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On the Generalized Chen's Conjecture on Biharmonic Submanifolds

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1. Biharmonic Submanifolds and the Generalized Chen's Conjecture

The generalized Chen's conjecture on biharmonic submanifolds asserts that any biharmonic submanifold of a nonpositively curved manifold is minimal. In this paper, we prove that this conjecture is false by constructing foliations of proper biharmonic hyperplanes in a 5-dimensional conformally flat space with negative sectional curvature. Many examples of proper biharmonic submanifolds of nonpositively curved spaces are also given.

All manifolds, maps, and tensor fields studied in this paper are assumed to be smooth. A *biharmonic map* is a map $\varphi\colon (M,g) \to (N,h)$ between Riemannian manifolds that is a local solution of the fourth-order partial differential equations

$$\tau^2(\varphi) := \mathrm{Trace}_g(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla^M})\tau(\varphi) - \mathrm{Trace}_g \mathrm{R}^N(\mathrm{d}\varphi, \tau(\varphi))\,\mathrm{d}\varphi = 0; \qquad (1)$$

here R^N denotes the curvature operator of (N,h) defined by

$$\mathrm{R}^N(X,Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z,$$

$\tau(\varphi) = \mathrm{Trace}_g \nabla\,\mathrm{d}\varphi$ is the tension field of φ, and $\tau(\varphi) = 0$ means that the map φ is harmonic. Clearly, it follows from (1) that any harmonic map is biharmonic. We refer to *non*harmonic biharmonic maps as *proper biharmonic maps.*

A submanifold M of (N,h) is called a *biharmonic submanifold* if the inclusion map $\mathbf{i}\colon (M, \mathbf{i}^*h) \to (N,h)$ is a biharmonic isometric immersion. It is well known that an isometric immersion is minimal if and only if it is harmonic. Hence a minimal submanifold is trivially biharmonic, and we refer to a *non*minimal biharmonic submanifold as a *proper biharmonic submanifold*.

Among the fundamental problems in the study of biharmonic maps are the following.

- *Existence problem.* Given two model spaces (e.g., some "good" spaces, such as spaces of constant sectional curvature or more general symmetric or homogeneous spaces), does there exist a proper biharmonic map from one space into another?

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- *Classification problem.* Classify all proper biharmonic maps between two model spaces where the existence is known.

A typical and challenging classification problem is the following.

CHEN'S CONJECTURE [Ch]. *Any biharmonic submanifold in a Euclidean space is minimal.*

The conjecture has been proved for biharmonic surfaces in $\mathbb{R}^3$ (by Jiang [J2] and independently by Chen and Ishikawa [ChIs]) and for biharmonic hypersurfaces in $\mathbb{R}^4$ [HV]. Dimitrić [D] showed that the conjecture is also true for any biharmonic curve, any biharmonic submanifold of finite type, any pseudo-umbilical biharmonic submanifold $M^m \subset \mathbb{R}^n$ with $m \neq 4$, and any biharmonic hypersurface in $\mathbb{R}^n$ with at most two distinct principal curvatures. However, the conjecture is still open in general.

In the same direction of classifying proper biharmonic submanifolds of nonpositively curved manifolds, Caddeo, Montaldo, and Oniciuc [CMO2] proved that any biharmonic submanifold in hyperbolic 3-space $H^3(-1)$ is minimal and that any pseudo-umbilical biharmonic submanifold $M^m \subset H^n$ with $m \neq 4$ is minimal. In [BMO1] it is shown that any biharmonic hypersurface of H^n with at most two distinct principal curvatures is minimal. All these results suggest the following generalized Chen's conjecture on biharmonic submanifolds, which was proposed in [CMO1].

THE GENERALIZED CHEN'S CONJECTURE. *Any biharmonic submanifold of (N, h) with $\mathrm{R}^N \leq 0$ is minimal* (see, e.g., [B1; B2; BMO1; BMO2; BMO3; CMO1; IInU; MO; Ou1; Ou2]).

The goal of this paper is to prove that the generalized Chen's conjecture for biharmonic submanifolds is false. We accomplish this by using the idea of constructing foliations of proper biharmonic hyperplanes in the conformally flat space given in [Ou1]. The idea is to determine a conformally flat metric on $\mathbb{R}^{m+1}$ such that a foliation by the hyperplanes defined by the graphs of linear functions becomes a proper biharmonic foliation. It turns out that when $m = 4$ the system of biharmonic equations reduces to a single equation that has infinitely many solutions—including counterexamples to the generalized Chen's conjecture.

