

Surfaces with Parallel Mean Curvature in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$

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1. Introduction

In 1968, J. Simons discovered a fundamental formula for the Laplacian of the second fundamental form of a minimal submanifold in a Riemannian manifold. He then used this formula to characterize certain minimal submanifolds of a sphere and Euclidean space (see [17]). One year later, K. Nomizu and B. Smyth generalized Simons's equation for hypersurfaces of constant mean curvature (cmc hypersurfaces) in a space form (see [15]). This was extended, in Smyth's work [18], to the more general case of a submanifold with parallel mean curvature vector (pmc submanifold) in a space form. Over the years such equations, called Simons-type equations, turned out to be very useful, and a great number of authors have used them in the study of cmc and pmc submanifolds (see e.g. [3; 10; 16]).

Nowadays, the study of cmc surfaces in Euclidean space and, more generally, in space forms is a classical subject in the field of differential geometry; well-known papers by H. Hopf [13] and S.-S. Chern [9] are representative examples from the literature on this topic. When the codimension is greater than 1, a natural generalization of cmc surfaces are pmc surfaces. These have been intensively studied in the last four decades, and among the first papers devoted to this subject are those by D. Ferus [11], B.-Y. Chen and G. D. Ludden [8], D. A. Hoffman [12], and S.-T. Yau [20]. All results in these papers were obtained in the case when the ambient space is a space form.

The next natural step was taken by U. Abresch and H. Rosenberg, who studied in [1; 2] cmc surfaces and obtained Hopf-type results in product spaces $M^2(c) \times \mathbb{R}$, where $M^2(c)$ is a complete simply connected surface with constant curvature c , as well as in the homogeneous 3-manifolds $\text{Nil}(3)$, $\widetilde{\text{PSL}}(2, \mathbb{R})$, and Berger spheres. Some of their results in [1] were extended to pmc surfaces in product spaces of $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is an n -dimensional space form, by H. Alencar, M. do Carmo, and R. Tribuzy [4; 5].

In a recent paper, M. Batista [7] derived a Simons-type equation involving the traceless part of the second fundamental form of a cmc surface in $M^2(c) \times \mathbb{R}$ and found several applications.

Received April 14, 2011. Revision received May 16, 2012.

The first author was supported by a Postdoctoral Fellowship "Pós-Doutorado Sênior (PDS)", E-26/102.878/2010, offered by FAPERJ, Brazil.

In this paper we compute the Laplacian of the squared norm of the traceless part ϕ of the second fundamental form σ of a pmc surface in a product space $M^3(c) \times \mathbb{R}$; then, using this Simons-type formula, we characterize some of these surfaces. The main results of our paper are the following.

THEOREM 5.2. *Let Σ^2 be an immersed pmc 2-sphere in $M^n(c) \times \mathbb{R}$ such that:*

- (1) $|T|^2 = 0$ or $|T|^2 \geq \frac{2}{3}$ and $|\sigma|^2 \leq c(2 - 3|T|^2)$ if $c < 0$;
- (2) $|T|^2 \leq \frac{2}{3}$ and $|\sigma|^2 \leq c(2 - 3|T|^2)$ if $c > 0$.

Here T is the tangent part of the unit vector ξ tangent to \mathbb{R} . Then Σ^2 is either a minimal surface in a totally umbilical hypersurface of $M^n(c)$ or a standard sphere in $M^3(c)$.

THEOREM 5.3. *Let Σ^2 be an immersed complete nonminimal pmc surface in $\bar{M} = M^3(c) \times \mathbb{R}$, with $c > 0$ and mean curvature vector H . Assume*

- (i) $|\phi|^2 \leq 2|H|^2 + 2c - \frac{5c}{2}|T|^2$ and
- (ii) *either*
 - (a) $|T| = 0$ or
 - (b) $|T|^2 > \frac{2}{3}$ and $|H|^2 \geq c|T|^2(1 - |T|^2)/(3|T|^2 - 2)$.

Then either

- (1) $|\phi|^2 = 0$ and Σ^2 is a round sphere in $M^3(c)$, or
- (2) $|\phi|^2 = 2|H|^2 + 2c$ and Σ^2 is a torus $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1/c - r^2})$, $r^2 \neq 1/2c$, in $M^3(c)$.

ACKNOWLEDGMENTS. The first author would like to thank the IMPA in Rio de Janeiro for providing a very stimulative work environment during the preparation of this paper.

2. Preliminaries

Let $M^n(c)$ be a simply connected n -dimensional manifold with constant sectional curvature c , and consider the product manifold $\bar{M} = M^n(c) \times \mathbb{R}$. The expression of the curvature tensor \bar{R} of such a manifold can be obtained from

$$\langle \bar{R}(X, Y)Z, W \rangle = c\{\langle d\pi Y, d\pi Z \rangle \langle d\pi X, d\pi W \rangle - \langle d\pi X, d\pi Z \rangle \langle d\pi Y, d\pi W \rangle\},$$

where $\pi: \bar{M} = M^n(c) \times \mathbb{R} \rightarrow M^n(c)$ is the projection map. A straightforward computation yields

$$\begin{aligned} \bar{R}(X, Y)Z &= c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y - \langle Y, \xi \rangle \langle Z, \xi \rangle X + \langle X, \xi \rangle \langle Z, \xi \rangle Y \\ &\quad + \langle X, Z \rangle \langle Y, \xi \rangle \xi - \langle Y, Z \rangle \langle X, \xi \rangle \xi\}, \end{aligned} \tag{2.1}$$

where ξ is the unit vector tangent to \mathbb{R} .

