Michigan Math. J. 61 (2012), 593–613

Moduli Spaces of Curves and Cox Rings

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1. Introduction

Let k be a field, X a smooth projective geometrically irreducible k-variety, and $\mathscr{C}$ a smooth projective geometrically irreducible k-curve of genus $g_{\mathscr{C}}$. Let $\mathscr{K}_X$ be the canonical class of X. For every element y of the dual $\mathrm{NS}(X)^{\vee}$ of the Néron–Severi group of X, let $\mathrm{Mor}(\mathscr{C}, X, y)$ denote the quasi-projective k-variety parameterizing the morphisms $f: \mathscr{C} \to X$ such that $[f_*\mathscr{C}] = y$. By [De, Sec. 2.11], every irreducible component of $\mathrm{Mor}(\mathscr{C}, X, y)$ has dimension at least $(1 - g_{\mathscr{C}}) \dim(X) + \langle y, -\mathscr{K}_X \rangle$. The latter quantity will be referred to as the *expected dimension* of $\mathrm{Mor}(\mathscr{C}, X, y)$. It is a natural though difficult question to ask for the number and dimension of the irreducible components of $\mathrm{Mor}(\mathscr{C}, X, y)$. Works addressing this question for specific families of varieties include [Ca; CS; dJS; HRS1; HRS2; KLO; KP; Pe; T].

In this paper we study the question using the so-called Cox ring of X, restricting ourselves to a class of varieties whose Cox ring has an especially simple presentation. It is known, at least when $\mathscr{C}$ is rational, that the Cox ring of a toric variety X provides a useful description of the moduli spaces $\mathrm{Mor}(\mathscr{C}, X, y)$ [Ba; Bo3; G]. Toric varieties may be characterized by the fact that their Cox ring is a polynomial ring; hence they are the simplest varieties from the viewpoint of the description of the Cox ring.

Here we consider varieties whose Cox ring may be presented by only one equation, which has moreover a kind of linearity property with respect to a certain subset of variables (see Definitions and Notation 2.1 for more precision). We will call such varieties *linear intrinsic hypersurfaces* (the terminology *intrinsic hypersurface* is borrowed from [BHau]). Let $\mathrm{Mor}(\mathscr{C}, X, y)^{\circ}$ denote the open set of $\mathrm{Mor}(\mathscr{C}, X, y)$ consisting of those morphisms that do not factor through the boundary—in other words, the union of the divisors of the sections used to present the Cox ring.

Our main result reads as follows (see Theorem 2.4 for a more precise statement).

Theorem 1.1. *Let X be smooth projective $\mathbf{Q}$-variety that is a linear intrinsic hypersurface. Assume that certain rational combinatoric series derived from the equation of the Cox ring fulfill some explicit analytic properties. Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve. For every $y \in \mathrm{NS}(X)^{\vee}$ lying*

Received March 11, 2011. Revision received September 30, 2011.

in an explicit truncation of an explicit subcone of the dual of the effective cone, $\mathrm{Mor}(\mathscr{C}, X, y)^\circ$ *is irreducible of the expected dimension and dense in* $\mathrm{Mor}(\mathscr{C}, X, y)$.

By *explicit* we mean explicit in terms of the data describing the Cox ring and the genus of the curve. A *truncation* of a polyedral cone $\mathcal{C}$ is a subpolyhedron of $\mathcal{C}$ defined by a finite number of affine inequalities $\langle x, \cdot \rangle \geq a$, where x lies in the dual of $\mathcal{C}$ and a is nonnegative. We emphasize that the needed properties of the combinatoric series alluded to in the theorem's statement can be checked by a computer algebra system once we have at our disposal an effective presentation of the Cox ring of X.

The basic strategy of the proof will be to count the number of points of the reduction of $\mathrm{Mor}(\mathscr{C}, X, y)$ modulo primes p with values in $\mathbf{F}_p$-extensions of large degree. We are thus reduced to a situation akin to the one encountered in the context of Manin's conjecture about the asymptotical behavior of curves of bounded degree, and we apply techniques similar to those used in [Bo1; Bo2; Bo4]. The main difference is that in this paper we fix the degree y and look at the asymptotic behavior of the number of points with value in an $\mathbf{F}_p$-extension of large degree whereas, in the context of Manin's conjecture, the $\mathbf{F}_p$-extension is fixed and the degree y becomes large. Our varieties are assumed to be defined over $\mathbf{Q}$ for the sake of simplicity and because all our examples of applications are, but by standard arguments the strategy could be applied over any field.

By the same method one can show the following theorem about toric varieties. We do not include the proof here because it is strongly similar to the one used in the case of a linear intrinsic hypersurface—as well as technically easier given that, in the toric case, the Cox ring has "no equation".

THEOREM 1.2. *Let X be a smooth projective split toric variety. Let $\mathscr{C}$ be a smooth projective geometrically irreducible* $\mathbf{Q}$*-curve. For every y lying in an explicit truncation of the dual of the effective cone,* $\mathrm{Mor}(\mathscr{C}, X, y)^\circ$ (*the open set parameterizing those morphisms that do not factor through the complement of the open orbit*) *is irreducible of the expected dimension and dense in* $\mathrm{Mor}(\mathscr{C}, X, y)$.

In case $\mathscr{C} = \mathbf{P}^1$, that $\mathrm{Mor}(\mathscr{C}, X, y)^\circ$ is irreducible of the expected dimension was proved in [Bo3].

At the end of the paper we give examples of linear intrinsic hypersurfaces for which Theorem 1.1 applies for a "positive proportion" of y (see Remark 2.6). One family of examples is drawn from [Bo4] and the other from Derenthal's list [D] of the minimal resolutions of those singular del Pezzo surfaces with a Cox ring that is presented by one equation. We will compare our results with those of [KLO], whose authors deal with the case of blow-ups of projective spaces.

The paper proceeds as follows. In Section 2, after introducing and defining the necessary objects, we state a more explicit version of our main result (Theorem 2.4). In Section 3 we recall some well-known facts about the connection between dimension and number of points in the reduction modulo a prime p. In Section 4 we recall from [Bo2] the expression for the number of points of

$\mathrm{Mor}(\mathscr{C}, X, y)$ over a finite field in terms of a presentation of the Cox ring. Section 5 is devoted to some technical lemmas, and in Section 6 we prove the main theorem. Finally, in Section 7 we give examples of applications.

I thank Bumsig Kim for answering questions about [KLO]. I also thank the referee for helpful remarks.

2. Statement of the Result

DEFINITIONS AND NOTATION 2.1. Let X be a smooth projective $\mathbf{Q}$-variety whose geometric Picard group is free and of finite rank with a trivial Galois action. Let $\mathrm{Eff}(X)$ be the effective cone of X. Assume, moreover, that X is a Mori dream space; in other words, the Cox ring $\mathrm{Cox}(X)$ of X (cf. e.g. [Has]) is finitely generated.

Let $(s_i)_{i\in\mathfrak{I}}$ be a finite family of nonconstant global sections generating the Cox ring, and let $\mathscr{E}_i$ be the divisor of s_i. The divisors $\{\mathscr{E}_i\}_{i\in\mathfrak{I}}$ span $\mathrm{Eff}(X)$. Later on, when considering a Mori dream space X, we shall always assume that such a family of sections has been chosen.

Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve and let $y \in \mathrm{NS}(X)^\vee$. We define a partition of $\mathrm{Mor}(\mathscr{C}, X, y)$ into locally closed subsets $\{\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)\}_{\mathfrak{I}_*\subset\mathfrak{I}}$ as follows: a morphism φ lies in $\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$ if and only if $\mathfrak{I}_*$ is the set of indices $i \in \mathfrak{I}$ such that φ does not factor through $\mathscr{E}_i$. Thus $\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I})$ is the open subset $\mathrm{Mor}(\mathscr{C}, X, y)^\circ$ of $\mathrm{Mor}(\mathscr{C}, X, y)$ parameterizing those morphisms $\mathscr{C} \to X$ that do not factor through the boundary $\bigcup_{i\in\mathfrak{I}} \mathscr{E}_i$.

Let $\mathscr{I}_{X,\mathfrak{I}}$ denote the kernel of the morphism $k[x_i]_{i\in\mathfrak{I}} \to \mathrm{Cox}(X)$ that maps x_i to s_i. This kernel is homogeneous with respect to the natural $\mathrm{Pic}(X)$-grading on $k[x_i]$. Let $\mathscr{I}_{X,\mathfrak{I}}^{\mathrm{hom}}$ denote the set of the homogeneous elements of $\mathscr{I}_{X,\mathfrak{I}}$.

DEFINITION 2.2. With notation as before, a Mori dream space X is said to be an *intrinsic hypersurface* if $\mathscr{I}_{X,\mathfrak{I}}$ is principal. Let I be a subset of $\mathfrak{I}$. The Mori dream space X is said to be a *linear intrinsic hypersurface* with respect to the pair $(\mathfrak{I}, I)$ if the classes of $\{\mathscr{E}_i\}_{i\in I}$ form a basis of $\mathrm{Pic}(X)$ and if $\mathscr{I}_{X,\mathfrak{I}}$ is principal, with generator a linear form with respect to the variables $\{x_i\}_{i\notin I}$, whose coefficients are pairwise coprime monomials in the $\{x_i\}_{i\in I}$. Thus a generator of $\mathscr{I}_{X,\mathfrak{I}}$ may be written as

$$\sum_{j\in\mathfrak{I}\setminus I} \alpha_j x_j \prod_{i\in I} x_i^{b_{i,j}}, \quad (\alpha_j) \in k^{\mathfrak{I}\setminus I},\ (b_{i,j}) \in \mathbf{N}^{I\times(\mathfrak{I}\setminus I)}, \tag{2.1}$$

where the sets $I_j = \{i \in I, b_{i,j} \neq 0\}$ are pairwise disjoint. The degree of the generator will be denoted by $\mathscr{D}_{\mathrm{tot}}$; thus, $\mathscr{D}_{\mathrm{tot}}$ lies in $\mathrm{Pic}(X)$.

Examples of linear intrinsic hypersurfaces will be given in Section 7.

REMARK 2.3. If X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$, then $\dim(X) = [\mathfrak{I} \setminus I] - 1$. Moreover, by [BHau, Prop. 8.5] there is an adjunction formula that allows us to compute the class of the canonical sheaf:

$$-\mathscr{K}_X = \sum_{i\in\mathfrak{I}} \mathscr{E}_i - \mathscr{D}_{\mathrm{tot}}. \tag{2.2}$$

Let X be a Mori dream space and let $y \in \mathrm{NS}(X)^\vee$. Since the divisors $\{\mathscr{E}_i\}_{i\in\mathfrak{I}}$ span $\mathrm{Eff}(X)$, if y does not lie in $\mathrm{Eff}(X)^\vee$ then $\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I})$ is empty; that is, every morphism of degree y has its image contained in the boundary. We are not interested in this kind of degeneracy and thus shall always assume that y lies in $\mathrm{Eff}(X)^\vee$. Let $\mathfrak{I}_*$ be a proper subset of $\mathfrak{I}$ such that $\bigcap_{i\notin\mathfrak{I}_*}\mathscr{E}_i$ is nonempty. For every $\varphi \in \mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$ and $j \in \mathfrak{I}_*$ such that $\mathscr{E}_j$ does not meet $\bigcap_{i\notin\mathfrak{I}_*}\mathscr{E}_i$, one must have $\varphi^*\mathscr{E}_j = 0$ and so $\langle y, \mathscr{E}_j\rangle = 0$. We will say that y satisfies the *degeneracy condition for* $\mathfrak{I}_*$ if the latter equalities hold; it is thus a necessary condition for the nonemptiness of $\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$.

