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Excess Porteous, Coherent Porteous, and the Hyperelliptic Locus in $\overline{\mathfrak{M}}_3$

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1. Introduction

In [HM, p. 162] the authors consider a family $\pi : X \to B$ of smooth curves of genus 3, not all of which are hyperelliptic, and a map $\sigma : \mathcal{E} \to \mathcal{F}$ of vector bundles (of ranks 3 and 2, respectively) on X. They show that this map fails to be surjective exactly at the hyperelliptic Weierstrass points of hyperelliptic fibers of π. They then use the Thom–Porteous formula for vector bundles to determine an expression for the class of

$$D_1(\sigma) = \{x \in X \mid \operatorname{rank}(\sigma_x) \le 1\}$$

in the Chow group of X. The authors then use this result to obtain an expression in $\operatorname{Pic}_{\text{fun}}(\mathfrak{M}_3)$ (the group of divisor classes on the moduli stack) for the class of the locus of hyperelliptic curves.

One would like to extend this technique to determine an expression in $\operatorname{Pic}_{\text{fun}}(\overline{\mathfrak{M}}_3)$ for the closure of the locus of hyperelliptic curves. Unfortunately, if one supposes that $\pi : X \to B$ is a family of stable curves of genus 3, then $\mathcal{F}$ will fail to be locally free at singular points of singular fibers of π (see [HM, Sec. 3.F] for details). Harris and Morrison are able to compute this class in $\operatorname{Pic}_{\text{fun}}(\overline{\mathfrak{M}}_3)$ using the method of test curves, but one would still like to extend the original technique to compute the class.

Diaz [D], by constructing a certain blow-up $g : X' \to X$ as well as a map $\sigma' : \mathcal{E}' \to \mathcal{F}'$ of vector bundles on X' that is related to the original map σ, is able to define the degeneracy class for a map of coherent sheaves. The author then applies this process in order to determine an expression in $\operatorname{Pic}_{\text{fun}}(\overline{\mathfrak{M}}_3)$ for the class of the closure of the hyperelliptic locus in $\overline{\mathfrak{M}}_3 \setminus \Delta_1$. Diaz points out that, at singular curves corresponding to general points of Δ_1, not only will $\mathcal{F}$ fail to be locally free at the singular points but also the map σ will have rank ≤ 1 at all points of the elliptic component of the fiber. The author suggests that one could combine the process for determining the degeneracy class of a map of coherent sheaves with the excess Porteous formula found in [F, Exm. 14.4.7] to compute an expression in $\operatorname{Pic}_{\text{fun}}(\overline{\mathfrak{M}}_3)$ for the class of the closure of the hyperelliptic locus in $\overline{\mathfrak{M}}_3$. We will do so in this paper.

To this end, we consider a family $\pi : X \to B$ of smooth, nonhyperelliptic curves degenerating to a general element of Δ_1 and also consider the map $\sigma' : \mathcal{E}' \to \mathcal{F}'$

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mentioned previously. After determining the scheme structure of $D_1(\sigma')$ in Sections 2 and 3, we then, in Section 4, use the excess Porteous formula to determine the number of times the standard Thom–Porteous formula counts a general element of Δ_1. Finally, in Section 5 we combine the result of Section 4 with that of [D] to determine an expression in $\mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3)$ for the class of the closure of the hyperelliptic locus in $\overline{\mathfrak{M}}_3$.

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2. The Rank of σ'

Let $\pi : X \to B$ be a 1-parameter family of smooth, nonhyperelliptic curves of genus 3 degenerating to a generic element of Δ_1. In other words, X_b is a smooth, nonhyperelliptic curve of genus 3 for $b \in B \setminus \{b_0\}$, and X_{b_0} is the union of a smooth elliptic curve E and a smooth curve C of genus 2 meeting transversely at one point P that is not a hyperelliptic Weierstrass point of C.

Let $\omega_{X/B}$ be the relative dualizing sheaf, $X_2 = X \times_B X$, p_1 and p_2 the projection maps, and $\Delta \subset X_2$ the diagonal with ideal sheaf $\mathcal{I}_\Delta$. Let $\mathcal{O}_{X_2} \to \mathcal{O}_{X_2}/\mathcal{I}_\Delta^2$ be the natural map. Tensoring both sides with $p_2^*\omega_{X/B}$ and pushing down via p_1, we obtain

$$\sigma : (p_1)_*(p_2^*\omega_{X/B}) \to (p_1)_*(p_2^*\omega_{X/B} \otimes \mathcal{O}_{X_2}/\mathcal{I}_\Delta^2).$$

Let $\mathcal{E}$ be the domain and $\mathcal{F}$ the target. Whereas $\mathcal{E}$ is a vector bundle on all of X, $\mathcal{F}$ fails to be locally free at P [HM, p. 169].

At a point Q on a fiber X_b, $\mathcal{E}_Q$ is the space of sections $H^0(X_b, \omega_b)$, where ω_b is the dualizing sheaf of X_b. If Q is a *smooth* point on X_b, then $\mathcal{F}_Q$ is the space of differentials in a neighborhood of Q in X_b modulo those vanishing to order 2 at Q. Thus, at a smooth point Q of a fiber X_b, σ_Q sends each global holomorphic differential on X_b to its truncated Taylor series at Q [HM, p. 163].

The following three propositions give the rank of σ_Q when Q is a smooth point of a fiber of π.