Now let Σ^2 be an immersed surface in \bar{M} and denote by R its curvature tensor. Then, from the Gauss equation

$$\langle R(X, Y)Z, W \rangle = \langle \bar{R}(X, Y)Z, W \rangle + \sum_{\alpha=3}^{n+1} \{ \langle A_\alpha Y, Z \rangle \langle A_\alpha X, W \rangle - \langle A_\alpha X, Z \rangle \langle A_\alpha Y, W \rangle \},$$

we obtain

$$\begin{aligned} R(X, Y)Z &= c\{ \langle Y, Z \rangle X - \langle X, Z \rangle Y - \langle Y, T \rangle \langle Z, T \rangle X + \langle X, T \rangle \langle Z, T \rangle Y \\ &\quad + \langle X, Z \rangle \langle Y, T \rangle T - \langle Y, Z \rangle \langle X, T \rangle T \} \\ &\quad + \sum_{\alpha=3}^{n+1} \{ \langle A_\alpha Y, Z \rangle A_\alpha X - \langle A_\alpha X, Z \rangle A_\alpha Y \}; \end{aligned} \quad (2.2)$$

here T is the component of ξ tangent to the surface, and A is the shape operator defined by the Weingarten equation

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V$$

for any vector field X tangent to Σ^2 and any vector field V normal to the surface. Here $\bar{\nabla}$ is the Levi-Civita connection on \bar{M} and ∇^\perp is the connection in the normal bundle; also, $A_\alpha = A_{E_\alpha}$, with $\{E_\alpha\}_{\alpha=3}^{n+1}$ a local orthonormal frame field in the normal bundle.

DEFINITION 2.1. If the mean curvature vector H of the surface Σ^2 is parallel in the normal bundle (i.e., if $\nabla^\perp H = 0$), then Σ^2 is called a *pmc surface*.

We conclude this section by recalling the Omori–Yau maximum principle, which will be used later.

THEOREM 2.2 [21]. *If M is a complete Riemannian manifold with Ricci curvature bounded from below, then for any smooth function $u \in C^2(M)$ with $\sup_M u < +\infty$ there exists a sequence of points $\{p_k\}_{k \in \mathbb{N}} \subset M$ satisfying*

$$\lim_{k \rightarrow \infty} u(p_k) = \sup_M u, \quad |\nabla u|(p_k) < \frac{1}{k}, \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}.$$

3. A Formula for pmc Surfaces in $M^n(c) \times \mathbb{R}$

Let Σ^2 be an immersed surface in $M^n(c) \times \mathbb{R}$ with mean curvature vector H . In this section we prove a formula for the Laplacian of the squared norm of A_V , where V is a normal vector field to the surface, such that V is parallel in the normal bundle; that is, $\nabla^\perp V = 0$ and $\text{trace } A_V = \text{constant}$.

LEMMA 3.1. *If U and V are normal vectors to the surface and if V is parallel in the normal bundle, then $[A_V, A_U] = 0$; in other words, A_V commutes with A_U .*

Proof. The conclusion follows easily from the Ricci equation

$$\langle R^\perp(X, Y)V, U \rangle = \langle [A_V, A_U]X, Y \rangle + \langle \bar{R}(X, Y)V, U \rangle,$$

since $R^\perp(X, Y)V = 0$ and (2.1) implies that $\langle \bar{R}(X, Y)V, U \rangle = 0$. \square

Now, from the Codazzi equation

$$\begin{aligned} \langle \bar{R}(X, Y)Z, V \rangle &= \langle \nabla_X^\perp \sigma(Y, Z), V \rangle - \langle \sigma(\nabla_X Y, Z), V \rangle - \langle \sigma(Y, \nabla_X Z), V \rangle \\ &\quad - \langle \nabla_Y^\perp \sigma(X, Z), V \rangle + \langle \sigma(\nabla_Y X, Z), V \rangle + \langle \sigma(X, \nabla_Y Z), V \rangle \end{aligned}$$

(where σ is the second fundamental form of Σ^2), we get

$$\begin{aligned} \langle \bar{R}(X, Y)Z, V \rangle &= X(\langle A_V Y, Z \rangle) - \langle \sigma(Y, Z), \nabla_X^\perp V \rangle - \langle A_V(\nabla_X Y), Z \rangle \\ &\quad - \langle A_V Y, \nabla_X Z \rangle - Y(\langle A_V X, Z \rangle) + \langle \sigma(X, Z), \nabla_Y^\perp V \rangle \\ &\quad + \langle A_V(\nabla_Y X), Z \rangle + \langle A_V X, \nabla_Y Z \rangle \\ &= \langle (\nabla_X A_V)Y - (\nabla_Y A_V)X, Z \rangle, \end{aligned}$$

since $\nabla^\perp V = 0$. Therefore, we can use (2.1) to obtain

$$(\nabla_X A_V)Y = (\nabla_Y A_V)X + c(V, N)((Y, T)X - \langle X, T \rangle Y), \quad (3.1)$$

where N is the normal part of ξ .

Next, we have

$$\frac{1}{2} \Delta |A_V|^2 = |\nabla A_V|^2 + \langle \nabla^2 A_V, A_V \rangle, \quad (3.2)$$

where we extended the metric $\langle \cdot, \cdot \rangle$ to the tensor space in the standard way. In order to calculate the second term on the right-hand side of (3.2), we shall use a method introduced in [15].

Let us write

$$C(X, Y) = \nabla_X(\nabla_Y A_V) - \nabla_{\nabla_X Y} A_V,$$

and note that the vanishing of the torsion of ∇ and the definition of the curvature tensor R on the surface together imply that

$$C(X, Y) = C(Y, X) + [R(X, Y), A_V]. \quad (3.3)$$

Now consider an orthonormal basis $\{e_1, e_2\}$ in $T_p \Sigma^2$ ($p \in \Sigma^2$), extend e_1 and e_2 to vector fields E_1 and E_2 in a neighborhood of p such that $\nabla E_i = 0$ at p , and let X be a tangent vector field such that $\nabla X = 0$. Obviously, at p we have

$$(\nabla^2 A_V)X = \sum_{i=1}^2 \nabla_{E_i}(\nabla_{E_i} A_V)X = \sum_{i=1}^2 C(E_i, E_i)X.$$