We can now state a more explicit version of Theorem 1.1.

THEOREM 2.4. *With notation as before, assume that X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$ (cf. Definition 2.2). Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve of genus $g_{\mathscr{C}}$. Let $y \in \mathrm{Eff}(X)^\vee$ satisfy the numerical inequality*

$$\left\langle y, \frac{1}{[\mathfrak{I}\setminus I]-1}\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j - \mathscr{D}_{\mathrm{tot}}\right\rangle \geq \mathrm{Sup}\left(1, \frac{4}{[\mathfrak{I}\setminus I]-1}\right) g_{\mathscr{C}} \dim(X). \tag{2.3}$$

(1) *Suppose Assumptions 5.7 hold for $(\mathfrak{I}, I)$. Then* $\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I})$ *is irreducible of the expected dimension.*
(2) *Let $\mathfrak{I}_*$ be a proper subset of $\mathfrak{I}$ such that $\bigcap_{i\notin\mathfrak{I}_*}\mathscr{E}_i$ is nonempty. Assume that y fulfills the degeneracy condition for $\mathfrak{I}_*$ and that the numerical inequalities*

$$\langle y, \mathscr{E}_i\rangle \geq g_{\mathscr{C}} \quad \forall i \in I\setminus\mathfrak{I}_* \tag{2.4}$$

hold. In case $\mathfrak{I}\setminus I \subset \mathfrak{I}_*$, suppose moreover that Assumptions 5.7 hold for $(\mathfrak{I}_*, I)$ and that at least one of the inequalities in (2.4) *is strict. Then*

$$\dim(\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)) < (1-g_{\mathscr{C}})\dim(X) + \langle y, -\mathscr{K}_X\rangle. \tag{2.5}$$

REMARK 2.5. We still postpone the description of Assumptions 5.7, which are the conditions on the combinatoric series alluded to in Theorem 1.1 but are technically somewhat cumbersome. It would naturally be desirable to omit them from the statement—or at least to relax them or reinterpret them in a more conceptual way. Doing so would allow us (a) to avoid the use of a computer algebra system and (b) to cover more cases of linear intrinsic hypersurfaces. (There are a few examples in Derenthal's list [D] for which the assumptions fail; see Section 7.)

REMARK 2.6. The subcone of $\mathrm{Eff}(X)^\vee$ in the statement of Theorem 1.1 is thus the dual of the cone generated by the effective cone and $\frac{1}{[\mathfrak{I}\setminus I]-1}\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j - \mathscr{D}_{\mathrm{tot}}$. Let us denote this subcone by $\mathcal{C}_I$ and denote by $\tilde{\mathcal{C}}$ the union of all the $\mathcal{C}_I$ for which X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$ and such that Assumptions 5.7 hold.

The result will be optimal if $\tilde{\mathcal{C}}$ coincides with the dual of the cone generated by the effective cone and $-\mathscr{K}_X$ and hence coincides with $\mathrm{Eff}(X)^\vee$ when $-\mathscr{K}_X$ lies in Eff(X) (thus, in the toric case, the result of Theorem 1.2 is optimal). In Section 7 we give examples of linear intrinsic hypersurfaces for which the result is either optimal or holds for a “positive proportion” of y; that is, $\mathcal{C}_I$ is of maximal dimension for at least one choice of I.

3. Reduction Modulo *p*

The following lemma is a standard application of the Weil conjectures (proved by Deligne).

LEMMA 3.1. *Let X be a $\mathbf{Q}$-variety. If p is a prime and r a positive integer, we denote by $X(\mathbf{F}_{p^r})$ the set of $\mathbf{F}_{p^r}$-points of the reduction of X modulo p, which is well-defined up to a finite number of primes. Assume that there exists an integer D and a positive integer N such that, for almost all primes p,*

$$\lim_{r\to+\infty} p^{-rD}[X(\mathbf{F}_{p^r})] = N \tag{3.6}$$

and (respectively)

$$\lim_{r\to+\infty} p^{-rD}[X(\mathbf{F}_{p^r})] = 0. \tag{3.7}$$

In the first case one has $\dim(X) = D$, *every irreducible component of X with dimension* $\dim(X)$ *is geometrically irreducible, and there are N such irreducible components. In the second case,* $\dim(X) < D$.

4. Counting Morphisms Using a Presentation of the Cox Ring

We retain our previous notation; until otherwise specified, X is only assumed to be a Mori dream space. If Y is a $\mathbf{Q}$-variety and p a prime number then we denote by Y_p the $\mathbf{F}_p$-variety obtained by reducing Y modulo p, which is well-defined up to a finite number of primes.

Let $\mathcal{C}_{\text{inc}}$ be the class of subsets $\mathfrak{I}_*$ of $\mathfrak{I}$ such that $\bigcap_{i\notin\mathfrak{I}_*}\mathscr{E}_i \neq \emptyset$. Let $\mathrm{T}_{\mathrm{NS}}(X) \stackrel{\text{def}}{=} \mathrm{Hom}(\mathrm{Pic}(X), \mathbf{G}_m)$. There exists a $\mathrm{T}_{\mathrm{NS}}(X)$-invariant morphism

$$\pi : \mathscr{T}_X \stackrel{\text{def}}{=} \mathrm{Spec}(\mathrm{Cox}(X)) \cap \bigcup_{\mathfrak{I}_*\in\mathcal{C}_{\text{inc}}} \left\{ \prod_{i\in\mathfrak{I}_*} x_i \neq 0 \right\} \to X \tag{4.8}$$

that makes $\mathscr{T}_X$ an X-torsor under $\mathrm{T}_{\mathrm{NS}}(X)$ (cf. [Bo4, Sec. 2.2]). For almost all primes p, (4.8) reduces to an X_p-torsor $\pi_p \colon \mathscr{T}_{X,p} \to X_p$ under $\mathrm{T}_{\mathrm{NS}}(X_p)$.

Let $\mathfrak{I}_* \in \mathcal{C}_{\text{inc}}$. We shall later give a formula (equation (4.20)) for the number of points in the reduction of $\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$ modulo a prime p. First we must introduce some definitions and notation. In order to motivate them, let us briefly sketch how (4.20) is obtained. Using the torsor (4.8) and adapting a proof of Cox (who addressed the toric setting in [Cox]), one can show that the map $\varphi \in \mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*) \mapsto \{(\varphi^*\mathscr{O}_X(\mathscr{E}_i), \varphi^* s_i)\}_{i\in\mathfrak{I}}$ induces a one-to-one correspondence between the set of points of $\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$ and the set of $\mathrm{T}_{\mathrm{NS}}(X)$-orbits of isomorphism classes of families $\{(\mathscr{L}_i, u_i)\}_{i\in\mathfrak{I}}$, where $\mathscr{L}_i$ is a line bundle on $\mathscr{C}$ and u_i is a global section of $\mathscr{L}_i$, such that:

- u_i is the zero section if and only if $i \notin \mathfrak{I}_*$;
- for every $(a_i) \in \mathbf{Z}^{\mathfrak{I}}$ such that $\sum a_i \mathscr{E}_i \sim 0$, the line bundle $\bigotimes_{i\in\mathfrak{I}} \mathscr{L}_i^{\otimes a_i}$ is trivial;
- for every $F \in \mathscr{I}^{\mathrm{hom}}_{X,\mathfrak{I}}$, we have $F((u_i)_{i\in\mathfrak{I}}) = 0$; and
- the sections $\left\{\prod_{i\in\mathfrak{K}} u_i\right\}_{\mathfrak{K}\in\mathcal{C}_{\mathrm{inc}}}$ do not vanish simultaneously, or (what amounts to the same thing as regards the first condition) the intersection of the supports of the divisors $\left\{\sum_{i\in\mathfrak{K}} \mathrm{div}(u_i)\right\}_{\mathfrak{K}\subset\mathfrak{I}_*,\,\mathfrak{K}\in\mathcal{C}_{\mathrm{inc}}}$ is empty.

We refer to [Bo2, Thm. 1.11] for a more precise statement. Now let

$$\mathfrak{I}_*^\circ \stackrel{\mathrm{def}}{=} \Big\{ i \in \mathfrak{I}_*,\ \mathscr{E}_i \cap \bigcap_{j\notin\mathfrak{I}_*} \mathscr{E}_j \neq \emptyset \Big\}. \tag{4.9}$$

Thus $y \in \mathrm{Eff}(X)^\vee \cap \mathrm{NS}(X)^\vee$ satisfies the degeneracy condition for $\mathfrak{I}_*$ if $\langle y, \mathscr{E}_i\rangle = 0$ for every $i \in \mathfrak{I}_* \setminus \mathfrak{I}_*^\circ$. One sees that the last listed condition on the data $\{(\mathscr{L}_i, u_i)\}$ is equivalent to the emptiness of the intersection of the supports of the divisors:

$$\Big\{ \sum_{i\in\mathfrak{K}} \mathrm{div}(u_i) \Big\}_{\mathfrak{K}\subset\mathfrak{I}_*^\circ,\ \mathfrak{K}\cup(\mathfrak{I}_*\setminus\mathfrak{I}_*^\circ)\in\mathcal{C}_{\mathrm{inc}}}. \tag{4.10}$$

We will perform a Möbius inversion in order to remove the conditions on the intersection of the supports.

DEFINITIONS AND NOTATION 4.1. Let $\mathfrak{I}_* \in \mathcal{C}_{\mathrm{inc}}$. Let $\mu^\circ_{X,\mathfrak{I}_*^\circ} : \mathbf{N}^{\mathfrak{I}_*^\circ} \to \mathbf{Z}$ be defined recursively by

$$\forall \boldsymbol{n} \in \mathbf{N}^{\mathfrak{I}_*^\circ}, \qquad \sum_{0\le \boldsymbol{m}\le \boldsymbol{n}} \mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{m}) = \begin{cases} 1 & \text{if} \quad \displaystyle\operatorname*{Inf}_{\substack{\mathfrak{K}\subset\mathfrak{I}_*^\circ \\ \mathfrak{K}\cup(\mathfrak{I}_*\setminus\mathfrak{I}_*^\circ)\in\mathcal{C}_{\mathrm{inc}}}} \Big(\sum_{i\in\mathfrak{K}} n_i\Big) = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

REMARK 4.2. For $i \in \mathfrak{I}_*^\circ$ one has $\mathfrak{I}_* \setminus \{i\} \in \mathcal{C}_{\mathrm{inc}}$. By definition, then, for $\boldsymbol{n} \in \mathbf{N}^{\mathfrak{I}_*^\circ}$ one has $\mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{n}) = 0$ as soon as $\sum_{i\in\mathfrak{I}_*^\circ} n_i = 1$.

