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Boundary Behavior of the Kobayashi Distance in Pseudoconvex Reinhardt Domains

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1. Introduction and Results

The problem of boundary behavior of the Kobayashi (pseudo)distance in pseudoconvex Reinhardt domains is connected with the study of their Kobayashi completeness. The qualitative condition for the k-completeness of a bounded domain D is

$$k_D(z_0, z) \to \infty \text{ as } z \to \partial D.$$

The main fact is that, if a pseudoconvex Reinhardt domain D is hyperbolic, then it is k-complete. At first Pflug [7] proved this for bounded complete domains. A second step was done by Fu [2] for bounded domains. The general case was finally solved by Zwonek [8].

Hence it is natural to ask about quantitative behavior of the function $k_D(z_0, \cdot)$. Forstnerič and Rosay estimated it from below on bounded strongly pseudoconvex domains. Namely, it was proved in [1] that

$$k_D(z_1, z_2) \geq -\frac{1}{2}\log d_D(z_1) - \frac{1}{2}\log d_D(z_2) + C$$

for z_j near two distinct points $\zeta_j \in \partial D$, $j = 1, 2$. In the same paper, the authors showed the opposite estimate for $\mathcal{C}^{1+\varepsilon}$-smooth domains with z_1, z_2 near $\zeta_0 \in \partial D$. This estimate in the bounded case follows from the inequality for the Lempert function of bounded $\mathcal{C}^{1+\varepsilon}$-smooth domains obtained by Nikolov, Pflug, and Thomas [6]:

$$\tilde{k}_D(z_1, z_2) \leq -\frac{1}{2}\log d_D(z_1) - \frac{1}{2}\log d_D(z_2) + C, \quad z_1, z_2 \in D.$$

It was also proved that this estimate fails in the $\mathcal{C}^1$-smooth case. The other general version of an upper estimate, for $\mathcal{C}^2$-smooth domains, can be found in [3]. The case of bounded convex domains was investigated by Mercer [5]. For such domains we have

$$-\frac{1}{2}\log d_D(z) + C' \leq k_D(z_0, z) \leq -\alpha \log d_D(z) + C$$

with $\alpha > \frac{1}{2}$ and z close to $\zeta_0 \in \partial D$ $\big($the constant α cannot be replaced with $\frac{1}{2}\big)$. The example

$$D_\beta := \{(z, w) \in \mathbb{C}^2 : |z|^\beta + |w|^\beta < 1\}, \quad 0 < \beta < 1,$$

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shows that the lower estimate by $-\alpha \log d_D(z) + C$, where $\alpha > 0$ (a constant independent on a domain), is not true for complete pseudoconvex Reinhardt domains. Easy calculations lead to

$$k_{D_\beta}((0,0),(z,0)) \le -\frac{\beta}{2} \log d_{D_\beta}(z,0) + C$$

if $0 < z < 1$ and $(z,0)$ tends to $(1,0)$.

In this paper we prove the following theorems.

THEOREM 1. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Fix $z_0 \in D$ and $\zeta_0 \in \partial D$. Then, for some constant C, the inequality*

$$k_D(z_0, z) \le -\log d_D(z) + C$$

holds if $z \in D$ tends to ζ_0. Additionally, for $\zeta_0 \in \mathbb{C}^n_*$ the estimate can be improved to

$$k_D(z_0, z) \le -\frac{1}{2} \log d_D(z) + C',$$

where C' is a constant.

THEOREM 2. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Fix $z_0 \in D$ and $\zeta_0 \in \partial D \cap \operatorname{int} \bar{D}$. Then, for some constant C, the inequality*

$$k_D(z_0, z) \le \frac{1}{2} \log(-\log d_D(z)) + C$$

holds if $z \in D$ tends to ζ_0.

THEOREM 3. *Let $D \subset \mathbb{C}^n$ be a $\mathcal{C}^1$-smooth pseudoconvex Reinhardt domain. Fix $z_0 \in D$ and $\zeta_0 \in \partial D$. Then, for some constant C, the inequality*

$$k_D(z_0, z) \ge -\frac{1}{2} \log d_D(z) + C$$

holds if $z \in D$ tends nontangentially to ζ_0.

2. Notation and Definitions

By D we denote a domain in $\mathbb{C}^n$. The *Kobayashi (pseudo)distance* is defined as

$$k_D(w,z) := \sup\{d_D(w,z) : (d_D) \text{ is a family of holomorphically invariant pseudodistances} \le \tilde{k}_D\},$$

where

$$\tilde{k}_D(w,z) := \inf\{p(\lambda,\mu) : \lambda,\mu \in \mathbb{D} \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) : f(\lambda) = w,\ f(\mu) = z\}$$

is the *Lempert function* of D, $\mathbb{D}$ is the unit disc in $\mathbb{C}$, and p is the Poincaré distance on $\mathbb{D}$. For general properties of functions k_D, one may refer to [3].

Let z_j denote the jth coordinate of point $z \in \mathbb{C}^n$. A domain D is called a *Reinhardt domain* if $(\lambda_1 z_1, \dots, \lambda_n z_n) \in D$ for all numbers $\lambda_1, \dots, \lambda_n \in \partial\mathbb{D}$ and points $z \in D$. A Reinhardt domain D is *complete in the jth direction* if

$$(\{1\}^{j-1} \times \bar{\mathbb{D}} \times \{1\}^{n-j}) \cdot D \subset D,$$

where $A \cdot B := \{(a_1b_1, \ldots, a_nb_n) : a \in A,\, b \in B\}$. Define subspaces $V_j^n := \{z \in \mathbb{C}^n : z_j = 0\}$ for $j = 1, \ldots, n$. If a Reinhardt domain D is complete in the jth direction for all j such that $D \cap V_j^n \neq \emptyset$, then D is called *relatively complete*.

Let $A_* := A \setminus \{0\}$ for a set $A \subset \mathbb{C}$ and let $\mathbb{C}_*^n := (\mathbb{C}_*)^n$. By $d_D(z)$ we denote a distance of a point $z \in D$ to ∂D (here, exceptionally, D can be a domain in $\mathbb{R}^n$), and by $\zeta_D(z)$ we denote one of the points admitting the distance of a point $z \in D$ to ∂D.

We will use the following main branch of the power $z^\alpha := e^{\alpha \log z} = e^{\alpha(\log|z| + i \operatorname{Arg} z)}$, where the main argument $\operatorname{Arg} z \in (-\pi, \pi]$. Define $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with $|z|^\alpha := |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}$ for $z \in \mathbb{C}_*^n$ and $\alpha \in \mathbb{R}^n$. Moreover, let $|z| := (|z_1|, \ldots, |z_n|)$ for $z \in \mathbb{C}^n$, $\log|z| := (\log|z_1|, \ldots, \log|z_n|)$ for $z \in \mathbb{C}_*^n$, and $\log D := \{\log|z| : z \in D \cap \mathbb{C}_*^n\}$—a *logarithmic image* of D. We use C to denote constants that need not be the same in different places. We write $f \lesssim g$ if there exists a $C > 0$ such that $f \leq Cg$; also, $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

We call D a $\mathcal{C}^k$*-smooth* domain if, for any point $\zeta_0 \in \partial D$, there exist its open neighborhood $U \subset \mathbb{C}^n$ and a $\mathcal{C}^k$-smooth function $\rho\colon U \to \mathbb{R}$ such that:

(i) $U \cap D = \{z \in U : \rho(z) < 0\}$;
(ii) $U \setminus \bar{D} = \{z \in U : \rho(z) > 0\}$; and
(iii) $\nabla\rho := \left(\frac{\partial\rho}{\partial\bar{z}_1}, \ldots, \frac{\partial\rho}{\partial\bar{z}_n}\right) \neq 0$ on U.

