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The Luzin Theorem for Higher-Order Derivatives

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1. Introduction

In 1917, Luzin ([2]; see also [4]) proved a surprising result: For any Lebesgue measurable function $f: \mathbb{R} \to \mathbb{R}$ there is a continuous a.e. differentiable function g such that $g' = f$ almost everywhere. This is surprising even for the function $f(x) = 1/x$ because the antiderivative of f is discontinuous and, in fact, unbounded at 0. In this case, we correct the antiderivative by adding continuous functions that are differentiable almost everywhere with derivative equal to zero but that are not constant (one example of such a function is a Cantor staircase).

The original proof due to Luzin is purely one-dimensional and offers no guidance toward a proof in higher dimensions. However, in 2008 Moonens and Pfeffer [3] proved the following generalization:

> *Let U be an open subset of $\mathbb{R}^N$. Given any Lebesgue measurable function $f: U \to \mathbb{R}^N$, there is an a.e. differentiable function $g \in C(\mathbb{R}^N)$ such that $\nabla g = f$ almost everywhere.*

The goal of this paper is to extend the results to include higher-order derivatives.

For an m-times differentiable function g defined in an open subset $U \subset \mathbb{R}^N$, we write

$$D^m g = (D^\alpha g)_{|\alpha|=m}$$

to denote the collection of all partial derivatives of order m. Our main result reads as follows.

Theorem 1.1. *Let $f = (f^\alpha)_{|\alpha|=m}$ be a Lebesgue measurable function defined in an open set $U \subset \mathbb{R}^n$. Then there is a function $g \in C^{m-1}(\mathbb{R}^n)$ that is m-times differentiable a.e. and such that*

$$D^m g = f \quad \text{a.e. in } U;$$

that is,

$$D^\alpha g = f^\alpha \quad \text{a.e. in } U \quad \text{for } |\alpha| = m.$$

Moreover, for any $\sigma > 0$, the function g may be chosen such that

$$\|D^\gamma g\|_\infty < \sigma \quad \text{for every } |\gamma| < m.$$

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The outline of the proof is as follows. If $f = (f^\alpha)_{|\alpha|=m}$ is continuous and bounded on an open set U of finite measure, then we can find a function $g \in C^m$ such that $D^m g$ *approximates* f on a large compact set. Using this approximation and a suitable limiting process, we can find $g \in C^m$ such that $D^m g$ *is equal to* f on a large compact set. We then show that the same holds more generally for the class of Lebesgue measurable functions because such functions are continuous and bounded when restricted to a large compact set. The final construction involves piecing together approximations of f using a compact exhaustion of $\mathbb{R}^n$, taking care to avoid any overlap that would cause the resulting approximation to lose its desired form. The proof requires careful estimates for the approximation, which is the main difficulty.

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2. Proofs of the Results

Throughout the paper, $|U|$ denotes the N-dimensional Lebesgue measure of a set U.

Given a continuous function $f = (f^\alpha)_{|\alpha|=m}$ defined in an open set $U \subset \mathbb{R}^n$ with $|U| < \infty$, our first task is to construct a compactly supported function $u \in C^m_c(U)$ such that $D^m u = f$ on a large compact subset of U. Toward this end we need the following approximation result. (For the case $m = 1$, see [1].)

LEMMA 2.1. *Fix $m \in \mathbb{N}$ and let $U \subset \mathbb{R}^N$ be open with $|U| < \infty$. Let $f = (f^\alpha)_{|\alpha|=m}$ be a continuous and bounded function on U. Then, for any $\varepsilon, \eta, \sigma > 0$, there exist a function $u \in C^\infty_c(U)$ and a compact set $K \subset U$ such that, for each $p \in [1, \infty]$, the following inequalities hold*:

(i) $|U \setminus K| < \varepsilon$;
(ii) $|D^m u(x) - f(x)| < \eta$ *for each* $x \in K$;
(iii) $\|D^m u\|_p \le C(m, N)(\varepsilon/|U|)^{1/p-m}\|f\|_p$;
(iv) $\|D^\gamma u\|_\infty < \sigma$ *for every* $|\gamma| < m$.