For almost all primes p and every positive integer r, let $\mathscr{C}_{p,r} \stackrel{\mathrm{def}}{=} \mathscr{C}_p \otimes \mathbf{F}_{p^r}$. Let $\mathscr{C}^{(0)}_{p,r}$ be the set of closed points of $\mathscr{C}_{p,r}$, let $\mathrm{Div}(\mathscr{C}_{p,r})$ be the group of its divisors, and let $\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})$ be the monoid of its effective divisors. For $P \in \mathscr{C}^{(0)}_{p,r}$ and $\boldsymbol{n} \in \mathbf{N}^{\mathfrak{I}_*^\circ}$ we set

$$\mu_{X,\mathfrak{I}_*^\circ,p,r}((n_i P)_{i\in\mathfrak{I}_*^\circ}) \stackrel{\mathrm{def}}{=} \mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{n}). \tag{4.12}$$

By additivity, we extend $\mu_{X,\mathfrak{I}_*^\circ,p,r}$ to a function $\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ} \to \mathbf{Z}$; this is the unique function satisfying, for all $\mathfrak{D} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}$,

$$\sum_{0\le\mathfrak{E}\le\mathfrak{D}} \mu_{X,\mathfrak{I}_*^\circ,p,r}(\mathfrak{E}) = \begin{cases} 1 & \text{if} \quad \displaystyle\operatorname*{Inf}_{\substack{\mathfrak{K}\subset\mathfrak{I}_*^\circ \\ \mathfrak{K}\cup(\mathfrak{I}_*\setminus\mathfrak{I}_*^\circ)\in\mathcal{C}_{\mathrm{inc}}}} \Big(\sum_{i\in\mathfrak{K}} \mathfrak{D}_i\Big) = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

DEFINITIONS AND NOTATION 4.3. We retain the notation described previously. For $j \in \mathfrak{I} \setminus I$, write $\mathscr{E}_j = \sum_{i\in I} a_{i,j}\mathscr{E}_i$ with $(a_{i,j}) \in \mathbf{Z}^I$.

Let $\mathfrak{I}_* \in \mathcal{C}_{\mathrm{inc}}$. Let p be a sufficiently large prime and r a positive integer. Abusing notation, we let $\mathrm{Pic}^\circ(\mathscr{C}_{p,r})$ denote a set of representatives in $\mathrm{Div}(\mathscr{C}_{p,r})$ of $\mathrm{Pic}^\circ(\mathscr{C}_{p,r})$. We fix a degree-1 divisor $\mathfrak{d}_1$ in $\mathrm{Div}(\mathscr{C}_{p,r})$. For $y \in \mathrm{Eff}(X)^\vee \cap \mathrm{NS}(X)^\vee$ satisfying the degeneracy condition for $\mathfrak{I}_*$ and for $\mathfrak{D} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ\cap I}$, $\mathcal{E} \in \mathrm{Div}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}$, $\mathfrak{E} \in \mathrm{Pic}^\circ(\mathscr{C}_{p,r})^{I\setminus\mathfrak{I}_*}$, and $K_1, K_2 \subset \mathfrak{I}_* \setminus I$, let

$$\mathcal{N}(y, \mathfrak{I}_*, I, K_1, K_2, p, r, \mathfrak{E}, \mathfrak{D}, \mathcal{E}) \tag{4.14}$$

denote the cardinality of the set of the elements $(t_j)_{j\in\mathfrak{I}_*\setminus I}$ of the product

$$\prod_{j\in\mathfrak{I}_*^\circ\setminus I} H^\circ\Bigg(\mathscr{C}_{p,r}, \mathscr{O}_{\mathscr{C}}\Bigg(-\mathcal{E}_j + \sum_{i\in\mathfrak{I}_*^\circ\cap I} a_{i,j}(\mathcal{D}_i + \mathcal{E}_i) + \sum_{i\in I\setminus\mathfrak{I}_*} a_{i,j}(\mathfrak{E}_i + \mathfrak{d}_1\langle y, \mathscr{E}_i\rangle)\Bigg)\Bigg) \times \prod_{j\in\mathfrak{I}_*\setminus(\mathfrak{I}_*^\circ\cup I)} H^\circ(\mathscr{C}_{p,r}, \mathscr{O}_{\mathscr{C}}) \tag{4.15}$$

that satisfy

$$t_j \neq 0 \quad \forall j \in K_1, \qquad t_j = 0 \quad \forall j \in K_2, \tag{4.16}$$

and

$$F((s_{\mathcal{D}_i}s_{\mathcal{E}_i})_{i\in\mathfrak{I}_*\cap I}, 0, \ldots, 0, (t_j s_{\mathcal{E}_j})_{j\in\mathfrak{I}_*\setminus I}, 0, \ldots, 0) = 0 \quad \forall F \in \mathscr{I}_{X,\mathfrak{I}}^{\mathrm{hom}}; \tag{4.17}$$

here we have set $s_{\mathcal{D}_i} = s_{\mathcal{E}_i} = s_{\mathcal{E}_j} = 1$ for $i \in I \cap (\mathfrak{I}_* \setminus \mathfrak{I}_*^\circ)$ and $j \in \mathfrak{I}_* \setminus (I \cup \mathfrak{I}_*^\circ)$.

For every subset K of $\mathfrak{I}_* \setminus I$ we set

$$\mathcal{N}(y, \mathfrak{I}_*, I, K, p, r) \stackrel{\mathrm{def}}{=} \sum_{\substack{\mathcal{E}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ} \\ \deg(\mathcal{E}_i)\le\langle y,\mathscr{E}_i\rangle,\, i\in\mathfrak{I}_*^\circ}} \mu_{X,\mathfrak{I}_*^\circ,p,r}(\mathcal{E}) \times \sum_{\substack{\mathfrak{E}\in\mathrm{Pic}^\circ(\mathscr{C}_{p,r})^{I\setminus\mathfrak{I}_*} \\ \mathfrak{D}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ\cap I} \\ \deg(\mathcal{D}_i)=\langle y,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i),\, i\in\mathfrak{I}_*^\circ\cap I}} \mathcal{N}(y, \mathfrak{I}_*, I, \emptyset, K, p, r, \mathfrak{E}, \mathfrak{D}, \mathcal{E}). \tag{4.18}$$

LEMMA 4.4. *With notation as before, let $y \in \mathrm{Eff}(X)^\vee \cap \mathrm{NS}(X)^\vee$ and $\mathfrak{I}_* \in \mathcal{C}_{\mathrm{inc}}$. Assume that y satisfies the degeneracy condition for $\mathfrak{I}_*$. Then, for almost all primes p and every positive integer r,*

$$[\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)(\mathbf{F}_{p^r})] = \sum_{K\subset\mathfrak{I}_*\setminus I} (-1)^{|K|}\mathcal{N}(y, \mathfrak{I}_*, I, K, p, r). \tag{4.19}$$

Proof. For almost all primes p, there is a natural isomorphism $\mathrm{NS}(X) \xrightarrow{\sim} \mathrm{NS}(X_p)$ and $\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)_p$ is isomorphic to $\mathrm{Mor}(\mathscr{C}_p, X_p, y, \mathfrak{I}_*)$. Thus from [Bo2, Sec. 1.2, Sec. 1.3] we have

$$[\mathrm{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)(\mathbf{F}_{p^r})] = \sum_{\substack{\mathcal{E} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ} \\ \deg(\mathcal{E}_i) \leq \langle y, \mathscr{E}_i \rangle,\, i \in \mathfrak{I}_*^\circ}} \mu_{X, \mathfrak{I}_*^\circ, p, r}(\mathcal{E}) \times \sum_{\substack{\mathfrak{E} \in \mathrm{Pic}^\circ(\mathscr{C}_{p,r})^{I \setminus \mathfrak{I}_*} \\ \mathfrak{D} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ \cap I} \\ \deg(\mathcal{D}_i) = \langle y, \mathscr{E}_i \rangle - \deg(\mathcal{E}_i),\, i \in \mathfrak{I}_*^\circ \cap I}} \mathcal{N}(y, \mathfrak{I}_*, I, \mathfrak{I}_* \setminus I, \emptyset, p, r, \mathfrak{E}, \mathfrak{D}, \mathcal{E}) \qquad (4.20)$$

and thus the result follows by inclusion-exclusion. Strictly speaking, in [Bo2] only the case $\mathfrak{I}_* = \mathfrak{I}$ is extensively treated; however, the general case may be addressed in the same way. □

Theorem 2.4 follows immediately from Lemma 3.1, Lemma 4.4, and the next proposition. Recall that, according to deformation theory, every irreducible component of $\mathrm{Mor}(\mathscr{C}, X, y)$ has dimension greater than or equal to the expected dimension $(1 - g_{\mathscr{C}}) \dim(X) + \langle y, -\mathscr{K}_X \rangle$.

PROPOSITION 4.5. *With notation as before, assume that X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$ (cf. Definition 2.2). Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve of genus $g_{\mathscr{C}}$ and let $y \in \mathrm{Eff}(X)^\vee$. Assume that y fulfills the numerical inequality*

$$\left\langle y, \frac{1}{[\mathfrak{I} \setminus I] - 1} \sum_{j \in \mathfrak{I} \setminus I} \mathscr{E}_j - \mathscr{D}_{\mathrm{tot}} \right\rangle \geq \mathrm{Sup}\left(1, \frac{4}{[\mathfrak{I} \setminus I] - 1}\right) g_{\mathscr{C}} \dim(X). \qquad (4.21)$$

(1) *For every nonempty subset K of $\mathfrak{I} \setminus I$,*

$$\lim_{r \to \infty} p^{-r[(1-g_{\mathscr{C}}) \dim(X) + \langle y, -\mathscr{K}_X \rangle]} \mathcal{N}(y, \mathfrak{I}, I, K, p, r) = 0. \qquad (4.22)$$

(2) *Let $\mathfrak{I}_* \in \mathcal{C}_{\mathrm{inc}}$ be such that y satisfies the degeneracy condition for $\mathfrak{I}_*$. Let K be a subset of $\mathfrak{I}_* \setminus I$. Assume that $\mathfrak{I}_* \setminus I$ is a proper subset of $\mathfrak{I} \setminus I$ or that K is nonempty, and assume that y satisfies the numerical inequalities*

$$\langle y, \mathscr{E}_i \rangle \geq g_{\mathscr{C}} \quad \forall i \in I \setminus \mathfrak{I}_*. \qquad (4.23)$$

Then

$$\lim_{r \to \infty} p^{-r[(1-g_{\mathscr{C}}) \dim(X) + \langle y, -\mathscr{K}_X \rangle]} \mathcal{N}(y, \mathfrak{I}_*, I, K, p, r) = 0. \qquad (4.24)$$

(3) *Let $\mathfrak{I}_*$ be an element of $\mathcal{C}_{\mathrm{inc}}$ such that $\mathfrak{I} \setminus I \subset \mathfrak{I}_*$ and y satisfies the degeneracy condition for $\mathfrak{I}_*$. Assume, moreover, that Assumptions 5.7 hold for $(\mathfrak{I}_*, I)$ and that y satisfies the numerical inequalities*

$$\langle y, \mathscr{E}_i \rangle \geq g_{\mathscr{C}} \quad \forall i \in I \setminus \mathfrak{I}_*. \qquad (4.25)$$

In case $\mathfrak{I}_*$ is a proper subset of $\mathfrak{I}$, assume that at least one of these inequalities is strict. Then

$$\lim_{r \to \infty} p^{-r[(1-g_{\mathscr{C}}) \dim(X) + \langle y, -\mathscr{K}_X \rangle]} \mathcal{N}(y, \mathfrak{I}_*, I, \emptyset, p, r) = \begin{cases} 1 & \text{if } \mathfrak{I}_* = \mathfrak{I}, \\ 0 & \text{otherwise.} \end{cases} \qquad (4.26)$$

Our task is now to prove Proposition 4.5. The next section contains some preliminary technical lemmas that will be needed during the proof.