The function ρ is called a *local defining function* for D at the point ζ_0.

For a $\mathcal{C}^1$-smooth domain D we define a *normal vector* to ∂D at a point $\zeta_0 \in \partial D$ as

$$\nu_D(\zeta_0) := \frac{\nabla\rho(\zeta_0)}{\|\nabla\rho(\zeta_0)\|},$$

where ρ is a local defining function for D at ζ_0. Clearly,

$$z = \zeta_D(z) - d_D(z)\nu_D(\zeta_D(z))$$

for $z \in D$ and

$$\lim_{D \ni z \to \zeta_0} \nu_D(\zeta_D(z)) = \nu_D(\zeta_0)$$

for every choice of $\zeta_D(z)$. To ease the notation we shorten the symbol $\nu_D(\zeta_D(z))$ to $\nu_D(z)$.

Defining a nontangential convergence requires the concept of a *cone* with a vertex $x_0 \in \mathbb{R}^n$, a semi-axis $\nu \in (\mathbb{R}^n)_*$ and an angle $\alpha \in \left(0, \frac{\pi}{2}\right)$. This cone is a set of $x \in \mathbb{R}^n \setminus \{x_0\}$ such that an angle between vectors ν and $x - x_0$ does not exceed α. Let D be a $\mathcal{C}^1$-smooth domain and let $\zeta_0 \in \partial D$. We say that $z \in D$ tends *nontangentially* to ζ_0 if there exist a cone $\mathcal{A} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ with a vertex ζ_0, a semi-axis $-\nu_D(\zeta_0)$, and an angle $\alpha \in \left(0, \frac{\pi}{2}\right)$ as well as an open neighborhood $U \subset \mathbb{C}^n$ of ζ_0 such that $U \cap \mathcal{A} \subset D$ and z tends to ζ_0 in $U \cap \mathcal{A}$.

We say that a Reinhardt domain D satisfies the *Fu condition* if, for any $j \in \{1, \ldots, n\}$, the following implication holds:

$$\partial D \cap V_j^n \neq \emptyset \implies D \cap V_j^n \neq \emptyset.$$

The following well-known properties of pseudoconvex Reinhardt domains will be used in the paper (see e.g. [4]).

FACT 1. A Reinhardt domain D is pseudoconvex if and only if $\log D$ is convex and D is relatively complete.

FACT 2. A $\mathcal{C}^1$-smooth Reinhardt domain satisfies the Fu condition.

3. Proofs

Proof of Theorem 1. We proceed as follows. The first step is to simplify the general case to "real" coordinates, after which we consider some parallelepipeds contained in the given domain and use the decreasing property of the Kobayashi distance. Finally, we explicitly calculate and estimate that distance in other domains: Cartesian products of a strip and annuli in $\mathbb{C}$. To improve the estimate for a boundary point with all nonzero coordinates, we use similar methods but with intervals instead of parallelepipeds.

Using some biholomorphism of the form

$$w \ni \mathbb{C}^n \mapsto (a_1 w_1, \dots, a_n w_n) \in \mathbb{C}^n,\ a \in \mathbb{C}_*^n,$$

and the triangle inequality for k_D, we can assume that $z_0 = (1, \dots, 1)$ and $|\zeta_{0j}| \neq 1$ for $j = 1, \dots, n$. Notice that the proof can be reduced to the case $z \in D \cap \mathbb{C}_*^n$ near ζ_0, and this case amounts to the situation

$$z \in D \cap (0, \infty)^n \ \text{ near } \zeta_0 \in \partial D \cap ([0, \infty) \setminus \{1\})^n.$$

Indeed, the first reduction follows from the continuity of k_D and the triangle inequality for k_D. Now, if $z \to \zeta_0$ then $|z| \to |\zeta_0| \in \partial D$ and

$$k_D(z_0, z) = k_D(\tilde{z}_0, |z|),$$

where

$$\tilde{z}_0 := \left(\frac{|z_1|}{z_1} z_{01}, \dots, \frac{|z_n|}{z_n} z_{0n} \right) \in T := \{(\lambda_1 z_{01}, \dots, \lambda_n z_{0n}) : \lambda_1, \dots, \lambda_n \in \partial\mathbb{D}\}.$$

The continuity of k_D gives

$$\max_{T \times T} k_D =: C < \infty$$

and therefore

$$k_D(\tilde{z}_0, |z|) \le k_D(\tilde{z}_0, z_0) + k_D(z_0, |z|) \le k_D(z_0, |z|) + C.$$

The property $d_D(|z|) = d_D(z)$ finishes this reduction. In what follows, we assume that points $z \in D \cap (0, \infty)^n$ are sufficiently close to $\zeta_0 \in \partial D \cap ([0, \infty) \setminus \{1\})^n$.

Observe that

$$d_{\log D}(\log z) \ge \varepsilon d_D(z)$$

for some $\varepsilon > 0$. Indeed, for $u \in \mathbb{R}^n$ such that $\|u\| < 1$ and $0 \le t \le \varepsilon d_D(z)$, where

$$\varepsilon := \frac{1}{3(\|\zeta_0\| + 1)},$$

we also have

$$\log z + tu \in \log D \iff (z_1 e^{tu_1}, \dots, z_n e^{tu_n}) \in D,$$

but $(z_1 e^{tu_1}, \dots, z_n e^{tu_n}) \in D$ follows from

$$\|(z_j e^{tu_j})_{j=1}^n - z\| \le \sqrt{\sum_{j=1}^n z_j^2 (2t)^2} \le 2t(\|\zeta_0\| + 1) < d_D(z).$$

Moreover, for $\zeta_0 = 0$, a similar consideration leads to

$$d_{\log D}(\log z) \ge \varepsilon' \frac{d_D(z)}{\|z\|}$$

for sufficiently small $\varepsilon' > 0$. Indeed, there exists an $\varepsilon' \in \left(0, \frac{1}{2}\right)$ such that the inequalities

$$|e^{tu_j} - 1| \le 2t, \quad j = 1, \dots, n,$$

hold for $0 \le t \le \varepsilon'$. Hence, for $0 \le t \le \varepsilon' \frac{d_D(z)}{\|z\|}$ we have

$$\|(z_j e^{tu_j})_{j=1}^n - z\| \le \sqrt{\sum_{j=1}^n z_j^2 (2t)^2} \le 2\varepsilon' \frac{d_D(z)}{\|z\|} \|z\| < d_D(z).$$

Now let

$$\tilde{d}_D(z) := \begin{cases} \varepsilon d_D(z) & \text{if } \zeta_0 \ne 0, \\ \varepsilon'' \frac{d_D(z)}{\|z\|} & \text{if } \zeta_0 = 0; \end{cases}$$

here $\varepsilon'' := \varepsilon' d_{\log D}(0)$. We define

$$m_z := \min\{0, \log z_1\}, \qquad M_z := \max\{0, \log z_1\}$$

and consider the domain

$$\begin{aligned} D_z := \Big\{ w \in \mathbb{C}^n : m_z - \tilde{d}_D(z) < \log|w_1| < M_z + \tilde{d}_D(z), \\ \frac{\log z_j}{\log z_1} \log|w_1| - \tilde{d}_D(z) < \log|w_j| \\ < \frac{\log z_j}{\log z_1} \log|w_1| + \tilde{d}_D(z),\ j = 2, \dots, n \Big\}. \end{aligned}$$