PROPOSITION 2.1. *Let Q be a closed point of X_b for $b \in B \setminus \{b_0\}$. Then σ_Q is surjective.*

Proof. Let ω_b be the canonical bundle on X_b. Since ω_b is base point free, the foregoing description of σ shows that σ_Q will fail to be surjective if and only if $h^0(\omega_b(-Q)) = h^0(\omega_b(-2Q))$—that is, iff Q is a hyperelliptic Weierstrass point. But by assumption, X_b is not hyperelliptic. □

PROPOSITION 2.2. *Suppose Q is a closed point of $E - P$. Then* $\mathrm{rank}(\sigma_Q) = 1$.

Proof. Let ω_{b_0}, ω_1, and ω_2 be the respective dualizing sheaves on X_{b_0}, E, and C. Again we see that σ_Q fails to be surjective iff $h^0(\omega_{b_0}(-2Q)) = h^0(\omega_{b_0}(-Q))$. Moreover, since ω_{b_0} is base point free away from P, the rank of σ_Q is always positive. Let P_i be the point on the curve of genus i lying over P in the normalization

$\widetilde{X_{b_0}}$, $i = 1, 2$. Using the description of the dualizing sheaf of X_{b_0} given in [HM, p. 82], we have

$$H^0(\omega_{b_0}(-Q)) = H^0(\omega_1(-Q + P_1)) \oplus H^0(\omega_2(P_2))$$

and

$$H^0(\omega_{b_0}(-2Q)) = H^0(\omega_1(-2Q + P_1)) \oplus H^0(\omega_2(P_2)).$$

Since sections of $\omega_1(P_1)$ are simply constants, if a section vanishes at Q then it vanishes to infinite order. Thus $H^0(\omega_1(-Q + P_1)) = H^0(\omega_1(-2Q + P_1))$. □

PROPOSITION 2.3. *Suppose Q is a closed point of $C - P$. Then* $\operatorname{rank}(\sigma_Q) = 1$ *if and only if Q is a hyperelliptic Weierstrass point of C; otherwise, σ_Q is surjective.*

Proof. With notation as before, we again see that, since ω_{b_0} is base point free away from P, it follows that σ_Q will always have positive rank and that σ_Q will fail to be surjective iff $h^0(\omega_{b_0}(-2Q)) = h^0(\omega_{b_0}(-Q))$. We have

$$H^0(\omega_{b_0}(-2Q)) = H^0(\omega_1(P_1)) \oplus H^0(\omega_2(-2Q + P_2))$$

and

$$H^0(\omega_{b_0}(-Q)) = H^0(\omega_1(P_1)) \oplus H^0(\omega_2(-Q + P_2)).$$

Thus σ_Q will fail to be surjective if and only if

$$h^0(\omega_2(-2Q + P_2)) = h^0(\omega_2(-Q + P_2)).$$

Since $H^0(\omega_2(P_2)) = H^0(\omega_2)$, the preceding equality holds if and only if

$$h^0(\omega_2(-2Q)) = h^0(\omega_2(-Q));$$

that is, it holds iff Q is a hyperelliptic Weierstrass point of C. □

We would now like to determine the behavior of σ at P. Because $\mathcal{F}$ fails to be locally free at P, we apply the process of [D] as follows.

The proof of [D, Lemma 2] trivially generalizes to show that the smallest nonzero Fitting ideal of $\mathcal{F}$ is the maximal ideal of P, which shows that $\mathcal{F}$ is not locally free at P. Let $g \colon X' \to X$ be the blow-up of X at P. By a slight abuse of notation we will continue to use E and C to represent the proper transforms in X' of the elliptic and genus-2 curves, respectively. We let E_0 be the exceptional divisor of the blow-up and P_i the point where the curve of genus i meets E_0, $i = 1, 2$. As described in [D], we pull back $\mathcal{E}$, $\mathcal{F}$, and σ and then take the double dual. Let $\mathcal{E}' = (g^*\mathcal{E})^{**}$, $\mathcal{F}' = (g^*\mathcal{F})^{**}$, and $\sigma' = (g^*\sigma)^{**}$. By [D, Thm. 1], σ' is a map of vector bundles.

Since g is an isomorphism away from P, we see that $\operatorname{rank}(\sigma'_Q) = 1$ for Q a point of $E - P_1$ or a hyperelliptic Weierstrass point of C and that $\operatorname{rank}(\sigma'_Q) = 2$ for all other points of $X' - E_0$. The following proposition describes the behavior of σ' along E_0.

PROPOSITION 2.4. *If Q is a closed point of E_0, then* $\operatorname{rank}(\sigma'_Q) = 1$.

Proof. Choose local coordinates x and y on X centered at P so that the map π is given locally by $xy = t$, where t is a local coordinate on B centered at b_0. At P,

$\mathcal{F}$ is simply the linearizations of differentials in a neighborhood of P. Locally, then, $\mathcal{F}$ is generated by 1, dx, and dy; however, since these are relative differentials, we have the nontrivial relation $dt = 0$. That is,

$$d(xy) = y\,dx + x\,dy = 0.$$

Thus we have the local presentation

$$F_1 \xrightarrow{[0\ \ y\ \ x]} F_0 \longrightarrow \mathcal{F}_P \longrightarrow 0$$

for F_0 the free module generated by $\{1, dx, dy\}$. We can define a map $\mathcal{E}_P \to F_0$ by sending a differential to its linearization. Moreover, it is clear that σ_P factors through this map. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} & & \mathcal{E}_P & & & & \\ & & \downarrow & \searrow^{\sigma_P} & & & \\ F_1 & \xrightarrow{[0\ \ y\ \ x]} & F_0 & \longrightarrow & \mathcal{F}_P & \longrightarrow & 0. \end{array}$$

