dimensional irreducible component over ∞ . This locus is isomorphic to $\bar{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)$.

F_{g_1,g_2} : $2g_1 + 1$ marked points map to 0 and $2g_2 + 1$ marked points map to ∞ (this includes the points that are already forced to map to 0 and ∞). The corresponding double cover contracts a genus- g_1 component over 0 and a genus- g_2 component over ∞ . This locus is isomorphic to $\bar{\mathcal{M}}_{0;2g_1+2,0}(\mathcal{B}\mathbb{Z}_2) \times \bar{\mathcal{M}}_{0;2g_2+2,0}(\mathcal{B}\mathbb{Z}_2)$.

The mirror analogue of F_g is not in the fixed locus because we require that at least two marked points map to 0.

The first integral evaluates on the two types of fixed loci to

$$\begin{aligned}
 F_g &: \frac{(-1)^i}{t^{m_1}} \int_{\overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)} \lambda_g \lambda_{g-i} \bar{\psi}^{\bar{m}} = \frac{(-1)^i}{t^{m_1}} L(g, i, \bar{m}), \\
 F_{g_1, g_2} &: \frac{2(-1)^i}{t^{m_1}} \sum_{k=1}^{i-1} \sum_{A \subseteq \{2, \dots, l-1\}} \binom{2g+1-l}{2g_1+1-|A|} (-1)^{k-|\bar{m}(A^c)|-1} \\
 &\quad \cdot \int_{\overline{\mathcal{M}}_{0;2g_1+2,0}(\mathcal{B}\mathbb{Z}_2)} \lambda_{g_1} \lambda_{g_1-i+k} \psi_1^{i-k-|\bar{m}(A)|-1} \bar{\psi}^{\bar{m}(A)} \psi_l^{m_l} \psi_{l+1}^{m_{l+1}} \\
 &\quad \cdot \int_{\overline{\mathcal{M}}_{0;2g_2+2,0}(\mathcal{B}\mathbb{Z}_2)} \lambda_{g_2} \lambda_{g_2-k} \psi_1^{k-|\bar{m}(A^c)|-1} \bar{\psi}^{\bar{m}(A^c)};
 \end{aligned}$$

here we sum only over subsets A that keep the powers of ψ -classes nonnegative. The subset A determines which ψ -classes map to 0, and the binomial coefficient corresponds to the number of ways of distributing the marked points with no corresponding ψ -class in the integral.

Now write $\bar{n}_{A,k}$ for the multi-index $\bar{m}(A^c)[k - |\bar{m}(A^c)| - 1]$. The vanishing of the integral and the foregoing computations yield the following relation:

$$\begin{aligned}
 L(g, i, \bar{m}) &= 2 \sum_{g_1} \sum_{k=1}^{i-1} \sum_{A \subseteq \{2, \dots, l-1\}} \binom{2g+1-l}{2g_1+1-|A|} (-1)^{k-|\bar{m}(A^c)|} \\
 &\quad \cdot L(g_1, i-k, \bar{m} - \bar{n}_{A,k}) \cdot L(g_2, k, \bar{n}_{A,k}). \tag{7}
 \end{aligned}$$

Evaluating the auxiliary integral for all genera and then packaging (7) in generating function form, we have

$$\begin{aligned}
 &\frac{d^{l-1}}{dx^{l-1}} \mathcal{L}_{i, \bar{m}} \\
 &= 2 \sum_{k=1}^{i-1} \sum_{A \subseteq \{2, \dots, l-1\}} (-1)^{k-|\bar{m}(A^c)|} \left(\frac{d^{l-1-|A|}}{dx^{l-1-|A|}} \mathcal{L}_{i-k, \bar{m} - \bar{n}_{A,k}} \right) \left(\frac{d^{|A|}}{dx^{|A|}} \mathcal{L}_{k, \bar{n}_{A,k}} \right). \tag{8}
 \end{aligned}$$

The second integral leads to a similar relation:

$$\begin{aligned}
 &\frac{d^{l-1}}{dx^{l-1}} \mathcal{L}(i, \bar{m}') \\
 &= 2 \sum_{k=1}^{i-1} \sum_{A \subseteq \{2, \dots, l-1\}} (-1)^{k-|\bar{m}'(A^c)|} \left(\frac{d^{l-1-|A|}}{dx^{l-1-|A|}} \mathcal{L}_{i-k, \bar{m}' - \bar{n}'_{A,k}} \right) \left(\frac{d^{|A|}}{dx^{|A|}} \mathcal{L}_{k, \bar{n}'_{A,k}} \right). \tag{9}
 \end{aligned}$$

By definition, $\bar{n}_{A,k} = \bar{n}'_{A,k}$; hence

$$\frac{d^{|A|}}{dx^{|A|}} \mathcal{L}_{k, \bar{n}_{A,k}} = \frac{d^{|A|}}{dx^{|A|}} \mathcal{L}_{k, \bar{n}'_{A,k}}. \tag{10}$$

Also, the induction hypothesis implies (because $k \geq 1$) that

$$\frac{d^{l-1-|A|}}{dx^{l-1-|A|}} \mathcal{L}_{i-k, \bar{m}-\bar{n}_{A,k}} = \frac{(m_l + m_{l+1})!}{m_l! m_{l+1}!} \frac{d^{l-1-|A|}}{dx^{l-1-|A|}} \mathcal{L}_{i-k, \bar{m}'-\bar{n}'_{A,k}}. \tag{11}$$

Thus the left-hand sides of (8) and (9) are term-by-term proportional; therefore,

$$\frac{d^{l-1}}{dx^{l-1}} \mathcal{L}_{i, \bar{m}} = \frac{(m_l + m_{l+1})!}{m_l! m_{l+1}!} \frac{d^{l-1}}{dx^{l-1}} \mathcal{L}_{i, \bar{m}'}. \tag{12}$$

Now recall that $l(\bar{m}) = l + 1$ and so, in order for $\int \lambda_g \lambda_{g-i} \bar{\psi}^{\bar{m}}$ to be defined, we must have at least $l + 1$ marked points in our moduli space. Hence obtaining a nontrivial integral requires that $2g + 2 \geq l + 1$. All coefficients of monomials of smaller degree than x^{l-1} in both generating functions vanish, so we conclude that

$$\begin{aligned} \mathcal{L}_{i, \bar{m}} &= \frac{(m_l + m_{l+1})!}{m_l! m_{l+1}!} \mathcal{L}_{i, \bar{m}'} \\ &= \frac{(m_l + m_{l+1})!}{m_l! m_{l+1}!} \binom{m_1 + \dots + m'_l}{m_1, \dots, m'_l} \frac{2^{i-1}}{i!} \mathcal{L}_{1, \emptyset}^i \\ &= \binom{m_1 + \dots + m_{l+1}}{m_1, \dots, m_{l+1}} \frac{2^{i-1}}{i!} \mathcal{L}_{1, \emptyset}^i; \end{aligned} \tag{13}$$

here we again use the induction hypothesis on the second equality. □

All $\mathcal{L}_{i, \bar{m}}$ can be further packaged in one jumbo generating function (with infinitely many symmetric variables q_i keeping track of all possible descendant insertions):

$$\mathcal{L}(x, \bar{q}) := \sum_{i, \bar{m}} \mathcal{L}_{i, \bar{m}} \bar{q}^{\bar{m}}. \tag{14}$$

COROLLARY 2.6.

$$\mathcal{L} = \frac{1}{2 \sum q_i} \exp\left(\left(2 \sum q_i\right) \mathcal{L}_{1, \emptyset}\right) = \frac{1}{2 \sum q_i} \sec^{2 \sum q_i} \left(\frac{x}{2}\right). \tag{15}$$

Proof. The first equality follows immediately from Theorem 2.3. The second is obtained by replacing $\mathcal{L}_{1, \emptyset}$ with the RHS of (4). □

3. Open Gromov–Witten Invariants of $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$

In this section we compute the open GW invariants of $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. We give the space a \mathbb{C}^* -action with (Calabi–Yau) weights as in Figure 4.

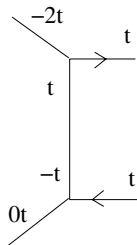


Figure 4 The web diagram for $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ along with the specialized toric weights

In local coordinates at the top vertex, the action is defined by $\lambda \cdot (z, u, v) = (\lambda \cdot z, \lambda^{-2} \cdot u, \lambda \cdot v)$, and similarly for the bottom vertex. The \mathbb{C}^* fixed maps are easily described and understood.

