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Betti Numbers of Smooth Schubert Varieties and the Remarkable Formula of Kostant, Macdonald, Shapiro, and Steinberg

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1. Introduction

Let G be a semi-simple linear algebraic group over $\mathbb{C}$, B a Borel subgroup of G, and $T \subset B$ a maximal torus. Let Φ denote the root system of the pair (G,T) and Φ^+ the set of positive roots determined by B. Let $\alpha_1,\dots,\alpha_\ell$ denote the basis of Φ associated to Φ^+, and recall the *height* of $\alpha = \sum k_i\alpha_i \in \Phi$ is defined to be $\operatorname{ht}(\alpha) = \sum k_i$. Finally, let $W = N_G(T)/T$ be the Weyl group of (G,T).

A remarkable formula—originally noticed by A. Shapiro and proved by Kostant [13] using the representation theory of the principal three-dimensional subgroup of G, by Macdonald [14] using the holomorphic Lefschetz formula, and by Steinberg [16] by verification—says that

$$\prod_{i=1}^{\ell}(1+t^2+\dots+t^{2m_i}) = \prod_{\alpha\in\Phi^+}\frac{1-t^{2\operatorname{ht}(\alpha)+2}}{1-t^{2\operatorname{ht}(\alpha)}}, \tag{1}$$

where $m_1,\dots,m_\ell$ are the exponents of G. The identity (1) can also be formulated combinatorically. Suppose h_i is the number of roots of height i where k is the height of the highest root. That is, $k+1$ is the Coxeter number of (G,T). Then $h_i \ge h_{i+1}$, so $(h_1,h_2,\dots,h_k)$ is a partition of $|\Phi^+|$. Then (1) is equivalent to saying that $(h_1,h_2,\dots,h_k)$ is conjugate to the partition determined by the exponents m_j of (G,T) (see Lemma 1).

A cohomological proof of (1), which we will generalize in this paper, goes as follows. First, by the well-known Borel picture of the cohomology algebra of G/B as the coinvariant algebra of W, the Poincaré polynomial $P(G/B,t)$ of the flag variety G/B has the expression

$$P(G/B,t) = \prod_{i=1}^{\ell}\frac{1-t^{2d_i}}{1-t^2}, \tag{2}$$

where $d_1,\dots,d_\ell$ are the degrees of the fundamental generators of the ring of W-invariant polynomials on the Lie algebra $\mathfrak{t}$ of T. By a different cohomological method, reviewed in Section 2 (cf. [1, Cor. 1]), one also obtains that

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$$P(G/B,t) = \prod_{\alpha\in\Phi^+} \frac{1-t^{2\,\mathrm{ht}(\alpha)+2}}{1-t^{2\,\mathrm{ht}(\alpha)}}. \tag{3}$$

Since the fundamental degrees and exponents are related by the identities $d_i = m_i + 1$ for each i with $1 \le i \le \ell$, the identity (1) follows immediately from (2) and (3). The proof of (3) is obtained from a theorem that is in the spirit of both the holomorphic Lefschetz formula and the principal three-dimensional subgroups of G. We will review this theorem in Section 2 and then present a generalization. Briefly, it says that if a two-dimensional solvable group $\mathfrak{B}$ acts algebraically on a smooth complex projective variety X so that the unipotent radical $\mathfrak{U}$ of $\mathfrak{B}$ has a unique fixed point, then the fixed point scheme of $\mathfrak{U}$ has the property that its coordinate algebra is a graded ring that is isomorphic with the cohomology algebra $H^*(X,\mathbb{C})$ with its natural grading. The formula for the Poincaré polynomial is then obtained from some basic commutative algebra.