5. Technical Lemmas

The following lemma is a slight variation of [Bo4, Lemme 12] but with an identical proof.

LEMMA 5.1. *Let n be a positive integer, $\rho > 1$ a real number, and $(a_{\boldsymbol{d}}) \in \mathbf{C}^{\mathbf{N}^n}$. Assume that the series $F(\boldsymbol{t}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{d}\in\mathbf{N}^n} a_{\boldsymbol{d}}\boldsymbol{t}^{\boldsymbol{d}}$ converges absolutely on a polydisc of multi-radius $(\rho^{-1+\nu}, \dots, \rho^{-1+\nu})$ with $\nu > 0$. For every positive real number η such that $\eta < \rho^{-1+\nu}$, denote by $\|F\|_\eta$ the quantity* $\operatorname{Sup}_{|t_1|=\dots=|t_n|=\eta}|F(\boldsymbol{t})|$.

Define $(b_{\boldsymbol{d}}) \in \mathbf{C}^{\mathbf{N}^n}$ by

$$\sum_{\boldsymbol{d}\in\mathbf{N}^n} b_{\boldsymbol{d}}\boldsymbol{t}^{\boldsymbol{d}} \stackrel{\text{def}}{=} \frac{F(\boldsymbol{t})}{(1-\rho t_1)\cdots(1-\rho t_n)}. \tag{5.27}$$

Then, for every $\varepsilon > 0$ such that $\varepsilon < \nu$, for all $\boldsymbol{d} \in \mathbf{N}^n$ we have

$$|b_{\boldsymbol{d}}| \le \frac{1+n\rho^{-\varepsilon}}{(1-\rho^{-\varepsilon})^n}\|F\|_{\rho^{-1+\varepsilon}}\rho^{|\boldsymbol{d}|} \tag{5.28}$$

and

$$|b_{\boldsymbol{d}} - F(\rho^{-1},\dots,\rho^{-1})\rho^{|\boldsymbol{d}|}| \le \frac{\rho^{-\varepsilon}}{(1-\rho^{-\varepsilon})^n}\|F\|_{\rho^{-1+\varepsilon}} \sum_{1\le i\le n} \rho^{(1-\varepsilon)d_i+\sum_{j\neq i} d_j}. \tag{5.29}$$

LEMMA 5.2. *Let p be a sufficiently large prime number. For $v \in \mathscr{C}_{p,r}^{(0)}$ denote by f_v the degree of the extension $\kappa_v/\mathbf{F}_{p^r}$, where κ_v is the residue field at v. Let $\theta\colon \mathbf{N}_{>0}\times\mathbf{N}_{>0} \to \mathbf{R}_{\ge 0}$ be an application such that there exist $C \ge 0$ and $\eta > 0$ satisfying*

$$\theta(r,f) \le Cp^{-rf(1+\eta)} \quad \forall (r,f) \in \mathbf{N}_{>0}\times\mathbf{N}_{>0}. \tag{5.30}$$

Then

$$\lim_{r\to\infty} \prod_{v\in\mathscr{C}_{p,r}^{(0)}} [1+\theta(r,f_v)] = 1. \tag{5.31}$$

Proof. Since

$$[\{v\in\mathscr{C}_{p,r}^{(0)},\ f_v = f\}] \le [\mathscr{C}(\mathbf{F}_{p^{rf}})]s = p^{rf} + \underset{f\to+\infty}{\mathcal{O}}(p^{rf/2}), \tag{5.32}$$

we have

$$[1+\theta(r,f)]^{[\{v\in\mathscr{C}_{p,r}^{(0)},\, f_v=f\}]} \le \exp\Big[Cp^{-rf\eta} + \underset{f\to+\infty}{\mathcal{O}}(p^{rf(-1/2+\eta)})\Big]; \tag{5.33}$$

hence the result follows by dominated convergence. □

LEMMA 5.3. *Let $\mathfrak{I}_* \in \mathcal{C}_{\text{inc}}$, $r \in \mathbf{N}$, and $\boldsymbol{d} \in \mathbf{R}^{\mathfrak{I}_*^\circ}$. We set*

$$\Theta(\mathfrak{I}_*, p, r, \boldsymbol{d}) \stackrel{\text{def}}{=} \sum_{\mathcal{E}\in\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}} |\mu_{X,\mathfrak{I}_*^\circ,p,r}(\mathcal{E})| p^{-r\sum_{i\in\mathfrak{I}_*^\circ} d_i \deg(\mathcal{E}_i)}. \tag{5.34}$$

Assume that, for every $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ}$ *such that* $\mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{e}) \neq 0$, *one has* $\sum_{i \in \mathfrak{I}_*^\circ} d_i e_i > 1$. *Then*

$$\lim_{r \to +\infty} \Theta(\mathfrak{I}_*, p, r, \boldsymbol{d}) = 1. \tag{5.35}$$

Proof. $\Theta(\mathfrak{I}_*, p, r, \boldsymbol{d}) - 1$ is nonnegative and bounded from above by

$$-1 + \prod_{v \in \mathscr{C}_{p,r}^{(0)}} \sum_{\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ}} |\mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{e})| p^{-r f_v \sum_{i \in \mathfrak{I}_*^\circ} d_i e_i}, \tag{5.36}$$

and one can conclude by using Lemma 5.2. □

REMARK 5.4. By Remark 4.2, the assumption on $\boldsymbol{d}$ holds, for example, if $d_i > \frac{1}{2}$ for all i.

The following crucial lemma gives an estimate of the number of sections satisfying the equation of the Cox ring of a linear intrinsic hypersurface. It follows easily from [Bo4, Prop. 14], whose proof rests on elementary linear algebra and the Riemann–Roch theorem for curves.

LEMMA 5.5. *With notation as before, assume that X is a linear intrinsic hypersurface with respect to* $(\mathfrak{I}, I)$ (*cf. Definition 2.2*). *Let* $\mathfrak{I}_* \in \mathcal{C}_{\text{inc}}$ *and let*

$$y \in \operatorname{Pic}(X)^\vee \cap \operatorname{Eff}(X)^\vee$$

satisfy the degeneracy condition for $\mathfrak{I}_*$. *Let* $\boldsymbol{\mathfrak{E}} \in \operatorname{Pic}^\circ(\mathscr{C}_{p,r})^{I \setminus \mathfrak{I}_*}$, $\boldsymbol{\mathcal{E}} \in \operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}$, *and* $\boldsymbol{\mathcal{D}} \in \operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ \cap I}$ *such that* $\deg(\mathcal{D}_i) = \langle y, \mathscr{E}_i \rangle - \deg(\mathcal{E}_i)$ *for all* $i \in \mathfrak{I}_*^\circ \cap I$. *We set* $\mathcal{D}_i = \mathcal{E}_i = \mathcal{E}_j = 0$ *for* $i \in I \setminus \mathfrak{I}_*^\circ$ *and* $j \in \mathfrak{I}_* \setminus (\mathfrak{I}_*^\circ \cup I)$.

(1) *Let* $K \subset \mathfrak{I}_* \setminus I$, *and let* $(\alpha_j)_{j \in \mathfrak{I}_* \setminus (I \cup K)}$ *be nonnegative real numbers such that* $\sum \alpha_j = 1$. *Then*

$$\begin{aligned} &\log_{p^r}[\mathcal{N}(\mathfrak{I}, I, \emptyset, K, p, r, \boldsymbol{\mathfrak{E}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}})] \\ &\qquad \leq [\mathfrak{I}_* \setminus (I \cup K)] - 1 + \sum_{j \in \mathfrak{I}_* \setminus (I \cup K)} (1 - \alpha_j)(\langle y, \mathscr{E}_j \rangle - \deg(\mathcal{E}_j)). \end{aligned} \tag{5.37}$$

(2) *Assume that* $\mathfrak{I} \setminus I \subset \mathfrak{I}_*$. *Then either*

$$\begin{aligned} &\log_{p^r}[\mathcal{N}(\mathfrak{I}_*, I, \emptyset, \emptyset, p, r, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}})] \\ &\qquad \leq [\mathfrak{I} \setminus I] - 1 + \Big\langle y, -\mathscr{D}_{\text{tot}} + \sum_{j \in \mathfrak{I} \setminus I} \mathscr{E}_j \Big\rangle - \sum_{j \in \mathfrak{I} \setminus I} \deg(\mathcal{E}_j) \\ &\qquad\quad + \deg\Big[\inf_{j \in \mathfrak{I} \setminus I} \Big(\sum_{i \in \mathfrak{I}_* \cap I} b_{i,j}(\mathcal{E}_i + \mathcal{D}_i) + \mathcal{E}_j \Big)\Big] \end{aligned} \tag{5.38}$$

or

$$\begin{aligned} &\log_{p^r}[\mathcal{N}(\mathfrak{I}_*, I, \emptyset, \emptyset, p, r, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}})] \\ &\qquad \leq [\mathfrak{I} \setminus I] - 2 + \left(1 - \frac{1}{[\mathfrak{I} \setminus I] - 1}\right) \sum_{j \in \mathfrak{I} \setminus I} (\langle y, \mathscr{E}_j \rangle - \deg(\mathcal{E}_j)). \end{aligned} \tag{5.39}$$

(3) *Assume that $\mathfrak{I} \setminus I \subset \mathfrak{I}_*$, and assume there exists a numbering of $\mathfrak{I} \setminus I$ such that, for all* $1 \leq j \leq [\mathfrak{I} \setminus I] - 1$,

$$\langle y, \mathscr{E}_j + \mathscr{E}_{j+1} - \mathscr{D}_{\mathrm{tot}} \rangle \geq \deg(\mathcal{E}_j) + \deg(\mathcal{E}_{j+1}) + 2g_{\mathscr{C}} - 1. \tag{5.40}$$

Then it follows that

$$\begin{aligned}\log_{p^r}&[\mathcal{N}(\mathfrak{I}_*, I, \emptyset, \emptyset, p, r, \mathfrak{D}, \mathcal{E})] \\ &= ([\mathfrak{I} \setminus I] - 1)(1 - g_{\mathscr{C}}) + \left\langle y, -\mathscr{D}_{\mathrm{tot}} + \sum_{j \in \mathfrak{I} \setminus I} \mathscr{E}_j \right\rangle - \sum_{j \in \mathfrak{I} \setminus I} \deg(\mathcal{E}_j) \\ &\quad + \deg\left[\inf_{j \in \mathfrak{I} \setminus I} \left(\sum_{i \in \mathfrak{I}_* \cap I} b_{i,j}(\mathcal{E}_i + \mathcal{D}_i) + \mathcal{E}_j \right) \right].\end{aligned} \tag{5.41}$$