Then $\log D_z$ is a domain in $\mathbb{R}^n$ containing points 0 and $\log z$ but contained in a convex domain $\log D$. Define also

$$\begin{aligned} G_z := \{ v \in \mathbb{C}^n : m_z - \tilde{d}_D(z) < \operatorname{Re} v_1 < M_z + \tilde{d}_D(z), \\ -\tilde{d}_D(z) < \log|v_j| < \tilde{d}_D(z),\ j = 2, \dots, n \}. \end{aligned}$$

Then the holomorphic map

$$f_z(v) := (e^{v_1}, v_2 e^{v_1(\log z_2 / \log z_1)}, \dots, v_n e^{v_1(\log z_n / \log z_1)}), \quad v \in G_z,$$

has values in D_z. Moreover,

$$w = f_z\left(\log w_1, \frac{w_2}{w_1^{\log z_2/\log z_1}}, \dots, \frac{w_n}{w_1^{\log z_n/\log z_1}}\right) \quad \text{for } w \in D_z.$$

Therefore,

$$\begin{aligned}
k_D(z_0, z) &\le k_{D_z}(z_0, z)\\
&= k_{D_z}\left(f_z(0,1,\dots,1), f_z\left(\log z_1, \frac{z_2}{z_1^{\log z_2/\log z_1}}, \dots, \frac{z_n}{z_1^{\log z_n/\log z_1}}\right)\right)\\
&= k_{D_z}(f_z(0,1,\dots,1), f_z(\log z_1, 1, \dots, 1))\\
&\le k_{G_z}((0,1,\dots,1), (\log z_1, 1, \dots, 1))\\
&= \max\{k_{S_z}(0, \log z_1), k_{A_z}(1,1), \dots, k_{A_z}(1,1)\} = k_{S_z}(0, \log z_1),
\end{aligned}$$

where

$$S_z := \{\lambda \in \mathbb{C} : m_z - \tilde{d}_D(z) < \operatorname{Re}\lambda < M_z + \tilde{d}_D(z)\}$$

and

$$A_z := \{\lambda \in \mathbb{C} : -\tilde{d}_D(z) < \log|\lambda| < \tilde{d}_D(z)\}.$$

Using suitable biholomorphisms allows us to calculate

$$k_{S_z}(0, \log z_1) = p\left(\frac{i - \exp \pi i P(z)}{i + \exp \pi i P(z)}, \frac{i - \exp \pi i Q(z)}{i + \exp \pi i Q(z)}\right),$$

where

$$P(z) := \frac{\tilde{d}_D(z) - m_z}{2\tilde{d}_D(z) + M_z - m_z}, \qquad Q(z) := \frac{\log z_1 + \tilde{d}_D(z) - m_z}{2\tilde{d}_D(z) + M_z - m_z}.$$

Analogously, changing the index 1 to any of $2, \dots, n$ yields

$$k_D(z_0, z) \le \min_{j=1,\dots,n} k_{S_z^{(j)}}(0, \log z_j),$$

where

$$k_{S_z^{(j)}}(0, \log z_j) = p\left(\frac{i - \exp \pi i P^{(j)}(z)}{i + \exp \pi i P^{(j)}(z)}, \frac{i - \exp \pi i Q^{(j)}(z)}{i + \exp \pi i Q^{(j)}(z)}\right)$$

and

$$S_z^{(j)} := \{\lambda \in \mathbb{C} : m_z^{(j)} - \tilde{d}_D(z) < \operatorname{Re}\lambda < M_z^{(j)} + \tilde{d}_D(z)\};$$

$$m_z^{(j)} := \min\{0, \log z_j\}, \quad M_z^{(j)} := \max\{0, \log z_j\}, \quad j = 1, \dots, n;$$

$$P^{(j)}(z) := \frac{\tilde{d}_D(z) - m_z^{(j)}}{2\tilde{d}_D(z) + M_z^{(j)} - m_z^{(j)}}, \qquad Q^{(j)}(z) := \frac{\log z_1 + \tilde{d}_D(z) - m_z^{(j)}}{2\tilde{d}_D(z) + M_z^{(j)} - m_z^{(j)}}.$$

Consider two cases, $\zeta_0 \ne 0$ and $\zeta_0 = 0$. If $\zeta_0 \ne 0$ then choose $j \in \{1, \dots, n\}$ such that $\zeta_{0j} \ne 0$ (recall that $\zeta_{0j} = |\zeta_{0j}| \ne 1$). In the case of $\zeta_{0j} > 1$ we obtain

$$\begin{aligned}
k_{S_z^{(j)}}(0, \log z_j) &= p\left(\frac{i - \exp \pi i T^{(j)}(z)}{i + \exp \pi i T^{(j)}(z)}, \frac{i - \exp \pi i U^{(j)}(z)}{i + \exp \pi i U^{(j)}(z)}\right)\\
&\le p\left(0, \frac{i - \exp \pi i T^{(j)}(z)}{i + \exp \pi i T^{(j)}(z)}\right) + p\left(0, \frac{i - \exp \pi i U^{(j)}(z)}{i + \exp \pi i U^{(j)}(z)}\right), \quad (1)
\end{aligned}$$

where

$$T^{(j)}(z) := \frac{\varepsilon d_D(z)}{2\varepsilon d_D(z) + \log z_j}, \qquad U^{(j)}(z) := \frac{\log z_j + \varepsilon d_D(z)}{2\varepsilon d_D(z) + \log z_j}.$$

By Taylor expansion, we have

$$\frac{i - \exp \pi i T^{(j)}(z)}{i + \exp \pi i T^{(j)}(z)} = i - \pi i T^{(j)}(z) + O(d_D(z)^2).$$

Hence

$$\begin{aligned}
p\left(0, \frac{i - \exp \pi i T^{(j)}(z)}{i + \exp \pi i T^{(j)}(z)}\right) &= p\big(0, i - \pi i T^{(j)}(z) + O(d_D(z)^2)\big) \\
&\le \frac{\log 2}{2} - \frac{1}{2}\log\big(1 - |i - \pi i T^{(j)}(z) + O(d_D(z)^2)|\big) \\
&\le \frac{\log 2}{2} - \frac{1}{2}\log\big(1 - |i - \pi i T^{(j)}(z)| - |O(d_D(z)^2)|\big) \\
&= \frac{\log 2}{2} - \frac{1}{2}\log\left(\pi \frac{\varepsilon d_D(z)}{2\varepsilon d_D(z) + \log z_j} - O(d_D(z)^2)\right) \\
&\le -\frac{1}{2}\log d_D(z) + C.
\end{aligned}$$

Similarly,

$$\frac{i - \exp \pi i U^{(j)}(z)}{i + \exp \pi i U^{(j)}(z)} = -i + \pi i T^{(j)}(z) + O(d_D(z)^2),$$

which gives the same estimation for the second summand.