Using equation (3.1) yields, at p ,

$$\begin{aligned} C(E_i, X)E_i &= (\nabla_{E_i}(\nabla_X A_V))E_i - (\nabla_{\nabla_{E_i} X} A_V)E_i \\ &= \nabla_{E_i}((\nabla_X A_V)E_i) - (\nabla_X A_V)(\nabla_{E_i} E_i) \\ &= \nabla_{E_i}((\nabla_{E_i} A_V)X) + c\nabla_{E_i}((V, N)((E_i, T)X - \langle X, T \rangle E_i)); \end{aligned}$$

then

$$\begin{aligned} C(E_i, X)E_i &= (\nabla_{E_i}(\nabla_{E_i} A_V))X + (\nabla_{E_i} A_V)(\nabla_{E_i} X) \\ &\quad - c\langle A_V E_i, T \rangle((E_i, T)X - \langle X, T \rangle E_i) \\ &\quad + c\langle V, N \rangle(\langle A_N E_i, E_i \rangle X - \langle A_N X, E_i \rangle E_i) \\ &= C(E_i, E_i)X - c\langle A_V E_i, T \rangle((E_i, T)X - \langle X, T \rangle E_i) \\ &\quad + c\langle V, N \rangle(\langle A_N E_i, E_i \rangle X - \langle A_N X, E_i \rangle E_i). \end{aligned} \quad (3.4)$$

Here we have used that $\sigma(E_i, T) = -\nabla_{E_i}^\perp N$ and $\nabla_{E_i} T = A_N E_i$, which follow because ξ is parallel.

We also have

$$C(X, E_i)E_i = \nabla_X((\nabla_{E_i} A_V)E_i); \quad (3.5)$$

then (3.3)–(3.5) yield

$$\begin{aligned} C(E_i, E_i)X &= C(E_i, X)E_i + c\langle A_V E_i, T \rangle (\langle E_i, T \rangle X - \langle X, T \rangle E_i) \\ &\quad - c\langle V, N \rangle (\langle A_N E_i, E_i \rangle X - \langle A_N X, E_i \rangle E_i) \\ &= C(X, E_i)E_i + [R(E_i, X), A_V]E_i \\ &\quad + c\langle A_V E_i, T \rangle (\langle E_i, T \rangle X - \langle X, T \rangle E_i) \\ &\quad - c\langle V, N \rangle (\langle A_N E_i, E_i \rangle X - \langle A_N X, E_i \rangle E_i), \end{aligned}$$

which means that

$$\begin{aligned} C(E_i, E_i)X &= \nabla_X((\nabla_{E_i} A_V)E_i) + [R(E_i, X), A_V]E_i \\ &\quad + c\langle A_V E_i, T \rangle (\langle E_i, T \rangle X - \langle X, T \rangle E_i) \\ &\quad - c\langle V, N \rangle (\langle A_N E_i, E_i \rangle X - \langle A_N X, E_i \rangle E_i). \end{aligned}$$

Since A_V is symmetric, it follows that also $\nabla_{E_i} A_V$ is symmetric. Hence by (3.1) we have

$$\begin{aligned} \left\langle \sum_{i=1}^2 (\nabla_{E_i} A_V)E_i, Z \right\rangle &= \sum_{i=1}^2 \langle E_i, (\nabla_{E_i} A_V)Z \rangle = \sum_{i=1}^2 \langle E_i, (\nabla_Z A_V)E_i \rangle \\ &\quad + c\langle V, N \rangle \sum_{i=1}^2 \langle E_i, \langle Z, T \rangle E_i - \langle E_i, T \rangle Z \rangle \\ &= \text{trace}(\nabla_Z A_V) + c\langle V, N \rangle \langle T, Z \rangle \\ &= Z(\text{trace } A_V) + c\langle V, N \rangle \langle T, Z \rangle \\ &= c\langle V, N \rangle \langle T, Z \rangle \end{aligned}$$

for any vector Z that is tangent to Σ^2 , since $\text{trace } A_V = \text{constant}$.

Therefore, at p ,

$$\begin{aligned} (\nabla^2 A_V)X &= \sum_{i=1}^2 C(E_i, E_i)X \\ &= c\nabla_X(\langle V, N \rangle T) + \sum_{i=1}^2 [R(E_i, X), A_V]E_i \\ &\quad + c \sum_{i=1}^2 \langle A_V E_i, T \rangle (\langle E_i, T \rangle X - \langle X, T \rangle E_i) \\ &\quad - c \sum_{i=1}^2 \langle V, N \rangle (\langle A_N E_i, E_i \rangle X - \langle A_N X, E_i \rangle E_i). \end{aligned}$$

Then, since $\bar{\nabla}_X \xi = 0$ implies $\sigma(X, T) = -\nabla_X^\perp N$ and $\nabla_X T = A_N X$, we have

$$\begin{aligned}
(\nabla^2 A_V)X &= \sum_{i=1}^2 [R(E_i, X), A_V]E_i \\
&\quad + c\{2\langle V, N \rangle A_N X - \langle A_V X, T \rangle T + \langle A_V T, T \rangle X \\
&\quad - \langle X, T \rangle A_V T - 2\langle H, N \rangle \langle V, N \rangle X\}. \tag{3.6}
\end{aligned}$$

From the Gauss equation (2.2) of the surface Σ^2 and Lemma 3.1, it now follows that

$$\begin{aligned}
\sum_{i=1}^2 R(E_i, X)A_V E_i &= c \sum_{i=1}^2 \{ \langle X, A_V E_i \rangle E_i - \langle E_i, A_V E_i \rangle X \\
&\quad - \langle X, T \rangle \langle A_V E_i, T \rangle E_i + \langle E_i, T \rangle \langle A_V E_i, T \rangle X \\
&\quad + \langle E_i, A_V E_i \rangle \langle X, T \rangle T - \langle X, A_V E_i \rangle \langle E_i, T \rangle T \} \\
&\quad + \sum_{i=1}^2 \sum_{\alpha=3}^{n+1} \{ \langle A_\alpha X, A_V E_i \rangle A_\alpha E_i - \langle A_\alpha E_i, A_V E_i \rangle A_\alpha X \},
\end{aligned}$$