Proof. Fix $\varepsilon, \eta, \sigma > 0$. By $Q(x, r)$ we denote the closed cube centered at x with side length r. Select a compact set $K' \subset U$ such that $|U \setminus K'| < \varepsilon/2$. Choose $\delta > 0$ so small that

$$Q(x, 4\delta) \subset U \quad \text{for all } \ x \in K'$$

and

$$(Q(z, \delta) \cap K' \neq \emptyset,\ (x, y) \in Q(z, \delta)) \implies |f(x) - f(y)| < \eta. \tag{2.1}$$

Cover $\mathbb{R}^N$ with a lattice of closed cubes of side length δ. Let $\{T_i\}_{i \in I}$ be the finite subcollection of cubes whose intersection with K' is nonempty. Clearly,

$$K' \subset \bigcup_{i \in I} T_i \subset U.$$

For each i, let Q_i be a closed cube concentric with T_i and side length $\left(1 - \frac{\varepsilon}{2N|U|}\right)\delta$. Denote the center of the cube by c_i. Then

$$K = \bigcup_{i \in I} Q_i$$

satisfies

$$|U \setminus K| = \left| U \setminus \bigcup_{i \in I} Q_i \right| \leq |U \setminus K'| + \sum_i |T_i \setminus Q_i| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2|U|} \sum_i |T_i| < \varepsilon.$$

If we define

$$a_i^\alpha = \fint_{T_i} f^\alpha(y)\, dy, \quad |\alpha| = m,$$

then the function

$$g_i(x) = \sum_{|\alpha|=m} \frac{a_i^\alpha}{\alpha!}(x - c_i)^\alpha$$

is a polynomial such that

$$D^\alpha g_i(x) = a_i^\alpha \quad \text{for } |\alpha| = m;$$

hence (2.1) yields that, for all $x \in T_i$ and $|\alpha| = m$,

$$|D^\alpha g_i(x) - f^\alpha(x)| \leq \fint_{T_i} |f^\alpha(y) - f^\alpha(x)|\, dy < \eta.$$

Let $\Phi_i \in C_c^\infty(T_i)$ with $\Phi_i \equiv 1$ on Q_i. If Φ_i is a cut-off function, then

$$u = \sum_{i \in I} \Phi_i g_i \in C_c^\infty(U)$$

satisfies

$$|D^m u(x) - f(x)| < \eta \quad \text{for all } x \in K.$$

We need only choose Φ_i carefully in order to guarantee the estimates (iii) and (iv). Let

$$T = \left[\frac{-1}{2}, \frac{1}{2}\right]^N \quad \text{and} \quad Q = \left[\frac{-1}{2} + \frac{\varepsilon}{4|U|N}, \frac{1}{2} - \frac{\varepsilon}{4|U|N}\right]^N;$$

in other words, Q is the cube concentric with T and with side length $1 - \frac{\varepsilon}{2N|U|}$.

Let $\zeta \in C_c^\infty(B^N(0,1))$ with $\zeta \geq 0$ and $\int_{\mathbb{R}^N} \zeta = 1$, and let $\zeta_\varepsilon(x) := \varepsilon^{-N}\zeta(x/\varepsilon)$ be a standard mollifier. For

$$\tilde{Q} = \left[\frac{-1}{2} + \frac{\varepsilon}{8|U|N}, \frac{1}{2} - \frac{\varepsilon}{8|U|N}\right]^N,$$

we define

$$\Phi = \chi_{\tilde{Q}} * \zeta_{(\varepsilon/16|U|N)}.$$

Clearly $\Phi \in C_c^\infty(T)$ with $\Phi = 1$ on Q and

$$|D^\alpha \Phi(x)| \leq C(m, N)(\varepsilon/|U|)^{-|\alpha|} \quad \text{for } |\alpha| \leq m \text{ and } x \in T.$$