Otherwise, if $\zeta_{0j} < 1$ then

$$k_{S_z^{(j)}}(0, \log z_j) = p\left(\frac{i - \exp \pi i V^{(j)}(z)}{i + \exp \pi i V^{(j)}(z)}, \frac{i - \exp \pi i W^{(j)}(z)}{i + \exp \pi i W^{(j)}(z)}\right), \tag{2}$$

where

$$V^{(j)}(z) := \frac{\varepsilon d_D(z) - \log z_j}{2\varepsilon d_D(z) - \log z_j}, \qquad W^{(j)}(z) := \frac{\varepsilon d_D(z)}{2\varepsilon d_D(z) - \log z_j}.$$

We see that the expression in (2) is the same as in (1) after substituting $\log z_j \rightsquigarrow -\log z_j$, and the estimates stay true.

Now assume that $\zeta_0 = 0$. Then, for $j = 1, \ldots, n$,

$$\begin{aligned}
k_{S_z^{(j)}}(0, \log z_j) &= p\left(\frac{i - \exp \pi i X^{(j)}(z)}{i + \exp \pi i X^{(j)}(z)}, \frac{i - \exp \pi i Y^{(j)}(z)}{i + \exp \pi i Y^{(j)}(z)}\right) \\
&\le p\left(0, \frac{i - \exp \pi i X^{(j)}(z)}{i + \exp \pi i X^{(j)}(z)}\right) + p\left(0, \frac{i - \exp \pi i Y^{(j)}(z)}{i + \exp \pi i Y^{(j)}(z)}\right),
\end{aligned}$$

where

$$X^{(j)}(z) := \frac{\varepsilon'' d_D(z)\|z\|^{-1} - \log z_j}{2\varepsilon'' d_D(z)\|z\|^{-1} - \log z_j}, \qquad Y^{(j)}(z) := \frac{\varepsilon'' d_D(z)\|z\|^{-1}}{2\varepsilon'' d_D(z)\|z\|^{-1} - \log z_j}.$$

Putting

$$\delta^{(j)}(z) := \frac{\varepsilon'' d_D(z)}{\|z\| \log z_j},$$

we have

$$X^{(j)}(z) = \frac{\delta^{(j)}(z) - 1}{2\delta^{(j)}(z) - 1}, \qquad Y^{(j)}(z) = \frac{\delta^{(j)}(z)}{2\delta^{(j)}(z) - 1},$$

and $\delta^{(j)}(z) \to 0$ as $z \to 0$. Calculations analogous to those in the $\zeta_0 \neq 0$ case give

$$\frac{i - \exp \pi i X^{(j)}(z)}{i + \exp \pi i X^{(j)}(z)} = -i + \pi i Y^{(j)}(z) + O(\delta^{(j)}(z)^2)$$

and

$$\frac{i - \exp \pi i Y^{(j)}(z)}{i + \exp \pi i Y^{(j)}(z)} = i - \pi i Y^{(j)}(z) + O(\delta^{(j)}(z)^2).$$

Therefore,

$$\begin{aligned} p\left(0, \frac{i - \exp \pi i X^{(j)}(z)}{i + \exp \pi i X^{(j)}(z)}\right) &\leq \frac{\log 2}{2} - \frac{1}{2} \log\left(\pi \frac{\delta^{(j)}(z)}{2\delta^{(j)}(z) - 1} - O(\delta^{(j)}(z)^2)\right) \\ &\leq -\frac{1}{2} \log(-\delta^{(j)}(z)) + C \end{aligned}$$

and similarly

$$p\left(0, \frac{i - \exp \pi i Y^{(j)}(z)}{i + \exp \pi i Y^{(j)}(z)}\right) \leq -\frac{1}{2} \log(-\delta^{(j)}(z)) + C.$$

Finally,

$$\begin{aligned} \min_{j=1,\dots,n} k_{S_z^{(j)}}(0, \log z_j) &\leq \min_{j=1,\dots,n} -\log(-\delta^{(j)}(z)) + C \\ &= -\log d_D(z) + \log\|z\| + \min_{j=1,\dots,n} \log(-\log z_j) + C \\ &= -\log d_D(z) + \log\|z\| + \log\left(-\log \max_{j=1,\dots,n} z_j\right) + C \\ &\leq -\log d_D(z) + \log\|z\| + \log(-\log\|z\|) + C \\ &\leq -\log d_D(z) + C. \end{aligned}$$

For improving the estimate in the case of $\zeta_0 \in \partial D \cap \mathbb{C}_*^n$, we may assume that $z_0 \in \mathbb{C}_*^n$ and $|z_{0j}|, |\zeta_{0j}| \neq 1$ for $j = 1, \dots, n$. Since $\log D$ is a convex domain, it follows that the interval

$$I_z := \{t \log|z| + (1-t) \log|z_0| : t \in (-\varepsilon(z), 1 + \delta(z))\}$$

is contained in $\log D$ for some positive numbers $\delta(z)$ and $\varepsilon(z)$. The number $\varepsilon(z)$ can be chosen as a sufficiently small positive constant ε independent of z. Indeed,

$$t \log|z| + (1-t) \log|z_0| = \log|z_0| + t(\log|z| - \log|z_0|)$$

and $\|\log|z| - \log|z_0|\|$ is bounded—say, by M. Hence

$$\varepsilon := \frac{d_{\log D}(\log|z_0|)}{2M}$$

is good. Analogously,

$$\frac{d_{\log D}(\log|z|)}{2M}$$

is a candidate for $\delta(z)$. We have

$$\frac{d_{\log D}(\log|z|)}{2M} \geq \delta d_D(z)$$

for some $\delta > 0$ (in fact, "$\geq$" can be replaced with "$\approx$"). Thus we can choose $\delta(z) := \delta d_D(z)$.

From the inclusion $I_z \subset \log D$ it follows that

$$\exp I_z \subset D;$$

that is,

$$\left(\left|\frac{z_1}{z_{01}}\right|^t |z_{01}|, \ldots, \left|\frac{z_n}{z_{0n}}\right|^t |z_{0n}|\right) \in D$$

for $t \in (-\varepsilon, 1 + \delta d_D(z))$. Hence the holomorphic map

$$f_z(\lambda) := \left(e^{i \operatorname{Arg} z_1}\left|\frac{z_1}{z_{01}}\right|^\lambda |z_{01}|, \ldots, e^{i \operatorname{Arg} z_n}\left|\frac{z_n}{z_{0n}}\right|^\lambda |z_{0n}|\right)$$

leading from the strip

$$S_z := \{\lambda \in \mathbb{C} : -\varepsilon < \operatorname{Re} \lambda < 1 + \delta d_D(z)\}$$

has values in D. Moreover $f_z(1) = z$ and $f_z(0)$ lies on the torus

$$T := \{(\lambda_1 z_{01}, \ldots, \lambda_n z_{0n}) : \lambda_1, \ldots, \lambda_n \in \partial\mathbb{D}\}.$$

Therefore,

$$\begin{aligned} k_D(z_0, z) &\leq k_D(z_0, f_z(0)) + k_D(f_z(0), z) \\ &\leq k_D(f_z(0), f_z(1)) + \max_{T \times T} k_D \leq k_{S_z}(0,1) + \max_{T \times T} k_D. \end{aligned}$$

Calculating $k_{S_z}(0,1)$ now yields

$$k_{S_z}(0,1) = p\left(\frac{i - \exp \pi i P^{(j)}(z)}{i + \exp \pi i P^{(j)}(z)}, \frac{i - \exp \pi i Q^{(j)}(z)}{i + \exp \pi i Q^{(j)}(z)}\right),$$

where

$$P^{(j)}(z) := \frac{\varepsilon}{1 + \varepsilon + \delta d_D(z)}, \qquad Q^{(j)}(z) := \frac{1 + \varepsilon}{1 + \varepsilon + \delta d_D(z)}.$$