which means that

$$\begin{aligned}
\sum_{i=1}^2 R(E_i, X)A_V E_i &= c\{A_V X - (\text{trace } A_V)X + (\text{trace } A_V)\langle X, T \rangle T \\
&\quad - \langle A_V X, T \rangle T - \langle X, T \rangle A_V T + \langle A_V T, T \rangle X\} \\
&\quad + \sum_{\alpha=3}^{n+1} \{A_V A_\alpha^2 X - (\text{trace}(A_V A_\alpha))A_\alpha X\}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^2 A_V R(E_i, X)E_i &= c \sum_{i=1}^2 \{ \langle X, E_i \rangle A_V E_i - \langle E_i, E_i \rangle A_V X \\
&\quad - \langle X, T \rangle \langle E_i, T \rangle A_V E_i + \langle E_i, T \rangle \langle E_i, T \rangle A_V X \\
&\quad + \langle E_i, E_i \rangle \langle X, T \rangle A_V T - \langle X, E_i \rangle \langle E_i, T \rangle A_V T \} \\
&\quad + \sum_{i=1}^2 \sum_{\alpha=3}^{n+1} \{ \langle A_\alpha X, E_i \rangle A_V A_\alpha E_i - \langle A_\alpha E_i, E_i \rangle A_V A_\alpha X \} \\
&= -c(1 - |T|^2)A_V X + \sum_{\alpha=3}^{n+1} \{A_V A_\alpha^2 X - (\text{trace } A_\alpha)A_V A_\alpha X\}.
\end{aligned}$$

Finally, replacing in equation (3.6), we find

$$\begin{aligned}
(\nabla^2 A_V)X &= c\{(2 - |T|^2)A_V X + 2\langle A_V T, T \rangle X - 2\langle A_V X, T \rangle T - 2\langle X, T \rangle A_V T \\
&\quad + 2\langle V, N \rangle A_N X - 2\langle H, N \rangle \langle V, N \rangle X \\
&\quad - (\text{trace } A_V)X + (\text{trace } A_V)\langle X, T \rangle T\} \\
&\quad + \sum_{\alpha=3}^{n+1} \{(\text{trace } A_\alpha)A_V A_\alpha X - (\text{trace}(A_V A_\alpha))A_\alpha X\};
\end{aligned}$$

a straightforward computation then yields

$$\begin{aligned} \langle \nabla^2 A_V, A_V \rangle &= \sum_{i=1}^2 \langle (\nabla^2 A_V) E_i, A_V E_i \rangle \\ &= c\{2 - |T|^2\} |A_V|^2 - 4|A_V T|^2 + 3(\text{trace } A_V) \langle A_V T, T \rangle \\ &\quad + 2(\text{trace}(A_N A_V)) \langle V, N \rangle - (\text{trace } A_V)^2 \\ &\quad - 2(\text{trace } A_V) \langle H, N \rangle \langle V, N \rangle \} \\ &\quad + \sum_{\alpha=3}^{n+1} \{(\text{trace } A_\alpha)(\text{trace}(A_V^2 A_\alpha)) - (\text{trace}(A_V A_\alpha))^2\}. \end{aligned}$$

Thus, from (3.2) we obtain the following result.

PROPOSITION 3.2. *Let Σ^2 be an immersed surface in $M^n(c) \times \mathbb{R}$. If V is a normal vector field, parallel in the normal bundle and with $\text{trace } A_V = \text{constant}$, then*

$$\begin{aligned} \frac{1}{2} \Delta |A_V|^2 &= |\nabla A_V|^2 + c\{2 - |T|^2\} |A_V|^2 - 4|A_V T|^2 + 3(\text{trace } A_V) \langle A_V T, T \rangle \\ &\quad + 2(\text{trace}(A_N A_V)) \langle V, N \rangle - (\text{trace } A_V)^2 \\ &\quad - 2(\text{trace } A_V) \langle H, N \rangle \langle V, N \rangle \} \\ &\quad + \sum_{\alpha=3}^{n+1} \{(\text{trace } A_\alpha)(\text{trace}(A_V^2 A_\alpha)) - (\text{trace}(A_V A_\alpha))^2\}, \quad (3.7) \end{aligned}$$

where $\{E_\alpha\}_{\alpha=3}^{n+1}$ is a local orthonormal frame field in the normal bundle.

COROLLARY 3.3. *If Σ^2 is an immersed nonminimal pmc surface in $M^n(c) \times \mathbb{R}$ and if ϕ_H is the operator defined by $\phi_H = \frac{1}{|H|} A_H - |H| \text{Id}$, then*

$$\begin{aligned} \frac{1}{2} \Delta |\phi_H|^2 &= |\nabla \phi_H|^2 + \{c(2 - 3|T|^2) + 4|H|^2 - |\sigma|^2\} |\phi_H|^2 \\ &\quad - 2c|H| \langle \phi_H T, T \rangle + \frac{2c}{|H|} \langle H, N \rangle \text{trace}(A_N \phi_H). \quad (3.8) \end{aligned}$$

Proof. From the definition of ϕ_H we have $\nabla \phi_H = \frac{1}{|H|} \nabla A_H$ as well as $|\phi_H|^2 = \frac{1}{|H|^2} |A_H|^2 - 2|H|^2$ and $\frac{1}{|H|^2} |A_H T|^2 = \frac{1}{2} |T|^2 |\phi_H|^2 + |H|^2 |T|^2 + 2|H| \langle \phi_H T, T \rangle$; here we have used that $|\phi_H T|^2 = \frac{1}{2} |T|^2 |\phi_H|^2$, which can be easily verified by working in a basis that diagonalizes ϕ_H while taking into account that $\text{trace } \phi_H = 0$.