The plan of this paper is to use the results just mentioned to generalize (1) to smooth Schubert varieties in an arbitrary flag variety G/B. Before stating our result, we will quickly set up the notation. Recalling that $\alpha_1,\dots,\alpha_\ell$ denote the simple roots, let $S \subset W$ be the associated set of simple relections r_{α_i}, and recall that (W,S) is a Coxeter system. Let $\ell(w)$ be the length of $w \in W$ and less than or equal to the Bruhat–Chevalley order. By well-known properties of the Bruhat decomposition $G = BWB$, every B-orbit on G/B has the form BwB/B for a unique $w \in W$. The Zariski closure X_w of BwB/B is called the Schubert variety associated to w. Each Schubert variety X_w is a projective B-variety such that $\dim X_w = \ell(w)$, and one has

$$X_w = \bigcup_{x\le w} BxB/B.$$

Furthermore, BwB/B is an affine cell isomorphic with $\mathbb{C}^{\ell(w)}$. Thus, the Poincaré polynomial of X_w, which we will denote by $P_w(t)$, has the expression

$$P_w(t) = \sum_{x\le w} t^{2\ell(x)}.$$

Not all Schubert varieties are smooth. In fact, smoothness is equivalent to having $\dim T_e(X_w) = \ell(w)$, where $T_e(X_w)$ is the Zariski tangent space to X_w at the identity coset e. A simple requirement given in terms of the Bruhat–Chevalley order is as follows. Let $\Phi^+(w) = \{\alpha > 0 \mid r_\alpha \le w\}$. Then, if X_w is smooth, $|\Phi^+(w)| = \ell(w)$, the reason being that each $\alpha \in \Phi^+(w)$ gives rise to a T-stable line in $T_e(X_w)$ having weight $-\alpha$ whereas, by [7, Sec. 2], $T_e(X_w)$ cannot contain more than $\dim T_e(X_w)$ T-stable lines.

The generalization of the identity (3) for a smooth Schubert variety X_w in G/B says

$$P_w(t) = \prod_{\alpha\in\Phi^+(w)} \frac{1-t^{2\,\mathrm{ht}(\alpha)+2}}{1-t^{2\,\mathrm{ht}(\alpha)}}. \tag{4}$$

For each $i > 0$, put $h_{w,i} = |\{\alpha \in \Phi^+(w) \mid \mathrm{ht}(\alpha) = i\}|$. We will show that $h_{w,i} \ge h_{w,i+1}$, so the $h_{w,i}$ form a nonincreasing partition η of $\ell(w) = |\Phi^+(w)|$. Let $d_{w,i} = h_{w,i} - h_{w,i+1}$. Then here is our result.

THEOREM 1. *Let X_w be a smooth Schubert variety in G/B, and let k denote the largest height occurring in $T_e(X_w)$. Then $d_{w,k} > 0$, and*

$$P_w(t) = \prod_{1 \le i \le k} (1 + t^2 + \cdots + t^{2i})^{d_{w,i}}. \tag{5}$$

If μ is the partition of $\ell(w)$ conjugate to η and $i \ge 1$, then $d_{w,i}$ is the number of times i occurs in μ.

REMARK 1. By definition, $\sum_{1 \le i \le k} d_{w,i} = h_{w,1}$ is the number of simple roots in $\Phi^+(w)$, so by (5), the second Betti number of X_w satisfies $b_2(X_w) = h_{w,1}$.

REMARK 2. What is notable about (5) is the factorization of $P_w(t)$ into polynomials of the form $\mu_i(t) = 1 + t^2 + \cdots + t^{2i}$. This doesn't hold for smooth Schubert varieties in G/P, for example. Indeed, the Poincaré polynomial $1+t^2+2t^4+t^6+t^8$ of the Grassmanian $\mathrm{Gr}(2,4)$ of two planes in $\mathbb{C}^4$ factors $(1+t^4)(1+t^2+t^4)$.

Different versions of (4) and (5) have appeared in several places. Formula (4) was stated for arbitrary smooth Schubert varieties in [7, Thm. I] and used by Billey [2] to derive (5) in type A. (We were unaware of this when the first version of this paper was written.) Gasharov [10] gave a purely combinatorial proof in type A for the assertion that $P_w(t)$ is palindromic if and only if there exist $i_1, \dots, i_k$ such that $P_w(t) = \mu_{i_1}(t) \cdots \mu_{i_k}(t)$, and Billey [2] showed that this is also true in types B and C.