Let $g\colon X' \to X$ be the blow-up of X at P. Pulling back this diagram gives

$$\begin{array}{ccccccc} & & g^*\mathcal{E}_P & & & & \\ & & \downarrow & \searrow^{g^*\sigma_P} & & & \\ g^*F_1 & \xrightarrow{[0\ \ xy\ \ x]} & g^*F_0 & \longrightarrow & g^*\mathcal{F}_P & \longrightarrow & 0 \end{array}$$

on one patch, and taking the dual then yields

$$\begin{array}{ccccccc} & & & & (g^*\mathcal{E}_P)^* & & \\ & & & \nearrow^{(g^*\sigma_P)^*} & \uparrow & & \\ 0 & \longrightarrow & (g^*\mathcal{F}_P)^* & \longrightarrow & (g^*F_0)^* & \xrightarrow{\begin{bmatrix} 0 \\ xy \\ x \end{bmatrix}} & (g^*F_1)^*. \end{array}$$

But since x is a nonzero divisor on $(g^*F_1)^*$, the bottom row of the following diagram is exact:

$$\begin{array}{ccccccccc} & & & & (g^*\mathcal{E}_P)^* & & & & \\ & & & \nearrow^{(g^*\sigma_P)^*} & \uparrow & & & & \\ 0 & \longrightarrow & (g^*\mathcal{F}_P)^* & \longrightarrow & (g^*F_0)^* & \xrightarrow{\begin{bmatrix} 0 \\ y \\ 1 \end{bmatrix}} & (g^*F_1)^* & \longrightarrow & 0. \end{array}$$

Again taking the dual gives

$$\begin{array}{ccccccccc} & & & & (g^*\mathcal{E}_P)^{**} & & & & \\ & & & & \downarrow & \searrow^{(g^*\sigma_P)^{**}} & & & \\ 0 & \longrightarrow & (g^*F_1)^{**} & \xrightarrow{[0\ \ y\ \ 1]} & (g^*F_0)^{**} & \longrightarrow & (g^*\mathcal{F}_P)^{**} & \longrightarrow & 0, \end{array}$$

where the bottom row is exact.

Thus we see that $(g^*\mathcal{F}_P)^{**}$ is generated by $\{1, dx, dy\}$ with the relation $y\,dx + dy = 0$. Furthermore, the map $(g^*\sigma_P)^{**}$ is given by sending a differential to its linearization and then substituting the relation $dy = -y\,dx$. Similarly, on the other patch of this blow-up, we have the relation $dx + x\,dy = 0$ and the map is given analogously.

Choose $\{\alpha_1, \alpha_2, \alpha_3\}$ as a basis for $H^0(\omega_{b_0})$ (with ω_{b_0} as before), where α_1 is a nonzero constant function on E, α_2 is a regular differential on C that does not vanish at P_2, and α_3 is a regular differential on C vanishing to order 1 at P_2. After all the α are multiplied by suitable constants, the map becomes

$$\sigma'(\alpha_1) = 0(1) + 1\,dx + 0\,dy,$$
$$\sigma'(\alpha_2) = 0(1) + 0\,dx + 1\,dy,$$
$$\sigma'(\alpha_3) = 0(1) + 0\,dx + 0\,dy. \qquad \square$$

In summary, we have proved the following result.

THEOREM 2.5. *The map* $\sigma' \colon \mathcal{E}' \to \mathcal{F}'$ *fails to be surjective at every point of* E, E_0, *and at the hyperelliptic Weierstrass points of* C. *Moreover,* σ' *has rank 1 at such points but has rank 2 at all other points of* X'.

3. The Scheme Structure of $D_1(\sigma')$

DEFINITION 3.1. Let $\sigma : \mathcal{E} \to \mathcal{F}$ be a homomorphism of vector bundles of ranks e and f on a variety X, and let $k \le \min(e, f)$. Then

$$D_k(\sigma) := \{x \in X \mid \operatorname{rank}(\sigma_x) \le k\}.$$

In the present context, $\sigma' : \mathcal{E}' \to \mathcal{F}'$ is a map of vector bundles of ranks 3 and 2 on the variety X'. By (2.5) we see that, *as a set,*

$$D_1(\sigma') = E \cup E_0 \cup \left(\bigcup_{i=1}^{6} Q_i \right),$$

where $Q_1, \dots, Q_6$ are the hyperelliptic Weierstrass points of C.

If Spec A is an open set over which $\mathcal{E}'$ and $\mathcal{F}'$ are both trivial then, over this open set, σ' is given by a 2×3 matrix with coefficients in A. Hence locally $D_1(\sigma')$ is given by the ideal I generated by the 2×2 minor determinants of this matrix. These local pictures patch together to give an ideal sheaf $\mathcal{I}$. Thus $D_1(\sigma')$ has a natural scheme structure given by $\mathcal{I}$. As a first step toward determining this structure, we prove the following statement.

PROPOSITION 3.2. $D_1(\sigma')$ *is reduced at the hyperelliptic Weierstrass points of* C.