Finally, we define

$$\Phi_i(x) = \Phi\left(\frac{x - c_i}{\delta}\right) \quad \text{and} \quad u = \sum_{i \in I} \Phi_i g_i.$$

Observe that for $x \in T_i$ and $|\beta|, |\gamma| \le m$ we have

$$|D^\beta g_i(x)| \le C(m,N)\|f\|_\infty \delta^{m-|\beta|},$$
$$|D^\gamma \Phi_i(x)| \le C(m,N)(\varepsilon/|U|)^{-|\gamma|}\delta^{-|\gamma|}.$$

Hence for any $|\alpha| \le m$ and $x \in Q_i$,

$$\begin{aligned}|D^\alpha u(x)| &= |D^\alpha(g_i \Phi_i)(x)| \\ &\le \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!} |D^\beta g_i(x)||D^\gamma \Phi_i(x)| \\ &\le C(m,N)\|f\|_\infty (\varepsilon/|U|)^{-|\alpha|}\delta^{m-|\alpha|}.\end{aligned}$$

Note that by choosing δ small enough we can ensure

$$C(m,N)\|f\|_\infty \sup_{|\alpha|<m} \delta^{m-|\alpha|}(\varepsilon/|U|)^{-|\alpha|} < \sigma,$$

which proves (iv). Considering the case $|\gamma| = m$, we see that the proof of (iii) is complete for the case $p = \infty$.

We are left with the case $1 \le p < \infty$ of (iii). Observe that, for $|\gamma| > 0$,

$$\operatorname{supp} D^\gamma \Phi_i \subset \overline{T_i \backslash Q_i}$$

and

$$\frac{|\overline{T_i \backslash Q_i}|}{|T_i|} = 1 - \left(1 - \frac{\varepsilon}{2N|U|}\right)^N < \frac{\varepsilon}{2|U|}$$

by Bernoulli's inequality. Hence for $|\alpha| = m$ and $x \in T_i$ we have

$$\begin{aligned}|D^\alpha u(x)| &= |D^\alpha(g_i \Phi_i)(x)| \\ &\le |D^\alpha g_i(x)||\Phi_i(x)| + C \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|>0}} |D^\beta g_i(x)||D^\gamma \Phi_i(x)| \\ &\le \left(⨍_{T_i} |f|\right)\chi_{T_i}(x) + C \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|>0}} \left(⨍_{T_i} |f|\right)\delta^{m-|\beta|}\left(\frac{\varepsilon}{|U|}\right)^{-|\gamma|}\delta^{-|\gamma|}\chi_{\overline{T_i \backslash Q_i}}(x) \\ &\le \left(\int_{T_i} |f|^p\right)^{1/p} |T_i|^{-1/p}\chi_{T_i}(x) \\ &\quad + C\left(\frac{\varepsilon}{|U|}\right)^{-m}\left(\int_{T_i} |f|^p\right)^{1/p} |T_i|^{-1/p}\chi_{\overline{T_i \backslash Q_i}}(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\|D^\alpha u\|_p &\le \|f\|_p + C(\varepsilon/|U|)^{-m}\left(\sum_{i\in I}\left(\int_{T_i}|f|^p\right)\frac{|\overline{T_i \backslash Q_i}|}{|T_i|}\right)^{1/p} \\ &\le \|f\|_p(1 + C(\varepsilon/|U|)^{1/p-m}) \le C'\|f\|_p(\varepsilon/|U|)^{1/p-m}.\end{aligned}$$ □

Let $V \subset \mathbb{R}^n$ be open with $|V| < \infty$. Recall that if $u_i \in C_c^m(V)$ satisfies both

$$u = \sum_{i=1}^{\infty} u_i \text{ converges uniformly in } V$$

and

$$\sum_{i=1}^{\infty} \|D^m u_i\|_\infty < \infty,$$

then $u \in C^m(V)$.

In the next lemma we will exhaust U by compact sets K_i and build a series $u_i \in C_c^m(U)$,

$$u = \sum_{i=1}^{\infty} u_i \in C_c^m(U).$$

The functions u_i will be constructed with the help of Lemma 2.1, so the partial sums of the series $D^m u_i$ will approximate a given continuous function $f = (f^\alpha)_{|\alpha|=m}$ on U. Then, for the limiting function u, $D^m u$ will coincide with f on a large compact set.