2. Foliations of Conformally Flat Spaces by Biharmonic Hyperplanes

Since conformally flat spaces play a central role in this paper, in this section we summarize some basic definitions and the relations between various curvatures of two Riemannian manifolds that are conformally related. Two Riemannian metrics g and $\bar{g}$ on M are *conformally equivalent* if $\bar{g} = e^{2\sigma}g$ for some function σ on M. A map $\varphi\colon (M, g) \to (N, h)$ between Riemannian manifolds is *conformal* if $\varphi^*h = e^{2\sigma}g$ for some function σ on M. We say that two Riemannian manifolds (M, g) and (N, h) are *conformally diffeomorphic* if there exists a conformal diffeomorphism from one space into the other. A Riemannian manifold (M^m, g) is a *conformally flat space* if for any point of M there exists a neighborhood that is conformally diffeomorphic to the Euclidean space $\mathbb{R}^m$. It is well known that any 2-dimensional Riemannian manifold is conformally flat owing to the existence of isothermal coordinates. For $m = 3$, (M^m, g) is conformally flat if and only if the Schouten tensor H satisfies $(\nabla_X H)(Y, Z) = (\nabla_Y H)(X, Z)$ for any vector fields X, Y, and Z on M. For $m \geq 4$, (M^m, g) is conformally flat if and only if the Weyl curvature vanishes identically. It is easy to see that a space of constant sectional curvature is conformally flat yet there exist many conformally flat spaces that are not of constant sectional curvature.

Let ∇, R, Ric, and K (resp., $\bar{\nabla}$, $\bar{\mathrm{R}}$, $\overline{\mathrm{Ric}}$, and $\bar{\mathrm{K}}$) denote the Levi–Civita connection, Riemannian curvature, Ricci curvature, and sectional curvature of the Riemannian metric g (resp., $\bar{g} = e^{2\sigma} g$). Then, it is not difficult to verify (cf. [Ha; W]) the following relations between the connections and curvatures of the two Riemannian metrics that are conformally equivalent:

$$\bar{\nabla}_X Y = \nabla_X Y + (X\sigma)Y + (Y\sigma)X - g(X, Y)\,\mathrm{grad}_g\,\sigma \tag{2}$$

for all $X, Y \in TM$;

$$\begin{aligned}
&\bar{\mathrm{R}}(W, Z, X, Y)\\
&\quad = e^{2\sigma}\{\mathrm{R}(W, Z, X, Y) + g(\nabla_X \nabla\sigma, Z)g(Y, W)\\
&\qquad\quad - g(\nabla_Y \nabla\sigma, Z)g(X, W) + g(X, Z)g(\nabla_Y \nabla\sigma, W)\\
&\qquad\quad - g(Y, Z)g(\nabla_X \nabla\sigma, W) + [(Y\sigma)(Z\sigma) - g(Y, Z)|\nabla\sigma|^2]g(X, W)\\
&\qquad\quad - [(X\sigma)(Z\sigma) - g(X, Z)|\nabla\sigma|^2]g(Y, W)\\
&\qquad\quad + [(X\sigma)g(Y, Z) - (Y\sigma)g(X, Z)]g(\nabla\sigma, W)\}
\end{aligned} \tag{3}$$

for any $W, Z, X, Y \in TM$.

With respect to local coordinates $\{x_i\}$ and the natural frame $\left\{\frac{\partial}{\partial x_i} = \partial_i\right\}$, equation (3) is equivalent to

$$\begin{aligned}
e^{-2\sigma}\bar{\mathrm{R}}_{ijkl} = \mathrm{R}_{ijkl} &+ g_{il}\sigma_{jk} - g_{ik}\sigma_{jl} + g_{jk}\sigma_{il} - g_{jl}\sigma_{ik}\\
&+ (g_{il}g_{jk} - g_{ik}g_{jl})|\nabla\sigma|^2,
\end{aligned} \tag{4}$$

where we have used the notation $\sigma_{jl} = \nabla_l \sigma_j - \sigma_l \sigma_j = \nabla_l \nabla_j \sigma - \sigma_l \sigma_j$. Contracting (4) now yields

$$\bar{\mathrm{R}}_{jk} = \mathrm{R}_{jk} - (n-2)\sigma_{jk} - g_{jk}[\Delta\sigma + (n-2)|\nabla\sigma|^2], \tag{5}$$

where Δ and ∇ denote (respectively) the Laplacian and the gradient operator defined by the metric g.