Next, from equation (3.6) with $V = H$ we get $\langle \nabla^2 A_H, \text{Id} \rangle = 0$; therefore, from Proposition 3.2 it follows that

$$\begin{aligned} \frac{1}{2} \Delta |\phi_H|^2 &= |\nabla \phi_H|^2 + c(2 - 3|T|^2) |\phi_H|^2 - 2c|H| \langle \phi_H T, T \rangle \\ &\quad + \frac{2c}{|H|} \langle H, N \rangle \text{trace}(A_N \phi_H) \\ &\quad + \sum_{\alpha=3}^{n+1} \{(\text{trace } A_\alpha)(\text{trace}(\phi_H^2 A_\alpha)) - (\text{trace}(\phi_H A_\alpha))^2\}. \quad (3.9) \end{aligned}$$

We now consider the local orthonormal frame field $\{E_3 = H/|H|, E_4, \dots, E_{n+1}\}$ in the normal bundle. One sees that $\text{trace } A_3 = 2|H|$ and $\text{trace } A_\alpha = 0$ for $\alpha > 3$, so

$$\sum_{\alpha=3}^{n+1} (\text{trace } A_\alpha)(\text{trace}(\phi_H^2 A_\alpha)) = 2|H|(\text{trace}(\phi_H^2 A_3)) = 2(\text{trace}(\phi_H^2 A_H)).$$

From the definition of ϕ_H it follows that $\phi_H^2 A_H = |H|\phi_H^3 + |H|^2\phi_H^2$ and, because $\text{trace } \phi_H^3 = 0$,

$$2(\text{trace } \phi_H^2 A_H) = 2|H| \text{trace } \phi_H^3 + 2|H|^2 \text{trace } \phi_H^2 = 2|H|^2|\phi_H|^2.$$

We have just proved that

$$\sum_{\alpha=3}^{n+1} (\text{trace } A_\alpha)(\text{trace}(\phi_H^2 A_\alpha)) = 2|H|^2|\phi_H|^2. \tag{3.10}$$

As we have seen in Lemma 3.1, since H is parallel, A_H commutes with A_U for any normal vector field U . Then, either there exists a basis that diagonalizes A_U , for all vectors U normal to Σ^2 , or the surface is pseudo-umbilical (i.e., $A_H = |H|^2 \text{Id}$). Moreover, since the map $p \in \Sigma^2 \rightarrow (A_H - \mu \text{Id})(p)$ for μ a constant is analytic, it follows that if H is an umbilical direction then this either holds on the whole surface or only on a closed set without interior points.

That the vector H is an umbilical direction everywhere implies that ϕ_H vanishes on the surface; hence (3.8) is verified and so we will study only the case when H is an umbilical direction on a closed set without interior points (which means that H is not umbilical in an open dense set). We will first work on this set and then extend our result throughout Σ^2 via continuity.

Let $\{e_1, e_2\}$ be a basis that diagonalizes A_U for all vectors U normal to the surface. Then, with respect to this basis and for $\alpha > 3$, since $\text{trace } A_\alpha = \text{trace } \phi_H = 0$ we have

$$A_\alpha = \begin{pmatrix} \mu_\alpha & 0 \\ 0 & -\mu_\alpha \end{pmatrix} \quad \text{and} \quad \phi_H = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix};$$

therefore,

$$(\text{trace}(\phi_H A_\alpha))^2 = 4a^2\mu_\alpha^2 = |A_\alpha|^2|\phi_H|^2. \tag{3.11}$$

We also obtain $\phi_H A_3 = \phi_H^2 + |H|\phi_H$, which leads to

$$(\text{trace}(\phi_H A_3))^2 = |\phi_H|^4 = (|A_3|^2 - 2|H|^2)|\phi_H|^2. \tag{3.12}$$

Finally, substituting (3.10)–(3.12) into (3.9), we get equation (3.8). □

REMARK 3.1. For the case of an immersed pmc submanifold in $\mathbb{S}^n(1)$, the Laplacian of $|\phi_H|^2$ was computed in [6; 19].

4. A Simons-Type Equation and Applications

In the following we shall derive a Simons-type equation for nonminimal pmc surfaces in $M^3(c) \times \mathbb{R}$ and then use that equation to characterize some of these surfaces. Throughout this section, Σ^2 will be an immersed nonminimal pmc surface

in a product space $M^3(c) \times \mathbb{R}$, with mean curvature vector H and the Gaussian curvature K .

Let us consider the local orthonormal frame field $\{E_3 = H/|H|, E_4\}$ in the normal bundle, and denote by $\phi_3 = A_3 - |H| \text{Id}$ and $\phi_4 = A_4$. The normal part of ξ can be written as $N = \nu_3 E_3 + \nu_4 E_4$, where $\nu_3 = \langle \xi, E_3 \rangle$ and $\nu_4 = \langle \xi, E_4 \rangle$.

Since H is parallel in the normal bundle, it follows that also E_4 is parallel in the normal bundle. We use the same argument as in the proof of Corollary 3.3 to see that either H is an umbilical direction on the whole surface or it is not umbilical on an open dense set. In both cases, it is easy to verify that

$$\begin{aligned} (\text{trace } A_3)(\text{trace } \phi_4^2 A_3) &= 2|H|^2|\phi_4|^2, \\ (\text{trace } (\phi_4 A_3))^2 &= (|A_3|^2 - 2|H|^2)|\phi_4|^2; \end{aligned}$$

then, since $(\text{trace } \phi_4^2)^2 = |\phi_4|^4$, $\text{trace } \phi_4 = 0$, and $|\phi_4 T|^2 = \frac{1}{2}|T|^2|\phi_4|^2$, Proposition 3.2 allows us to derive the following formula for the Laplacian of $|\phi_4|^2$:

$$\begin{aligned} \frac{1}{2} \Delta |\phi_4|^2 &= |\nabla \phi_4|^2 + \{c(2 - 3|T|^2) + 4|H|^2 - |\sigma|^2\}|\phi_4|^2 \\ &\quad + 2c\nu_4 \text{trace}(A_N \phi_4). \end{aligned} \tag{4.1}$$

Next, let ϕ be the traceless part of the second fundamental form σ of the surface

$$\phi(X, Y) = \sigma(X, Y) - \langle X, Y \rangle H = \sum_{\alpha=3}^4 \langle \phi_\alpha X, Y \rangle E_\alpha.$$

We have $|\phi|^2 = |\phi_3|^2 + |\phi_4|^2 = |\sigma|^2 - 2|H|^2$ and then, using (3.8) and (4.1), we can state our next proposition.