- The source curve consists of a genus-0 (possibly nodal) closed curve along with attached disks.
- The noncontracted irreducible components of the closed curve must be multiple covers of the torus-invariant \mathbb{P}^1 .
- The disks must map to the fixed fibers of the trivial bundle with prescribed windings at the Lagrangians.

Analyzing the obstruction theory via the normalization sequence of the source curve, one sees that the 0 weight at the bottom vertex limits the possible contributing maps in the following ways.

- Maps with positive-dimensional components contracting to the bottom vertex do not contribute.
- Maps with nodes mapping to the bottom vertex contribute only if the node connects a d -fold cover of the invariant \mathbb{P}^1 to a disk with winding d .

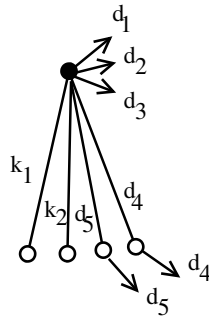


Figure 5 The open localization graphs have bi-colored vertices to keep track of the components to which each vertex contracts; labeled arrows represent disks mapping with the indicated winding

Fixed loci F_Γ are indexed by localization graphs as in Figure 5. The combinatorial data is given by three multi-indices as follows.

- k_1, \dots, k_l , the degrees of the multiple covers of the invariant \mathbb{P}^1 that do not attach to a disk at the bottom vertex.
- d_1, \dots, d_m , the winding profile of the disks with origin mapping to the top vertex.
- d_{m+1}, \dots, d_n , the winding profile of the disks with origin mapping to the bottom vertex. Equivalently, if $n > 1$ then these are the degrees of the multiple covers of the invariant \mathbb{P}^1 that do attach to a disk at the bottom vertex.

If $n = 1$, then maps from a single disk may map the origin to the bottom vertex; we label the locus of such maps Γ' .

With the given multi-indices, the fixed locus F_Γ is isomorphic to a finite quotient of $\overline{\mathcal{M}}_{0,n+1}$ (where we interpret $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ as points). Define the contribution from a fixed locus Γ to be

$$\text{OGW}(\Gamma) := \int_{F_\Gamma} \frac{i^*[\overline{\mathcal{M}}]^{\text{vir}}}{e(N_{\text{vir}})}, \tag{16}$$

where $i^*[\overline{\mathcal{M}}]^{\text{vir}}$ is the restriction of the virtual fundamental class (proposed in [KaL]) to the fixed locus and N_{vir} denotes the virtual normal bundle of F_Γ in the moduli space of stable maps.

In order to package the invariants in the Gromov–Witten potential, we assign the following formal variables:

- q tracks the degree of the map on the base \mathbb{P}^1 ;
- $y_i^{(t)}$ tracks the number of disks with winding i at the top vertex;
- $y_i^{(b)}$ tracks the number of disks with winding i at the bottom vertex;
- x tracks insertions of the nontrivial cohomology class (conveniently, this class is a divisor).

The open potential is computed by adding the contributions of all fixed loci:

$$\begin{aligned} \text{OGW}_{\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}}(x, q, y_i^{(t)}, y_i^{(b)}) &= \sum_{\Gamma'} \text{OGW}(\Gamma') y_d^{(b)} \\ &+ \sum_{\Gamma \neq \Gamma'} \text{OGW}(\Gamma) (qe^x)^{k+d_{m+1}+\dots+d_n} y_{d_1}^{(t)} \dots y_{d_m}^{(t)} y_{d_{m+1}}^{(b)} \dots y_{d_n}^{(b)}. \end{aligned} \tag{17}$$

In (17), Γ' denotes graphs consisting of a single white vertex and an arrow labeled with winding d . For nondegenerate graphs $\Gamma \neq \Gamma'$, we denote by $\text{OGW}(\Gamma)$ the contribution to the potential from the fixed locus indexed by Γ , including invariants with any number of divisor insertions. Following the obstruction theory for open invariants proposed in [KaL], the $\text{OGW}(\Gamma)$ are computed using the following ingredients: the Euler class of the push–pull of the tangent bundle, the Euler class of the normal bundle of F_Γ in the moduli space of stable maps, and all relevant automorphisms of the map

$$\frac{1}{|\text{glob. aut.}|} \int_{F_\Gamma} \frac{e(-R^*\pi_*f^*T_{\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}}) \cdot (\text{inf. aut.})}{(\text{smoothing of nodes})} \tag{18}$$

For convenience, we organize the computation on each locus Γ into three parts: closed curves, disks, and nodes.

1. *Closed curve*: This consists of a closed curve contracting to the upper vertex as well as multiple covers of the torus-fixed \mathbb{P}^1 . We choose not to include the d -covers of the fixed line that are attached to a disk mapping with winding d to the bottom vertex. The contracted component contributes $(-2t^3)^{-1}$ from the push–pull of the tangent bundle, and each k -cover contributes

$$\frac{-t}{k^2} \frac{eH^1(\mathcal{O}(-2k))}{eH^0(\mathcal{O})eH^0(\mathcal{O}(2k))} = \frac{(-1)^k}{tk^2} \binom{2k-1}{k}. \tag{19}$$

Here we have included both the global automorphism of the $k : 1$ cover and the infinitesimal automorphism at the point ramified over the bottom vertex.

2. *Disks*: A disk can be mapped either to the top or the bottom vertex. Following Katz and Liu [KaL], the contribution of a disk mapping to the top vertex with winding d is given by

$$\frac{1}{d} \frac{eH^1(N(d))}{eH^0(L(2d))} = \frac{(-1)^{d+1}}{td} \binom{2d-1}{d}, \tag{20}$$

where $L(2d)$ and $N(d)$ are defined in [KaL, Exm. 3.4.3, Exm. 3.4.4]. We have divided the contributions in a way that the contribution of a disk mapping to the bottom vertex also includes the contribution of the multiple cover attaching it to the contracted component. The reason for this is that the combined contribution becomes

$$\begin{aligned} & \frac{1}{d^2} \frac{eH^1(\mathcal{O}(-2d))}{eH^0(\mathcal{O})eH^0(\mathcal{O}(2d))} \frac{eH^1(N(d))}{eH^0(L(2d))} \frac{eH^0(N_{y/Y})}{t/d - t/d} \\ &= \frac{1}{d^2} \frac{(-1)^{d+1}}{t^2} \binom{2d-1}{d} \frac{1}{t} \frac{-0t^3}{t/d - t/d} \\ &= \frac{(-1)^{d+1}}{td} \binom{2d-1}{d}, \end{aligned} \tag{21}$$

which is the same as the contribution of the disk at the top vertex.

REMARK 3.1. In order to interpret the quotient $\frac{0}{i-1}$ in (21), recall that this term arises as $\frac{s_1 s_2 s_3}{s_1 + s_2}$, where the s_i sum to 0. As $s_3 \rightarrow 0$, the quotient tends to $-s_1 s_2$.

3. *Nodes:* Since we have already accounted for the nodes at the bottom vertex (those attaching winding- d disks to $d : 1$ covers), this piece contains only the contribution from nodes at the top vertex. For each such node connecting either a disk of winding d or a curve of degree d to the contracted component, we get a contribution of $-2t^3$ from the push-pull of the tangent sheaf and a contribution of $(t/d - \psi_i)^{-1}$ from node smoothing.

Putting these three parts together yields

$$\begin{aligned} \text{OGW}(\Gamma) &= \frac{1}{|\text{Aut}(\Gamma)|} \prod_{i=1}^l \frac{(-1)^{k_i}}{tk_i^2} \binom{2k_i-1}{k_i} \prod_{i=1}^n \frac{(-1)^{d_i+1}}{td_i} \binom{2d_i-1}{d_i} \\ &\cdot (-2t^3)^{l+n-1} \int_{\overline{M}_{0,n+l}} \frac{1}{\prod (\frac{t}{k_i} - \psi_i) \prod (\frac{t}{d_i} - \psi_{i+l})}, \end{aligned} \tag{22}$$

where $\text{Aut}(\Gamma)$ is the product of the automorphisms of the ordered tuples (k_1, \dots, k_l) , (d_1, \dots, d_m) , and (d_{m+1}, \dots, d_n) .