Let P be a section spanned by an orthonormal basis X, Y with respect to g (hence $\bar{X} = e^{-\sigma}X$, $\bar{Y} = e^{-\sigma}Y$ form an orthonormal basis with respect to $\bar{g}$). Then we may express the relationship between sectional curvatures with respect to metrics $\bar{g}$ and g as

$$\begin{aligned}
e^{2\sigma}\bar{\mathrm{K}}(P) = \mathrm{K}(P) &- (g(\nabla_X \nabla\sigma, X) + g(\nabla_Y \nabla\sigma, Y))\\
&- (|\nabla\sigma|^2 - (X\sigma)^2 - (Y\sigma)^2).
\end{aligned} \tag{6}$$

We also need the following theorem, which will be used to prove our main theorem about biharmonic hypersurfaces in a conformally flat space.

THEOREM 2.1 [Ou1]. *Let $\varphi\colon M^m \to N^{m+1}$ be an isometric immersion of codimension 1 with mean curvature vector $\eta = H\xi$. Then φ is biharmonic if and only if*

$$\begin{aligned} \Delta_g H - H|A|^2 + H\,\mathrm{Ric}^N(\xi,\xi) = 0,\\ 2A(\mathrm{grad}_g\, H) + \frac{m}{2}\,\mathrm{grad}_g\, H^2 - 2H(\mathrm{Ric}^N(\xi))^{\top} = 0. \end{aligned} \tag{7}$$

Here Δ_g and grad_g are (respectively) the Laplacian and gradient operators of the hypersurface, $\mathrm{Ric}^N\colon T_qN \to T_qN$ denotes the Ricci operator of the ambient space defined by $\langle \mathrm{Ric}^N(Z), W\rangle = \mathrm{Ric}^N(Z,W)$, and A is the shape operator of the hypersurface with respect to the unit normal vector ξ.

Now we are ready to prove one of the main theorems of this paper.

THEOREM 2.2. *For a positive integer $m \geq 2$, let a_i $(i = 1,2,\ldots,m)$ and c be constants. Then, for $h = f^{-2}(z)\big(\sum_{i=1}^m \mathrm{d}x_i^2 + \mathrm{d}z^2\big)$ and $\varphi(x_1,\ldots,x_m) = \big(x_1,\ldots,x_m, \sum_{i=1}^m a_i x_i + c\big)$, the isometric immersion $\varphi\colon \mathbb{R}^m \to (\mathbb{R}^{m+1}, h)$ into the conformally flat space is biharmonic if and only if one of the following three cases occurs*:

(i) $f' = 0$, *in which case φ is minimal* (*actually, totally geodesic*); *or*

(ii) $m = 4$ *and f is a solution of the equation*

$$\sum_{i=1}^4 a_i^2 f^2 f''' + \left(4 - \sum_{i=1}^4 a_i^2\right) ff'f'' - 4\left(2 + \sum_{i=1}^4 a_i^2\right)(f')^3 = 0; \tag{8}$$

or

(iii) $a_i = 0$ *for $i = 1,\ldots,m$ and $f(z) = \frac{1}{Az+B}$, where A and B are constants. In this case each hyperplane is a proper biharmonic hypersurface.* (This recovers a result (Theorem 3.1) obtained in [Ou1].)

Proof. Using the notation $\partial_i = \frac{\partial}{\partial x_i}$ $(i = 1,2,\ldots,m)$ and $\partial_{m+1} = \frac{\partial}{\partial z}$, we can easily check that $\{\bar{e}_\alpha = f(z)\partial_\alpha, \alpha = 1,2,\ldots,m+1\}$ constitute an orthonormal frame on the conformally flat space $(\mathbb{R}^{m+1}, h)$. One can also check that

$$\begin{aligned} \varphi(\partial_i) &= \frac{1}{f}(\bar{e}_i + a_i\bar{e}_{m+1}), \quad i = 1,2,\ldots,m,\\ \eta &= \frac{1}{f}\left(\sum_{j=1}^m a_j\bar{e}_j - \bar{e}_{m+1}\right) \end{aligned} \tag{9}$$

constitute a natural frame adapted to the hypersurface, where η is a normal vector.

Applying Gram–Schmidt orthonormalization to the natural frame

$$\begin{aligned} &\bar{e}_i + a_i\bar{e}_{m+1}, \quad i = 1,2,\ldots,m,\\ &\sum_{j=1}^m a_j\bar{e}_j - \bar{e}_{m+1} \end{aligned} \tag{10}$$

or by a straightforward checking, one can verify the following claim.

Claim I. Let $k_i = 1/\sqrt{1+\sum_{l=1}^{i} a_l^2}$ for $i = 1, \ldots, m$ with $k_0 = 1$ and $a_0 = 0$. Then the vector fields

$$\begin{aligned} e_i &= -a_i k_i k_{i-1} \sum_{l=1}^{i-1} a_l \bar{e}_l + \frac{k_i}{k_{i-1}} \bar{e}_i + a_i k_i k_{i-1} \bar{e}_{m+1}, \quad i = 1, 2, \ldots, m, \\ e_{m+1} &= \sum_{l=1}^{m} a_l k_m \bar{e}_l - k_m \bar{e}_{m+1} \end{aligned} \tag{11}$$

form an orthonormal frame adapted to the hypersurface $z = \sum_{i=1}^{m} a_i x_i + c$, where $\xi = e_{m+1}$ is the unit normal vector field.