More recently, Oh, Postnikov, and Yoo [15] found a surprising expression for the Poincaré polynomial of a smooth Schubert variety $X_w \subset \mathrm{SL}(n,\mathbb{C})/B$ in terms of a certain arrangement associated to w. Let the inversion arrangement associated to $X(w)$ be the hyperplane arrangement $\mathcal{A}_w$ in $\mathbb{R}^n$ defined by $\{\alpha \in \Phi^+ \mid w^{-1}(\alpha) < 0\}$. Then they showed $P_w(t)$ is palindromic if and only if it equals the wall-crossing polynomial $R_w(t)$ associated to $\mathcal{A}_w$. See Section 4 for the definition of this polynomial. This result was the motivation for us to reconsider the factorization (4) in the smooth case. We will discuss some further questions about the connection between the inversion arrangement and smoothness of X_w in Section 4.

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2. Regular Actions and the Product Formula

In fact, Theorem 1 follows immediately from a general product formula that holds for certain $\mathfrak{B}$-varieties. Let X denote a smooth complex projective variety with an algebraic action $\mathfrak{B} \circlearrowright X$ such that the unipotent radical $\mathfrak{U}$ of $\mathfrak{B}$ has exactly one fixed point, say $X^{\mathfrak{U}} = \{o\}$. We will stick with the terms "regular action" for $(\mathfrak{B}, X)$ and "$\mathfrak{B}$-regular variety" for X used in [8]. The following facts, proved in [6], will be needed in the sequel.

(a) The fixed point set of a maximal torus of $\mathfrak{B}$ is finite. Moreover, if $\mathfrak{T}$ is the maximal torus on the diagonal of $\mathfrak{B}$ then $o \in X^{\mathfrak{T}}$.
(b) If $\lambda\colon \mathbb{C}^* \to \mathfrak{T}$ is the one-parameter subgroup $\lambda(s) = \operatorname{diag}[s, s^{-1}]$, then the Bialynicki–Birula cell

$$X_o = \left\{ x \in X \mid \lim_{s \to \infty} \lambda(s) \cdot x = o \right\} \tag{6}$$

is a dense open subset of X. Consequently, the weights of the natural action of λ on $T_o(X)$ are negative integers, say $b_1 > b_2 > \cdots > b_k$.
(c) X_o is $\mathfrak{T}$-equivariantly isomorphic with the Zariski tangent space $T_o(X)$.

Consequently, $\mathbb{C}[T_o(X)]$ and $\mathbb{C}[X_o]$ are isomorphic rings graded by the $\mathfrak{T}$-action. Putting $a_i = -b_i$ for each i, we have $0 < a_1 < a_2 < \cdots < a_k$.

Let $M_{b_i} \subset T_o(X)$ denote the $\mathfrak{T}$-weight space corresponding to b_i and let $\mu_i = \dim M_{b_i}$ so that

$$T_o(X) = M_{b_1} \oplus M_{b_2} \oplus \cdots \oplus M_{b_k}. \tag{7}$$

The main result on regular varieties (see [6] and also [1]) says that if X is a $\mathfrak{B}$-regular variety then the cohomology algebra $H^*(X, \mathbb{C})$ is isomorphic with $\mathbb{C}[X_o]/I$, where I is the ideal of the fixed point scheme of $\mathfrak{U}$, which is a punctual scheme supported by o. Note that this isomorphism doubles degrees. As shown in [1], this gives rise to a product representation for the Poincaré polynomial $P(X,t)$ of X. Namely,

$$P(X,t) = \prod_{1 \le i \le k} \left(\frac{1 - t^{a_i+2}}{1 - t^{a_i}} \right)^{\mu_i}. \tag{8}$$

The exponents grow by 2 because the induced Lie($\mathfrak{B}$)-module action on $T_o(X)$ has the property that $v(M_{b_i}) \subset M_{b_i+2}$, where

$$v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

As in [1], we say a $\mathfrak{B}$-regular variety is *homogeneous* when $\ker v = M_{b_1}$. Homogeneity has a number of nice consequences. For example, v is injective on M_{b_i} for all $i > 1$, so the weight spaces have nonincreasing dimension: $\dim M_{b_j} \le \dim M_{b_i}$ if $i \le j$. The following key result is proved in [1, Thm. 3].