Next we introduce some combinatoric series derived from the Cox ring equation of a linear intrinsic hypersurface. We assume that X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$ (cf. Definition 2.2). Let $\mathfrak{I}_*$ be an element of $\mathcal{C}_{\mathrm{inc}}$ such that $\mathfrak{I} \setminus I \subset \mathfrak{I}_*$. For $\boldsymbol{e} \in \mathbf{N}^{\mathfrak{I}_*^\circ}$, we set

$$\begin{aligned}&F(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) \\ &\quad \stackrel{\mathrm{def}}{=} \sum_{\boldsymbol{d} \in \mathbf{N}^{\mathfrak{I}_*^\circ \cap I}} \rho^{\mathrm{Inf}_{j \in \mathfrak{I} \setminus I}\left(e_j + \sum_{i \in \mathfrak{I}_*^\circ \cap I} b_{i,j}(d_i + e_i)\right)} \boldsymbol{t}^{\boldsymbol{d}} \in k[[\rho, (t_i)_{i \in \mathfrak{I}_*^\circ \cap I}]]\end{aligned} \tag{5.42}$$

(where $e_j = 0$ for $j \notin I \cup \mathfrak{I}_*^\circ$) and

$$\tilde{F}(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) \stackrel{\mathrm{def}}{=} \left(\prod_{i \in \mathfrak{I}_*^\circ \cap I} 1 - t_i \right) F(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}). \tag{5.43}$$

Remark 5.6. Recall that the sets $I_j = \{i \in I, b_{i,j} \neq 0\}$ are assumed to be pairwise disjoint. Let m be the lowest common multiple of the $b_{i,j}$ that are positive. After partitioning $\mathbf{N}^{\mathfrak{I}_*^\circ \cap I}$ in accordance with the various remainders of the d_i modulo $m/b_{i,j}$ for $i \in I_j$, one sees easily that $F(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t})$ is rational; in fact, it is possible to obtain thereby an explicit formula that allows for calculations by a computer algebra program. Arguing as in the proof of [Bo4, Prop. 57], one can show that

$$\prod_{(i_j) \in \prod_{j \in \mathfrak{I} \setminus I} I_j} \left(1 - \rho^m \prod t_i^{m/b_{i_j,j}} \right) \prod_{i \in I} (1 - t_i) F(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) \tag{5.44}$$

is a polynomial with explicitly bounded degrees in the t_i; however, this approach does not seem well suited to computational purposes.

Let us write $\tilde{F}(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) = \sum_{\boldsymbol{d} \in \mathbf{N}^{\mathfrak{I}_*^\circ \cap I}} P(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}} \boldsymbol{t}^{\boldsymbol{d}}$, where $P(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ is polynomial with respect to the variable ρ. We need the following assumptions on the series $\tilde{F}(\rho, \mathfrak{I}_*, I, \boldsymbol{e}, \boldsymbol{t})$.

Assumptions 5.7. (1) For every sufficiently small $\eta > 0$ and every $\boldsymbol{d} \neq 0$,

$$(1 - \eta)|\boldsymbol{d}| \geq 1 + \eta + \deg_\rho P(\mathfrak{I}_*, I, 0, \boldsymbol{d}, \rho). \tag{5.45}$$

(2) For every $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ}$ one can write

$$\tilde{F}(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) = \left(1 + \sum_{\boldsymbol{d} \neq 0} Q(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}} \boldsymbol{t}^{\boldsymbol{d}}\right) R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}), \tag{5.46}$$

where $Q(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ is polynomial with respect to ρ and $R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t})$ is polynomial with respect to ρ and $\boldsymbol{t}$. Moreover, for every sufficiently small $\eta > 0$ and every $\boldsymbol{d} \neq 0$,

$$(1-\eta)|\boldsymbol{d}| \geq 1 + \eta + \deg_\rho Q(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}. \tag{5.47}$$

(3) For $\boldsymbol{e} \in \mathbf{N}^{\mathfrak{I}_*^\circ}$, let $R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ denote the coefficient of $\boldsymbol{t}^{\boldsymbol{d}}$ in $R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t})$ and set

$$C(\mathfrak{I}_*, I, \boldsymbol{e}) \stackrel{\text{def}}{=} \operatorname*{Sup}_{\substack{\boldsymbol{d} \in \mathbf{N}^{\mathfrak{I}_*^\circ \cap I} \\ R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}} \neq 0}} [\deg_\rho R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}} - |\boldsymbol{d}|]. \tag{5.48}$$

Then, for all $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ} \setminus \{(0,\ldots,0)\}$,

$$\mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{e}) \neq 0 \implies C(\mathfrak{I}_*, I, \boldsymbol{e}) - |\boldsymbol{e}| \leq -2. \tag{5.49}$$

REMARK 5.8. From (5.42) and (5.43) we see immediately that, for every $\boldsymbol{d}$, the ρ-polynomial $P(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ has at most $[\mathfrak{I}_*^\circ \cap I]$ nonzero coefficients, whose absolute values are bounded by $[\mathfrak{I}_*^\circ \cap I]$.

Moreover, if Assumption 5.7(2) holds then, for every $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*}$ and letting $D(\mathfrak{I}_*, I, \boldsymbol{e})$ denote the degree with respect to $\boldsymbol{t}$ of $R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t})$, it is straightforward to check that, for every sufficiently small $\eta > 0$, the degree with respect to ρ of $P(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ is bounded by $|\boldsymbol{d}|(1-\eta) + C(\mathfrak{I}_*, I, \boldsymbol{e}) + \eta D(\mathfrak{I}_*, I, \boldsymbol{e})$.

Hence if Assumptions 5.7(1) and 5.7(2) both hold, then letting p denote a sufficiently large prime and setting

$$c(\mathfrak{I}_*, I, p, \eta) \stackrel{\text{def}}{=} [\mathfrak{I}_*^\circ \cap I]^2 \left[\prod_{i \in \mathfrak{I}_*^\circ \cap I} \frac{1}{1 - p^{-\eta/2}} - 1\right], \tag{5.50}$$

for all positive integers r and f and every sufficiently small $\eta > 0$ one has

$$\|-1 + \tilde{F}(p^{rf}, \mathfrak{I}_*, I, 0, \boldsymbol{t})\|_{p^{r(-1+\eta/2)}} \leq c(\mathfrak{I}_*, p, \eta) . p^{rf(-1-\eta)} \tag{5.51}$$

and, for every $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ}$,

$$\|\tilde{F}(p^{rf}, \mathfrak{I}_*, I, \boldsymbol{e}, \boldsymbol{t})\|_{p^{r(-1+\eta/2)}} \leq c(\mathfrak{I}_*, I, p, \eta) . p^{rf[C(\mathfrak{I}_*, I, \boldsymbol{e}) + \eta D(\mathfrak{I}_*, I, \boldsymbol{e})]}. \tag{5.52}$$

In particular, for every sufficiently small $\eta > 0$ there exists a positive integer R_η such that, for all $r \geq R_\eta$ and all f,

$$\operatorname*{Inf}_{|t_1| = \cdots = |t_n| = p^{r(-1+\eta/2)}} |\tilde{F}(p^{rf}, \mathfrak{I}_*, I, 0, \boldsymbol{t})| \geq \frac{1}{2}. \tag{5.53}$$

6. Proof of the Main Theorem

In this section we prove Proposition 4.5—and hence also Theorem 2.4, as remarked before the statement of the proposition.

Let p be a sufficiently large prime number. Thanks to the Riemann hypothesis for abelian varieties, one has

$$[\mathrm{Pic}^{\circ}(\mathscr{C}_{p,r})] \underset{r\to+\infty}{\sim} p^{rg_{\mathscr{C}}}. \tag{6.54}$$

We write the Hasse–Weil zeta function of $\mathscr{C}_{p,r}$ as $\frac{P_{p,r}(t)}{(1-t)(1-p^r t)}$. By the Riemann hypothesis for curves, there exists a positive constant c depending only on $g_{\mathscr{C}}$ and such that, for every t satisfying $|t| < p^{-r/2}$,

$$|P_{p,r}(t)| \leq 1 + c|t|p^{r/2}. \tag{6.55}$$

From this one easily deduces that, for every sufficiently small positive ε,

$$\limsup_{r\to+\infty} \left\| \frac{P_{p,r}(t)}{1-t} \right\|_{p^{r(-1+\varepsilon)}} \leq 1. \tag{6.56}$$

Let us denote by $\mathfrak{N}(p,r,d)$ the cardinality of $\{\mathfrak{D} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r}),\, \deg(\mathfrak{D}) = d\}$. From Lemma 5.1 we obtain that, for every sufficiently large r,

$$\mathfrak{N}(p,r,d) \leq 2p^{rd}. \tag{6.57}$$

In what follows we shall ease the notation by systematically dropping the indices y, $\mathfrak{I}_*$, I, and p from the name of the previously introduced functions, since they may be assumed to remain fixed throughout the proof. Also, we use $h_{p,r}$ to denote the cardinality of $[\mathrm{Pic}^{\circ}(\mathscr{C}_{p,r})]$. By (6.54), if $\mathfrak{I}_*$ is a subset of $\mathfrak{I}$ such that (4.25) holds then

$$\lim_{r\to+\infty} h_{p,r}^{[I\setminus\mathfrak{I}_*]} p^{-r\sum_{i\in I\setminus\mathfrak{I}_*}\langle y,\mathscr{E}_i\rangle} = \begin{cases} 0 & \text{if at least one of the inequalities in (4.25) is strict,} \\ 1 & \text{otherwise.} \end{cases} \tag{6.58}$$

We first prove part (1) of the proposition. Here $\mathfrak{I}_* = \mathfrak{I}$ and we set

$$\mathcal{N}_2(r,(\beta_j)_{j\in\mathfrak{I}\setminus I}) \overset{\mathrm{def}}{=} \sum_{\substack{\mathcal{E}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}} \\ \deg(\mathcal{E}_i)\leq\langle y,\mathscr{E}_i\rangle,\, i\in\mathfrak{I}}} |\mu_{X,r}(\mathcal{E})| p^{r\left[[\mathfrak{I}\setminus I]-2+\sum_{j\in\mathfrak{I}\setminus I}(1-\beta_j)(\langle y,\mathscr{E}_j\rangle-\deg(\mathcal{E}_j))\right]} \times \prod_{i\in I} \mathfrak{N}(r,\langle y,\mathscr{E}_i\rangle - \deg(\mathcal{E}_i)). \tag{6.59}$$

Let K be a nonempty subset of $\mathfrak{I}\setminus I$ and let $j_0 \in K$. From (5.37) we deduce the inequality

$$\begin{aligned}\mathcal{N}(K,r) &\leq \mathcal{N}_2\left(r, \left(\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\neq j_0}, (1)_{j_0}\right) \\ &\leq \mathcal{N}_2\left(r, \left(\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\in\mathfrak{I}\setminus I}\right).\end{aligned} \tag{6.60}$$

Recall from Remark 2.3 that $\dim(X) = [\mathfrak{I}\setminus I]-1$ and $-\mathscr{K}_X = \sum_{i\in\mathfrak{I}}\mathscr{E}_i - \mathscr{D}_{\mathrm{tot}}$. Thus, thanks to (6.57), for r large enough one has the inequality

$$p^{-r[(1-g_{\mathscr{C}})\dim(X)+\langle y,-\mathscr{K}_X\rangle]}\mathcal{N}_2\left(r,\left(\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\in\mathfrak{I}\setminus I}\right)$$
$$\leq 2^{[I]}p^{r\left[g_{\mathscr{C}}\dim(X)-1-\left\langle y,\sum_{j\in\mathfrak{I}\setminus I}\frac{1}{[\mathfrak{I}\setminus I]-1}\mathscr{E}_j-\mathscr{D}_{\text{tot}}\right\rangle\right]}$$
$$\times\,\Theta\left(r,(1)_{i\in I},\left(1-\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\in\mathfrak{I}\setminus I}\right). \quad (6.61)$$

Owing to (4.21), (6.54), Lemma 5.3, and Remark 5.4, we obtain

$$\lim_{r\to\infty}p^{-r[(1-g_{\mathscr{C}})\dim(X)+\langle y,-\mathscr{K}_X\rangle]}\mathcal{N}(K,r)=0, \quad (6.62)$$

thus proving part (1) of Proposition 4.5.