Certainly, the first of the preceding arguments of the function p tends to some point from the unit disc; for the second argument, we have

$$\frac{i - \exp \pi i Q^{(j)}(z)}{i + \exp \pi i Q^{(j)}(z)} = -i + \pi i \frac{\delta d_D(z)}{1 + \varepsilon + \delta d_D(z)} + O(d_D(z)^2).$$

As a result,

$$\begin{aligned} p\left(0, \frac{i - \exp \pi i Q^{(j)}(z)}{i + \exp \pi i Q^{(j)}(z)}\right) &\leq \frac{\log 2}{2} - \frac{1}{2}\log\left(\pi \frac{\delta d_D(z)}{1 + \varepsilon + \delta d_D(z)} - O(d_D(z)^2)\right) \\ &\leq -\frac{1}{2}\log d_D(z) + C. \end{aligned}$$

The triangle inequality for p finishes the proof. □

Proof of Theorem 2. This proof is based on the decreasing and product properties of the Kobayashi distance. We must consider some cases that will lead to an induction.

Note that if $E \subset \mathbb{R}^n$ is a convex domain then $E = \operatorname{int} \bar{E}$. The condition $\zeta_0 \in \partial D \cap \operatorname{int} \bar{D}$ implies $\zeta_0 \notin \mathbb{C}_*^n$. To see this, assume that $\zeta_0 \in \mathbb{C}_*^n$. An easy topological argument shows that

$$\log|\zeta_0| \in (\partial \log D) \cap \operatorname{int} \overline{\log D} = (\partial \log D) \cap \log D = \emptyset.$$

Assume without loss of generality that

$$\zeta_0 = (\zeta_{01}, \dots, \zeta_{0k}, 0, \dots, 0),$$

where $0 \le k \le n-1$ and $\zeta_{0j} \ne 0$ for $j \le k$. Let $r > 0$ be such that an open polydisc $P(\zeta_0, r)$ is contained in $\bar{D}$. Then $\log P(\zeta_0, r) \subset \log \bar{D}$. Taking interiors of both sides, we obtain

$$\log P(\zeta_0, r) \subset \operatorname{int} \log \bar{D} = \operatorname{int} \overline{\log D} = \log D.$$

Therefore,

$$P(\zeta_0, r) \cap \mathbb{C}_*^n \subset D. \tag{3}$$

Clearly, for fixed small r we have

$$P(\zeta_0, r) \cap \mathbb{C}_*^n = \mathbb{D}(\zeta_{01}, r) \times \cdots \times \mathbb{D}(\zeta_{0k}, r) \times (r\mathbb{D}_*)^{n-k},$$

where $\mathbb{D}(\zeta_{0j}, r)$ is a disc in $\mathbb{C}$ centered at ζ_{0j} and with radius r. Hence, choosing any $z_0 \in P(\zeta_0, r) \cap \mathbb{C}_*^n$, we have

$$k_D(z_0, z) \le \max\Big\{ \max_{j=1,\dots,k} k_{\mathbb{D}(\zeta_{0j}, r)}(z_{0j}, z_j),\ \max_{j=k+1,\dots,n} k_{r\mathbb{D}_*}(z_{0j}, z_j) \Big\}$$

for $z \in D \cap \mathbb{C}_*^n$ near ζ_0. For $j = 1, \dots, k$ the numbers z_j tend to ζ_{0j}, so the first of these maxima is bounded by a constant. The well-known estimate for the punctured disc gives us

$$k_{r\mathbb{D}_*}(z_{0j}, z_j) \le \frac{1}{2} \log(-\log d_{r\mathbb{D}_*}(z_j)) + C = \frac{1}{2} \log(-\log|z_j|) + C$$

for $j = k+1, \dots, n$. Therefore,

$$k_D(z_0, z) \le \frac{1}{2} \log\Big(-\log \min_{j=k+1,\dots,n} |z_j| \Big) + C. \tag{4}$$

We can improve on the estimate (4). Let $z' := (z_1, \dots, z_k)$ and note that

$$(z', 0, \dots, 0) \in \partial D. \tag{5}$$

Indeed, $(z', 0, \dots, 0) \in \bar{D}$. If $(z', 0, \dots, 0) \in D$ then D is complete in the directions $k+1, \dots, n$ (by Fact 1). Moreover, $(\zeta_{01}, \dots, \zeta_{0k}, r/2, \dots, r/2) \in D$, which implies $(\zeta_{01}, \dots, \zeta_{0k}, 0, \dots, 0) \in D$—a contradiction.

We claim that, for all $k+1 \le p < q \le n$,

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0) \in \partial D \quad \text{or} \quad (z', 0, \dots, 0, \underline{z_q}, 0, \dots, 0) \in \partial D; \tag{6}$$

here $\underline{z_j}$ denotes that z_j is on the jth place. If (6) does not hold then both points belong to D (recall that $P(\zeta_0, r) \subset \bar{D}$). Hence D is complete in the directions $k+1, \dots, n$ and $(z', 0, \dots, 0) \in D$, which contradicts (5).

Therefore all points

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0), \quad p = k+1, \dots, n,$$

except possibly one, belong to ∂D. Consider the following cases.

Case 1.1: One of these points (*say,* $(z', 0, \dots, 0, z_n)$) *does not belong to* ∂D. Then it belongs to D and hence D is complete in the directions $k+1, \dots, n-1$. Now the inclusion (3) can be improved to

$$P(\zeta_0, r) \cap (\mathbb{C}^{n-1} \times \mathbb{C}_*) \subset D$$

and

$$P(\zeta_0, r) \cap (\mathbb{C}^{n-1} \times \mathbb{C}_*) = \mathbb{D}(\zeta_{01}, r) \times \cdots \times \mathbb{D}(\zeta_{0k}, r) \times (r\mathbb{D})^{n-k-1} \times r\mathbb{D}_*.$$

The estimate for $k_D(z_0, z)$ is improved to

$$\max\left\{ \max_{j=1,\dots,k} k_{\mathbb{D}(\zeta_{0j}, r)}(z_{0j}, z_j), \max_{j=k+1,\dots,n-1} k_{r\mathbb{D}}(z_{0j}, z_j), k_{r\mathbb{D}_*}(z_{0n}, z_n) \right\}$$
$$= k_{r\mathbb{D}_*}(z_{0n}, z_n) \le \frac{1}{2} \log(-\log|z_n|) + C.$$

It remains to observe that

$$(z', z_{k+1}, \dots, z_{n-1}, 0) \in \partial D,$$

for otherwise the domain D would be complete in the nth direction and the property $(z', 0, \dots, 0, z_n) \in D$ would imply $(z', 0, \dots, 0) \in D$, in contradiction with (5). Thus

$$d_D(z) \le \|z - (z', z_{k+1}, \dots, z_{n-1}, 0)\| = |z_n|,$$

which allows us to estimate

$$\frac{1}{2} \log(-\log|z_n|) + C \le \frac{1}{2} \log(-\log d_D(z)) + C.$$

Case 1.2: All the points

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0), \quad p = k+1, \dots, n,$$

belong to ∂D. We claim that, for all $k+1 \le p < q \le n$ and $k+1 \le p' < q' \le n$ with $\{p, q\} \ne \{p', q'\}$,

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0, \underline{z_q}, 0, \dots, 0) \in \partial D \quad \text{or}$$
$$(z', 0, \dots, 0, \underline{z_{p'}}, 0, \dots, 0, \underline{z_{q'}}, 0, \dots, 0) \in \partial D.$$

Analogously as before we use an argument of completeness in the suitable directions to get

$$(z', 0, \dots, 0, \underline{z_j}, 0, \dots, 0) \in D$$

for some $j \in \{p, q, p', q'\}$—a contradiction with the assumption of this case. Therefore, all points

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0, \underline{z_q}, 0, \dots, 0), \quad k+1 \le p < q \le n,$$

except possibly one, belong to ∂D. Again we consider two cases.