THEOREM 2. *Suppose X is a $\mathfrak{B}$-regular homogeneous variety. Then $a_i = 2i$ for each $i = 1, \ldots, k$.*

Therefore, by (8),

$$P(X,t) = \prod_{1 \le i \le k} \left(\frac{1 - t^{2i+2}}{1 - t^{2i}} \right)^{\mu_i}. \tag{9}$$

Define *defects* $d_i = \mu_i - \mu_{i+1}$ for each $i = 1, \ldots k$, where $\mu_{k+1} = 0$. Thus $\sum_{i=1}^{k} d_i = \mu_1$ and $d_k > 0$. Now we have the main result.

THEOREM 3. *Let X denote a homogeneous $\mathfrak{B}$-regular variety with defects $d_1, \ldots, d_k$. Then*

$$P(X,t) = \prod_{1\le i\le k} (1 + t^2 + \cdots + t^{2i})^{d_i}. \tag{10}$$

Consequently, $b_2(X) = \mu_1$, so the nonzero defects form a not necessarily decreasing partition of $b_2(X)$.

Proof. The right-hand side of (9) is

$$\left(\frac{1-t^4}{1-t^2}\right)^{\mu_1}\left(\frac{1-t^6}{1-t^4}\right)^{\mu_2}\cdots\left(\frac{1-t^{2k}}{1-t^{2k-2}}\right)^{\mu_{k-1}}\left(\frac{1-t^{2k+2}}{1-t^{2k}}\right)^{\mu_k},$$

which after a little algebra becomes the right-hand side of (10). The assertion about $b_2(X)$ follows from a straightforward calculation. □

Before stating the next corollary, we make a well-known and useful remark on partitions.

LEMMA 1. *Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$ be a nonincreasing partition with $\mu_k > 0$, and put $\delta_i = \mu_i - \mu_{i+1}$ where $\mu_{k+1} = 0$. Then the partition*

$$(k, \ldots, k, k-1, \ldots, k-1, \ldots, 1, \ldots, 1), \tag{11}$$

where i is repeated δ_i times, is conjugate to $\boldsymbol{\mu}$.

Proof. Indeed, consider the Ferrers diagram of $\boldsymbol{\mu}$ and observe that the first μ_k columns have k boxes, the next $\mu_{k-1} - \mu_k$ columns have $k-1$ boxes, and so on. □

Hence, if X is regular and homogeneous then the lemma gives a direct connection among the weight decomposition of $T_o(X)$, the Poincaré polynomial of X, and a polynomial associated to the partition conjugate to the partition associated to the dimensions of the weight spaces. This gives an interesting expression for the Euler characteristic $\chi(X) = P(X, 1)$ of X.

COROLLARY 1. *If X is as in Theorem 3, then the Euler number $\chi(X)$ is given by*

$$\chi(X) = \prod_{1\le i\le k} (i+1)^{d_i}. \tag{12}$$

Consequently, $\chi(X)$ is divisible by $1+k$. Moreover, the number of nontrivial terms in this factorization is $b_2(X)$.

Proof. This follows immediately by setting $t = 1$ in (10). □

3. Schubert Varieties in G/B As Homogeneous $\mathfrak{B}$-Regular Varieties

Let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{t}$ denote the Lie algebras of G, B, and T. Since the identity coset eB is fixed under the natural action of B on G/B by left translation, $T_e(G/B)$ is a $\mathfrak{b}$-module, which, as is well known, is isomorphic to $\mathfrak{g}/\mathfrak{b}$. Choosing a weight vector $e_{-\alpha} \in \mathfrak{g}$ for each $\alpha \in \Phi^+$, one also has an isomorphism

$$T_e(G/B) \cong \sum_{\alpha>0} \mathbb{C}e_{-\alpha}$$

in the sense of $\mathfrak{t}$-modules. To see that G/B is $\mathfrak{B}$-regular, consider the principal nilpotent