Proof. Let $\pi : X \to B$ be a 1-parameter family of smooth, nonhyperelliptic curves of genus 3 degenerating to a general member of Δ_1, where both X and B are smooth. Let $X_{b_0} = E \cup C$ denote the special fiber of this family and let $\omega_{X/B}$ be the relative dualizing sheaf, with $\omega_{X/B}(E) := \omega_{X/B} \otimes \mathcal{O}_X(E)$. On smooth fibers, $\omega_{X/B}(E)$ restricts to the canonical bundle; however, on the special fiber we see that $\omega_{X/B}(E)$ restricts to $\omega_2(2P)$ on C and to the trivial bundle on E, where ω_2 is the canonical bundle on C and $P = E \cap C$. Since X_{b_0} is a general member of Δ_1, we can assume that P is not a hyperelliptic Weierstrass point of C. Thus $\omega_{X/B}(E)$ determines a map

$$\varphi\colon X \to \mathbb{P}^2 \times B$$

that embeds each of the smooth fibers as a planar quartic and maps the special fiber to a cuspidal quartic whose normalization is C. The transform E is collapsed to the cusp.

Since the proposition is clearly local on B, we can assume $B = \operatorname{Spec} A$ for some 1-dimensional ring A; in addition, we can shrink B to a smaller affine open set if necessary and will do so without comment. Let t be a local parameter at $b \in B$. Then φ maps X to $\mathbb{P}^2_A$, and we can choose homogeneous coordinates so that the image of X has the form

$$Y^2Z^2 + Z \sum_{i+j=3} \alpha_{i,j} X^i Y^j + \sum_{i+j=4} \beta_{i,j} X^i Y^j + tG(X,Y,Z) = 0,$$

where G is a homogenous polynomial of degree 4 in $A[X,Y,Z]$.

Since we are interested in the behavior of σ' at the hyperelliptic Weierstrass points of C and since φ is an embedding away from E, it is enough to determine the behavior of σ' on $\varphi(X)$. Moreover, it is clear by the construction of σ' that $D_1(\sigma')$ will be reduced at $Q_1, \dots, Q_6$ if and only if $D_1(\sigma)$ is. Thus we will consider $\sigma : \mathcal{E} \to \mathcal{F}$ on $\varphi(X)$.

Let s_X, s_Y, and s_Z be as in Lemma 3.4 (to follow). In order to give the map σ in local coordinates at a smooth point x of a fiber of π, we simply determine local equations for s_X, s_Y, and s_Z in a neighborhood of x and then consider their linearizations. Presently we are interested in hyperelliptic Weierstrass points of C. Since X_{b_0} is a general point of Δ_1, we can assume that P is not such a point. Because the family remains unchanged away from E under φ, we can make our computations on $\varphi(X)$.

The hyperelliptic Weierstrass points of C can be determined by looking at lines through the cusp of $\varphi(C)$. A point of $\varphi(C)$ whose tangent line passes through the cusp is a hyperelliptic Weierstrass point. Since the line given by $Y = 0$ intersects the cusp with multiplicity 3, we see that no hyperelliptic Weierstrass points lie along this line. Hence it is enough to consider the affine open set of $\varphi(X)$ given by $Y \neq 0$. The total space of our family on this open set is given in affine coordinates by

$$z^2 + z \sum_{i+j=3} \alpha_{i,j} x^i + \sum_{i+j=4} \beta_{i,j} x^i + tG(x,1,z) = 0.$$

On this open set, the local equations for s_X, s_Y, and s_Z are x, 1, and zt, respectively. The linearizations of these sections at a point (x_0, z_0, t_0) are

$$\begin{aligned} x &= x_0 + dx, \\ 1 &= 1, \\ zt &= z_0 t_0 + t_0\, dz + z_0\, dt. \end{aligned}$$

Since the family is parameterized by t, we must have $dt = 0$. But then we also have

$$\begin{aligned} 0 &= d\Big(z^2 + z\sum_{i+j=3}\alpha_{i,j}x^i + \sum_{i+j=4}\beta_{i,j}x^i + tG(x,1,z)\Big) \\ &= \Big(2z + \sum_{i+j=3}\alpha_{i,j}x^i + t\frac{\partial}{\partial z}G(x,1,z)\Big)\,dz \\ &\quad + \Big(z\sum_{i+j=3} i\alpha_{i,j}x^{i-1} + \sum_{i+j=4} i\beta_{i,j}x^{i-1} + t\frac{\partial}{\partial x}G(x,1,z)\Big)\,dx. \end{aligned}$$

Suppose $(x_0, z_0, 0)$ is a point on $\varphi(C)$ such that

$$z_0\sum_{i+j=3} i\alpha_{i,j}x_0^{i-1} + \sum_{i+j=4} i\beta_{i,j}x_0^{i-1} = 0.$$

Then the tangent line to $\varphi(C)$ at this point is given by $z - z_0 = 0$ or, in homogeneous coordinates, $Z - z_0Y = 0$. But this line does not pass through the cusp of $\varphi(C)$, so $(x_0, z_0, 0)$ cannot be a hyperelliptic Weierstrass point. Thus it suffices to consider the open set given by $z\sum_{i+j=3} i\alpha_{i,j}x^{i-1} + \sum_{i+j=4} i\beta_{i,j}x^{i-1} + t\frac{\partial}{\partial z}G(x,1,z) \neq 0$, in which case we have

$$dx = \frac{-\big(2z + \sum_{i+j=3}\alpha_{i,j}x^i + t\frac{\partial}{\partial z}G(x,1,z)\big)}{z\sum_{i+j=3} i\alpha_{i,j}x^{i-1} + \sum_{i+j=4} i\beta_{i,j}x^{i-1} + t\frac{\partial}{\partial x}G(x,1,z)}\,dz.$$