After we apply the string equation to the integral and then simplify, (22) becomes

$$\begin{aligned} \text{OGW}(\Gamma) &= \frac{-2^{l+n-1}}{|\text{Aut}(\Gamma)|} \left[\prod_{i=1}^l \frac{(-1)^{k_i+1}}{k_i} \binom{2k_i-1}{k_i} \right] \\ &\cdot \left[\prod_{i=1}^n (-1)^{d_i} \binom{2d_i-1}{d_i} \right] (d+k)^{l+n-3}, \end{aligned} \tag{23}$$

where $d = \sum d_i$ and $k = \sum k_i$.

Recall now that the contribution of a disk is the same regardless of whether it maps to the top or bottom Lagrangian. Therefore, letting $\Gamma(\vec{d}; \vec{k})$ denote all $\Gamma \neq \Gamma'$ with winding profile $\vec{d} = (d_1, \dots, d_n)$ and fixed $\vec{k} = (k_1, \dots, k_l)$, we can attach the formal variables and compute

$$\begin{aligned} & \sum_{\Gamma \in \Gamma(\vec{d}; \vec{k})} \overline{\text{OGW}}(\Gamma) \\ &= \frac{-2^{l+n-1}}{|\text{Aut}(\vec{d})|} \prod_{i=1}^n (y_{d_i}^{(t)} + y_{d_i}^{(b)}(qe^x)^{d_i}) \prod_{i=1}^n (-1)^{d_i} \binom{2d_i - 1}{d_i} \\ & \cdot \frac{1}{|\text{Aut}(\vec{k})|} \prod_{i=1}^l \frac{(-1)^{k_i+1}(qe^x)^{k_i}}{k_i} \binom{2k_i - 1}{k_i} (d+k)^{l+n-3}. \end{aligned} \tag{24}$$

We now sum over all \vec{k} with $\sum k_i = k$. In order to do this, set

$$\begin{aligned} F(X, Y) &:= \exp\left(\sum_{\kappa \geq 1} \frac{(-1)^{\kappa+1}}{\kappa} \binom{2\kappa - 1}{\kappa} X^\kappa Y\right) \\ &= \sum_{l, k} \sum_{\vec{k}} \frac{1}{|\text{Aut}(\vec{k})|} \left[\prod_{i=1}^l \frac{(-1)^{k_i+1}}{k_i} \binom{2k_i - 1}{k_i} \right] X^k Y^l, \end{aligned} \tag{25}$$

where the second sum is over all l -tuples $\vec{k} = (k_1, \dots, k_l)$ with $\sum k_i = k$. The sum of all contributions with fixed winding profile (d_1, \dots, d_n) and with (k_1, \dots, k_l) satisfying $\sum k_i = k$ is obtained by specializing $Y = 2(d+k)$ and then multiplying the coefficient of X^k by an appropriate factor:

$$\begin{aligned} \sum_{|\vec{k}|=k} \sum_{\Gamma \in \Gamma(\vec{d}; \vec{k})} \overline{\text{OGW}}(\Gamma) &= \frac{-2^{n-1}}{|\text{Aut}(\vec{d})|} \prod_{i=1}^n (y_{d_i}^{(t)} + y_{d_i}^{(b)}(qe^x)^{d_i}) \prod_{i=1}^n (-1)^{d_i} \binom{2d_i - 1}{d_i} \\ & \cdot (qe^x)^k (d+k)^{n-3} [F(X, 2(d+k))]_{X^k}. \end{aligned} \tag{26}$$

To handle (26), we find a closed-form expression for F . Start with the known generating function

$$\sum_{k \geq 1} \binom{2k - 1}{k} (-1)^k X^k = \frac{1}{2} \cdot \frac{1 - \sqrt{1 + 4X}}{\sqrt{1 + 4X}}. \tag{27}$$

If we divide by $-X$ and formally integrate term by term (while requiring that the constant term be 0), we obtain

$$\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \binom{2k - 1}{k} X^k = \ln\left(\frac{1}{2}(1 + \sqrt{1 + 4X})\right). \tag{28}$$

Finally, we can write

$$\begin{aligned} F &= \exp\left(Y \ln\left(\frac{1}{2}(1 + \sqrt{1 + 4X})\right)\right) \\ &= \left[\frac{1}{2}(1 + \sqrt{1 + 4X})\right]^Y. \end{aligned} \tag{29}$$

Several remarks are worth making at this point, as follow.

- Setting $G := \frac{1}{2}(1 + \sqrt{1 + 4X})$, we see that $G = 1 + X \cdot C(X)$ for $C(X)$ the generating function for the Catalan numbers.
- G satisfies the recursive relation $G^n = G^{n-1} + XG^{n-2}$.
- It is easy to see that the recursion and the relation between G and the Catalan numbers are equivalent to the array of coefficients of G^i taking on a slight variation of two classical combinatorial objects, as illustrated in Figure 6. Here “slight variation” is probably best described by looking at the first few terms in Table 1.

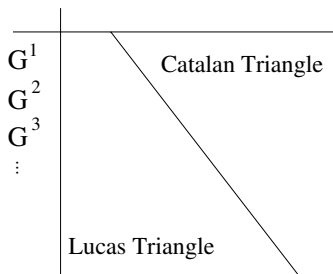


Figure 6 The coefficients of G^n as classical combinatorial numbers

Using the recursion and induction, one easily proves the following lemma.

LEMMA 3.2. *If $d > 0$, then the X^k -coefficient of $G^{2(d+k)}$ is*

$$\binom{k + (2d - 1)}{2d - 1} \frac{d + k}{d}. \tag{30}$$

The X^k -coefficient of G^{2k} is 2.

These are precisely the coefficients we need. We can therefore draw the following conclusions.

- By equation (26), if $(d_1, \dots, d_n) \neq \emptyset$ then

$$\begin{aligned} & \sum_{|\bar{k}|=k} \sum_{\Gamma \in \Gamma(\bar{d}; \bar{k})} \overline{\text{OGW}}(\Gamma) \\ &= \frac{-2^{n-1}}{d \cdot |\text{Aut}(\bar{d})|} \prod_{i=1}^n (y_{d_i}^{(r)} + y_{d_i}^{(b)}(qe^x)^{d_i}) \\ & \quad \cdot \prod_{i=1}^n (-1)^{d_i} \binom{2d_i - 1}{d_i} \sum_{k \geq 0} (d + k)^{n-2} \binom{k + (2d - 1)}{2d - 1} (qe^x)^k. \end{aligned} \tag{31}$$

- Also by (26), if $(d_1, \dots, d_n) = \emptyset$ and $(k_1, \dots, k_l) \neq \emptyset$ then

$$\sum_{|\bar{k}|=k} \sum_{\Gamma \in \Gamma(\emptyset; \bar{k})} \overline{\text{OGW}}(\Gamma) = \frac{-1}{k^3} (qe^x)^k. \tag{32}$$

Here we have recovered the Aspinwall–Morrison formula for $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.