Let $\bar{h} = \sum_{i=1}^{m} \mathrm{d}x_i^2 + \mathrm{d}z^2$ denote the Euclidean metric on $\mathbb{R}^{m+1}$. Then $h = e^{-2\sigma}\bar{h}$ with $\sigma = \ln f(z)$. It follows that

$$\operatorname{grad}_h \sigma = \bar{e}_{m+1}(\sigma)\bar{e}_{m+1} = f'\bar{e}_{m+1}. \tag{12}$$

Using that $\bar{\nabla}_{\partial_\alpha}\partial_\beta = 0$ for all $\alpha, \beta = 1, 2, \ldots, m+1$ together with the equality

$$\bar{\nabla}_X Y = \nabla_X Y + (X\sigma)Y + (Y\sigma)X - h(X, Y)\operatorname{grad}_h \sigma,$$

we can compute the Levi–Civita connection ∇ of the conformally flat metric h with respect to the orthonormal frame $\{\bar{e}_i\}$ to obtain

$$(\nabla_{\bar{e}_\alpha}\bar{e}_\beta) = \begin{pmatrix} f'\bar{e}_{m+1} & 0 & \cdots & 0 & -f'\bar{e}_1 \\ 0 & f'\bar{e}_{m+1} & \cdots & 0 & -f'\bar{e}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f'\bar{e}_{m+1} & -f'\bar{e}_m \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{(m+1)\times(m+1)}. \tag{13}$$

A further computation using (13) yields

$$\begin{aligned} \nabla_{e_i} e_i &= k_i^2 k_{i-1}^2 \nabla_{(-a_i\sum_{l=1}^{i-1} a_l\bar{e}_l + (1/k_{i-1}^2)\bar{e}_i + a_i\bar{e}_{m+1})}\left(-a_i \sum_{l=1}^{i-1} a_l \bar{e}_l + \frac{1}{k_{i-1}^2}\bar{e}_i + a_i \bar{e}_{m+1}\right) \\ &= k_i^2 k_{i-1}^2 f'\left[\left(a_i^2 \sum_{l=1}^{i-1} a_l^2 + \frac{1}{k_{i-1}^4}\right)\bar{e}_{m+1} + a_i^2 \sum_{l=1}^{i-1} a_l \bar{e}_l - \frac{a_i}{k_{i-1}^2}\bar{e}_i\right], \end{aligned} \tag{14}$$

$$\begin{aligned} \nabla_{e_i} e_{m+1} &= \nabla_{(-a_i k_i k_{i-1}\sum_{l=1}^{i-1} a_l\bar{e}_l + (k_i/k_{i-1})\bar{e}_i + a_i k_i k_{i-1}\bar{e}_{m+1})}\left(\sum_{l=1}^{m} a_l k_m \bar{e}_l - k_m \bar{e}_{m+1}\right) \\ &= k_m f'\left[a_i k_i k_{i-1}\bar{e}_{m+1} + \frac{k_i}{k_{i-1}}\bar{e}_i - a_i k_i k_{i-1}\sum_{l=1}^{i-1} a_l \bar{e}_l\right] \\ &= k_m f' e_i, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \nabla_{e_{m+1}} e_{m+1} &= \nabla_{\sum_{l=1}^{m} a_l k_m \bar{e}_l - k_m \bar{e}_{m+1}}\left(\sum_{l=1}^{m} a_l k_m \bar{e}_l - k_m \bar{e}_{m+1}\right) \\ &= (1 - k_m^2) f'\bar{e}_{m+1} + \sum_{l=1}^{m} a_l k_m^2 f'\bar{e}_l. \end{aligned} \tag{16}$$

On the other hand, one can use the relations $e_i = C_i^\alpha \bar{e}_\alpha = fC_i^\alpha \partial_\alpha$ and $\xi = C_{m+1}^\alpha \bar{e}_\alpha = fC_{m+1}^\alpha \partial_\alpha$ along with the Ricci curvature (5) to verify the following claim.