The linearizations of our sections at a point (x_0, z_0, t_0) of the open set being considered are then

$$\begin{aligned} x &= x_0 + \frac{-\big(2z_0 + \sum_{i+j=3}\alpha_{i,j}x_0^i + t_0\frac{\partial}{\partial z}G(x_0,1,z_0)\big)}{z_0\sum_{i+j=3} i\alpha_{i,j}x_0^{i-1} + \sum_{i+j=4} i\beta_{i,j}x_0^{i-1} + t_0\frac{\partial}{\partial x}G(x_0,1,z_0)}\,dz, \\ 1 &= 1, \\ zt &= z_0t_0 + t_0\,dz. \end{aligned}$$

We have thus shown that, in local coordinates, at a point of this open set the map $\sigma : \mathcal{E} \to \mathcal{F}$ is given by the matrix

$$\begin{bmatrix} x & 1 & zt \\ \dfrac{-\big(2z + \sum_{i+j=3}\alpha_{i,j}x^i + t\frac{\partial}{\partial z}G(x,1,z)\big)}{z\sum_{i+j=3} i\alpha_{i,j}x^{i-1} + \sum_{i+j=4} i\beta_{i,j}x^{i-1} + t\frac{\partial}{\partial x}G(x,1,z)} & 0 & t \end{bmatrix}.$$

The ideal generated by the 2×2 minors of this matrix is

$$\begin{aligned} I &= \Big(t, 2z + \sum_{i+j=3}\alpha_{i,j}x^i\Big) \\ &\subset \frac{A[x,z]}{\big(z^2 + z\sum_{i+j=3}\alpha_{i,j}x^i + \sum_{i+j=4}\beta_{i,j}x^i + tG(x,1,z)\big)} \end{aligned}$$

or, equivalently,

$$I=\left(2z+\sum_{i+j=3}\alpha_{i,j}x^i,z^2+z\sum_{i+j=3}\alpha_{i,j}x^i+\sum_{i+j=4}\beta_{i,j}x^i\right)\subset\mathbb{C}[x,z].$$

Substituting $z=-\frac{1}{2}\sum_{i+j=3}\alpha_{i,j}x^i$ into the second equation gives

$$I=\left(2z+\sum_{i+j=3}\alpha_{i,j}x^i,h(x)\right)\subset\mathbb{C}[x,z],$$

where $h(x)$ is a polynomial of degree 6. Since there are six hyperelliptic Weierstrass points on C, we see that $h(x)$ must have distinct roots and so I is the ideal of these six points. This shows that $D_1(\sigma')$ is reduced at these points. □

The following construction was used implicitly in the preceding proof.

Consider the divisors $\{X=0\}$, $\{Y=0\}$, and $\{Z=0\}$ on $\varphi(X)$. Since the canonical bundle on a smooth planar quartic is $\mathcal{O}(1)$, these will restrict to the canonical divisor on each smooth fiber; moreover, the associated sections will give a basis for the space of sections of the canonical bundle on a smooth fiber.

We consider the pull-backs of these divisors to X. Since $\{Z=0\}$ does not pass through the cusp of the central fiber, it pulls back isomorphically to a divisor that we will call D_Z. Also, $\varphi^*\{X=0\}$ is supported on E and on an irreducible curve that we call D_X; with this notation we have $\varphi^*\{X=0\}=D_X+2E$. Similarly, $\varphi^*\{Y=0\}$ is supported on E and on an irreducible curve that we call D_Y; we then have $\varphi^*=D_Y+3E$. Clearly, these pull-backs still restrict to the canonical bundle on smooth fibers of π. Yet because the map on the special fiber to $\mathbb{P}^2$ is given by K_C+2P, we see that these pull-backs must restrict to K_C+2P on C and are linearly equivalent to 0 on E.

Since $E.(E+C)\sim 0$, $C.(E+C)\sim 0$, and $E.C\sim P$, it follows that $E.E\sim -P$ and $C.C\sim -P$. Thus we have the following lemma.

LEMMA 3.3. *The divisors D_X+E, D_Y+2E, and D_Z+C restrict to (divisor classes linearly equivalent to) the canonical divisor on smooth fibers of π, to P on E, and to K_C+P on C, respectively.*

By the description of the dualizing sheaf given in [HM, p. 82], the invertible sheaf associated to such divisors will be $\omega_{X/B}$. The hope is that the global sections associated to these divisors will restrict to the desired basis on each fiber. Our next lemma, which was used in the proof of Proposition 3.2, states this claim formally.

LEMMA 3.4. *Let s_X, s_Y, and s_Z be the sections of $\omega_{X/B}$ associated to D_X+E, D_Y+2E, and D_Z+C, respectively. Then, on each fiber of π, the sections s_X, s_Y, and s_Z restrict to a basis for the space of global sections of the dualizing sheaf.*

Proof. Since X, Y, and Z give a basis for $\mathcal{O}(1)$ on $\mathbb{P}^2$, the statement is clear for smooth fibers of π. We therefore consider the restrictions of s_X, s_Y, and s_Z to X_{b_0}. Because these sections have different vanishing orders along E, they are linearly independent. Moreover, the space of global sections of the dualizing sheaf of $C\cup E$ has rank 3. This completes the proof. □

To determine the remaining scheme structure of $D_1(\sigma')$, we explicitly construct a family of smooth nonhyperelliptic curves of genus 3 degenerating to a general member of Δ_1. Using the proof of Proposition 3.2 as a guide, we begin with a family of smooth planar quartics over (an open subset of) $\mathbb{A}^1_t$ degenerating to a cuspidal quartic. We then explicitly compute the stable reduction of such a family.