LEMMA 2.2. *Fix $m \in \mathbb{N}$ and let $U \subset \mathbb{R}^N$ be open with $|U| < \infty$. Let $f = (f^\alpha)_{|\alpha|=m}$ be a continuous and bounded function on U. For any $\varepsilon, \sigma > 0$, there exist a function $u \in C_c^m(U)$ and a compact set $K \subset U$ such that the following statements hold*:

(i) $|U \setminus K| < \varepsilon$;
(ii) $D^m u(x) = f(x)$ *for each* $x \in K$;
(iii) $\|D^m u\|_p \le C(m,N)(\varepsilon/|U|)^{1/p-m}\|f\|_p$ *for all* $1 \le p \le \infty$;
(iv) $\|D^\gamma u\|_\infty < \sigma$ *for* $|\gamma| < m$.

Proof. We can assume that $f \neq 0$. Then the function

$$\varphi(p) = |U|^{1/p}\|f\|_p^{-1}, \quad p \in [1,\infty),$$

is continuous and $\varphi(p) \to \|f\|_\infty^{-1}$ as $p \to \infty$, so φ is bounded and hence

$$0 < A := \sup_{1 \le p < \infty} |U|^{1/p}\|f\|_p^{-1} < \infty.$$

Let $\eta_0 = \|f\|_\infty$ and $\eta_i = 2^{-(m+1)i}A^{-1}$, $i = 1, 2, \ldots$. Then

$$\sum_{i=1}^{\infty} 2^{mi}\eta_i = A^{-1}.$$

Let $V \subset\subset U$ be open with $|U \setminus V| < \varepsilon/2$. Let $f_1 = f|_V$. Applying Lemma 2.1, we select a compact subset K_1 of V and $u_1 \in C_c^m(V)$ such that

$$|V \setminus K_1| < 2^{-2}\varepsilon,$$

$$|D^m u_1(x) - f_1(x)| < \eta_1 \quad \text{for } x \in K_1,$$

$$\|D^m u_1\|_p \le C(m,N)(\varepsilon/|U|)^{1/p-m}\|f_1\|_p,$$

$$\|D^\gamma u_1\|_\infty < 2^{-1}\sigma \quad \text{for } |\gamma| < m.$$

We will recursively construct sequences f_n, $K_n \subset U$ compact, and $u_n \in C_c^m(V)$ such that

(I) $|V \setminus K_n| < 2^{-(n+1)}\varepsilon$,
(II) $|D^m u_n(x) - f_n(x)| < \eta_n$ for each $x \in K_n$,
(III) $\|D^m u_n\|_p \le C(m,N)(2^{-n}\varepsilon/|U|)^{1/p-m}\|f_n\|_p$, and
(IV) $\|D^\gamma u_n\|_\infty < 2^{-n}\sigma$ for $|\gamma| < m$.

Assume that f_{n-1}, K_{n-1}, and u_{n-1} have been selected to satisfy (I)–(IV). Define a function $\tilde{f}_n$ by

$$\tilde{f}_n(x) = f_{n-1}(x) - D^m u_{n-1}(x), \quad x \in \bigcap_{i=1}^{n-1} K_i.$$

Applying the Teizte extension theorem to $\tilde{f}_n$ yields a continuous function f_n on U, which by (II) satisfies

$$\|f_n\|_\infty \le \eta_{n-1}.$$

By Lemma 2.1, there is a compact set K_n and a $u_n \in C_c^m(V)$ satisfying (I)–(IV).