Claim II. For a hypersurface in the conformally flat space $(\mathbb{R}^{m+1}, h)$, where $h = e^{-2\sigma}\big(\sum_{i=1}^m \mathrm{d}x_i^2 + \mathrm{d}z^2\big)$, we have

$$\mathrm{Ric}(\xi,\xi) = \Delta\sigma + (m-1)[\mathrm{Hess}(\sigma)(\xi,\xi) - (\xi\sigma)^2 + |\mathrm{grad}_h\,\sigma|^2], \tag{17}$$

$$(\mathrm{Ric}(\xi))^T = (m-1)[\mathrm{grad}_g(\xi\sigma) - \xi(\sigma)\,\mathrm{grad}_g\,\sigma + A(\mathrm{grad}_g\,\sigma)]; \tag{18}$$

here grad_g is the gradient defined by the induced metric on the hypersurface. Substituting

$$\Delta\sigma = \sum_{\alpha=1}^{m+1}[\bar{e}_\alpha\bar{e}_\alpha(\sigma) - (\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)(\sigma)] = ff'' - m(f')^2,$$

$$\begin{aligned}\mathrm{Hess}(\sigma)(\xi,\xi) &= e_{m+1}e_{m+1}(\sigma) - (\nabla_{e_{m+1}}e_{m+1})(\sigma)\\ &= k_m^2 ff'' - (1-k_m^2)(f')^2,\end{aligned}$$

$$\mathrm{grad}_h\,\sigma = f'\bar{e}_{m+1}, \quad |\mathrm{grad}_h\,\sigma|^2 = (f')^2, \quad \xi(\sigma) = -k_m f'$$

into (17), we obtain

$$\begin{aligned}\mathrm{Ric}(\xi,\xi) &= \Delta\sigma + (m-1)[\mathrm{Hess}(\sigma)(\xi,\xi) - (\xi\sigma)^2 + |\mathrm{grad}_h\,\sigma|^2]\\ &= [1+(m-1)k_m^2]ff'' - m(f')^2.\end{aligned} \tag{19}$$

Given that $\xi = e_{m+1}$ is the unit normal vector field, we can easily compute the components of the second fundamental form to get

$$h(e_i,e_i) = \langle\nabla_{e_i}e_i, e_{m+1}\rangle = -\langle\nabla_{e_i}e_{m+1}, e_i\rangle = -k_m f',$$
$$h(e_i,e_j) = \langle\nabla_{e_i}e_j, e_{m+1}\rangle = -\langle\nabla_{e_i}e_{m+1}, e_j\rangle = 0, \quad i \neq j.$$

From this we conclude that each hyperplane $z = \sum_{i=1}^m a_i x_i + c$ is a totally umbilical hypersurface in the conformally flat space and that all principal normal curvatures are equal to

$$H = \xi(\sigma) = -k_m f'. \tag{20}$$

It follows that

$$\begin{aligned}[\mathrm{Ric}(\xi)]^T &= (m-1)[\mathrm{grad}_g(\xi\sigma) - \xi(\sigma)\,\mathrm{grad}_g\,\sigma + A(\mathrm{grad}_g\,\sigma)]\\ &= (m-1)\,\mathrm{grad}_g\,H,\end{aligned} \tag{21}$$

and the norm of the second fundamental form is given by

$$|A|^2 = \sum_{i=1}^m\langle\nabla_{e_i}\xi, \nabla_{e_i}\xi\rangle^2 = mk_m^2(f')^2. \tag{22}$$

A further computation yields

$$\begin{aligned} e_i(H) &= \left(-a_i k_i k_{i-1} \sum_{l=1}^{i-1} a_l \bar{e}_l + \frac{k_i}{k_{i-1}} \bar{e}_i + a_i k_i k_{i-1} \bar{e}_{m+1}\right)(-k_m f') \\ &= -a_i k_i k_{i-1} k_m f f'', \end{aligned}$$

$$\operatorname{grad}_g H = \sum_{i=1}^{m} e_i(H) e_i = -\sum_{i=1}^{m} a_i k_i k_{i-1} k_m f f'' e_i, \tag{23}$$

$$\begin{aligned} e_i e_i(H) &= -e_i(a_i k_i k_{i-1} k_m f f'') \\ &= -a_i^2 k_i^2 k_{i-1}^2 k_m (f^2 f''' + f f' f''), \end{aligned} \tag{24}$$

$$\begin{aligned} (\nabla_{e_i} e_i)(H) &= -k_i^2 k_{i-1}^2 k_m f' \left(a_i^2 \sum_{l=1}^{i-1} a_l \bar{e}_l - \frac{a_i}{k_{i-1}^2} \bar{e}_i\right)(f') \\ &= a_i^2 k_i^2 k_{i-1}^2 k_m f f' f'', \end{aligned} \tag{25}$$

and

$$\begin{aligned} \triangle^M H &= \sum_{i=1}^{m} [e_i e_i(H) - (\nabla_{e_i}^M e_i)(H)] \quad \text{(by the Gauss formula)} \\ &= \sum_{i=1}^{m} [e_i e_i(H) - (\nabla_{e_i} e_i)(H) + h(e_i, e_i)\xi(H)] \\ &= -\sum_{i=1}^{m} a_i^2 k_i^2 k_{i-1}^2 k_m [(2-m) f f' f'' + f^2 f'''] \\ &= -(1 - k_m^2) k_m [(2-m) f f' f'' + f^2 f''']. \end{aligned} \tag{26}$$