PROPOSITION 4.1. *If Σ^2 is an immersed nonminimal pmc surface in $M^3(c) \times \mathbb{R}$ and if ϕ is the traceless part of its second fundamental form, then*

$$\begin{aligned} \frac{1}{2} \Delta |\phi|^2 &= |\nabla \phi_3|^2 + |\nabla \phi_4|^2 - |\phi|^4 + \{c(2 - 3|T|^2) + 2|H|^2\}|\phi|^2 \\ &\quad - 2c\langle \phi(T, T), H \rangle + 2c|\nu_3 \phi_3 + \nu_4 \phi_4|^2 \end{aligned} \tag{4.2}$$

or, equivalently,

$$\begin{aligned} \frac{1}{2} \Delta |\phi|^2 &= |\nabla \phi_3|^2 + |\nabla \phi_4|^2 - |\phi|^4 + \{c(2 - 3|T|^2) + 2|H|^2\}|\phi|^2 \\ &\quad - 2c\langle \phi(T, T), H \rangle + 2c|A_N|^2 - 4c\langle H, N \rangle^2. \end{aligned}$$

THEOREM 4.2. *Let Σ^2 be an immersed complete nonminimal pmc surface in $\bar{M} = M^3(c) \times \mathbb{R}$, with $c < 0$, such that*

$$\sup_{\Sigma^2} |\phi|^2 < 2|H|^2 + c(4 - 5|T|^2) \quad \text{and} \quad \langle \phi(T, T), H \rangle \geq 0.$$

Then $|\phi|^2 = 0$ and Σ^2 is a round sphere in $M^3(c)$.

Proof. We start by using the Schwarz inequality to obtain

$$|\nu_3 \phi_3 + \nu_4 \phi_4|^2 \leq (\nu_3^2 + \nu_4^2)(|\phi_3|^2 + |\phi_4|^2) = (1 - |T|^2)|\phi|^2;$$

this expression, together with (4.2) and the hypothesis, leads to

$$\begin{aligned} \frac{1}{2} \Delta |\phi|^2 &\geq -|\phi|^4 + \{c(4 - 5|T|^2) + 2|H|^2\} |\phi|^2 - 2c \langle \phi(T, T), H \rangle \\ &\geq \{-|\phi|^2 + c(4 - 5|T|^2) + 2|H|^2\} |\phi|^2 \\ &\geq 0. \end{aligned} \tag{4.3}$$

Next we show that the Gaussian curvature K of the surface is bounded from below. Indeed, by (2.2) we have

$$\begin{aligned} 2K &= 2c(1 - |T|^2) + 4|H|^2 - |\sigma|^2 \\ &= 2c(1 - |T|^2) + 2|H|^2 - |\phi|^2 > c(3|T|^2 - 2) \geq c. \end{aligned}$$

Therefore, since Σ^2 is complete, the Omori–Yau maximum principle holds on the surface.

We take $u = |\phi|^2$ in Theorem 2.2, from which it follows that there exists a sequence of points $\{p_k\}_{k \in \mathbb{N}} \subset \Sigma^2$ such that

$$\lim_{k \rightarrow \infty} |\phi|^2(p_k) = \sup_{\Sigma^2} |\phi|^2 \quad \text{and} \quad \Delta |\phi|^2(p_k) < \frac{1}{k}.$$

By (4.3), $\lim_{k \rightarrow \infty} |\phi|^2(p_k) = 0$ and then $\sup_{\Sigma^2} |\phi|^2 = 0$, which implies $|\phi|^2 = 0$. Since $\phi_3 = A_3 - |H| \text{Id} = 0$, it follows that H is an umbilical direction; this, in turn, implies that Σ^2 is a totally umbilical surface in $M^3(c)$ (see [5, Lemma 3]). Therefore, Σ^2 is a horosphere or a round sphere. But $|\phi|^2 < 2|H|^2 + 4c$ implies that $K > -c > 0$, so Σ^2 cannot be flat; hence we conclude that the surface is a round sphere. \square

THEOREM 4.3. *Let Σ^2 be an immersed complete nonminimal pmc surface in $\bar{M} = M^3(c) \times \mathbb{R}$, with $c > 0$, such that*

$$|\phi|^2 \leq 2|H|^2 + c(2 - 3|T|^2) \quad \text{and} \quad \langle \phi(T, T), H \rangle \leq 0.$$

Then ξ is normal to the surface, and either

- (1) $|\phi|^2 = 0$ and Σ^2 is a round sphere in $M^3(c)$ or
- (2) $|\phi|^2 = 2|H|^2 + 2c$ and Σ^2 is a torus $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1/c - r^2})$, $r^2 \neq 1/2c$, in $M^3(c)$.

Proof. From the Gauss equation (2.2) of the surface it follows that, since $|\phi|^2 \leq 2|H|^2 + c(2 - 3|T|^2)$, we have

$$2K = 2c(1 - |T|^2) + 4|H|^2 - |\sigma|^2 = 2c(1 - |T|^2) + 2|H|^2 - |\phi|^2 \geq c|T|^2 \geq 0;$$

a result of Huber [14] then implies that Σ^2 is a parabolic space.

On the other hand, by (4.2) we have $\Delta |\phi|^2 \geq 0$ and so $|\phi|^2$ is a bounded subharmonic function on a parabolic space. Hence $|\phi|^2$ is a constant and, again using (4.2), we obtain

$$\begin{aligned} \{-|\phi|^2 + c(2 - 3|T|^2) + 2|H|^2\} |\phi|^2 &= 0, \\ \langle \phi(T, T), H \rangle &= 0, \quad \text{and} \quad \nu_3 \phi_3 + \nu_4 \phi_4 = 0. \end{aligned}$$

Now we can split our study in two cases.

Case I: $|\phi|^2 = 0$. This case can be handled exactly as in the proof of Theorem 4.2.