Table 1 First Coefficients of the Series of G^n

	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
G	1	1	-1	2	-5	14	-42	132	-429	1430
G^2	1	2	-1	2	-5	14	-42	132	-429	1430
G^3	1	3	0	1	-3	9	-28	90	-297	1001
G^4	1	4	2	0	-1	4	-14	48	-165	572
G^5	1	5	5	0	0	1	-5	20	-75	275
G^6	1	6	9	2	0	0	-1	6	-27	110
G^7	1	7	14	7	0	0	0	1	-7	35
G^8	1	8	20	16	2	0	0	0	-1	8
G^9	1	9	27	30	9	0	0	0	0	1
G^{10}	1	10	35	50	25	2	0	0	0	0

Finally, recall that:

- if both $\bar{d} = \emptyset$ and $\bar{k} = \emptyset$, then the locus consists of the degree-0 maps with only divisor insertions that can be computed via localization to be

$$\frac{-x^3}{12}; \tag{33}$$

- the contribution from a locus Γ' consisting of a single disk mapping to the bottom vertex with winding d is given by

$$\frac{1}{d^2} y_d^{(b)}. \tag{34}$$

After adding all contributions, we conclude that

$$\begin{aligned} & \text{OGW}_{\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}}(x, q, y_i^{(t)}, y_i^{(b)}) \\ &= \frac{-1}{2} \frac{x^3}{3!} + \sum_{k \geq 1} \frac{-1}{k^3} (qe^x)^k + \sum_{d \geq 1} \frac{1}{d^2} y_d^{(b)} \\ &+ \sum_{(d_1, \dots, d_n) \neq \emptyset} \left[\frac{-2^{n-1}}{d \cdot |\text{Aut}(\bar{d})|} \prod_{i=1}^n (y_{d_i}^{(t)} + y_{d_i}^{(b)} (qe^x)^{d_i}) \prod_{i=1}^n (-1)^{d_i} \binom{2d_i - 1}{d_i} \right. \\ &\quad \left. \cdot \sum_{k \geq 0} (d+k)^{n-2} \binom{k + (2d-1)}{2d-1} (qe^x)^k \right]. \tag{35} \end{aligned}$$

In a neighborhood of $x = -\infty$, we have:

$$\sum_{k \geq 0} \binom{k + (2d-1)}{2d-1} (qe^x)^{d+k} = \frac{(qe^x)^d}{(1 - qe^x)^{2d}}. \tag{36}$$

We can now use (36) to express (35) as

$$\sum_{k \geq 0} (d+k)^{n-2} \binom{k + (2d-1)}{2d-1} (qe^x)^k = \frac{1}{(qe^x)^d} \frac{d^{n-2}}{dx^{n-2}} \left(\frac{(qe^x)^d}{(1 - qe^x)^{2d}} \right), \tag{37}$$

where differentiation/integration is computed formally termwise. When $n \geq 2$, there is no ambiguity since $\frac{d^{n-2}}{dx^{n-2}}$ is a derivative. When $n = 1$, we must exercise some caution because the integral is defined only up to translation. Observe that

$$\lim_{x \rightarrow -\infty} \sum_{k \geq 0} \frac{1}{k+d} \binom{k+(2d-1)}{2d-1} (qe^x)^{k+d} = 0; \tag{38}$$

hence we use

$$\int \frac{(qe^x)^d}{(1-qe^x)^{2d}} dx$$

to denote the antiderivative with limit 0 as x approaches $-\infty$.

We conclude this section by putting the open potential in its simplest form.

PROPOSITION 3.3. *The open Gromov–Witten potential (sans fundamental class insertions) for $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is*

$$\begin{aligned} & \text{OGW}_{\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}}(x, q, y_i^{(t)}, y_i^{(b)}) \\ &= \frac{-1}{12} x^3 + \sum_{k \geq 1} \frac{-1}{k^3} (qe^x)^k \\ &+ \sum_{d \geq 1} \left[\frac{1}{d^2} y_d^{(b)} + \frac{(-1)^{d+1}}{d} (y_d^{(t)} + y_d^{(b)} (qe^x)^d) \binom{2d-1}{d} \right. \\ &\quad \left. \cdot \frac{1}{(qe^x)^d} \int \frac{(qe^x)^d}{(1-qe^x)^{2d}} dx \right] \\ &+ \sum_{d_1, \dots, d_n (n \geq 2)} \left[\frac{-2^{n-1}}{d \cdot |\text{Aut}(\bar{d})|} \left[\prod_{i=1}^n (-1)^{d_i} (y_{d_i}^{(t)} + y_{d_i}^{(b)} (qe^x)^{d_i}) \binom{2d_i-1}{d_i} \right] \right. \\ &\quad \left. \cdot \frac{1}{(qe^x)^d} \frac{d^{n-2}}{dx^{n-2}} \left(\frac{(qe^x)^d}{(1-qe^x)^{2d}} \right) \right]. \end{aligned}$$

The first two lines capture the closed contribution, the next two lines are the contribution from curves with one boundary component, and the final two lines are the contribution from curves with more than one boundary component.

4. Open Orbifold Gromov–Witten Invariants of $[\mathbb{C}^3/\mathbb{Z}_2]$

In this section we compute the open orbifold GW invariants of $[\mathbb{C}^3/\mathbb{Z}_2]$, following [BrC]. We define a \mathbb{C}^* -action on the orbifold with weights described in Figure 7.

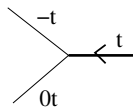


Figure 7 Toric diagram for $[\mathbb{C}^3/\mathbb{Z}_2]$ and \mathbb{C}^* -weights

We characterize the \mathbb{C}^* fixed maps as follows.

- The source curve consists of a genus-0 closed curve along with attached disks. The closed component can carry (possibly twisted) marks, whereas a disk can carry a mark only at the origin (and then only if it is not attached to a closed component). The attaching points of the nodes must carry inverse twisting.
- The closed curve must contract to the vertex.
- The disks must map to the twisted \mathbb{C} with prescribed windings at the Lagrangian.

Since we are working with a \mathbb{Z}_2 quotient, we simply refer to points as twisted or untwisted as there is no ambiguity. A careful analysis of the obstruction theory via the normalization sequence of the source curve shows that the 0 weight conveniently kills all contributions where a disk attaches to a contracted component at an untwisted node. For dimensional reasons, all other marks must be twisted.

Combinatorially, the fixed loci Λ are indexed by

- m , the number of insertions of the twisted sector; and
- d_1, \dots, d_n , the winding profile of the disks.

REMARK 4.1. Since all nodes and marked points are twisted, the maps restricted to the contracted component (maps into $\mathcal{B}\mathbb{Z}_2$) classify double covers of the contracted component with simple ramification over $m + n$ points. Since such a cover exists only if $m + n$ is even, it follows that the loci are nonempty only when $m + n$ is even.

If we let z and w_d be formal variables tracking the twisted sector insertions and the winding d disks, then the open orbifold potential can be computed as

$$\text{OGW}_{[\mathbb{C}^3/\mathbb{Z}_2]}(z, w_i) = \sum_{\Lambda} \text{OGW}(\Lambda) \frac{z^m}{m!} w_{d_1} \cdots w_{d_n}. \tag{39}$$

As before, we group the computation of $\text{OGW}(\Lambda)$ into three components.

1. *A closed curve:* The closed curve contracted to the vertex essentially carries the information of a map into $\mathcal{B}\mathbb{Z}_2$ along with the weights of the \mathbb{C}^* -action on the three normal directions. This classifies a double cover of the source curve. Analogously to [CaC, Sec. 2.1], the contribution from the closed component is the equivariant Euler class of two copies of the dual of the Hodge bundle on the cover twisted by the weights of the action on the untwisted fixed fibers:

$$e(\mathbb{E}_{-1}^{\vee}(-1) \oplus \mathbb{E}_{-1}^{\vee}(0)). \tag{40}$$

We also get a contribution of t^{-1} from the weight of the action on the twisted sector.

2. *Disks:* The disk contribution is laid out in [BrC, Sec. 2.2.3]. This contribution is a combinatorial function that depends on the winding at the Lagrangian and the twisting at the origin of the disk. The localization step simplifies the disk contribution to two cases: either the origin of the disk is marked and twisted (possibly a node) or the origin is unmarked. For the particular case at hand, a disk with winding d and with twisting at the origin contributes

$$\frac{1}{2d} \frac{(2d - 1)!!}{(2d)!!} \tag{41}$$

whereas a disk with no mark and no twisting at the origin contributes

$$\frac{1}{2d^2}. \tag{42}$$

3. *Nodes*: We consider the nodes that attach a winding- d disk to the closed component. Each node gets a t from the weight of the action on the twisted sector. Smoothing the node contributes $(t/2d - \psi_i/2)^{-1}$.