Define $K = \bigcap_{i=1}^\infty K_i$. Clearly K is compact and

$$|U \setminus K| \le |U \setminus V| + |V \setminus K| < \varepsilon.$$

Define $u = \sum_{i=1}^\infty u_i$. To show (iii), for $p \in [1,\infty)$ we estimate

$$\begin{aligned}
\sum_{i=1}^\infty \|D^m u_i\|_p &\le C(m,N)(\varepsilon/|U|)^{1/p-m} \sum_{i=1}^\infty (2^{m-1/p})^i \|f_i\|_p \\
&\le 2^m C(m,N)(\varepsilon/|U|)^{1/p-m}\|f\|_p \left(1 + \|f\|_p^{-1} \sum_{i=2}^\infty 2^{m(i-1)}\|f_i\|_p\right) \\
&\le 2^m C(m,N)(\varepsilon/|U|)^{1/p-m}\|f\|_p \left(1 + \frac{|U|^{1/p}}{\|f\|_p} \sum_{i=2}^\infty 2^{m(i-1)}\|f_i\|_\infty\right) \\
&\le 2^m C(m,N)(\varepsilon/|U|)^{1/p-m}\|f\|_p \left(1 + A \sum_{i=2}^\infty (2^m)^{i-1}\eta_{i-1}\right) \\
&\le 2^{m+1} C(m,N)(\varepsilon/|U|)^{1/p-m}\|f\|_p. \qquad (2.2)
\end{aligned}$$

Now we claim that $u \in C_c^m(U)$. By (IV), for $|\gamma| < m$ we have

$$\sum_{i=1}^\infty \|D^\gamma u_i\|_\infty < \sigma,$$

which implies the uniform convergence of the series in U. Moreover, note that since $|U| < \infty$ we can let $p \to \infty$ in (2.2); therefore,

$$\sum_{i=1}^\infty \|D^m u_i\|_\infty \le C'(m,N)(\varepsilon/|U|)^{-m}\|f\|_\infty.$$

As we remarked previously, this implies $u \in C^m(U)$. Since each u_i is supported in V and since $V \subset\subset U$, we have $u \in C_c^m(U)$. Hence (iii) and (iv) follow.

We are left with the proof of (ii). Fix $x \in K$. An easy inductive argument shows that

$$f_n(x) = f(x) - \sum_{i=1}^{n-1} D^m u_i(x).$$

Hence, for every n,

$$\left| f(x) - \sum_{i=1}^{n} D^m u_i(x) \right| = |f_n(x) - D^m u_n(x)| < \eta_n.$$

Thus

$$\begin{aligned} |f(x) - D^m u(x)| &\leq \left| f(x) - \sum_{i=1}^{n} D^m u_i(x) \right| + \sum_{i=n+1}^{\infty} \|D^m u_i\|_\infty \\ &\leq \eta_n + \sum_{i=n+1}^{\infty} \|D^m u_i\|_\infty \to 0 \text{ as } n \to \infty. \end{aligned}$$ □

Any Lebesgue measurable function on a set $U \subset \mathbb{R}^n$ with $|U| < \infty$ is continuous and bounded outside a set of small measure. This fact allows us to prove a result similar to Lemma 2.2 without the restrictions that f be bounded or continuous. We simply isolate the region where f is badly behaved.

LEMMA 2.3. *Let f be a Lebesgue measurable function on an open set $U \subset \mathbb{R}^N$ with $|U| < \infty$. Then, for any $\varepsilon > 0$, there exist a compact set $K \subset U$ and a continuous, bounded function $\tilde{f}$ on U such that*

(i) $|U \setminus K| < \varepsilon$,
(ii) $f = \tilde{f}$ *on* K, *and*
(iii) $\|\tilde{f}\|_p \leq 2\|f\|_p$ *for all* $p \in [1, \infty]$.