Case 2.1: One of these points (say, $(z', 0, \dots, 0, z_{n-1}, z_n)$) does not belong to ∂D. Then it belongs to D. We see, much as in Case 1.1, that

$$P(\zeta_0, r) \cap (\mathbb{C}^{n-2} \times \mathbb{C}_*^2) \subset D,$$

$$k_D(z_0, z) \le \frac{1}{2} \log\Bigl(-\log \min_{j=n-1,n} |z_j|\Bigr) + C,$$

$$(z', z_{k+1}, \dots, z_{n-2}, z_{n-1}, 0), (z', z_{k+1}, \dots, z_{n-2}, 0, z_n) \in \partial D,$$

$$d_D(z) \le \min_{j=n-1,n} |z_j|.$$

Case 2.2: All the points

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0, \underline{z_q}, 0, \dots, 0), \quad k+1 \le p < q \le n,$$

belong to ∂D. We see, by induction, that in the sth step ($s = 3, \dots, n-k-1$) all points

$$(z', 0, \dots, 0, \underline{z_{p_1}}, 0, \dots, 0, \underline{z_{p_s}}, 0, \dots, 0), \quad k+1 \le p_1 < \cdots < p_s \le n,$$

except possibly one, belong to ∂D.

If one of these points (say, $(z', 0, \dots, 0, z_{n-s+1}, \dots, z_n)$) does not belong to ∂D, then it belongs to D and

$$P(\zeta_0, r) \cap (\mathbb{C}^{n-s} \times \mathbb{C}_*^s) \subset D,$$

$$k_D(z_0, z) \le \frac{1}{2} \log\Bigl(-\log \min_{j=n-s+1,\dots,n} |z_j|\Bigr) + C,$$

$$(z', z_{k+1}, \dots, z_{n-s}, z_{n-s+1}, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_n) \in \partial D, \quad j = n-s+1, \dots, n,$$

$$d_D(z) \le \min_{j=n-s+1,\dots,n} |z_j|,$$

which finishes the proof in the case $s.1$.

If all the points

$$(z', 0, \dots, 0, \underline{z_{p_1}}, 0, \dots, 0, \underline{z_{p_s}}, 0, \dots, 0), \quad k+1 \le p_1 < \cdots < p_s \le n,$$

belong to ∂D, then we "jump" from the case $s.2$ to the case $(s+1).1$ and finally obtain

$$(z', 0, z_{k+2}, \dots, z_n) \in D,$$

$$P(\zeta_0, r) \cap (\mathbb{C}^{k+1} \times \mathbb{C}_*^{n-k-1}) \subset D,$$

$$k_D(z_0, z) \le \frac{1}{2} \log\Bigl(-\log \min_{j=k+2,\dots,n} |z_j|\Bigr) + C,$$

$$(z', z_{k+1}, z_{k+2}, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_n) \in \partial D, \quad j = k+2, \dots, n,$$

$$d_D(z) \le \min_{j=k+2,\dots,n} |z_j|$$

in the case $(n-k-1).1$ or

$$(z', z_{k+1}, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_n) \in \partial D, \quad j = k+1, \dots, n,$$

in the case $(n-k-1).2$. This property allows us to estimate $d_D(z)$ from above by $\min_{j=k+1,\dots,n}|z_j|$ and then use (4) to finish the proof. $\square$

Proof of Theorem 3. The proof has two main parts. We first prove the claim for $\zeta_0 \in \partial D \cap \mathbb{C}_*^n$ thanks to the effective formulas for the Kobayashi distance in special domains. The second part amounts to the lower-dimensional situation with a boundary point having all nonzero coordinates.

Let $\zeta_0 \in \partial D \cap \mathbb{C}_*^n$ and consider $z \in D \cap \mathbb{C}_*^n$ close to ζ_0. From the convexity of the set $\log D$ there exist $\alpha \in \mathbb{R}^n$ and $c > 0$ such that the hyperplane

$$\{x \in \mathbb{R}^n : \langle \alpha, x \rangle_{\mathbb{R}^n} = \log c\}$$

contains the point $\log|\zeta_0|$ and, moreover, $\log D$ lies on the one side of this hyperplane. Assume without loss of generality that this side is $\{x \in \mathbb{R}^n : \langle \alpha, x \rangle_{\mathbb{R}^n} < \log c\}$, since in the case of $\log D \subset \{x \in \mathbb{R}^n : \langle \alpha', x \rangle_{\mathbb{R}^n} > \log c'\}$ it suffices to define

$$\alpha := -\alpha' \quad \text{and} \quad c := 1/c'.$$

Therefore,

$$\{(e^{x_1}, \dots, e^{x_n}) : x \in \log D\} \subset \{w \in \mathbb{C}^n : |w|^\alpha < c\} =: D_{\alpha,c}$$

(these sets are called *elementary Reinhardt domains*), where by a point satisfying the condition $|w|^\alpha < c$ we mean such a point w whose coordinate w_j is nonzero when $\alpha_j < 0$ (and satisfies $|w|^\alpha < c$ in the usual sense). To affirm that $D \subset D_{\alpha,c}$, we must check that this restriction for points w does not remove from D points with some zero coordinates. Indeed, if there is no such inclusion, we can assume that the order of zero coordinates of point $w \in D$ and negative terms of the sequence α is as follows:

$$w_1, \dots, w_k \neq 0, \quad w_{k+1}, \dots, w_n = 0,$$

$$\alpha_{k+1}, \dots, \alpha_l \geq 0, \quad \alpha_{l+1}, \dots, \alpha_n < 0;$$

here $1 \leq k \leq l < n$. In some neighborhood of the point w contained in D, there exist points $v \in \mathbb{C}_*^n$ with coordinates v_j such that

$$|v_1|, \dots, |v_l| > \varepsilon > 0$$

and $|v_{l+1}|, \dots, |v_n|$ arbitrarily close to zero (i.e., moved from w in the direction of subspace $\{0\}^l \times \mathbb{C}^{n-l}$ and then moved by a constant vector in the direction $\mathbb{C}^l \times \{0\}^{n-l}$). Hence there exist points $u \in \log D$ whose coordinates u_j satisfy

$$u_1, \dots, u_l > \log \varepsilon > -\infty,$$

although $u_{l+1}, \dots, u_n$ are arbitrarily close to $-\infty$. However, this contradicts the fact that values of the expression

$$\sum_{j=l+1}^{n} \alpha_j u_j$$

are, for these points u, bounded from above by a constant $\log c - \sum_{j=1}^{l} \alpha_j \log \varepsilon$.