Putting together the three parts just described, we find that

$$\begin{aligned} \text{OGW}(\Lambda) &= \frac{1}{|\text{Aut}(\bar{d})|} \left[\prod_{i=1}^n \frac{1}{2d} \frac{(2d_i - 1)!!}{(2d_i)!!} \right] \int (2)^n t^{n-1} \frac{e^{\text{eq.}(\mathbb{E}_{-1}^{\vee}(-1) \oplus \mathbb{E}_{-1}^{\vee}(0))}}{\prod_{i=1}^n (t/d_i - \psi_i)} \\ &= \frac{1}{|\text{Aut}(\bar{d})|} \left[\prod_{i=1}^n \frac{(2d_i - 1)!!}{(2d_i)!!} \right] \sum_{i=1}^{g-1} \sum_{|\bar{j}|=i-1} \int \lambda_g \lambda_{g-i} (\bar{d}\bar{\psi})^{\bar{j}}, \end{aligned}$$

where the integral is taken over $\bar{\mathcal{M}}_{0,m+n,0}(\mathcal{B}\mathbb{Z}_2)$, $g = \frac{m+n-2}{2}$ (the genus of the cover of the closed curve), and $(\bar{d}\bar{\psi})^{\bar{j}}$ and $|\bar{j}|$ are as defined in Section 2.3.

Summing over all m (equivalently g) and specializing $q_i = d_i$ in Theorem 2.6, we see that the contribution to the open potential from all maps with a fixed winding profile d_1, \dots, d_n is given by

$$\frac{1}{|\text{Aut}(\bar{d})|} \left[\prod_{i=1}^n \frac{(2d_i - 1)!!}{(2d_i)!!} \right] \frac{d^{n-2} \sec^{2d}(z/2)}{dz^{n-2} 2d}. \tag{43}$$

There is no ambiguity for $n \geq 2$, but we must again be careful when $n < 2$.

If $n = 1$ then formula (43) still holds; however, since integrals are defined only up to translation, we must make sure to get the correct constant term. The constant term corresponds to the contribution from maps with one boundary component and no marked points. The only type of map in the fixed locus that satisfies this criterion is a disk with no marked points mapping with winding d . We've seen that the contribution from such a map is $1/2d^2$.

When $n = 0$, we must compute the closed contribution. The maps must have at least three marked points to be stable, but any map into $\mathcal{B}\mathbb{Z}_2$ must have an even number of twisted points (see Remark 4.1). Because there are no disk or node smoothing factors, the contribution is

$$H(z) = \sum_{g \geq 1} \int_{\bar{\mathcal{M}}_{0,2g+2,0}(\mathcal{B}\mathbb{Z}_2)} \lambda_g \lambda_{g-1} \frac{z^{2g+2}}{(2g+2)!}; \tag{44}$$

hence the now classical $\lambda_g \lambda_{g-1}$ result of Faber and Pandharipande [FaP] implies that $\frac{d^2}{dz^2} H(z) = \log(\sec(z/2))$.

Pulling together everything from this discussion enables us to prove the following result.

PROPOSITION 4.2. *The open orbifold Gromov–Witten potential (sans fundamental class insertions) of $[\mathbb{C}^3/\mathbb{Z}_2]$ is*

$$\begin{aligned} \text{OGW}_{[\mathbb{C}^3/\mathbb{Z}_2]}(z, w_i) &= H(z) + \sum_{d \geq 1} \left(\frac{1}{2d^2} + \frac{(2d-1)!!}{(2d)!!} \int \frac{\sec^{2d}(z/2)}{2d} dz \right) w_d \\ &+ \sum_{d_1, \dots, d_n (n \geq 2)} \frac{1}{|\text{Aut}(\vec{d})|} \left(\prod_{i=1}^n \frac{(2d_i-1)!!}{(2d_i)!!} \right) \left(\frac{d^{n-2}}{dz^{n-2}} \frac{\sec^{2d}(z/2)}{2d} \right) w_{d_1} \cdots w_{d_n}, \end{aligned} \tag{45}$$

where the antiderivative is chosen to vanish at $z = 0$.

5. An Example of the Open Crepant Resolution Conjecture

Now that we have computed the open potentials for $[\mathbb{C}^3/\mathbb{Z}_2]$ and its crepant resolution $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, we show that there is a change of variables that equates the stable terms of the two potentials. We start with the contribution from a given winding profile on the orbifold and then consider all contributions on the resolution with that same winding profile to show that the change of variables equates these contributions. More specifically, we demonstrate the following.

THEOREM 5.1. *Under the change of variables*

$$\begin{aligned} q &\rightarrow -1, \\ x &\rightarrow iz, \\ y_d^{(b)} &\rightarrow \frac{i}{2} w_d, \\ y_d^{(t)} &\rightarrow \frac{i}{2} w_d (-e^{iz})^d, \end{aligned} \tag{46}$$

the open GW potential of $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ analytically continues to the open GW potential of $[\mathbb{C}^3/\mathbb{Z}_2]$ up to unstable terms.

Proof. On the closed portion of the potential, we essentially (up to a harmless weight factor) have the result of [BG, Sec. 3.2].

Now consider the winding- d disk contribution on the resolution:

$$\frac{1}{d^2} y_d^{(b)} + \frac{(-1)^{d+1}}{d} (y_d^{(t)} + y_d^{(b)} (qe^x)^d) \binom{2d-1}{d} \frac{1}{(qe^x)^d} \left[\int \frac{(qe^x)^d}{(1 - qe^x)^{2d}} dx \right]. \tag{47}$$

After the change of variables, (47) becomes

$$\begin{aligned} & \frac{i}{2d^2} w_d + \frac{(-1)^{d+1}}{d} \left(\frac{i}{2} w_d (-e^{iz})^d + \frac{i}{2} w_d (-e^{iz})^d \right) \binom{2d-1}{d} \\ & \quad \cdot \frac{1}{(-e^{iz})^d} \left[i \int \frac{(-e^{iz})^d}{(1+e^{iz})^{2d}} dz \right] \\ & = \frac{i}{2d^2} w_d + \frac{i(-1)^{d+1}}{d} w_d \binom{2d-1}{d} \left[i \int \frac{(-e^{iz})^d}{(1+e^{iz})^{2d}} dz \right] \\ & = \frac{i}{2d^2} w_d + \frac{-i}{d} w_d \binom{2d-1}{d} \cdot \left[i \int \frac{\sec^{2d}(z/2)}{(2^{2d})} dz \right]. \end{aligned}$$

Here we ignore the constant terms in the antiderivatives because they correspond to unstable terms about which we make no claims. Hence we obtain

$$\frac{1}{d} \binom{2d-1}{d} w_d \int \frac{\sec^{2d}(z/2)}{(2^{2d})} dz = \frac{(2d-1)!!}{(2d)!!} w_d \int \frac{\sec^{2d}(z/2)}{2d} dz, \tag{48}$$

the disk potential computed on the orbifold.

Finally, consider a general term in the open potential of the resolution with winding profile d_1, \dots, d_n :

$$\begin{aligned} & \frac{-2^{n-1}}{d \cdot |\text{Aut}(\vec{d})|} \left[\prod_{i=1}^n (-1)^{d_i} (y_{d_i}^{(t)} + y_{d_i}^{(b)} (q e^x)^{d_i}) \binom{2d_i-1}{d_i} \right] \\ & \quad \cdot \frac{1}{(q e^x)^d} \frac{d^{n-2}}{dx^{n-2}} \left(\frac{(q e^x)^d}{(1-q e^x)^{2d}} \right). \tag{49} \end{aligned}$$

Making the change of variables, we obtain

$$\begin{aligned} & \frac{-2^{n-1}}{d \cdot |\text{Aut}(\vec{d})|} (i)^n \prod_{i=1}^n w_{d_i} \binom{2d_i-1}{d_i} \frac{1}{i^{n-2}} \frac{d^{n-2}}{dz^{n-2}} \frac{1}{2d} \sec^{2d} \left(\frac{z}{2} \right) \\ & = \frac{1}{2d \cdot |\text{Aut}(\vec{d})|} \prod_{i=1}^n \frac{w_{d_i}}{2^{2d_i-1}} \binom{2d_i-1}{d_i} \left(\frac{d^{n-2}}{dz^{n-2}} \sec^{2d} \left(\frac{z}{2} \right) \right) \\ & = \frac{1}{|\text{Aut}(\vec{d})|} \prod_{i=1}^n \frac{(2d-1)!!}{(2d)!!} \left(\frac{d^{n-2}}{dz^{n-2}} \frac{\sec^{2d}(\frac{z}{2})}{2d} \right) w_{d_1} \cdots w_{d_n}; \end{aligned}$$

this final expression coincides with the contribution on the orbifold. □

6. Gluing Open Invariants

In this section we develop rules for gluing open GW invariants to obtain closed GW invariants. For nonorbifold invariants, we develop a general rule for gluing invariants from trivalent vertices with any compatible torus actions. For orbifold invariants, we specialize to the case of the \mathbb{Z}_2 quotient with the specific torus action introduced in the previous sections.