$$e = \sum_{i=1}^{\ell} e_{\alpha_i}, \tag{13}$$

where $\alpha_1, \dots, \alpha_\ell$ are the simple roots determined by B. Let h be the unique element of $\mathfrak{t}$ such that $\alpha_i(h) = 2$ for each i. Thus, $\alpha(h) = 2\,\mathrm{ht}(\alpha)$ for any $\alpha \in \Phi$. Let $\mathfrak{B}$ denote the solvable subgroup of B corresponding to the two-dimensional solvable subalgebra $\mathbb{C}h \oplus \mathbb{C}e$. Then the action $\mathfrak{B} \circlearrowright G/B$ is regular with $o = eB$. Indeed, if we identify G/B with the variety of Borel subalgebras of $\mathfrak{g}$ [12], then the fixed point set of $\mathfrak{U}$ consists of all Borel subalgebras containing $\mathfrak{u} = \mathrm{Lie}(\mathfrak{U})$. However, since e is a regular nilpotent (i.e., its centralizer has dimension ℓ), it follows that e and hence $\mathfrak{u}$ is contained in just one Borel subalgebra—namely, $\mathfrak{b}$.

Since $[h, e_{-\alpha}] = -\alpha(h)e_{-\alpha}$, the $\mathbb{C}h$-weight subspaces of $T_e(G/B)$ take the form

$$M_{2i} = \mathrm{span}\{e_{-\alpha} \mid \alpha \in \Phi^+,\ \mathrm{ht}(\alpha) = 2i\},$$

where $i = 1, \dots, k$, k being the height of the highest root. Clearly $\ker e = M_2$, so $\mathfrak{B} \circlearrowright G/B$ is homogeneous. Recalling the notation $\dim M_{2i} = h_i$ and $d_i = h_i - h_{i+1}$, Theorem 3 gives

$$P(G/B, t) = \prod_{1\le i\le k} \left(\frac{1-t^{2i+2}}{1-t^{2i}}\right)^{h_i} = \prod_{1\le i\le k} (1 + t^2 + t^4 + \cdots + t^{2i})^{d_i}, \tag{14}$$

which implies the KMSS identity (1) by Lemma 1.

Corollary 1 gives a variation of a well-known expression for the order of the Weyl group: namely,

$$|W| = \prod_{i=1}^{k} (i+1)^{d_i}.$$

In what follows we will generalize this to Weyl group intervals corresponding to smooth Schubert varieties. For example, if $G = \mathrm{SL}(n, \mathbb{C})$, then $W = S_n$, $k = n-1$, and each $d_i = 1$, so

$$P(\mathrm{SL}(n, \mathbb{C})/B, t) = \prod_{i=1}^{n-1} (1 + t^2 + \cdots + t^{2i}),$$

which gives another proof of the trivial fact that $|S_n| = n!$.

Since Schubert varieties in G/B are B-stable, a smooth Schubert variety X_w is $\mathfrak{B}$-regular and automatically homogeneous. Assume from now on that X_w is in fact smooth. By the discussion preceding (4),

$$T_e(X_w) = \bigoplus_{\alpha\in\Phi^+(w)} \mathbb{C}e_{-\alpha}. \tag{15}$$

It follows that the $\mathbb{C}h$-weight subspaces of $T_e(X_w)$ have the form

$$M(w)_{2i} = \operatorname{span}\{e_{-\alpha} \mid \operatorname{ht}(\alpha) = i,\ r_\alpha \le w\}.$$

Fixing notation, let $h_{w,i} = \dim M(w)_{2i}$ and put $d_{w,i} = \dim M(w)_{2i} - \dim M(w)_{2i+2}$. Define k_w to be the height of the highest root (or roots) in $\Phi^+(w)$, and put $h_w = \dim M(w)_{2k_w}$. Finally, set $\chi_w = \chi(X_w)$. Theorems 2 and 3 then yield the following result.

THEOREM 4. *A smooth Schubert variety X_w in G/B is a homogeneous $\mathfrak{B}$-regular variety. Consequently, the $\mathfrak{T}$-weights on $T_e(X_w)$ form a string of even negative integers $-2 \ge -4 \ge \cdots \ge -2k_w$. Furthermore, recalling that $P_w(t) = P(X_w, t)$, we have*

$$P_w(t) = \prod_{i=1}^{k_w} (1 + t^2 + \cdots + t^{2i})^{d_{w,i}}. \tag{16}$$

In particular, since $h_w = d_{w,k_w} \ge 1$, $(1 + t^2 + \cdots + t^{2k_w})^{h_w}$ is a nontrivial factor of $P_w(t)$. Moreover,

$$\chi_w = |[e, w]| = \prod_{i=1}^{k_w} (i + 1)^{d_{w,i}}. \tag{17}$$

Because $b_2(X_w) = \dim M(w)_2$ is the number of simple reflections in $[e, w]$, it follows that χ_w is a product of $b_2(X_w)$ integers between 2 and $(k_w + 1)^{h_w}$.