To begin, let

$$F(X,Y,Z) = Y^2Z^2 + Z\sum_{i+j=3}\alpha_{i,j}X^iY^j + \sum_{i+j=4}\beta_{i,j}X^iY^j,$$

where the $\alpha_{i,j}$ and $\beta_{i,j}$ are such that $F(X,Y,Z)=0$ is nonsingular away from $[0,0,1]$. Observe that, if the coordinates are chosen properly, then any cuspidal quartic can be given by such an equation. Moreover, we will assume that $\alpha_{3,0} = -1$. Let C_t be the family of curves parameterized by $t \in \mathbb{C}$ and given by

$$F(X,Y,Z) - at^2XZ^3 - bt^3Z^4 = 0,$$

where $a,b \in \mathbb{C}$ are such that $a,b \neq 0$ and $4a^3 + 27b^2 \neq 0$. The special fiber C_0 is the cuspidal quartic given by $F(X,Y,Z)=0$.

If we consider the specific planar curve given by

$$Y^2Z^2 - X^3Z + Y^4 - XZ^3 - Z^4 = 0,$$

then one easily checks that this curve is nonsingular. For general choices of $\alpha_{i,j}$, $\beta_{i,j}$, a, b, and t, such a curve is nonsingular; thus, for general choices of $\alpha_{i,j}$, $\beta_{i,j}$, a, and b, all but finitely many fibers of the family C_t will be smooth. So we will assume that we have chosen $\alpha_{i,j}$, $\beta_{i,j}$, a, and b in such a manner. By restricting t to an open neighborhood of $t=0$, we can also assume that all fibers other than C_0 are smooth.

The astute reader may well wonder why we have chosen to consider such a general family rather than a simple pencil, like the one considered in [HM, pp. 122–128]. The answer is a practical one. It will turn out that $D_1(\sigma')$ contains nonreduced points on a general member of Δ_1. Since the member of Δ_1 resulting from the stable reduction of a pencil of smooth quartics degenerating to a cuspidal quartic will always have an elliptic tail with j-invariant equal to 0, it is not enough to show that $D_1(\sigma')$ contains these nonreduced points on such a family.

The elliptic curve that will appear in the stable limit will lie over the cusp of $F(X,Y,Z)=0$. As a result, for our purposes it will suffice to consider the family $f(x,y) - at^2x - bt^3 = 0$, where $f(x,y) = F(x,y,1)$.

By [Ha, III.10.1(c)], the total space of our family is smooth away from the singular point of C_0. However, the total space of this family does have a singularity at the origin. We will simultaneously resolve the singularity in the total space and the cusp in the central fiber with four successive blow-ups.

First Blow-up. First, we blow up along the linear subspace $x=y=0$ in $\mathbb{A}^3_{(x,y,t)}$ and take the proper transform of our family. This gives us two patches to consider as follows.

On the first patch (which we will call (P1)), we make the substitution $y = xy$. The exceptional divisor (by which we will always mean the exceptional divisor in the ambient affine space restricted to the total space of our family), which we call E_1, is then given by $x = 0$, and the proper transform of C_t is given by $f(x, xy) - at^2x - bt^3 = 0$. (We will continue to call this C_t.)

On the second patch (P2), we make the substitution $x = xy$. Here E_1 is given by $y = 0$ and the proper transform of C_t is given by $f(xy, y) - at^2xy - bt^3 = 0$.

On both patches, the special fiber consists of the union of C (the normalization of $f(x, y) = 0$) and E_1, which has multiplicity 2. On the first patch, these two components are tangent at $x = y = 0$. (They are disjoint on the second.) Both patches contain a codimension-1 singularity along E_1.

Second Blow-up. Next, we blow up (P1) along $x = y = t = 0$. This action yields three patches to consider.

On the first patch (P1-1), we make the substitutions $y = xy$ and $t = xt$. The exceptional divisor E_2 is given by $x = 0$, and the proper transform of C_t is given by $\frac{1}{x^3} f(x, x^2y) - at^2 - bt^3 = 0$. The special fiber is given by $xt = 0$ and consists of the union of C and E_2, which do not meet on this patch. The total space of the family on this patch is nonsingular.

On the second patch (P1-2), we make the substitutions $x = xy$ and $t = yt$. Here E_2 is given by $y = 0$, and the proper transform of C_t is given by $\frac{1}{y^3} f(xy, xy^2) - at^2x - bt^3 = 0$. The special fiber is given by $yt = 0$ and consists of the union of C, E_1 (which appears with multiplicity 2), and E_2. The total space of the family is still singular along E_1.

On the third patch (P1-3), we make the substitutions $x = xt$ and $y = yt$; now E_2 is given by $t = 0$, and the proper transform of C_t is given by $\frac{1}{t^3} f(xt, xyt^2) - ax - b = 0$. Since $x \neq 0$ on this patch, we see that it is contained in (P1-1); thus we can ignore it.

Note that E_2 is the union of three rational curves (which are distinct by the restrictions placed on a, b) meeting at one point (contained in (P1-2)).

Third Blow-up. Next, in an effort to obtain a special fiber that is supported on a nodal curve, we will blow up the point in (P1-2) where E_1, E_2, and C all meet. Again we have three patches to consider.

On the first patch (P1-2-1), we make the substitutions $y = xy$ and $t = xt$. The exceptional divisor E_3 is given by $x = 0$, and the proper transform of C_t is given by $\frac{1}{x^6y^3} f(x^2y, x^3y^2) - at^2 - bt^3 = 0$. The special fiber is given by $x^2yt = 0$ and consists of the union of C, E_2 (which is now the *disjoint* union of three rational curves that we call E_2', E_2'', and E_2'''), and E_3 (which appears with multiplicity 2). The total space of the family is nonsingular on this patch.