6.1. Nonorbifold Gluing

In the spirit of the topological vertex [AKMV], we show in this section that the open invariants defined by Katz and Liu can be glued to obtain closed invariants

of a smooth toric Calabi–Yau 3-fold. Any smooth toric Calabi–Yau 3-fold can be equipped with a \mathbb{C}^* -action so that the three (Calabi–Yau) weights at any vertex of the web diagram sum to 0. The torus action can be lifted to the moduli space of stable maps, and the fixed loci consist of maps that contract components to the vertices and map rational components to the compact edges via multiple covers fully ramified over the vertices. The Gromov–Witten potential is then computed as a sum over contributions coming from these fixed loci.

Placing a Lagrangian along each compact edge of the web diagram, we can “cut” each fixed locus into a locus of open maps at each vertex. In this section, we show that the contribution of the fixed locus to the usual Gromov–Witten potential can be obtained essentially by multiplying the corresponding open Gromov–Witten invariants. The standard procedure for localization computations of Gromov–Witten invariants shows that we need only check that the contribution from a multiple cover of a compact edge can be recovered from the disk contributions on each half-edge. Specifically, we show that the degree- d multiple cover contribution of $\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k - 2)$ can be obtained essentially by multiplying winding- d disk contributions on each of the vertices.

PROPOSITION 6.1. *Closed GW invariants of a smooth toric Calabi–Yau 3-fold are obtained by first computing the open invariants at each vertex and then contracting the winding- d contributions along the edges with a factor of*

$$\begin{aligned} &(-1)^{dk+1}d \quad \text{if the half-edges have the same orientation,} \\ &(-1)^{dk+d}d \quad \text{if the half-edges have opposite orientation.} \end{aligned}$$

Proof. Figure 8 gives arbitrary Calabi–Yau weights for a neighborhood of a general fixed line in a toric Calabi–Yau 3-fold. Assume that k and a are positive. One computes the winding- d disk invariant on the left vertex to be

$$\frac{(-1)^{d+1}}{a^{d-1}d(d!)} \prod_{i=1}^{d-1} (bd - ai) \tag{50}$$

and the winding- d disk invariant on the right vertex to be

$$\frac{1}{a^{d-1}d(d!)} \prod_{i=1}^{d-1} (bd - akd + ai). \tag{51}$$



Figure 8 A general edge in the web diagram of a toric Calabi–Yau 3-fold has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k - 2)$

If we multiply the two disk invariants, the result is

$$\frac{(-1)^{d+1}}{a^{2d-2}d^2(d!)^2} \prod_{i=1}^{d-1} (bd - ai)(bd - akd + ai). \tag{52}$$

We now compare (52) with the contribution to the closed GW potential of $\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k - 2)$ given by a degree- d multiple cover. If $k = 1$ then we immediately obtain the same expression (up to the appropriate sign factor). For $k \geq 2$, the contribution is

$$\frac{(-1)^{d+1}}{a^{2d-2}d(d!)^2} \frac{\prod_{i=1}^{dk-1} (-bd + ai)}{\prod_{i=0}^{d(k-2)} (d(b-a) - ia)}. \tag{53}$$

We can use that

$$bd - ai = d(b - a) - ja \iff i = d + j \tag{54}$$

to write (53) as

$$\begin{aligned} &\frac{(-1)^{d+1}(-1)^{dk-1}}{a^{2d-2}d(d!)^2} \prod_{i=1}^{d-1} (bd - ai) \frac{\prod_{i=d}^{dk-d} (bd - ai)}{\prod_{j=0}^{d(k-2)} (d(b-a) - ja)} \prod_{i=dk-d+1}^{dk-1} (bd - ai) \\ &= \frac{(-1)^{d+1}(-1)^{dk-1}}{a^{2d-2}d(d!)^2} \prod_{i=1}^{d-1} (bd - ai)(bd - a(dk - d + i)). \end{aligned}$$

Reversing the index on the second term in the product yields

$$\frac{(-1)^{d(k+1)}}{a^{2d-2}d(d!)^2} \prod_{i=1}^{d-1} (bd - ai)(bd - akd + ai). \tag{55}$$

Comparing (52) with (55) proves Proposition 6.1 when the half-edges have the same orientation. We conclude the proof by recalling that changing the orientation affects the disk invariants by a factor of $(-1)^{d+1}$. \square

6.2. Orbifold Gluing

Gluing orbifold disks has another level of complexity that arises from the twisting at the ramification points of the multiple covers. Here we simplify the scenario and show that we can glue disk contributions of $[\mathbb{C}^3/\mathbb{Z}_2]$ to obtain multiple cover contributions of $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ when using the weights in Figure 9.

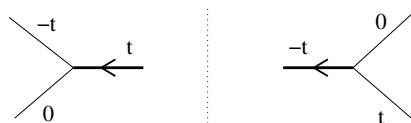


Figure 9 Special weights used to check the gluing of orbifold disk invariants

PROPOSITION 6.2. *Orbifold GW invariants of $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ are obtained by contracting open invariants with the same winding at each vertex and then scaling the orbifold Poincaré pairing by a factor of $(-1)^d d$.*

REMARK 6.3. The last part of this proposition means that, when open invariants with a twisted (resp. untwisted) origin on one side of the edge are multiplied by invariants with a twisted (resp. untwisted) origin on the other side, the product is scaled by $(-1)^d 2d$. Then the two products are added to obtain the total contribution.

Proof of Proposition 6.2. In Section 3 we computed disk invariants for the left vertex. The right vertex with the given weights and orientation gives the same invariants multiplied by a factor of $(-1)^d$. In order to glue two orbifold disk invariants, we must have matching windings and *inverse* twisting at the ramification points. For the \mathbb{Z}_2 case, this means that either both origins are twisted or both are untwisted. The 0 weight at each vertex reduces the circumstance to two cases: either the origins of the disks are marked and twisted; or they are both unmarked (and hence untwisted). If we multiply two winding- d disk invariants that are *twisted* at the origin, we get

$$(-1)^d \left(\frac{1}{2d} \frac{(2d-1)!!}{(2d)!!} \right)^2; \tag{56}$$

if we multiply two winding- d disk invariants that are *untwisted* at the origin, we get

$$(-1)^d \left(\frac{1}{2d^2} \right)^2. \tag{57}$$

We compare (56) and (57) to the contribution of $d : 1$ covers of the twisted \mathbb{P}^1 in the orbifold $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$.

First consider a $d : 1$ cover that is fully ramified over 0 and ∞ with twisted marks at the ramification points. Since f maps into $\mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2$, it classifies a double cover of the source curve fully ramified over the twisted marked points. Pulling back the tangent bundle to this double cover and considering only the weights of \mathbb{Z}_2 -invariant sections, we can compute the contribution as

$$\frac{1}{2d} \frac{eH^1(\mathcal{O}(-2d) \oplus \mathcal{O}(-2d))}{eH^0(\mathcal{O}(2d))} = \frac{1}{2d} \left(\frac{(2d-1)!!}{(2d)!!} \right)^2, \tag{58}$$

where the $2d$ in the denominator corresponds to the global automorphisms of the covers. Now consider a $d : 1$ cover fully ramified over 0 and ∞ with no marked points. Such a map classifies a double cover of the source curve with no ramification (i.e., two disjoint copies of the source curve). If we pull back the tangent bundle to the cover, then the \mathbb{Z}_2 -invariant weights are the weights for one of the disjoint copies. Taking into account global automorphisms and the infinitesimal automorphisms at the ramified points of the source curve, one computes the contribution to be

$$\frac{1}{2d} \frac{eH^1(\mathcal{O}(-d) \oplus \mathcal{O}(-d))}{eH^0(\mathcal{O}(2d))} \frac{t}{d} \frac{-t}{d} = \frac{1}{2d^3}. \tag{59}$$

The proof is concluded by comparing (56) with (58) and (57) with (59). □

7. The Closed Crepant Resolution Conjecture via Gluing

In this section we deduce the Ruan–Bryan–Graber crepant resolution conjecture for the orbifold $\mathfrak{X} = [\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ and its crepant resolution $Y = \mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$ from the results established in previous sections.