Proof. Fix $\varepsilon > 0$. Suppose first that f is essentially unbounded. Then there exists an $R > 0$ such that

$$0 < |\{|f| > R\}| < \varepsilon/2.$$

Let $K \subset \{|f| \leq R\}$ be a compact set such that $f|_K$ is continuous and

$$|\{|f| \leq R\} \setminus K| < |\{|f| > R\}| < \varepsilon/2.$$

Let $\tilde{f}$ be the Tietze extension of $f|_K$. Clearly $\|\tilde{f}\|_\infty \leq R$. We have

$$U \setminus K \subset (\{|f| \leq R\} \setminus K) \cup \{|f| > R\}.$$

Hence $|U \setminus K| < \varepsilon$. Also $\tilde{f} = f$ on K. We are left with the estimate for the L^p norm:

$$\begin{aligned} \int_U |\tilde{f}|^p &\leq \int_K |f|^p + \int_{\{|f| \leq R\} \setminus K} R^p + \int_{\{|f| > R\}} R^p \\ &\leq \int_K |f|^p + 2 \int_{\{|f| > R\}} R^p \leq 2 \int_U |f|^p. \end{aligned}$$

Now suppose that f is essentially bounded—say, $\|f\|_\infty = M > 0$. If $|\{|f| = M\}| = 0$, then the proof follows from the previous argument because we can find $0 < R < M$ with

$$0 < |\{|f| > R\}| < \varepsilon/2.$$

Thus we may suppose that $|\{|f| = M\}| > 0$. Let $K \subset U$ be compact such that $f|_K$ is continuous and

$$|U \setminus K| < \min\{\varepsilon, |\{|f| = M\}|\}.$$

Let $\tilde{f}$ be the Tietze extension of $f|_K$. Clearly $\|\tilde{f}\|_\infty \le M$.

As before, we estimate the L^p norm:

$$\begin{aligned}\int_U |\tilde{f}|^p &\le \int_K |f|^p + \int_{U\setminus K} M^p \\ &\le \int_K |f|^p + \int_{\{|f|=M\}} M^p \\ &= \int_K |f|^p + \int_{\{|f|=M\}} |f|^p \le 2\int_U |f|^p.\end{aligned}$$ □

As a consequence we have the following immediate result.

THEOREM 2.4. *Fix $m \in \mathbb{N}$ and let $U \subset \mathbb{R}^n$ be open with $|U| < \infty$. Let $f = (f^\alpha)_{|\alpha|=m}$ be Borel. Then for any $\varepsilon, \sigma > 0$ there is a function $u \in C_c^m(U)$ and a compact set $K \subset U$ such that, for each $p \in [1, \infty]$, the following hold*:

(i) $|U \setminus K| < \varepsilon$;
(ii) $D^m u(x) = f(x)$ *for each* $x \in K$;
(iii) $\|D^m u\|_p \le C(m, N)(\varepsilon/|U|)^{1/p-m}\|f\|_p$;
(iv) $\|D^\gamma u\|_\infty < \sigma$ *for* $|\gamma| < m$.

To prove Theorem 2.4 we simply observe that, by Lemma 2.3, we can replace f with $\tilde{f}$ (which is bounded and continuous) and then apply Lemma 2.2, noting that $\tilde{f} = f$ on a large compact set and that $\|\tilde{f}\|_p \le 2\|f\|_p$.

3. The Luzin Theorem for Higher-Order Derivatives

Now we come to the main result. We no longer require that the open set U have finite measure.

THEOREM 3.1. *Let U be open in $\mathbb{R}^n$ and let $f = (f^\alpha)_{|\alpha|=m}$ be a Lebesgue measurable function defined on U. Then, for any $\sigma > 0$, there is a $u \in C^{m-1}(\mathbb{R}^n)$ that is m-times differentiable almost everywhere and such that*

$$D^m u(x) = f(x) \quad \text{for a.e. } x \in U,$$

$$\|D^\gamma u\|_\infty \le \sigma \quad \text{for each } |\gamma| < m.$$

Proof. Let $U_1 = U \cap B(0, 1)$. We claim that there exist a compact set $K_1 \subset U_1$ and $u_1 \in C_c^m(U_1)$ such that

$$D^m u_1(x) = f(x) \quad \text{for } x \in K_1,$$
$$|U_1 \setminus K_1| < 2^{-1},$$
$$|D^\gamma u_1(x)| < 2^{-1}\sigma \min\{1, \operatorname{dist}^2(x, U_1^c)\}, \quad x \in \mathbb{R}^N, \ |\gamma| < m. \tag{3.1}$$