One can infer a little more information about χ_w from Theorem 4 as follows.

COROLLARY 2. *If X_w is smooth and m denotes the number of reflections $t \le w$ such that $\ell(t) = 3$, then $b_2(X_w) - m \le d_1$ and so $2^{(b_2(X_w)-m)}$ divides $|[e, w]|$.*

Proof. This follows from the claim that a root α of height 2 gives a reflection r_α of length 3 (but not conversely), which can be verified by a direct check. Alternatively, one can note that in a root system with simple roots α and β such that $\alpha + \beta$ is a root, either $\langle \alpha, \beta \rangle$ or $\langle \beta, \alpha \rangle = -1$, where, as usual, $\langle \alpha, \beta \rangle = 2(\alpha \cdot \beta)/(\beta \cdot \beta)$. □

The expression (17) for $|[e, w]|$ provides an easy-to-check but remarkably effective necessary condition for the smoothness of X_w because all it requires knowing is the height of each element of $\Phi^+(w)$ and the Euler number $|[e, w]|$. These issues can be dealt with using a reduced expression for w. Consider the following example.

EXAMPLE 1. Consider C_3/B and let 1, 2, and 3 denote the simple reflections r_1, r_2, and r_3 and α_1, α_2, and α_3 the corresponding simple roots. Here we assume α_1 is the long root. We claim $w = 21232$ is singular. Indeed, by the useful tables compiled by Goresky [11], one sees that $\chi_w = 18$. The reflections t such that $t \le w$ are 1, 2, 3, 212, and 232. In C_3, 212 corresponds to the root $\alpha_1 + 2\alpha_2$ of height 3 while 232 corresponds to the root $\alpha_2 + \alpha_3$ of height 2. Hence, $h_{w,1} = 3$ and $h_{w,2} = h_{w,3} = 1$. If X_w is smooth, then Theorem 4 implies χ_w is divisible by 4. But $\chi_w = 18$ [11], so X_w is singular. Similarly, if $v = 12132123$ then $\chi_v = 44$. But the highest root $\alpha_1 + 2\alpha_2 + 2\alpha_3$ in $\Phi(C_3)$ lies in $\Phi^+(v)$ since its reflection $t = 32123$ satisfies $t \leq v$. Consequently, similar reasoning says X_v is also singular.

One can also find a lower bound on χ_w in terms of $\Phi^+(w)$. Let $\Phi^+(w)_1$ denote the number of simple roots in $\Phi^+(w)$.

THEOREM 5. *If X_w is smooth then $\chi_w \geq |\Phi^+(w)_1|(|\Phi^+(w)|-1)+2$, with equality if and only if $\ell(w) = 1$.*

Proof. Since X_w is smooth projective and has vanishing odd Betti numbers, the hard Lefschetz property implies the inequality

$$\chi_w \geq b_2(X_w)(\dim X_w - 1) + 2.$$

Equating $b_2(X_w)$ with $|\Phi^+(w)_1|$ and $\dim X_w$ with $|\Phi^+(w)|$ gives the inequality stated in the theorem. Now suppose equality holds. Then it follows that $b_{2i}(X_w) = b_2(X_w)$ for each i with $1 \leq i \leq \dim X_w$. The only possibility for this is that $P_w(t) = 1 + t^2 + \cdots + t^{2\ell(w)}$. But then we see that $b_2(X_w) = 1$, so $|\Phi^+(w)_1| = 1$ as well; hence $|\Phi^+(w)| = 1$. Conversely, if $|\Phi^+(w)| = 1$ then equality clearly holds. $\square$