The last equality in (26) is obtained via the identity

$$\sum_{i=1}^{m} a_i^2 k_i^2 k_{i-1}^2 = (1 - k_m^2), \tag{27}$$

which can be proved by mathematical induction on $m \geq 2$.

Substituting (19), (20), (21), (22), (23), (26), and $A(\operatorname{grad}_g H) = H \operatorname{grad}_g H$ into the biharmonic equation (7) allows us to conclude that the isometric immersion φ is biharmonic if and only if

$$\begin{aligned} -(1 - k_m^2) f^2 f''' - [3 - m + (2m - 3) k_m^2] f f' f'' + m(1 + k_m^2)(f')^3 = 0, \\ (m - 4) f f' f'' a_i = 0, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{28}$$

The second equality in (28), and hence the system (28) itself, can be solved by considering the following three cases.

Case 1: $f' = 0$ (which implies $H = -k_m f' = 0$) gives the trivial solution. In this case, φ is actually totally geodesic because its image is a hyperplane in a space that is homothetic to a Euclidean space.

Case 2: $a_i f f'' = 0$ for $i = 1, 2, \ldots, m$ (which, together with (23), implies that $\operatorname{grad}_g H = 0$). In this case we can use $\triangle^M H = 0$ and (26) to reduce the first equality in (28) to

$$[1+(m-1)k_m^2]ff''-m(1+k_m^2)(f')^2=0. \tag{29}$$

If $ff''=0$ then equation (29) reduces to $f'=0$, which again gives the trivial solution (i.e., the hypersurfaces are minimal). If $ff''\neq 0$ then, for all $i=1,2,\dots,m$, we have $a_i=0$ and hence $k_m=1$. Thus (29) reduces to $ff''-2(f')^2=0$, which has solutions $f(z)=\frac{1}{Az+B}$ for constants A and B.

Case 3: $m=4$. In this case the biharmonic equation (28) reduces to

$$-(1-k_4^2)f^2f'''-(5k_4^2-1)ff'f''+4(1+k_4^2)(f')^3=0,$$

from which we obtain equation (8).

Theorem 2.2 follows when we combine the preceding results. □

3. The Generalized Chen's Conjecture on Biharmonic Submanifolds Is False

In this section we show that equation (8) has many solutions, including counterexamples to the generalized Chen's conjecture.

LEMMA 3.1. *Let $A>0$, $B>0$, and c be constants, let*

$$\mathbb{R}^5_+=\{(x_1,\dots,x_4,z)\in\mathbb{R}^5 : z>0\}$$

be the upper half-space, and let $f\colon\mathbb{R}^5_+\to\mathbb{R}$ with $f(x_1,\dots,x_4,z)=f(z)=(Az+B)^t$. Then, for any $t\in(0,1/2)$ and any $(a_1,a_2,a_3,a_4)\in S^3\big(\sqrt{2t/(1-2t)}\,\big)$, the isometric immersion

$$\varphi\colon\mathbb{R}^4\to\left(\mathbb{R}^5_+,\ h=f^{-2}(z)\left[\sum_{i=1}^4 \mathrm{d}x_i^2+\mathrm{d}z^2\right]\right) \tag{30}$$

with $\varphi(x_1,\dots,x_4)=\big(x_1,\dots,x_4,\ \sum_{i=1}^4 a_ix_i+c\big)$ is proper biharmonic into the conformally flat space.

Proof. We seek special solutions of (8) that have the form $f(z)=(Az+B)^t$. In this case, we have $f'=tA(Az+B)^{t-1}$, $f''=t(t-1)A^2(Az+B)^{t-2}$, and $f'''=t(t-1)(t-2)A^3(Az+B)^{t-3}$. Making these substitutions in (8) and assuming that $A,B>0$, we have

$$(t-1)(t-2)\sum_{i=1}^4 a_i^2+\left(4-\sum_{i=1}^4 a_i^2\right)t(t-1)-4t^2\left(2+\sum_{i=1}^4 a_i^2\right)=0,$$

which is equivalent to

$$\sum_{i=1}^4 a_i^2=\frac{2t}{1-2t}. \tag{31}$$

Solving the inequality $2t/(1-2t)>0$, we conclude that any $t\in(0,1/2)$ and $(a_1,a_2,a_3,a_4)\in S^3\big(\sqrt{2t/(1-2t)}\,\big)$ will solve equation (31) and hence the biharmonic equation (8). From this we obtain the lemma. □