Next we prove part (2). Let $\mathfrak{I}_*$ be a subset of $\mathfrak{I}$ such that $\mathfrak{I}_*\setminus I$ is a nonempty proper subset of $\mathfrak{I}\setminus I$ and $K\subset\mathfrak{I}_*\setminus I$. Then $\mathcal{N}(K,r)\leq\mathcal{N}(\emptyset,r)$. Arguing as before, we obtain the inequality

$$p^{-r[(1-g_{\mathscr{C}})\dim(X)+\langle y,-\mathscr{K}_X\rangle]}\mathcal{N}(K,r)$$
$$\leq 2^{[I\cap\mathfrak{I}_*^\circ]}p^{rg_{\mathscr{C}}\dim(X)-r\left\langle y,\sum_{j\in\mathfrak{I}\setminus(I\cup\mathfrak{I}_*)}\mathscr{E}_j+\frac{1}{[\mathfrak{I}_*\setminus I]}\sum_{j\in\mathfrak{I}_*\setminus I}\mathscr{E}_j-\mathscr{D}_{\text{tot}}\right\rangle}p^{-r\left\langle y,\sum_{i\in I\setminus\mathfrak{I}_*}\mathscr{E}_i\right\rangle}$$
$$\times\, h_{p,r}^{[I\setminus\mathfrak{I}_*]}\Theta\left(r,(1)_{i\in\mathfrak{I}_*\cap I},\left(1-\frac{1}{[\mathfrak{I}_*\setminus I]}\right)_{j\in\mathfrak{I}_*\setminus I}\right). \quad (6.63)$$

Since $[\mathfrak{I}_*\setminus I]\leq[\mathfrak{I}\setminus I]-1$, we can use (4.21), (6.58), and Lemma 5.3 to obtain

$$\lim_{r\to\infty}p^{-r[(1-g_{\mathscr{C}})\dim(X)+\langle y,-\mathscr{K}_X\rangle]}\mathcal{N}(K,r)=0. \quad (6.64)$$

The case where $\mathfrak{I}_*\setminus I=\mathfrak{I}\setminus I$ and $K\neq\emptyset$ is similar. If $\mathfrak{I}_*\setminus I=\emptyset$, then $\mathcal{N}(K,r)=1$ and the result is straightforward. This establishes Proposition 4.5(2).

Finally, we prove part (3). Set

$$\varphi(\mathfrak{D},\mathfrak{F},\mathfrak{G})\overset{\text{def}}{=}\left\langle y,-\mathscr{D}_{\text{tot}}+\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j\right\rangle-\sum_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j)$$
$$+\deg\left[\inf_{j\in\mathfrak{I}\setminus I}\left(\sum_{i\in I\cap\mathfrak{I}_*^\circ}b_{i,j}(\mathcal{E}_i+\mathcal{D}_i)+\mathcal{E}_j\right)\right], \quad (6.65)$$

$$\mathcal{N}_0(r)\overset{\text{def}}{=}h_{p,r}^{[I\setminus\mathfrak{I}_*]}\sum_{\substack{\mathfrak{D}\in\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{I\cap\mathfrak{I}_*^\circ}\\ \deg(\mathcal{D}_i)=\langle y,\mathscr{E}_i\rangle,\ i\in I\cap\mathfrak{I}_*^\circ}}p^{r[([\mathfrak{I}\setminus I]-1)(1-g_{\mathscr{C}})+\varphi(\mathfrak{D},0,0)]}, \quad (6.66)$$

and

$$\mathcal{N}_1^*(r)\overset{\text{def}}{=}\sum_{\substack{\boldsymbol{\mathcal{E}}\in\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}\setminus\{(0,\dots,0)\}\\ \deg(\mathcal{E}_i)\leq\langle y,\mathscr{E}_i\rangle,\ i\in\mathfrak{I}}}|\mu_{X,r}(\boldsymbol{\mathcal{E}})|h_{p,r}^{[I\setminus\mathfrak{I}_*]}$$
$$\times\sum_{\substack{\mathfrak{D}\in\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{I\cap\mathfrak{I}_*^\circ}\\ \deg(\mathcal{D}_j)=\langle y,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i),\ i\in I\cap\mathfrak{I}_*^\circ}}p^{r[([\mathfrak{I}\setminus I]-1)(1-g_{\mathscr{C}})+\varphi(\mathfrak{D},\mathfrak{F},\mathfrak{G})]}. \quad (6.67)$$

We fix a numbering of $\mathfrak{I}\setminus I$ and, for $j_0\in\mathfrak{I}\setminus I$, set

$$\mathcal{N}_{1,j_0}(r) \stackrel{\text{def}}{=} \sum_{\substack{\mathcal{E}\in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ} \\ \deg(\mathcal{E}_i)\leq \langle y,\mathscr{E}_i\rangle,\, i\in\mathfrak{I}_*^\circ \\ \langle y,\mathscr{E}_{j_0}+\mathscr{E}_{j_0+1}-\mathscr{D}_{\mathrm{tot}}\rangle \leq \deg(\mathcal{E}_{j_0})+\deg(\mathcal{E}_{j_0+1})+2g_{\mathscr{C}}-2}} |\mu_{X,r}(\mathcal{E})| h_{p,r}^{[I\setminus\mathfrak{I}_*]} \times \sum_{\substack{\mathcal{D}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{I\cap\mathfrak{I}_*^\circ} \\ \deg(\mathcal{D}_j)\leq\langle y,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i),\, i\in I\cap\mathfrak{I}_*^\circ}} p^{r[([\mathfrak{I}\setminus I]-1)+\varphi(y,\mathcal{D},\mathcal{F},\mathcal{G})]}. \tag{6.68}$$

For $j_0, j_1 \in \mathfrak{I}\setminus I$ with $j_0 \neq j_1$,

$$\mathscr{G}_{j_0}+\mathscr{G}_{j_1}-\mathscr{D}_{\mathrm{tot}} = \sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j - ([\mathfrak{I}\setminus I]-1)\mathscr{D}_{\mathrm{tot}} + \sum_{j\in\mathfrak{I}\setminus(I\cup\{j_0,j_1\})}(\mathscr{D}_{\mathrm{tot}}-\mathscr{E}_j). \tag{6.69}$$

By definition of $\mathscr{D}_{\mathrm{tot}}$, we have $\mathscr{D}_{\mathrm{tot}}-\mathscr{E}_j \in \mathrm{Eff}(X)$. Using (4.21) now yields the inequality

$$\langle y, \mathscr{E}_{j_0}+\mathscr{E}_{j_1}-\mathscr{D}_{\mathrm{tot}}\rangle \geq 4g_{\mathscr{C}}\dim(X). \tag{6.70}$$

Hence from parts (2) and (3) of Lemma 5.5 we deduce the inequality

$$|\mathcal{N}(\emptyset,r)-\mathcal{N}_0(r)| \leq \mathcal{N}_1^*(r) + \mathcal{N}_2\left(r, \left(\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\in\mathfrak{I}\setminus I}\right) + \sum_{1\leq j\leq[\mathfrak{I}\setminus I]-1}\mathcal{N}_{1,j}(r). \tag{6.71}$$

We first show that

$$\lim_{r\to+\infty} p^{-r[\dim(X)(1-g_{\mathscr{C}})+\langle y,-\mathscr{K}_X\rangle]}\mathcal{N}_1^*(r) = 0. \tag{6.72}$$

The quantity involved in (6.72) equals

$$\sum_{\substack{\mathcal{E}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}\setminus\{(0,\dots,0)\} \\ \deg(\mathcal{E}_i)\leq\langle y,\mathscr{E}_i\rangle,\, i\in\mathfrak{I}_*^\circ}} |\mu_{X,r}(\mathcal{E})| p^{-r\sum_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j)} \times p^{-r\langle y,\sum_{i\in I}\mathscr{E}_i\rangle} h_{p,r}^{[I\setminus\mathfrak{I}_*]} a(r,\mathcal{E},(\langle y,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i))_{i\in I\cap\mathfrak{I}_*^\circ}); \tag{6.73}$$

here, for $\boldsymbol{d}\in\mathbf{N}^{I\cap\mathfrak{I}_*^\circ}$, we have set

$$a(r,\mathcal{E},\boldsymbol{d}) \stackrel{\text{def}}{=} \sum_{\substack{\mathcal{D}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{I\cap\mathfrak{I}_*^\circ} \\ \deg(\mathcal{D}_i)=d_i,\, i\in I\cap\mathfrak{I}_*^\circ}} p^{r\deg\left[\inf_{j\in\mathfrak{I}\setminus I}\left(\sum_{i\in I\cap\mathfrak{I}_*^\circ} b_{i,j}(\mathcal{E}_i+\mathcal{D}_i)+\mathcal{E}_j\right)\right]}. \tag{6.74}$$

Setting $G(r,\mathcal{E},\boldsymbol{t}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{d}\in\mathbf{N}^{I\cap\mathfrak{I}_*^\circ}} a(r,\mathcal{E},\boldsymbol{d})\boldsymbol{t}^{\boldsymbol{d}}$, we have

$$G(r,\mathcal{E},\boldsymbol{t}) = \prod_{v\in\mathscr{C}_{p,r}^{(0)}} F(v(\mathcal{E}),p^{rf_v},\boldsymbol{t}^{f_v}) = \prod_{i\in I\cap\mathfrak{I}_*^\circ}\frac{P_{p,r}(t_i)}{(1-t_i)(1-p^r t_i)}\prod_{v\in\mathscr{C}_{p,r}^{(0)}}\tilde{F}(v(\mathcal{E}),p^{rf_v},\boldsymbol{t}^{f_v}). \tag{6.75}$$