The estimate of Theorem 4 improves a well-known result for a projective variety X with a torus action having isolated fixed points, which says $\chi(X) \geq \dim X + 1$ [4, Prop. 13.5]. According to a result of Björner and Ekedahl [3], the inequality $\chi_w \geq b_2(X_w)(\dim X_w - 1) + 2$ holds for arbitrary Schubert varieties as long as $\ell(w)$ is sufficiently large. Since $\chi_w = |[e, w]|$, this amounts to a lower bound on the size of a Bruhat interval starting at the identity with the length proviso. Notice that, in Example 1, if $w = 21232$ then the estimate says $\chi_w = 18 \geq 3(5-1) + 2 = 14$; however, if $v = 12132123$ then it says $\chi_v = 44 \geq 3(8-1) + 2 = 23$.

Finally, the assertion about conjugate partitions from Lemma 1 translates into Schubert variety terms as follows.

COROLLARY 3. *Suppose X_w is smooth. Then the partition $h_{w,1} \geq h_{w,2} \geq \cdots \geq h_w$ of $\ell(w)$ is dual to the partition* (11) *where each i, $1 \leq i \leq k_w$, is repeated $d_{w,i}$ times.*

This suggests that the conjugate partition for a smooth Schubert variety X_w may have a geometric interpretation analogous to the relation between the heights of roots and exponents.

4. Palindromicity of $P_w(t)$ and the Inversion Arrangement

Palindromicity of the Poincaré polynomial of a Schubert variety is a necessary condition for smoothness. Surprisingly, however, it turns out by a theorem of

Peterson that, for Schubert varieties in the flag variety of a simply laced group G, the converse is also true: $X(w)$ is smooth if and only if $P_w(t)$ is palindromic (see [9] for a proof). Thus the product formula (16) holds in the simply laced setting as long as $P_w(t)$ is palindromic. The following example shows that palindromicity doesn't guarantee that (16) holds in general, however.

EXAMPLE 2. Let $\Phi(B_2)$ be the root system of type B_2 with two simple roots α and β, where α is long and β is short, and with positive roots α, β, $\alpha+\beta$, and $\alpha+2\beta$. All Schubert varieties in B_2/B have the palindromicity property, but the Schubert variety X_w corresponding to $w = r_\beta r_\alpha r_\beta = r_{\alpha+2\beta}$ is singular. This is obvious since $\Phi^+(w) = \{\alpha, \beta, \alpha+2\beta\}$, so $h_{w,2} = 0$. Here, the right-hand side of product formula (16) is

$$\frac{(1+t^2)(1+t^2)(1-t^8)}{(1-t^6)},$$

which isn't even a polynomial. The product factorization of [2] is

$$P_w(t) = 1 + 2t^2 + 2t^4 + t^6 = (1+t^2)(1+t^2+t^4).$$

The following result classifies which Schubert varieties in G/B with the palindromicity property are smooth.

THEOREM 6. *Suppose G is semi-simple and doesn't contain G_2-factors, and let X_w be a Schubert variety in G/B with the palindromicity property* (*cf.* [7]). *Then X_w is smooth if and only if $\Phi^+(w)$ has the following property: if $\alpha \in \Phi^+(w)$ and $\beta \in \Phi^+$ are such that $\alpha - \beta \in \Phi^+$, then $\alpha - \beta \in \Phi^+(w)$. This is equivalent to the condition that*

$$\sum_{\alpha \in \Phi^+(w)} \mathbb{C} e_{-\alpha}$$

is a B-submodule of $T_e(X_w)$.

It seems to be an interesting question whether a Schubert variety $X(w)$ in G/B for which the product formula (16) holds is smooth. Unfortunately, the following counterexample shows this isn't true in general.

EXAMPLE 3. Let α and β denote (respectively) the long and short simple roots for G_2 corresponding to B, and let $r = r_\alpha$ and $s = r_\beta$ be the corresponding reflections. Let $w = srsrs$. Then

$$\Phi^+(w) = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta\}.$$

The heights in $\Phi^+(w)$ are 1, 2, 3, and 4, and the defects are $d_1 = 1$, $d_2 = d_3 = 0$, and $d_4 = 1$. Thus, the right-hand side of (16) is

$$(1+t^2)(1+t^2+t^4+t^6+t^8),$$

which is indeed $P_w(t)$. But it is well known that X_w is singular. In fact, X_w is a counterexample to Theorem 6 if one eliminates the G_2 hypothesis. It would be interesting to know whether any such examples occur in types B or C.