We will use effective formulas for the Kobayashi distance in domains $D_{\alpha,c}$ [9]. Define

$$l := \#\{j = 1, \dots, n : \alpha_j < 0\}$$

and

$$\tilde{\alpha} := \min\{\alpha_j : \alpha_j > 0\} \quad \text{if } l < n.$$

We first consider when $l < n$. The formula in this case gives

$$k_D(z_0, z) \geq k_{D_{\alpha,c}}(z_0, z) \geq p\left(0, \frac{|z|^{\alpha/\tilde{\alpha}}}{c^{1/\tilde{\alpha}}}\right) + C.$$

Yet

$$z = \zeta_{D_{\alpha,c}}(z) - d_{D_{\alpha,c}}(z)\nu_{D_{\alpha,c}}(z)$$

and hence

$$|z|^{\alpha/\tilde{\alpha}} = \prod_{j=1}^{n} |\zeta_{D_{\alpha,c}}(z)_j - d_{D_{\alpha,c}}(z)\nu_{D_{\alpha,c}}(z)_j|^{\alpha_j/\tilde{\alpha}} = c^{1/\tilde{\alpha}} - \rho(z) d_{D_{\alpha,c}}(z)$$

for some bounded positive function ρ. Thus

$$\begin{aligned} p\left(0, \frac{|z|^{\alpha/\tilde{\alpha}}}{c^{1/\tilde{\alpha}}}\right) &= p\left(0, 1 - \frac{\rho(z)}{c^{1/\tilde{\alpha}}} d_{D_{\alpha,c}}(z)\right) \geq -\frac{1}{2}\log\left(\frac{\rho(z)}{c^{1/\tilde{\alpha}}} d_{D_{\alpha,c}}(z)\right) \\ &\geq -\frac{1}{2}\log d_{D_{\alpha,c}}(z) + C. \end{aligned}$$

We will show that

$$d_{D_{\alpha,c}}(z) \approx d_D(z) \text{ as } z \to \zeta_0 \text{ nontangentially.}$$

By definition there exists a cone $\mathcal{A}$ with vertex ζ_0 and semi-axis $-\nu_{D_{\alpha,c}}(\zeta_0)$ that contains considered points z. By the $\mathcal{C}^1$-smoothness of D we have a cone $\mathcal{B}$, with vertex ζ_0 and semi-axis $-\nu_{D_{\alpha,c}}(\zeta_0)$, whose intersection with some neighborhood of the point ζ_0 is contained in D and contains in its interior the cone $\mathcal{A}$. Therefore,

$$\begin{aligned} 1 &\geq \frac{d_D(z)}{d_{D_{\alpha,c}}(z)} = \frac{\|z - \zeta_D(z)\|}{\|z - \zeta_{D_{\alpha,c}}(z)\|} \geq \frac{\|z - \zeta_D(z)\|}{\|z - \zeta_0\|} \geq \frac{\|z - \zeta_{\mathcal{B}}(z)\|}{\|z - \zeta_0\|} \\ &= \sin\angle(z, \zeta_0, \zeta_{\mathcal{B}}(z)) \geq \sin\theta; \end{aligned}$$

here $\angle(X, Y, Z)$ is an angle with vertex Y and with arms that contain points X and Z, and θ is the angle between these generatrices of cones $\mathcal{A}$ and $\mathcal{B}$ that lie in one plane with the axis of both cones. (In other words, θ is a difference of angles appearing in the definitions of the cones $\mathcal{B}$ and $\mathcal{A}$.)

In the second case, $l = n$, we have

$$k_D(z_0, z) \geq k_{D_{\alpha,c}}(z_0, z) \geq p\left(0, \frac{|z|^{\alpha}}{c}\right) + C.$$

Similarly as before,

$$|z|^{\alpha} = \prod_{j=1}^{n} |\zeta_{D_{\alpha,c}}(z)_j - d_{D_{\alpha,c}}(z)\nu_{D_{\alpha,c}}(z)_j|^{\alpha_j} = c - \sigma(z) d_{D_{\alpha,c}}(z)$$

with a bounded positive function σ. Hence

$$p\left(0, \frac{|z|^\alpha}{c}\right) \geq -\frac{1}{2}\log d_{D_{\alpha,c}}(z) + C \geq -\frac{1}{2}\log d_D(z) + C.$$

Now take $\zeta_0 \in \partial D \setminus \mathbb{C}_*^n$. We may assume that the first k coordinates of ζ_0 are nonzero and that the last $n-k$ are zero, where $0 \leq k \leq n-1$. Notice that $k \neq 0$; indeed, the assumption $k = 0$ is equivalent to $0 \in \partial D$. Using Facts 1 and 2, we see that the $\mathcal{C}^1$-smoothness of D implies (by the Fu condition) $D \cap V_j^n \neq \emptyset$ for $j = 1, \dots, n$. Hence D is complete and so $0 \in D$—a contradiction. Finally, point ζ_0 has the form

$$\zeta_0 = (\zeta_{01}, \dots, \zeta_{0k}, 0, \dots, 0), \quad \zeta_{0j} \neq 0, \quad 1 \leq j \leq k \leq n-1.$$

Consider the projection $\pi_k \colon \mathbb{C}^n \to \mathbb{C}^k$; that is,

$$\pi_k(z) = (z_1, \dots, z_k).$$

We will show that $D_k := \pi_k(D)$ is a $\mathcal{C}^1$-smooth pseudoconvex Reinhardt domain. A Reinhardt property is clear for D_k. To affirm the pseudoconvexity of D_k, it suffices to show that

$$D_k \times \{0\}^{n-k} = D \cap (\mathbb{C}^k \times \{0\}^{n-k}).$$

Inclusion,

$$D_k \times \{0\}^{n-k} \supset D \cap (\mathbb{C}^k \times \{0\}^{n-k}),$$

is obvious. To prove the opposite inclusion, we again use Facts 1 and 2. We have $D \cap V_j^n \neq \emptyset$ for $j = k+1, \dots, n$, so D is complete in jth direction for $j = k+1, \dots, n$. Take some $z \in D_k \times \{0\}^{n-k}$. Then $z = (z_1, \dots, z_k, 0, \dots, 0)$ and $(z_1, \dots, z_k, \tilde{z}_{k+1}, \dots, \tilde{z}_n) \in D$ for some $\tilde{z}_{k+1}, \dots, \tilde{z}_n \in \mathbb{C}$. Thus

$$(z_1, \dots, z_k, 0, \dots, 0) \in D;$$

that is, $z \in D \cap (\mathbb{C}^k \times \{0\}^{n-k})$.

The local defining function for D_k at point $\zeta \in \partial D_k$ is

$$\tilde{\rho}(z_1, \dots, z_k) := \rho(z_1, \dots, z_k, 0, \dots, 0), \quad (z_1, \dots, z_k) \in \pi_k(U) \cap D_k,$$

where $\rho \colon U \to \mathbb{R}$ is the local defining function for D at point $(\zeta, 0, \dots, 0)$. Indeed, $\nabla\tilde{\rho} \neq 0$ because

- $\nabla\rho \neq 0$,
- $\frac{\partial\tilde{\rho}}{\partial\bar{z}_j} = \frac{\partial\rho}{\partial\bar{z}_j}$ for $j = 1, \dots, k$, and
- $\frac{\partial\rho}{\partial\bar{z}_j} = 0$ for $j = k+1, \dots, n$.

However, the two remaining conditions for a defining function follow easy from the definition of $\tilde{\rho}$.