Hence for $r\geq 1$ and $\mathcal{E}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}}$ one may write

$$\tilde{G}(r,\mathcal{E},\boldsymbol{t}) \overset{\text{def}}{=} \Bigl(\prod_{i\in I\cap\mathfrak{I}_*^\circ} 1-p^r t_i\Bigr) G(r,\mathcal{E},\boldsymbol{t}) = \tilde{G}(r,0,\boldsymbol{t}) \prod_{\substack{v\in\mathscr{C}_{p,r}^{(0)}\\ v(\mathcal{E})\neq 0}} \frac{\tilde{F}(v(\mathcal{E}),p^{rf_v},\boldsymbol{t}^{f_v})}{\tilde{F}(0,p^{rf_v},\boldsymbol{t}^{f_v})}. \tag{6.76}$$

We next show that, for every sufficiently small $\eta > 0$,

$$\limsup_{r\to\infty} \|\tilde{G}(r,0,\boldsymbol{t})\|_{p^{-r(1+\eta)}} \leq 1. \tag{6.77}$$

Indeed, we have

$$\|\tilde{G}(r,0,\boldsymbol{t})\|_{p^{-r(1+\eta)}} \leq \left(\frac{\|P_{p,r}\|_{p^{-r(1+\eta)}}}{1-p^{-r(1+\eta)}}\right)^{[I\cap\mathfrak{I}_*^\circ]} \prod_{v\in\mathscr{C}_{p,r}^{(0)}} \|\tilde{F}(0,p^{rf_v},\boldsymbol{t}^{f_v})\|_{p^{-r(1+\eta)}}. \tag{6.78}$$

By (5.51), (6.55), and Lemma 5.2, we are done. It can similarly be shown that

$$\lim_{r\to\infty} \tilde{G}(r,0,(p^{-r},\dots,p^{-r})) = 1. \tag{6.79}$$

Now, owing to (5.52) and (5.53), for every sufficiently small $\eta > 0$ and every sufficiently large r we have

$$\|\tilde{G}(r,\mathcal{E},\boldsymbol{t})\|_{p^{r(-1+\eta/2)}} \leq \|\tilde{G}(r,0,\boldsymbol{t})\|_{p^{r(-1+\eta/2)}} \prod_{\substack{v\in\mathscr{C}_{p,r}^{(0)}\\ v(\mathcal{E})\neq 0}} 2c(\eta)p^{rf_v[C(v(\mathcal{E}))+\eta D(v(\mathcal{E}))]}. \tag{6.80}$$

We apply Lemma 5.1 to obtain that, for all $\boldsymbol{d}\in\mathbf{N}^{I\cap\mathfrak{I}_*^\circ}$, every sufficiently large r, and every $\mathcal{E}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}}$,

$$|a(r,\mathcal{E},\boldsymbol{d})| \leq p^{r|\boldsymbol{d}|}\frac{1+[I\cap\mathfrak{I}_*^\circ].p^{-r\eta}}{(1-p^{-r\eta})^{[I\cap\mathfrak{I}_*^\circ]}} \|\tilde{G}(r,0,\boldsymbol{t})\|_{p^{r(-1+\eta/2)}} \prod_{\substack{v\in\mathscr{C}_{p,r}^{(0)}\\ v(\mathcal{E})\neq 0}} 2c(\eta)p^{rf_v[C(v(\mathcal{E}))+\eta D(v(\mathcal{E}))]}. \tag{6.81}$$

Thus $p^{-r[\dim(X)(1-g_{\mathscr{C}})+\langle y,-\mathscr{K}_X\rangle]}\mathfrak{N}_1^*(r)$ is bounded from above by

$$\frac{1+[I\cap\mathfrak{I}_*^\circ]\,p^{-r\eta}}{(1-p^{-r\eta})^{[I\cap\mathfrak{I}_*^\circ]}} \|\tilde{G}(0,\boldsymbol{t})\|_{p^{r(-1+\eta/2)}} h_{p,r}^{[I\setminus\mathfrak{I}_*]}.p^{-r\sum_{i\in I\setminus\mathfrak{I}_*}\langle y,\mathscr{E}_i\rangle} \times\Bigl(-1+\prod_{v\in\mathscr{C}_{p,r}^{(0)}} 1+2c(\eta)\sum_{\boldsymbol{e}\in\{0,1\}^{\mathfrak{I}_*^\circ}\setminus(0,\dots,0)} |\mu_X^\circ(\boldsymbol{e})|p^{rf_v[C(\boldsymbol{e})+\eta D(\boldsymbol{e})-|\boldsymbol{e}|]}\Bigr). \tag{6.82}$$

Thanks to Assumption 5.7(3) and Lemma 5.2, the last factor in (6.82) tends to 0 as r approaches $+\infty$. Hence, by (6.77) and (6.58), we have proved (6.72).

Next we show that, for $j_0\in\mathfrak{I}\setminus I$,

$$\lim_{r\to+\infty} p^{-r[\dim(X)(1-g_{\mathscr{C}})+\langle y,-\mathscr{K}_X\rangle]}\mathfrak{N}_{1,j_0}(r) = 0. \tag{6.83}$$

The quantity involved in (6.83) equals

$$p^{r\dim(X)g_{\mathscr{C}}} \sum_{\substack{\mathcal{E}\in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}\\ \deg(\mathcal{E}_i)\leq \langle y,\mathscr{E}_i\rangle,\, i\in\mathfrak{I}_*^\circ \\ \langle y, \mathscr{E}_{j_0}+\mathscr{E}_{j_0+1}-\mathscr{D}_{\mathrm{tot}}\rangle \leq \deg(\mathcal{E}_{j_0})+\deg(\mathcal{E}_{j_0+1})+2g_{\mathscr{C}}-2}} |\mu_{X,r}(\mathcal{E})| p^{-r\sum_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j)} \times h_{p,r}^{[I\setminus\mathfrak{I}_*]} p^{-r\langle y,\sum_{i\in I}\mathscr{E}_i\rangle} a(r,\mathcal{E},(\langle y,\mathscr{E}_i\rangle - \deg(\mathcal{E}_i))_{i\in I\cap\mathfrak{I}_*^\circ}). \quad (6.84)$$

Using the last inequality in the description of the summation domain togther with (6.70), we obtain

$$-\frac{1}{4}[\deg(\mathcal{E}_{j_0})+\deg(\mathcal{E}_{j_0+1})] \leq -\dim(X)g_{\mathscr{C}} + \frac{g_{\mathscr{C}}}{2} - \frac{1}{2}; \quad (6.85)$$

we also see that (6.84) is bounded from above by

$$p^{-r(g_{\mathscr{C}}/2)} \sum_{\substack{\mathcal{E}\in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}\\ \deg(\mathcal{E}_i)\leq \langle y,\mathscr{E}_i\rangle,\, i\in\mathfrak{I}_*^\circ}} |\mu_{X,r}(\mathcal{E})| p^{-r\left[\sum_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j)-\frac{1}{4}[\deg(\mathcal{E}_{j_0})+\deg(\mathcal{E}_{j_0+1})]\right]} \times h_{p,r}^{[I\setminus\mathfrak{I}_*]} p^{\langle y,\sum_{i\in I}\mathscr{E}_i\rangle} a(r,\mathcal{E},(\langle y,\mathscr{E}_i\rangle - \deg(\mathcal{E}_i))_{i\in I\cap\mathfrak{I}_*^\circ}). \quad (6.86)$$

Now arguing as in the case of $\mathcal{N}_1^*(r)$, we see that $p^{-r[\dim(X)(1-g_{\mathscr{C}})+\langle y,-\mathscr{K}_X\rangle]}\mathcal{N}_{1,j_0}(r)$ is bounded from above by

$$p^{-r(g_{\mathscr{C}}+1)/2}.\frac{1+[I\cap\mathfrak{I}_*^\circ]p^{-r\eta}}{(1-p^{-r\eta})^{[I\cap\mathfrak{I}_*^\circ]}} \|\tilde{G}(r,0,\boldsymbol{t})\|_{p^{r(-1+\eta/2)}} h_{p,r}^{[I\setminus\mathfrak{I}_*]}.p^{-r\sum_{i\in I\setminus\mathfrak{I}_*}\langle y,\mathscr{E}_i\rangle} \times \prod_{v\in\mathscr{C}_{p,r}^{(0)}} 1+2c(\eta) \sum_{\boldsymbol{e}\in\{0,1\}^{\mathfrak{I}_*^\circ}\setminus(0,\ldots,0)} |\mu^\circ_{X,\mathfrak{I}}(\boldsymbol{e})| p^{rf_v\left[C(\boldsymbol{e})+\eta D(\boldsymbol{e})-|\boldsymbol{e}|+\frac{1}{4}(e_{j_0}+e_{j_0+1})\right]}. \quad (6.87)$$

By (6.77), Assumption 5.7(3), Lemma 5.2, and (6.58), this proves (6.83).

Finally we show that

$$\lim_{r\to+\infty} p^{-r[\dim(X)(1-g_{\mathscr{C}})+\langle y,-\mathscr{K}_X\rangle]}\mathcal{N}_0(r) = \begin{cases} 1 & \text{if } \mathfrak{I}_* = \mathfrak{I}, \\ 0 & \text{otherwise.}\end{cases} \quad (6.88)$$

Note that

$$\mathcal{N}_0(r) = p^{r\left[\dim(X)(1-g_{\mathscr{C}})+\left\langle y,-\mathscr{D}_{\mathrm{tot}}+\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j\right\rangle\right]} h_{p,r}^{[I\setminus\mathfrak{I}_*]} a(r,0,(\langle y,\mathscr{E}_i\rangle)_{i\in I\cap\mathfrak{I}_*^\circ}). \quad (6.89)$$

We can use Lemma 5.1 to deduce that

$$\begin{aligned} &|p^{-r[\dim(X)(1-g_{\mathscr{C}})+\langle y,-\mathscr{K}_X\rangle]}\mathcal{N}_0(r) - h_{p,r}^{[I\setminus\mathfrak{I}_*]}p^{-r\sum_{i\in I\setminus\mathfrak{I}_*}\langle y,\mathscr{E}_i\rangle}\tilde{G}(r,0,(p^{-r},\ldots,p^{-r}))| \\ &\quad\leq \frac{p^{-r(\eta/2)}}{(1-p^{-r(\eta/2)})^{[I\cap\mathfrak{I}_*^\circ]}} \|\tilde{G}(r,0,\boldsymbol{t})\|_{p^{r(-1+\eta/2)}} h_{p,r}^{[I\setminus\mathfrak{I}_*]} p^{-r\sum_{i\in I\setminus\mathfrak{I}_*}\langle y,\mathscr{E}_i\rangle} \\ &\quad\times \sum_{i\in I\cap\mathfrak{I}_*^\circ} p^{-r(\eta/2)\langle y,\mathscr{E}_i\rangle}. \end{aligned} \quad (6.90)$$

Thanks to (6.77) and (6.58), the right-hand side of (6.90) tends to 0 as r approaches $+\infty$. By (6.79) this concludes the proof of (6.88), which in turn, by (6.71), proves Proposition 4.5(3).

7. Examples

7.1. A Family of Intrinsic Quadrics

There is a family of smooth projective varieties $(X_n)_{n\geq 3}$ that satisfies the following properties (cf. [Bo4, Sec. 4.3]; the third property is easily deduced from [Bo4, Rem. 44]).