Case II: $|\phi|^2 = c(2 - 3|T|^2) + 2|H|^2$. Since $|\phi|^2$ is a constant, it follows that $|T|^2$ is a constant and thus that $\langle \nabla_X T, T \rangle = 0$ for any vector X tangent to Σ^2 . Since $\bar{\nabla}_X \xi = 0$ implies $\nabla_X T = A_N X$, we have $\langle A_N X, T \rangle = 0$. But $\nu_3 \phi_3 + \nu_4 \phi_4 = 0$ means that $A_N = \langle H, N \rangle \text{Id}$, which implies $\langle H, N \rangle \langle X, T \rangle = 0$ for any tangent vector X . Therefore, ξ is orthogonal to Σ^2 either at any point or only on a closed set without interior points. In the latter case, $\langle H, \xi \rangle = 0$ holds on an open dense set in which we also have $\langle \bar{\nabla}_T H, \xi \rangle = -\langle A_H T, T \rangle = 0$. We have just shown that $\langle A_3 T, T \rangle = 0$ on an open dense set; therefore, since also

$$\langle \phi(T, T), H \rangle = |H| \langle \phi_3 T, T \rangle = |H| (\langle A_3 T, T \rangle - |H| |T|^2) = 0,$$

it follows that $T = 0$ on the whole surface. Hence Σ^2 lies in $M^3(c)$ and its Gaussian curvature is $K = \frac{c}{2} |T|^2 = 0$. We use a similar argument to that in the proof of [3, Thm. 1.5] (see also [16]) to conclude. \square

5. A Gap Theorem for pmc Surfaces with Nonnegative Gaussian Curvature

In this section we will prove our main results, Theorem 5.2 and Theorem 5.3. In order to do so, let us consider an immersed pmc surface Σ^2 in $M^n(c) \times \mathbb{R}$. We begin by computing the Laplacian of $|T|^2$.

Let $\{e_1, e_2\}$ be an orthonormal in $T_p \Sigma^2$ ($p \in \Sigma^2$), and extend e_1, e_2 to vector fields E_1, E_2 in a neighborhood of p such that $\nabla E_i = 0$ at p . At p , we have

$$\begin{aligned} \frac{1}{2} \Delta |T|^2 &= \sum_{i=1}^2 (\langle \nabla_{E_i} T, \nabla_{E_i} T \rangle + \langle \nabla_{E_i} \nabla_{E_i} T, T \rangle) \\ &= |A_N|^2 + \sum_{i=1}^2 \langle \nabla_{E_i} A_N E_i, T \rangle; \end{aligned}$$

then, since $\nabla_X A_N$ is symmetric,

$$\begin{aligned} \sum_{i=1}^2 \langle \nabla_{E_i} A_N E_i, T \rangle &= \sum_{i=1}^2 \langle (\nabla_{E_i} A_N) E_i, T \rangle \\ &= \sum_{i=1}^2 \langle (\nabla_{E_i} A_N) T, E_i \rangle = \sum_{i=1}^2 \langle \nabla_{E_i} A_N T - A_N \nabla_{E_i} T, E_i \rangle \\ &= \sum_{i=1}^2 \langle \nabla_{E_i} \nabla_T T - \nabla_{\nabla_{E_i} T} T, E_i \rangle \\ &= \sum_{i=1}^2 \langle \nabla_{E_i} \nabla_T T + \nabla_{[T, E_i]} T, E_i \rangle = \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^2 (\langle \nabla_T \nabla_{E_i} T, E_i \rangle - \langle R(T, E_i)T, E_i \rangle) \\
&= |T|^2 K + \sum_{i=1}^2 \langle \nabla_T A_N E_i, E_i \rangle \\
&= |T|^2 K + \sum_{i=1}^2 T(\langle A_N E_i, E_i \rangle) = |T|^2 K + T(\text{trace } A_N) \\
&= |T|^2 K + 2T(\langle H, N \rangle) = |T|^2 K - 2\langle \sigma(T, T), H \rangle \\
&= c|T|^2(1 - |T|^2) - \frac{1}{2}|T|^2|\phi|^2 - 2\langle \phi(T, T), H \rangle - |T|^2|H|^2.
\end{aligned}$$

Here we used that $\nabla_X T = A_N X$ and $\nabla_X^\perp N = -\sigma(X, T)$, and ϕ is the traceless part of the second fundamental form σ of the surface.

Now we can state the following result.

PROPOSITION 5.1. *If Σ^2 is an immersed pmc surface in $M^n(c) \times \mathbb{R}$, then*

$$\begin{aligned}
\frac{1}{2}\Delta|T|^2 &= |A_N|^2 - \frac{1}{2}|T|^2|\phi|^2 - 2\langle \phi(T, T), H \rangle \\
&\quad + c|T|^2(1 - |T|^2) - |T|^2|H|^2.
\end{aligned} \tag{5.1}$$

In [4] the authors introduced a holomorphic differential defined on pmc surfaces in $M^n(c) \times \mathbb{R}$ (when $n = 2$ this is just the Abresch–Rosenberg differential defined in [1]). This holomorphic differential is the $(2, 0)$ -part of the quadratic form Q , which is given by

$$Q(X, Y) = 2\langle \sigma(X, Y), H \rangle - c\langle X, \xi \rangle \langle Y, \xi \rangle.$$

Using this result and Proposition 5.1, we can characterize pmc 2-spheres Σ^2 immersed in $M^n(c) \times \mathbb{R}$ and whose second fundamental form satisfies a certain condition.

THEOREM 5.2. *Let Σ^2 be an immersed pmc 2-sphere in $M^n(c) \times \mathbb{R}$ such that:*

- (1) $|T|^2 = 0$ or $|T|^2 \geq \frac{2}{3}$ and $|\sigma|^2 \leq c(2 - 3|T|^2)$ if $c < 0$;
- (2) $|T|^2 \leq \frac{2}{3}$ and $|\sigma|^2 \leq c(2 - 3|T|^2)$ if $c > 0$.

Then Σ^2 is either a minimal surface in a totally umbilical hypersurface of $M^n(c)$ or a standard sphere in $M^3(c)$.