EXAMPLE 1. Let $A > 0$, $B > 0$, and c be constants, and let $t = 1/6$ and $\big(\sqrt{2}/4, \sqrt{2}/4, \sqrt{2}/4, \sqrt{2}/4\big) \in S^3\big(\sqrt{2}/2\big)$. Then, by Lemma 3.1, we have a proper biharmonic isometric immersion

$$\varphi\colon \mathbb{R}^4 \to \left(\mathbb{R}^5_+,\ h = (Az+B)^{-1/3}\left(\sum_{i=1}^{4} \mathrm{d}x_i^2 + \mathrm{d}z^2\right)\right)$$

with $\varphi(x_1, \ldots, x_4) = \big(x_1, \ldots, x_4, \big(\sqrt{2}/4\big)(x_1 + x_2 + x_3 + x_4) + c\big)$.

LEMMA 3.2. *For constants $A, B > 0$ and $t \in (0,1)$, the conformally flat space $\big(\mathbb{R}^5_+, h = (Az+B)^{-2t}\big(\sum_{i=1}^4 \mathrm{d}x_i^2 + \mathrm{d}z^2\big)\big)$ has negative sectional curvature.*

Proof. Let $f(z) = (Az + B)^t$. As in the proof of Theorem 2.2, we use $\bar{e}_i = f(z)\partial_i$, $i = 1, \ldots, 5$, to denote the orthonormal frame on $\big(\mathbb{R}^5_+, h = (Az+B)^{-2t}\big(\sum_{i=1}^4 \mathrm{d}x_i^2 + \mathrm{d}z^2\big)\big)$. Let P be a plane section at any point, and suppose that P is spanned by an orthonormal basis X, Y. Then, we have $X = \sum_{i=1}^5 a_i \bar{e}_i$, $Y = \sum_{i=1}^5 b_i \bar{e}_i$. Using the sectional curvature relation (6) and given that the sectional curvature $\bar{\mathrm{K}}(p)$ of $\big(\mathbb{R}^5_+, \bar{h} = \sum_{i=1}^4 \mathrm{d}x_i^2 + \mathrm{d}z^2\big)$ vanishes identically, we find the sectional curvature of the conformally flat space to be

$$\begin{aligned}
\mathrm{K}(P) &= (h(\nabla_X \nabla\sigma, X) + h(\nabla_Y \nabla\sigma, Y)) + (|\nabla\sigma|^2 - (X\sigma)^2 - (Y\sigma)^2)\\
&= X(X\sigma) + Y(Y\sigma) - (\nabla_X X)(\sigma) - (\nabla_Y Y)(\sigma)\\
&\quad + (|\nabla\sigma|^2 - (X\sigma)^2 - (Y\sigma)^2),
\end{aligned}$$

where $\sigma = \ln f(z)$. A straightforward computation now gives

$$\begin{aligned}
X\sigma &= \sum_{i=1}^5 a_i \bar{e}_i(\sigma) = a_5 f',\\
X(X\sigma) &= \sum_{i=1}^5 a_i \bar{e}_i(a_5 f') = \sum_{i=1}^5 a_i \bar{e}_i(a_5) f' + a_5^2 f f'',\\
\nabla_X X &= \nabla_{\sum_{i=1}^5 a_i \bar{e}_i}\left(\sum_{j=1}^5 a_j \bar{e}_j\right)\\
&= \sum_{i=1}^5 a_i \bar{e}_i\left(\sum_{j=1}^5 a_j\right)\bar{e}_j + \sum_{i=1}^4 a_i^2 f' \bar{e}_5 - \sum_{i=1}^4 a_i a_5 f' \bar{e}_i,\\
(\nabla_X X)(\sigma) &= \sum_{i=1}^5 a_i \bar{e}_i(a_5) f' + \sum_{i=1}^4 a_i^2 (f')^2,\\
X(X\sigma) - (\nabla_X X)(\sigma) &= a_5^2 f f'' - \sum_{i=1}^4 a_i^2 (f')^2.
\end{aligned}$$