On the second patch (P1-2-2), we make the substitutions $x = xy$ and $t = yt$. Here E_3 is given by $y = 0$, and the proper transform of C_t is given by $\frac{1}{y^6} f(xy^2, xy^3) - at^2x - bt^3 = 0$. The special fiber is given by $y^2t = 0$ and consists of the union of C, E_1 (which appears with multiplicity 2), and E_3 (which also appears with multiplicity 2). The total space of the family is still singular along E_1.

On the third patch (P1-2-3), we make the substitutions $x = xt$ and $y = yt$; now E_3 is given by $t = 0$, and the proper transform of C_t is given by $\frac{1}{y^3t^6} f(xyt^2, xy^2t^3) - ax - b = 0$. Since $x \neq 0$ on this patch, it is contained in (P1-2-1); hence we can ignore it.

Fourth Blow-up. Finally, we resolve the singularity in the total space by blowing up along E_1. Points of E_1 occur only in (P2) and (P1-2-2), so we need only blow up these two patches. We blow up (P2) first, which we do by blowing up $\mathbb{A}^3$ along $y = t = 0$ and then taking the proper transform of C_t. There are two patches to consider as follows.

On the first patch (P2-1), we make the substitution $y = yt$. The exceptional divisor E_4 is given by $t = 0$ and the proper transform of C_t is given by $\frac{1}{t^2} f(xyt, yt) - atxy - bt = 0$. The special fiber is given by $t = 0$ and consists only of E_4 (which appears with multiplicity 2). The total space is nonsingular on this patch.

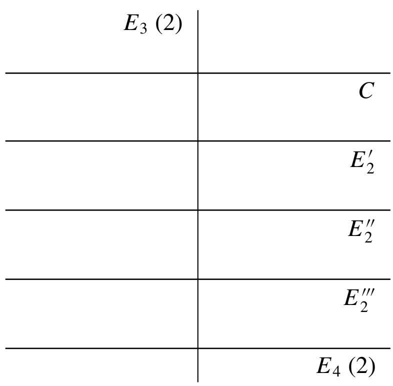
On the second patch (P2-2), we make the substitution $t = yt$. Here E_4 is given by $y = 0$ and the proper transform of C_t is given by $\frac{1}{y^2} f(xy, y) - at^2xy - bt^3y = 0$. The special fiber is given by $yt = 0$ and consists only of C. Again, the total space is nonsingular.

Fifth Blow-up. Next we blow up (P1-2-2). Again this is done by blowing up $\mathbb{A}^3$ along $x = t = 0$ and taking the proper transform of C_t. There are two patches to consider.

On the first patch (P1-2-2-1), we make the substitution $t = xt$. The exceptional divisor E_4 is given by $x = 0$, which does not meet this patch, and the proper transform of C_t is given by $\frac{1}{x^2y^6} f(xy^2, xy^3) - at^2x - bt^3x = 0$. The special fiber is given by $xy^2t = 0$ and consists of the union of C and E_3 (which appears with multiplicity 2). The total space of the family is nonsingular on this patch.

On the second patch (P1-2-2-2), we make the substitution $x = xt$; now E_4 is given by $t = 0$, and the proper transform of C_t is given by $\frac{1}{y^6t^2} f(xy^2t, xy^3t) - atx - bt = 0$. The special fiber is given by $y^2t = 0$ and consists of the union of E_4 (which appears with multiplicity 2) and E_3 (which also appears with multiplicity 2). The total space of the family is nonsingular.

The total space of our family is now nonsingular; the special fiber is supported on a nodal curve, but it contains components of multiplicity 2 that must be dealt with. A schematic drawing of the special fiber is given below.



We deal with the components of multiplicity 2 by making a base change of order 2 branched over $t = 0$. This base change will introduce new singularities into the total space, so we package it with the normalization of the resulting surface. The effect of this will be to take the branched cover of the total space branched along the union of C, E_2', E_2'', and E_2''' (see [HM, pp. 124–125]). Since this branch divisor is smooth, the resulting surface will be smooth as well. Since E_3 meets the branch locus in four points, its inverse image will be a double cover of $E_3 \cong \mathbb{P}^1$ branched at four points—that is, a single elliptic curve that we will call E. On the other hand, E_4 is disjoint from the branch locus and so its inverse image will be an unramified double cover of $E_4 \cong \mathbb{P}^1$: two disjoint rational curves that we call E_4' and E_4''.

The pull-back of the special fiber to the new family will then be

$$2\tilde{C} + 2E_2' + 2E_2'' + 2E_2''' + 2E_4' + 2E_4'' + 2E.$$

But the special fiber of the new family is exactly half of this divisor. Thus the special fiber is

$$\tilde{C} + E_2' + E_2'' + E_2''' + E_4' + E_4'' + E.$$

Since E_2', E_2'', E_2''', E_4', and E_4'' are all rational curves with self-intersection -1, they can be blown down. The special fiber then becomes the union of E and C meeting transversely at one point, as desired.

For our purposes, we are only interested in points of E. Hence we will explicitly compute the base change just described only on those patches containing points of E_3—specifically, (P1-2-1), (P1-2-2-1), and (P1-2-2-2). Furthermore, since all the points of E_3 in (P1-2-2-1) are contained in one of the other two open sets, we do not need to consider (P1-2-2-1). For ease of notation we will rename the open sets (P1-2-1) and (P1-2-2-2) as U and V, respectively.