Proof. If ξ is orthogonal to the surface in an open connected subset, then this subset lies in $M^n(c)$ and, by analyticity, it follows that Σ^2 lies in $M^n(c)$. In this case we use [20, Thm. 4] to conclude.

Next, let us assume that we are not in the previous case. Then we can choose an orthonormal frame $\{e_1, e_2\}$ on the surface such that $e_1 = T/|T|$. Since Σ^2 is a sphere and since the $(2, 0)$ -part of Q is holomorphic, it follows that the $(2, 0)$ -part of Q vanishes on the surface. This means that $Q(e_1, e_1) = Q(e_2, e_2)$ and $Q(e_1, e_2) = 0$. From $Q(e_1, e_1) = Q(e_2, e_2)$ we obtain $2\langle \sigma(e_1, e_1) - \sigma(e_2, e_2), H \rangle = c|T|^2$. Then

$$\begin{aligned} \langle \phi(T, T), H \rangle &= \langle \sigma(T, T), H \rangle - |T|^2 |H|^2 \\ &= \frac{1}{2} |T|^2 \langle \sigma(e_1, e_1) - \sigma(e_2, e_2), H \rangle = \frac{1}{4} c |T|^4, \end{aligned}$$

after which (5.1) becomes

$$\frac{1}{2} \Delta |T|^2 = |A_N|^2 + \frac{1}{2} |T|^2 (-|\sigma|^2 + c(2 - 3|T|^2)) \geq 0.$$

Since Σ^2 is a sphere and $|T|^2$ a bounded subharmonic function, it follows that $|T|^2$ is constant and hence that $|A_N|^2 = 0$ and $|T|^2 (-|\sigma|^2 + c(2 - 3|T|^2)) = 0$. Because $A_N = 0$ and ξ is parallel, we have $\nabla_X T = 0$ for any tangent vector X . This implies that $K = 0$ —a contradiction, since our surface is a sphere. Therefore, Σ^2 lies in $M^n(c)$ and, again using [20, Thm. 4] (see also [5, Thm. 2]), we come to the conclusion. \square

Next we assume that Σ^2 is an immersed nonminimal pmc surface in $M^3(c) \times \mathbb{R}$. Then (4.2) and (5.1) yield

$$\begin{aligned} \frac{1}{2} \Delta (|\phi|^2 - c|T|^2) &= |\nabla \phi_3|^2 + |\nabla \phi_4|^2 + \left\{ -|\phi|^2 + \frac{c}{2} (4 - 5|T|^2) + 2|H|^2 \right\} |\phi|^2 \\ &\quad + c|A_N|^2 - 4c \langle H, N \rangle^2 + c|T|^2 |H|^2 - c^2 |T|^2 (1 - |T|^2). \end{aligned} \quad (5.2)$$

We shall use this equation to prove the following theorem.

THEOREM 5.3. *Let Σ^2 be an immersed complete nonminimal pmc surface in $\bar{M} = M^3(c) \times \mathbb{R}$, with $c > 0$. Assume*

- (i) $|\phi|^2 \leq 2|H|^2 + 2c - \frac{5c}{2}|T|^2$ and
- (ii) either
 - (a) $|T| = 0$ or
 - (b) $|T|^2 > \frac{2}{3}$ and $|H|^2 \geq c|T|^2(1 - |T|^2)/(3|T|^2 - 2)$.

Then either

- (1) $|\phi|^2 = 0$ and Σ^2 is a round sphere in $M^3(c)$, or
- (2) $|\phi|^2 = 2|H|^2 + 2c$ and Σ^2 is a torus $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1/c - r^2})$, $r^2 \neq 1/2c$, in $M^3(c)$.

Proof. If $|T|^2 = 0$, then it is easy to see that (5.2) implies

$$\frac{1}{2} \Delta (|\phi|^2 - c|T|^2) \geq \{-|\phi|^2 + 2c + 2|H|^2\} |\phi|^2 \geq 0.$$

We therefore assume that $|T|^2 > \frac{2}{3}$ and $|H|^2 \geq c|T|^2(1 - |T|^2)/(3|T|^2 - 2)$. Since

$$|A_N|^2 - 2 \langle H, N \rangle^2 = |\nu_3 \phi_3 + \nu_4 \phi_4|^2 \geq 0$$

and since

$$\langle H, N \rangle^2 \leq |N|^2 |H|^2 = (1 - |T|^2) |H|^2 \quad (5.3)$$

by the Schwarz inequality, it follows from (5.2) that

$$\begin{aligned} \frac{1}{2} \Delta(|\phi|^2 - c|T|^2) &\geq \left\{ -|\phi|^2 + \frac{c}{2}(4 - 5|T|^2) + 2|H|^2 \right\} |\phi|^2 \\ &\quad + c(3|T|^2 - 2)|H|^2 - c^2|T|^2(1 - |T|^2) \\ &\geq 0. \end{aligned}$$

The Gaussian curvature of the surface satisfies

$$2K = 2c(1 - |T|^2) + 2|H|^2 - |\phi|^2 \geq \frac{1}{2}c|T|^2 \geq 0,$$

which means that Σ^2 is a parabolic space. Now, as a bounded subharmonic function, $|\phi|^2 - c|T|^2$ is constant. Therefore, either $|\phi|^2 = 0$ or $|\phi|^2 = 2|H|^2 + 2c - \frac{5c}{2}|T|^2$. The first case can be handled exactly as in the proof of Theorem 4.2. For the second case, by (5.2) we have $\nu_3\phi_3 + \nu_4\phi_4 = 0$, which means that $A_N = \langle H, N \rangle \text{Id}$ and that the equality holds in (5.3); in other words, either $N = \nu_3H$ or ξ is tangent to the surface. If ξ is tangent to the surface then, by (5.2) and the hypothesis, Σ^2 is a minimal surface—a contradiction. Hence $N = \nu_3H$ and we obtain $A_H = |H|^2 \text{Id}$, which (again using [5, Lemma 3.1]) implies that the surface lies in $M^3(c)$. We then arrive at the conclusion in the same way as in the proof of the second part of Theorem 4.3. \square

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