Similarly, we have

$$Y(Y\sigma) = b_5 f', \qquad (\nabla_Y Y)(\sigma) = \sum_{i=1}^{5} b_i \bar{e}_i(b_5) f' + \sum_{i=1}^{4} b_i^2 (f')^2,$$

$$Y(Y\sigma) - (\nabla_Y Y)(\sigma) = b_5^2 f f'' - \sum_{i=1}^{4} b_i^2 (f')^2,$$

from which we derive

$$\begin{aligned}\mathrm{K}(P) &= X(X\sigma) + Y(Y\sigma) - (\nabla_X X)(\sigma) - (\nabla_Y Y)(\sigma) \\ &\quad + (|\nabla\sigma|^2 - (X\sigma)^2 - (Y\sigma)^2) \\ &= (a_5^2 + b_5^2) f f'' - (f')^2.\end{aligned}$$

For $f = (Az+B)^t$ we have $f' = tA(Az+B)^{t-1}$ and $f'' = t(t-1)A^2(Az+B)^{t-2}$, so it follows that

$$\begin{aligned}\mathrm{K}(P) &= (a_5^2 + b_5^2) f f'' - (f')^2 \\ &= A^2(Az + B)^{t-2}[(a_5^2 + b_5^2)t(t-1) - t^2],\end{aligned}$$

which is strictly negative because $[(a_5^2 + b_5^2)t(t-1) - t^2] < 0$ for $0 < t < 1$ and $A^2(Az + B)^{t-2} > 0$ for $z > 0$. From this we obtain the lemma. □

Combining Lemma 3.1 and Lemma 3.2 yields our next theorem.

THEOREM 3.3. *Let $A > 0$, $B > 0$, and c be constants, let $\mathbb{R}^5_+ = \{(x_1, \dots, x_4, z) \in \mathbb{R}^5 : z > 0\}$ be the upper half-space, and let $f : \mathbb{R}^5_+ \to \mathbb{R}$ with $f(z) = (Az + B)^t$. Then, for any $t \in (0, 1/2)$ and any $(a_1, a_2, a_3, a_4) \in S^3\big(\sqrt{2t/(1-2t)}\big)$, the isometric immersion*

$$\varphi : \mathbb{R}^4 \to \left(\mathbb{R}^5_+, h = f^{-2}(z)\Big[\sum_{i=1}^{4} \mathrm{d}x_i^2 + \mathrm{d}z^2\Big]\right) \tag{32}$$

with $\varphi(x_1, \dots, x_4) = \big(x_1, \dots, x_4, \sum_{i=1}^4 a_i x_i + c\big)$ gives a proper biharmonic hypersurface into the conformally flat space with strictly negative sectional curvature. These solutions provide infinitely many counterexamples to the generalized Chen's conjecture on biharmonic submanifolds.

The following corollary can be used to construct proper biharmonic submanifolds of any codimension in a nonpositively curved manifold.

COROLLARY 3.4. *For any positive integer k there exists a proper biharmonic submanifold of codimension k in a nonpositively curved space. Thus, the generalized Chen's conjecture is false.*

Proof. Let

$$\varphi : \mathbb{R}^4 \to \left(\mathbb{R}^5_+, h = f^{-2}(z)\Big[\sum_{i=1}^{4} \mathrm{d}x_i^2 + \mathrm{d}z^2\Big]\right), \tag{33}$$

with $\varphi(x_1,\dots,x_4) = \big(x_1,\dots,x_4,\ \sum_{i=1}^4 a_i x_i + c\big)$, be one of the proper biharmonic hypersurfaces given in Theorem 3.3; let $\psi\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^{k-1} \equiv (\mathbb{R}^{n+k-1}, h_0)$, with $\psi(y) = (y, 0)$, be the totally geodesic embedding of a subspace into a Euclidean space. Then the isometric embedding $\phi\colon \mathbb{R}^4 \times \mathbb{R}^n \to (\mathbb{R}^5_+ \times \mathbb{R}^{n+k-1}, h + h_0)$ with $\phi(x, y) = (\varphi(x), \psi(y))$ gives a submanifold of codimension k. Since ϕ is biharmonic with respect to each variable separately and is proper biharmonic with respect to x (by Theorem 3.3), we can use [Ou2, Prop. 2.1] to conclude that ϕ is a proper biharmonic embedding. Thus, the image of ϕ provides a proper biharmonic submanifold of codimension k. By Lemma 3.2, the conformally flat space $\big(\mathbb{R}^5_+,\ h = (Az + B)^{-2t}\big(\sum_{i=1}^4 \mathrm{d}x_i^2 + \mathrm{d}z^2\big)\big)$ has negative sectional curvature and the Euclidean space $(\mathbb{R}^{n+k-1}, h_0)$ has zero curvature; hence their product $(\mathbb{R}^5_+ \times \mathbb{R}^{n+k-1}, h + h_0)$ gives a space of nonpositive curvature, which proves the corollary.

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