We saw in Section 6.2 that there is symmetry in computing open invariants at the two vertices of $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ with the given \mathbb{C}^* -action. In other words, the open potential for the right vertex in Figure 9 can be obtained from the open potential of the left vertex under the change of variables

$$\begin{aligned} z &\rightarrow \tilde{z} \\ w_d &\rightarrow -\tilde{w}_d. \end{aligned}$$

REMARK 7.1. Throughout the rest of this section, variables with a tilde correspond to formal variables on the right sides of the diagrams.

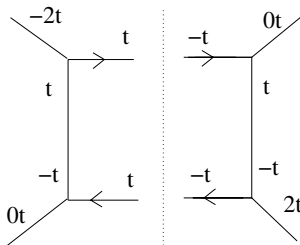


Figure 10 Symmetry in the open potentials of the two open sets of $\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$

Refer to Figure 10 for the resolution. Computing disk invariants for the right half of the diagram with the given orientations and weights leads to the exact same disk invariants computed in Section 3. Therefore, the open potential on the right can be obtained from the open potential on the left by the change of variables

$$\begin{aligned} q &\rightarrow \tilde{q}, \\ x &\rightarrow \tilde{x}, \\ y_d^{(b)} &\rightarrow \tilde{y}_d^{(t)}, \\ y_d^{(t)} &\rightarrow \tilde{y}_d^{(b)}. \end{aligned} \tag{60}$$

The setup for the crepant resolution conjecture is as follows. The Chen–Ruan orbifold cohomology of $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ has two generators in degree 2: the fiber over a point of \mathbb{P}^1 and the class of the constant function on the twisted \mathbb{P}^1 .

We assign the formal variables W and Z to correspond (respectively) to insertions of these classes. Any map into the orbifold is classified by the degree on the twisted \mathbb{P}^1 , so we need only one degree variable P . On the resolution, we have two insertion variables that correspond to the fiber over a point in each \mathbb{P}^1 ; let these be denoted X and Y . We also have two degree variables corresponding to the degree of a map on each \mathbb{P}^1 ; denote them Q and U , where Q corresponds to the \mathbb{P}^1 that is dual to the divisor corresponding to X .

THEOREM 7.2. *Under the change of variables*

$$\begin{aligned} Q &\rightarrow -1, \\ U &\rightarrow -P, \\ X &\rightarrow iZ, \\ Y &\rightarrow iZ + W, \end{aligned} \tag{61}$$

the genus-0 GW potential of $\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$ transforms to the genus-0 GW potential of $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ up to unstable terms.

First we express the two potentials as a sum over the same set of labeled trees. We then describe how one can extract the contribution to the GW potential from each tree by multiplying vertex and edge contributions. The open crepant resolution statement proved in Section 5 verifies that the change of variables equates the vertex contributions and edge contributions.

It is immediate (from the closed computation in Section 5) that the portion of the computation corresponding to degree 0 maps into the orbifold. We therefore focus on contributions with nonzero powers of U and P .

7.1. Closed Invariants of $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$

The closed potential of the orbifold can be expressed as a sum over localization trees:

- black (white) vertices of the tree correspond to components contracting to the left (right) orbifold vertex;
- edges of the tree correspond to multiple covers of the twisted \mathbb{P}^1 obtained by gluing disks, where each edge is labeled with a positive integer denoting the degree of the multiple cover.

By the gluing results of Section 6.2, closed GW invariants of the orbifold are obtained by gluing open invariants along half-edges. For a given localization tree T with more than one edge, the corresponding contribution to the GW potential is given by

$$\text{GW}_{\mathfrak{X}}(T) = \prod_{\text{black vertices}} V(v) \prod_{\text{edges } e} E(e) \prod_{\text{white vertices}} \tilde{V}(v). \tag{62}$$

In this formula, $V(v)$ and $\tilde{V}(v)$ are the open invariants with winding profile corresponding to the edges meeting at v (with the formal variables z and \tilde{z} replaced

by Z). If v is univalent, then only the contribution from disks with twisted origin is taken. The edge contribution is

$$E(e) = \frac{(-1)^d 2d (Pe^W)^d}{w_d \tilde{w}_d}, \tag{63}$$

where e is an edge marked with d . The Pe^W term is from applying the divisor equation to the new divisor class obtained by gluing, and the $(-1)^d 2d$ term is the gluing factor from Section 6.2.

In the case where T' is the tree with a unique edge labeled d , one must also take into account the contribution from gluing two unmarked disks. That contribution in this case is

$$GW_{\tilde{x}}(T') = V(v_1)E(e)\tilde{V}(v_2) + \frac{1}{2d^3} (Pe^W)^d. \tag{64}$$

7.2. Closed Invariants of $\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$

Again, the Gromov–Witten potential is expressed as a sum over localization graphs. For each graph, collapsing all “vertical” edges (i.e. edges corresponding to multiple covers of the vertical fixed fibers) produces essentially a tree as in Section 7.1, with the extra labeling of a subset S of the edges corresponding to edges mapping to the top invariant line. We forget this extra labeling to organize the potential as a sum over the same trees of Section 7.1.

By the results in Section 6.1, the contribution to the GW potential from all loci corresponding to a given labeled tree T is

$$GW_Y(T) = \sum_{S \subset \{\text{edges}\}} \left(\prod_{\text{black vertices}} V^{(S)}(v) \prod_{\text{edges } e} E'(e) \prod_{\text{white vertices}} \tilde{V}^{(S)}(v) \right). \tag{65}$$

In (65), $V^{(S)}(v)$ and $\tilde{V}^{(S)}(v)$ are the open GW contributions from all fixed loci with winding profile determined by the edges meeting v (we replace the formal variables q, \tilde{q} with Q and x, \tilde{x} with X). If an adjacent edge is in S , this corresponds to a disk mapping to the upper Lagrangian (and vice versa). Also,

$$E'(e) = \begin{cases} \frac{-d(Ue^Y)^d}{y_d^{(t)} \tilde{y}_d^{(t)}} & \text{if } e \in S, \\ \frac{-d(Ue^Y)^d}{y_d^{(b)} \tilde{y}_d^{(b)}} & \text{if } e \notin S; \end{cases} \tag{66}$$

here e is an edge labeled with d . The $-d$ term is the gluing factor of Section 6.1, and the Ue^Y term comes from applying the divisor equation to the new divisor class created by gluing.

Let $V'(v)$ and $\tilde{V}'(v)$ denote the open contributions corresponding to all fixed loci with winding profile (d_1, \dots, d_n) given by the edges (e_1, \dots, e_n) meeting v (summing over all possibilities for the disks to map to the top edge or the bottom edge). Using (24), we have:

$$V^{(S)}(v) = \begin{cases} \frac{y_d^{(t)}}{y_d^{(t)} + y_d^{(b)}(Qe^X)^d} \left(V'(v) - \frac{1}{d^2} y_d^{(b)} \right), & v \text{ univalent, } e \in S, \\ \frac{y_d^{(b)}(Qe^X)^d}{y_d^{(t)} + y_d^{(b)}(Qe^X)^d} \left(V'(v) - \frac{1}{d^2} y_d^{(b)} \right) + \frac{1}{d^2} y_d^{(b)}, & v \text{ univalent, } e \notin S, \\ \frac{\prod_{e_i \in S} y_{d_i}^{(t)} \prod_{e_i \notin S} y_{d_i}^{(b)} (Qe^X)^{d_i}}{\prod_{i=1}^n (y_{d_i}^{(t)} + y_{d_i}^{(b)} (Qe^X)^{d_i})} V'(v), & \text{otherwise;} \end{cases}$$

$$\tilde{V}^{(S)}(v) = \begin{cases} \frac{\tilde{y}_d^{(t)}(Qe^X)^d}{\tilde{y}_d^{(t)} + \tilde{y}_d^{(b)}(Qe^X)^d} \left(V'(v) - \frac{1}{d^2} \tilde{y}_d^{(t)} \right) + \frac{1}{d^2} \tilde{y}_d^{(t)}, & v \text{ univalent, } e \in S, \\ \frac{\tilde{y}_d^{(b)}}{\tilde{y}_d^{(t)} + \tilde{y}_d^{(b)}(Qe^X)^d} \left(V'(v) - \frac{1}{d^2} \tilde{y}_d^{(t)} \right), & v \text{ univalent, } e \notin S, \\ \frac{\prod_{e_i \in S} \tilde{y}_{d_i}^{(t)} (Qe^X)^{d_i} \prod_{e_i \notin S} \tilde{y}_{d_i}^{(b)}}{\prod_{i=1}^n (\tilde{y}_{d_i}^{(b)} + \tilde{y}_{d_i}^{(t)} (Qe^X)^{d_i})} V'(v), & \text{otherwise.} \end{cases}$$

REMARK 7.3. In each of these formulas for the vertex contributions, the third case is the generic case and the other two are adjusted to take into account the Γ' loci of (17).