If z tends to ζ_0 nontangentially in a cone $\mathcal{A} \subset \mathbb{C}^n$, then $\pi_k(z)$ tends to $\pi_k(\zeta_0) \in \mathbb{C}_*^k$ nontangentially in a cone $\pi_k(\mathcal{A}) \subset \mathbb{C}^k$. From the case $\zeta_0 \in \partial D \cap \mathbb{C}_*^n$ already shown, we have

$$k_D(z_0, z) \geq k_{D_k}(\pi_k(z_0), \pi_k(z)) \geq -\frac{1}{2}\log d_{D_k}(\pi_k(z)) + C.$$

Hence, to finish the proof it suffices to show that

$$d_{D_k}(\pi_k(z)) \lesssim d_D(z).$$

Consider a cone $\mathcal{B}$, with vertex ζ_0 and semi-axis $-\nu_{D_\alpha}(\zeta_0)$, whose intersection with some neighborhood of the point ζ_0 is contained in D and contains in its interior the cone $\mathcal{A}$. Then

$$1 \geq \frac{d_{\mathcal{B}}(z)}{d_D(z)} = \frac{\|z-\zeta_{\mathcal{B}}(z)\|}{\|z-\zeta_D(z)\|} \geq \frac{\|z-\zeta_{\mathcal{B}}(z)\|}{\|z-\zeta_0\|} = \sin\angle(z,\zeta_0,\zeta_{\mathcal{B}}(z)) \geq \sin\theta,$$

where θ is again the angle between these generatrices of the cones $\mathcal{A}$ and $\mathcal{B}$ that lie in one plane with the axis of both cones. Analogously,

$$1 \geq \frac{d_{\pi_k(\mathcal{B})}(\pi_k(z))}{d_{D_k}(\pi_k(z))} \geq \sin\theta',$$

where θ' depends only on $\mathcal{B}$. Hence

$$\frac{d_{D_k}(\pi_k(z))}{d_D(z)} \approx \frac{d_{\pi_k(\mathcal{B})}(\pi_k(z))}{d_{\mathcal{B}}(z)}.$$

However,

$$\begin{aligned}\frac{d_{\pi_k(\mathcal{B})}(\pi_k(z))}{d_{\mathcal{B}}(z)} &= \frac{\|\pi_k(z)-\zeta_{\pi_k(\mathcal{B})}(\pi_k(z))\|}{\|z-\zeta_{\mathcal{B}}(z)\|}\\ &= \frac{\|\pi_k(z)-\pi_k(\zeta_0)\|\sin\angle(\pi_k(z),\pi_k(\zeta_0),\zeta_{\pi_k(\mathcal{B})}(\pi_k(z)))}{\|z-\zeta_0\|\sin\angle(z,\zeta_0,\zeta_{\mathcal{B}}(z))}\\ &\leq \frac{\|\pi_k(z)-\pi_k(\zeta_0)\|}{\|z-\zeta_0\|\sin\theta} \leq \frac{1}{\sin\theta}.\end{aligned}$$ □

PROPOSITION. *The estimate from below by $-\frac{1}{2}\log d_D + C$ for the Carathéodory (pseudo)distance c_D is not true even for a smooth bounded complete pseudoconvex Reinhardt domain D and its boundary point $\zeta_0 \in \mathbb{C}_*^n$.*

Proof. Consider a domain

$$D := \{(z_1,z_2)\in\mathbb{C}^2 : |z_1| < R_1,\ |z_2| < R_2,\ |z_1||z_2|^\alpha < R_3\},$$

where $R_1,R_2,R_3 > 0$, $\alpha \in (\mathbb{R}\setminus\mathbb{Q})_+$, and $R_1R_2^\alpha > R_3$. Fix $\zeta_0 \in \partial D$ such that $|\zeta_{01}| < R_1$ and $|\zeta_{02}| < R_2$. This domain is not smooth. Since

$$\log D = \{(x_1,x_2)\in\mathbb{R}^2 : x_1 < \log R_1,\ x_2 < \log R_2,\ x_1+\alpha x_2 < \log R_3\},$$

it is easy to construct a smooth bounded convex domain $E \subset \mathbb{R}^2$ such that $\log D \subset E$ and ∂E contains the skew segment

$$(\partial\log D)\cap\{(x_1,x_2)\in\mathbb{R}^2 : x_1+\alpha x_2 = \log R_3\}.$$

Let $\tilde{D} \subset \mathbb{C}^2$ be a complete Reinhardt domain such that $\log\tilde{D} = E$. Then $\tilde{D}$ is bounded, smooth, and (thanks to Fact 1) pseudoconvex. Moreover, $D \subset \tilde{D}$ and there is a neighborhood U of their common boundary point ζ_0 such that $D\cap U = \tilde{D}\cap U$.

From [9, Prop. 4.3.2] it follows that

$$a_\lambda := \frac{g_D(\lambda\zeta_0, 0)}{\log|\lambda|} \to \infty \text{ as } \lambda \to \partial\mathbb{D},$$

where g_D is the pluricomplex Green function. (For general properties of the Carathéodory (pseudo)distance and the pluricomplex Green function, see e.g. [3; 9].) Certainly,

$$d_{\tilde{D}}(\lambda\zeta_0) = d_D(\lambda\zeta_0) \approx 1 - |\lambda| \text{ as } |\lambda| \to 1$$

and

$$c_D(\lambda\zeta_0, 0) \le \tanh^{-1} \exp g_D(\lambda\zeta_0, 0).$$

So if there exists a constant $C > 0$ such that

$$c_{\tilde{D}}(\lambda\zeta_0, 0) \ge -\frac{1}{2}\log d_{\tilde{D}}(\lambda\zeta_0) + C, \; |\lambda| \to 1$$

then, for $|\lambda| \to 1$,

$$c_D(\lambda\zeta_0, 0) \ge -\frac{1}{2}\log d_D(\lambda\zeta_0) + C,$$

$$-\frac{1}{2}\log(1 - |\lambda|) + C \le \tanh^{-1}|\lambda|^{a_\lambda},$$

$$\frac{1}{1 - |\lambda|} \le \frac{C'}{1 - |\lambda|^{a_\lambda}}$$

with a constant $C' > 0$. For $|\lambda|$ sufficiently close to 1 we have $a_\lambda \ge C' + 1$. Hence

$$\frac{1}{1 - |\lambda|} \le \frac{C'}{1 - |\lambda|^{C'+1}}$$

or, equivalently,

$$\frac{1 - |\lambda|^{C'+1}}{1 - |\lambda|} \le C'.$$

The left-hand side tends to $C' + 1$ as $|\lambda| \to 1$. □

4. Open Problems

We conclude by describing three open problems as follows.

1. Can we improve the estimate from Theorem 1 to $-\frac{1}{2}\log d_D(z) + C$?

2. Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain and let $\zeta_0 \in \partial D \cap \mathbb{C}^n_*$. Does it follow that, for some constant C, the inequality

$$k_D(z_0, z) \ge -\frac{1}{2}\log d_D(z) + C$$

holds if $z \in D$ tends to ζ_0?

3. Is it true for pseudoconvex Reinhardt domains $D \subset \mathbb{C}^n$ that if

$$\#\{j : \zeta_{0j} = 0 \text{ and } D \cap V_j^n = \emptyset\} = 0$$

then

$$k_D(z_0, z) \geq -\frac{1}{2}\log d_D(z) + C$$

and that otherwise

$$k_D(z_0, z) \geq \frac{1}{2}\log(-\log d_D(z)) + C$$

for $z \in D$ near $\zeta_0 \in \partial D$?

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