(1) $\dim(X_n) = n - 1$.
(2) The Cox ring of X_n may be generated by sections $\{s_i\}_{0\leq i\leq 2n}$ with divisors $\{\mathscr{E}_i\}_{0\leq i\leq 2n}$ and such that $(\mathscr{E}_0, \dots, \mathscr{E}_n)$ is a basis of $\mathrm{Pic}(X)$; the ideal of relations in the Cox ring is generated by $\sum_{1\leq i\leq n} s_i s_{i+n}$ and, for $i \in \{1, \dots, n\}$, we have $\mathscr{E}_{i+n} \sim -\mathscr{E}_i + \sum_{0\leq i'\leq n} \mathscr{E}_{i'}$.
(3) The maximal subsets J of $\{0, \dots, n\}$ such that $\bigcap_{i\in J} \mathscr{E}_i \neq \emptyset$ are those of the shape $\{0, \dots, n\} \setminus \{i_0, i_1\}$, where i_0, i_1 are distinct elements of $\{1, \dots, n\}$.

Hence X is a linear intrinsic hypersurface with respect to $(\{0, \dots, 2n\}, \{0, \dots, n\})$. It is straightforward to check that $\frac{1}{n-1}\sum_{1\leq i\leq n} \mathscr{E}_{i+n} - \mathscr{D}_{\mathrm{tot}}$ lies in $\mathrm{Eff}(X)$. Yet we must show that Assumptions 5.7 are satisfied for every $\mathfrak{I}_* \subset \{0, \dots, 2n\}$ such that $\{n+1, \dots, 2n\} \subset \mathfrak{I}_*$ and $\bigcap_{i\notin \mathfrak{I}_*} \mathscr{E}_i \neq \emptyset$. In view of property (3), the necessary arguments are contained in the proof of [Bo4, Thm. 47]. Thus we obtain the following statement.

THEOREM 7.1. *For every $n \geq 3$ and every y lying in a trunction of* $\mathrm{Eff}(X_n)^\vee$, $\mathrm{Mor}(\mathscr{C}, X_n, y, \mathfrak{I})$ *is irreducible of the expected dimension and also is dense in* $\mathrm{Mor}(\mathscr{C}, X_n, y)$.

7.2. Minimal Resolution of Singular del Pezzo Surfaces

If X is an intrinsic hypersurface with a chosen set of generating sections $\{s_i\}_{i\in\mathfrak{I}}$, then we shall call a subset $I \subset \mathfrak{I}$ *admissible* if X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$. For I admissible, we use $\mathcal{C}_I$ to denote the dual of the cone generated by the effective cone and $\frac{1}{[\mathfrak{I}\setminus I]-1}\sum_{j\in\mathfrak{I}\setminus I} \mathscr{E}_j - \mathscr{D}_{\mathrm{tot}}$ and use $\tilde{\mathcal{C}}$ to denote the union of the cone $\mathcal{C}_I$ when I ranges over the admissible subsets of $\mathfrak{I}$.

Derenthal [D] classified all the singular del Pezzo surfaces of degree at least 3 whose minimal resolution is an intrinsic hypersurface, giving in each case an explicit presentation of the Cox ring. There are 21 such surfaces, and among these it turns out that there are 20 for which there exists at least one admissible subset I (the exception is one of the cubic surfaces with a $\mathbf{D}_4$ singularity). We are interested in those surfaces for which there exists at least one admissible subset I such that $\mathcal{C}_I$ is of maximal dimension (i.e., those surfaces for which—once the ad hoc assumptions on the combinatoric series are satisfied—there is a "positive proportion" of y for which Theorem 2.4 guarantees that $\mathrm{Mor}(\mathscr{C}, X, y)$ is irreducible of the expected dimension; see Remark 2.6). It turns out that these 20 surfaces may be divided into three classes as follows.

(1) For each subset $\{i_1, i_2, i_3\}$ of $\mathfrak{I}_*$ with cardinality 3, $\mathscr{E}_{i_1} \cap \mathscr{E}_{i_2} \cap \mathscr{E}_{i_3}$ is empty. In this case there are several choices of admissible I, and for each of them $\mathcal{C}_I$ has maximal dimension (5 surfaces).

(2) There is exactly one subset $\{i_1, i_2, i_3\}$ of $\mathfrak{I}_*$ with cardinality 3 such that $\mathscr{E}_{i_1} \cap \mathscr{E}_{i_2} \cap \mathscr{E}_{i_3}$ is a point and $I = \mathfrak{I}_* \setminus \{i_1, i_2, i_3\}$ is admissible. In this case the only admissible subset for which $\mathcal{C}_I$ has maximal dimension is $\mathfrak{I}_* \setminus \{i_1, i_2, i_3\}$ (6 surfaces).
(3) There is exactly one subset $\{i_1, i_2, i_3\}$ of $\mathfrak{I}_*$ with cardinality 3 such that $\mathscr{E}_{i_1} \cap \mathscr{E}_{i_2} \cap \mathscr{E}_{i_3}$ is a point and $I = \mathfrak{I}_* \setminus \{i_1, i_2, i_3\}$ is *not* admissible. In this case there are no admissible subsets for which $\mathcal{C}_I$ has maximal dimension (9 surfaces).

(For some surfaces in the third class, it can certainly be shown that $\mathrm{Mor}(\mathscr{C}, X, y)$ is irreducible of the expected dimension for a positive proportion of y; this may be done by using a similar strategy and a counting lemma akin to the one used in [Bo4, Sec. 5] to prove the geometric Manin's conjecture for the sextic del Pezzo surface with an $\mathbf{A}_2$ singularity, which belongs to the third class.)

For each of the surfaces in the first two classes, we use Maple software to check whether the assumptions on the combinatoric series hold. This happens to be the case for each of them except one (the other cubic surface with a $\mathbf{D}_4$ singularity). For each of the 10 remaining surfaces, we estimate the "proportion" of those y for which Theorem 2.4 guarantees that $\mathrm{Mor}(\mathscr{C}, X, y)$ is irreducible of the expected dimension by computing the ratio $\frac{\mathrm{Vol}_{-\mathscr{K}_X}(\tilde{\mathcal{C}})}{\mathrm{Vol}_{-\mathscr{K}_X}(\mathrm{Eff}(X)^\vee)}$, where $\mathrm{Vol}_{-\mathscr{K}_X}$ is the volume of the intersection with the affine hyperplane $\langle \cdot, -\mathscr{K}_X \rangle = 1$. Using the description of the surface as a blowing up of the projective plane, we also compute the ratio $\frac{\mathrm{Vol}_{-\mathscr{K}_X}(\mathcal{C}_{\mathrm{KLO}})}{\mathrm{Vol}_{-\mathscr{K}_X}(\mathrm{Eff}(X)^\vee)}$; here $\mathcal{C}_{\mathrm{KLO}}$ is a subcone $\mathcal{C}_{\mathrm{KLO}}$ of $\mathrm{Eff}(X)^\vee$ described in [KLO] and such that, according to the main theorem of that paper, $\mathrm{Mor}(\mathbf{P}^1, X, y)$ is irreducible of the expected dimension for every $y \in \mathrm{NS}(X)^\vee \cap \mathcal{C}_{\mathrm{KLO}}$. The authors of [KLO] use deformation-theoretic arguments, and it seems likely that similar arguments could yield the same result in higher genus after replacing $\mathcal{C}_{\mathrm{KLO}}$ with an adequate truncation. The results of the computations, for which we benefited from [F], are presented in Table 1. Each surface is identified by its degree and the type of the singularity. (Knowing that the minimal resolution of each surface in the table is an intrinsic hypersurface and having ruled out the case of the $\mathbf{D}_4$ singularity in degree 3, this information determines completely the isomorphism class of the surface according to the results of [D].) We give generators of $\mathcal{C}_{\mathrm{KLO}}^\vee$ in terms of the boundary divisors, for which we use the notation in [D]. Note that the desingularization of the sextic with an $\mathbf{A}_1$ singularity is isomorphic to the variety X_3 of Theorem 7.1.

Of course, since the authors of [KLO] address the case of general blow-ups of projective space, their method covers a wide range of varieties that are not amenable to our approach. Even so, for the examples in Table 1 (where both methods apply), the numerical constraints reported here are weaker than theirs (one can check that this is also true in general for toric varieties that are blow-ups of a projective space). Cox rings might prove helpful for the understanding of the geometry of the moduli spaces of morphisms—at least when those rings have a sufficiently simple presentation. A similar philosophy prevails in the context of Manin's conjecture about the asymptotic behavior of rational points/curves of bounded height/degree.

Table 1

Degree	Singularities	$\frac{\mathrm{Vol}(\tilde{\mathcal{C}})}{\mathrm{Vol}(\mathrm{Eff}(X)^\vee)}$	Generators of $\mathcal{C}_{\mathrm{KLO}}^\vee$	$\frac{\mathrm{Vol}(\mathcal{C}_{\mathrm{KLO}})}{\mathrm{Vol}(\mathrm{Eff}(X)^\vee)}$	$\frac{\mathrm{Vol}(\mathcal{C}_{\mathrm{KLO}})}{\mathrm{Vol}(\tilde{\mathcal{C}})}$
6	$\mathbf{A}_1$	1	E_1, E_2, E_3, E_4	1	1
5	$\mathbf{A}_1$	23/36	$E_2, E_3, E_4, E_5, E_1 - E_5$	1/4	≈ 0.391
5	$\mathbf{A}_2$	3/8	$E_2, E_3, E_4, E_5, E_1 - 2E_5$	3/14	≈ 0.571
4	$3\mathbf{A}_1$	31/72	$E_3, E_5, E_6, E_7, E_9,$ $E_2 - E_3 - 3E_6$	1/28	≈ 0.083
4	$\mathbf{A}_2 + \mathbf{A}_1$	65/288	$E_1, E_4, E_6, E_7, E_9,$ $E_2 - E_4 - 3E_9$	3/80	≈ 0.166
4	$\mathbf{A}_3$	3/32	$E_2, E_3, E_4, E_5, E_8,$ $E_1 - E_2 - 4E_5 - E_8$	1/70	≈ 0.152
4	$\mathbf{A}_3 + \mathbf{A}_1$	1/8	$E_3, E_4, E_5, E_6, E_7,$ $A_1 - E_3 - 3E_4 - 6E_5 - E_6$	1/35	≈ 0.228
3	$2\mathbf{A}_2 + \mathbf{A}_1$	2567/23760	$E_1, E_2, E_5, E_6, E_7, E_{10},$ $E_3 - E_1 - 3E_6 - E_7 - 2E_{10}$	1/686	≈ 0.014
3	$\mathbf{A}_3 + 2\mathbf{A}_1$	181/3888	$E_1, E_2, E_3, E_5, E_6, E_8,$ $E_4 - E_1 - E_3 - 3E_5 - 4E_8$	2/2205	≈ 0.019
3	$\mathbf{A}_4 + \mathbf{A}_1$	5/288	$E_1, E_3, E_4, E_6, E_7, E_8,$ $A - E_1 - E_3 - 3E_4 - 6E_6 - 3E_8$	1/441	≈ 0.131

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