7.3. The Crepant Resolution Transformation

We shall verify the Ruan–Bryan–Graber crepant resolution conjecture by showing that, after the prescribed change of variables,

$$\text{GW}_Y(T) \rightarrow \text{GW}_{\mathfrak{X}}(T) \tag{67}$$

for every labeled tree T .

Even though our formulas for the vertex and edge contributions of $\text{GW}_{\mathfrak{X}}(T)$ and $\text{GW}_Y(T)$ involve winding variables, these variables cancel in the product. Hence we can make any substitution for the winding variables without affecting the overall product. Motivated by the open crepant resolution transformation, in the previous formulas for $\text{GW}_Y(T)$ we make the following substitutions:

$$\begin{aligned}
 y_d^{(b)} &\rightarrow \frac{i}{2} w_d, & \tilde{y}_d^{(b)} &\rightarrow \frac{i}{2} (e^{iZ})^d \tilde{w}_d, \\
 y_d^{(t)} &\rightarrow \frac{i}{2} (-e^{iZ})^d w_d, & \tilde{y}_d^{(t)} &\rightarrow (-1)^d \frac{i}{2} \tilde{w}_d, \\
 Q &\rightarrow -1, & U &\rightarrow -P, \\
 X &\rightarrow iZ & Y &\rightarrow iZ + W.
 \end{aligned}$$

By Theorem 5.1, this change of variables leads to $V'(v) \rightarrow V(v)$ and $\tilde{V}'(v) \rightarrow \tilde{V}(v)$. So for any $S \subset \{\text{edges}\}$, we have:

$$V^{(S)}(v) \rightarrow \begin{cases} \frac{1}{2} V(v) - \frac{i}{4d^2} w_d, & v \text{ univalent, } e \in S, \\ \frac{1}{2} V(v) + \frac{i}{4d^2} w_d, & v \text{ univalent, } e \notin S, \\ \frac{1}{2^n} V(v), & \text{otherwise;} \end{cases} \tag{68}$$

similarly,

$$\tilde{V}^{(S)}(v) \rightarrow \begin{cases} \frac{1}{2}\tilde{V}(v) + \frac{i}{4d^2}\tilde{w}_d, & v \text{ univalent, } e \in S, \\ \frac{1}{2}\tilde{V}(v) - \frac{i}{4d^2}\tilde{w}_d, & v \text{ univalent, } e \notin S, \\ \frac{1}{2^n}\tilde{V}(v), & \text{otherwise.} \end{cases} \quad (69)$$

Under this change of variables we also have

$$E'(e) \rightarrow 2E(e). \quad (70)$$

Given any tree T with more than one edge, the extra terms on the univalent vertices cancel by summing over all contributions $e \in S$ and $e \notin S$. Therefore, from (68)–(70) it follows that

$$\begin{aligned} \text{GW}_Y(T) &= \\ \sum_{S \subset \{\text{edges}\}} \prod \frac{1}{2} V(v) \prod 2E(e) \prod \frac{1}{2} \tilde{V}(v) &= 2^{\#\{\text{edges}\}} \frac{\prod V(v) \prod E(e) \prod \tilde{V}(v)}{2^{\#\{\text{edges}\}}} \\ &= \text{GW}_{\mathfrak{X}}(T). \end{aligned} \quad (71)$$

If T' is the tree with a unique edge labeled d , then

$$\text{GW}_Y(T') = V(v_1)E(e)\tilde{V}(v_2) + \frac{1}{2d^3}(Pe^Y)^d = \text{GW}_{\mathfrak{X}}(T'). \quad (72)$$

Equations (71) and (72) establish Theorem 7.2.

References

- [AKMV] M. Aganagic, A. Klemm, M. Mariño, and C. Vafa, *The topological vertex*, Comm. Math. Phys. 254 (2005), 425–478.
- [AKV] M. Aganagic, A. Klemm, and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, Z. Naturforsch. A 57 (2002), 1–28.
- [BeCT] A. Bertram, R. Cavalieri, and G. Todorov, *Evaluating tautological classes using only Hurwitz numbers*, Trans. Amer. Math. Soc. 360 (2008), 6103–6111.
- [Br] A. Brini, *Open topological strings and integrable hierarchies: Remodeling the A-model*, Comm. Math. Phys. 312 (2012), 735–780.
- [BrC] A. Brini and R. Cavalieri, *Open orbifold Gromov–Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_n]$: Localization and mirror symmetry*, Selecta Math. (N.S.) 17 (2011), 879–933.
- [BCaY] J. Bryan, C. Cadman, and B. Young, *The orbifold topological vertex*, Adv. Math. 229 (2012), 531–595.
- [BGh] J. Bryan and A. Gholampour, *Hurwitz–Hodge integrals, the E_6 and D_4 root systems, and the crepant resolution conjecture*, Adv. Math. 221 (2009), 1047–1068.
- [BG] J. Bryan and T. Graber, *The crepant resolution conjecture*, Algebraic geometry (Seattle, 2005), Proc. Sympos. Pure Math., 80, pp. 23–42, Amer. Math. Soc., Providence, RI, 2009.
- [BGP] J. Bryan, T. Graber, and R. Pandharipande, *The orbifold quantum cohomology of $\mathbb{C}^2/\mathbb{Z}_3$ and Hurwitz–Hodge integrals*, J. Algebraic Geom. 17 (2008), 1–28.

- [BP] J. Bryan and R. Pandharipande, *The local Gromov–Witten theory of curves*, J. Amer. Math. Soc. 21 (2008), 101–136.
- [CaC] C. Cadman and R. Cavalieri, *Gerby localization, Z_3 -Hodge integrals and the GW theory of $[\mathbb{C}^3/Z_3]$* , Amer. J. Math. 131 (2009), 1009–1046.
- [C1] R. Cavalieri, *Hodge-type integrals on moduli spaces of admissible covers*, The interaction of finite type and Gromov–Witten invariants (BIRS 2003), Geom. Topol. Monogr., 8, pp. 167–194, Geom. Topol. Publ., Coventry, 2006.
- [C2] ———, *Generating functions for Hurwitz–Hodge integrals*, Adv. Math. 218 (2008), 1419–1429.
- [CoR] T. Coates and Y. Ruan, *Quantum cohomology and crepant resolutions: A conjecture*, preprint, 2007.
- [DF] D.-E. Diaconescu and B. Florea, *Localization and gluing of topological amplitudes*, Comm. Math. Phys. 257 (2005), 119–149.
- [FaP] C. Faber and R. Pandharipande, *Logarithmic series and Hodge integrals in the tautological ring*, Michigan Math. J. 48 (2000), 215–252.
- [Gi] W. D. Gillam, *Hyperelliptic Gromov–Witten theory*, Ph.D. thesis, Columbia Univ., ProQuest LLC, Ann Arbor, MI, 2008.
- [KaL] S. Katz and M. Liu, *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disk*, Adv. Theor. Math. Phys. 5 (2001), 1–49.
- [Li+] J. Li, C.-C. M. Liu, K. Liu, and J. Zhou, *A mathematical theory of the topological vertex*, Geom. Topol. 13 (2009), 527–621.
- [Ro] D. Ross, *Localization and gluing of orbifold amplitudes, The Gromov–Witten orbifold vertex*, Trans. Amer. Math. Soc. (to appear).

R. Cavalieri
 Department of Mathematics
 Colorado State University
 Fort Collins, CO 80523
 renzo@math.colostate.edu

D. Ross
 Department of Mathematics
 Colorado State University
 Fort Collins, CO 80523
 ross@math.colostate.edu