Indeed, let $V_1 \subset\subset U_1$ with $|U_1 \setminus V_1| < 1/4$. According to Theorem 2.4, for any $\eta > 0$ there exist a compact set $K_1 \subset V_1$ and $u_1 \in C_c^m(V_1)$ such that

$$|V_1 \setminus K_1| < 1/4 \quad \text{(and hence } |U_1 \setminus K_1| < 1/2),$$
$$D^m u_1 f(x) = f(x) \quad \text{for } x \in K_1,$$
$$|D^\gamma u_1(x)| < \eta, \quad x \in \mathbb{R}^N, \ |\gamma| < m. \tag{3.2}$$

Since $\operatorname{dist}(\bar{V}_1, U_1) > 0$, if we take η small enough then (3.2) implies (3.1).

Next we construct a sequence of compact sets K_n and functions u_n by induction. Suppose that $K_1, \ldots, K_{n-1}$ and $u_1, \ldots, u_{n-1}$ have been defined. Let $U_n = U \cap B(0, n) \setminus (K_1 \cup \cdots \cup K_{n-1})$. Using a similar argument as before, we may find a compact set $K_n \subset U_n$ and a $u_n \in C_c^m(U_n)$ such that

$$D^m u_n(x) = f(x) - \sum_{i=1}^{n-1} D^m u_i(x) \quad \text{for } x \in K_n, \tag{3.3}$$
$$|U_n \setminus K_n| < 2^{-n},$$
$$|D^\gamma u_n(x)| < 2^{-n} \min\{1, \operatorname{dist}^2(x, U_n^c)\}, \quad x \in \mathbb{R}^N, \ |\gamma| < m.$$

Now let $C = \bigcup_{n=1}^\infty K_n$. It is easy to see that $|U \setminus C| = 0$. We will show that

$$u = \sum_{n=1}^\infty u_n$$

satisfies the claim of the theorem.

First, note that clearly $\operatorname{supp}(u) \subset \bar{U}$. Since for $|\gamma| < m$ we have

$$\sum_{n=1}^\infty \|D^\gamma u_n\|_\infty < \sigma,$$

it follows that $u \in C^{m-1}(\mathbb{R}^n)$ and

$$\|D^\gamma u\|_\infty \le \sigma \quad \text{for } |\gamma| < m.$$

It remains to show that, for $x \in C$, u is m-times differentiable at x and $D^m u(x) = f(x)$.

Let $x \in C$. Then $x \in K_n$ for some n. Observe that (3.3) implies that

$$\sum_{j=1}^n D^m u_j(x) = f(x).$$

Thus it remains to show that the function

$$\sum_{j>n} u_j \tag{3.4}$$

is m-times differentiable at x and that the mth derivative at x is 0. The function (3.4) is clearly of class C^{m-1}; hence it suffices to show that, for $|\gamma| = m-1$,

$$Dg(x) := D\Big(\sum_{j>n} D^\gamma u_j\Big)(x) = 0.$$

Since the functions $D^\gamma u_j$ are supported in U_j and since $x \notin U_j$ for $j > n$, we have

$$g(x) = \sum_{j>n} D^\gamma u_j(x) = 0.$$

Let $h \in \mathbb{R}^N$. If $x + h \notin U_j$, then $|D^\gamma u_j(x+h)| = 0$. On the other hand, if $x + h \in U_j$ then, since $x \notin U_j$ for $j > n$,

$$|D^\gamma u_j(x+h)| < 2^{-j}\sigma \min\{1, \operatorname{dist}^2(x+h, U_j^c)\} \le 2^{-j}\sigma|h|^2.$$

Therefore,

$$\begin{aligned} |g(x+h) - g(x)| &= |g(x+h)| \\ &\le \sum_{j>n} |D^\gamma u_j(x+h)| \\ &\le \sigma|h|^2 \sum_{j>n} 2^{-j} < \sigma|h|^2. \end{aligned}$$

Hence $Dg(x) = 0$, which completes the proof. □

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