We will now make some comments about the inversion arrangement. Let $w \in W$, and recall that the *inversion set for* w is the set $I(w)$ consisting of the positive roots α for which $w^{-1}(\alpha) < 0$. The hyperplanes $\alpha = 0$ in the $\mathbb{R}$-span of Φ in $\mathfrak{t}^*$ are called the *inversion hyperplanes,* and the associated hyperplane arrangement $\mathcal{A}_w$ is called the *inversion arrangement.* The *wall-crossing polynomial* $R_w(t)$ associated to $\mathcal{A}_w$ is defined as follows: after fixing an arbitrary chamber C_0 of $\mathcal{A}_w$, put

$$R_w(t) = \sum_C t^{2d(C,C_0)},$$

where the sum is over all chambers C of $\mathcal{A}_w$ and $n(C, C_0)$ is the number of walls of $\mathcal{A}_w$ that must be crossed when going from C_0 to C. The polynomial $R_w(t)$ is palindromic for all w, and by [15], when $w \in S_n$, $P_w(t)$ is palindromic if and only if $P_w(t) = R_w(t)$. This result says that one can determine in terms of $I(w)$ alone when $P_w(t)$ is palindromic.

PROBLEM. Using $I(w)$ alone, determine a necessary and sufficient condition for $P_w(t)$ to be palindromic. Similarly, determine a necessary and sufficient condition for X_w to be smooth.

References

[1] E. Akyildiz and J. B. Carrell, *A generalization of the Kostant–Macdonald identity,* Proc. Nat. Acad. Sci. U.S.A. 86 (1989), 3934–3937.

[2] S. Billey, *Pattern avoidance and rational smoothness of Schubert varieties,* Adv. Math. 139 (1998), 141–156.

[3] A. Björner and T. Ekedahl, *On the shape of Bruhat intervals,* Ann. of Math. (2) 170 (2009), 799–817.

[4] A. Borel, *Linear algebraic groups,* 2nd ed., Grad. Texts in Math., 126, Springer-Verlag, New York, 1991.

[5] J. B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties,* Algebraic groups and their generalizations: Classical methods (University Park, 1991), Proc. Sympos. Pure Math., 56, part 1, pp. 53–61, Amer. Math. Soc., Providence, RI, 1994.

[6] ———, *Deformation of the nilpotent zero scheme and the intersection ring of invariant subvarieties,* J. Reine Angew. Math. 460 (1995), 37–54.

[7] ———, *B-submodules of* $\mathfrak{g}/\mathfrak{b}$ *and smooth Schubert varieties in* G/B, Transform. Groups 16 (2011), 673–680.

[8] J. B. Carrell and K. Kaveh, *On the equivariant cohomology of subvarieties of a* $\mathfrak{B}$*-regular variety,* Transform. Groups 13 (2008), 495–505.

[9] J. B. Carrell and J. Kuttler, *Singular points of* T*-varieties in* G/P *and the Peterson map,* Invent. Math. 151 (2003), 353–379.

[10] V. Gasharov, *Factoring the Poincaré polynomials for the Bruhat order on* S_n, J. Combin. Theory Ser. A 83 (1998), 159–164.

[11] M. Goresky, *Kazhdan Lusztig polynomials for finite reflection group D4,* http://www.math.ias.edu/~goresky/D4.txt.

[12] J. Humphreys, *Linear algebraic groups,* Grad. Texts in Math., 21, Springer-Verlag, New York, 1975.

[13] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group,* Amer. J. Math. 81 (1959), 973–1032.
[14] I. G. Macdonald, *The Poincaré series of a Coxeter group,* Math. Ann. 199 (1972), 161–174.
[15] S. Oh, A. Postnikov, and H. Yoo, *Bruhat order, smooth Schubert varieties, and hyperplane arrangements,* J. Combin. Theory Ser. A 115 (2008), 1156–1166.
[16] R. Steinberg, *Finite reflection groups,* Trans. Amer. Math. Soc. 91 (1959), 493–504.

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