Recall that the total space on U is given by $\frac{1}{x^6y^3} f(x^2y, x^3y^2) - at^2 - bt^3 = 0$. The special fiber is given by $x^2yt = 0$ and consists of the union of $\tilde{C}$, E_2', E_2'', E_2''', and E_3 (which appears with multiplicity 2).

The total space on V is given by $\frac{1}{y^6t^2} f(xy^2t, xy^3t) - atx - bt = 0$. The special fiber is given by $y^2t = 0$ and consists of the union of E_4 (which appears with multiplicity 2) and E_3 (which also appears with multiplicity 2).

We now explicitly perform the calculations described before on these two open sets.

After the base change of order 2 on U, the total space of the family is given by

$$\left\{\frac{1}{x^6y^3} f(x^2y, x^3y^2) - at^2 - bt^3 = 0\right\} \cap \{u^2 - x^2yt = 0\} \subseteq \mathbb{A}^4_{(x,y,t,u)},$$

where the special fiber is given by $u = 0$. Normalizing by blowing up, we set $u = xv$. This gives

$$\left\{\frac{1}{x^6y^3} f(x^2y, x^3y^2) - at^2 - bt^3 = 0\right\} \cap \{v^2 - yt = 0\} \subseteq \mathbb{A}^4_{(x,y,t,v)},$$

where the special fiber is given by $xv = 0$. We continue to call this open set U.

On the open subset $\{t \neq 0\} \subset U$ we have $y = v^2/t$, so the total space is given by

$$\left\{\frac{t^3}{x^6v^6} f(x^2v^2t^{-1}, x^3v^4t^{-2}) - at^2 - bt^3 = 0\right\} \cap \{t \neq 0\} \subseteq \mathbb{A}^3_{(x,t,v)},$$

with the special fiber still given by $xv = 0$. In this case, $x = 0$ is a local equation for E and $v = 0$ is a local equation for $E_2' \cup E_2'' \cup E_2'''$. Hence these rational curves can be blown down using the relation $xv \mapsto x$. We thus arrive at

$$\left\{\frac{t^3}{x^6} f(x^2t^{-1}, x^3vt^{-2}) - at^2 - bt^3 = 0\right\} \cap \{t \neq 0\} \subseteq \mathbb{A}^3_{(x,t,v)};$$

here the special fiber is given by $x = 0$, which is a local equation for E. We will call this open set U_0.

Let U_1 denote the open subset $\{y \neq 0\} \subset U$. On this open set we have $t = v^2/y$, so the total space is given by

$$\left\{\frac{1}{x^6} f(x^2y, x^3y^2) - ayv^4 - bv^6 = 0\right\} \cap \{y \neq 0\} \subseteq \mathbb{A}^3_{(x,y,v)}$$

with the special fiber once again given by $xv = 0$. In this case, $x = 0$ is a local equation for E and $v = 0$ is a local equation for $\tilde{C}$. On this open set we can blow down using the relation $xy \mapsto x$. We then arrive at

$$\left\{y^4 + y^3 \sum_{i+j=3} \alpha_{i,j} x^j + y^2 \sum_{i+j=4} \beta_{i,j} x^{j+2} - ayv^4 - bv^6 = 0\right\} \cap \{y \neq 0\} \subseteq \mathbb{A}^3_{(x,y,v)}.$$

After the base change of order 2 on V, the total space of the family is given by

$$\left\{\frac{1}{y^6t^2} f(xy^2t, xy^3t) - atx - bt = 0\right\} \cap \{u^2 - y^2t = 0\} \subseteq \mathbb{A}^4_{(x,y,t,u)},$$

where the special fiber is given by $u = 0$. We normalize by setting $u = xyv$. This gives

$$\left\{\frac{1}{y^6t^2} f(xy^2t, xy^3t) - atx - bt = 0\right\} \cap \{x^2v^2 - t = 0\} \subseteq \mathbb{A}^4_{(x,y,t,v)};$$

here the special fiber is given by $xyv = 0$. Clearly this can be simplified to

$$\left\{\frac{1}{x^6y^6v^4} f(x^3y^2v^2, x^3y^3v^2) - axv^2 - bv^2 = 0\right\} \subseteq \mathbb{A}^3_{(x,y,v)}.$$

On this open set, $x = 0$ is a local equation for $E_4' \cup E_4''$ and $y = 0$ is a local equation for E. We can therefore blow down using the relation $xy \mapsto y$. The result is

$$\left\{\frac{1}{y^6v^4} f(xy^2v^2, y^3v^2) - axv^2 - bv^2 = 0\right\} \subseteq \mathbb{A}^3_{(x,y,v)}$$

with special fiber given by $yv = 0$, where $y = 0$ is a local equation for E.

The question remains as to which elements of Δ_1 can appear in this stable limit. On the open set U_0, the elliptic curve is given by the equation $v^2 - t - at^3 - bt^4 = 0$ in the (t, v)-plane and the double cover of $\mathbb{P}^1$ that appears in the stable reduction process is given by $(t, v) \mapsto t$, where we consider t as an affine coordinate on $\mathbb{P}^1$. Moreover, U_0 contains all but three points of the elliptic curve (i.e., $C \cap E$, $E_4' \cap E$, and $E_4'' \cap E$), and it is clear where these points map to: $C \cap E$ maps to 0 and the other two points map to infinity. Thus we see that the map is branched at the points of $\mathbb{P}^1$ satisfying $t + at^3 + bt^4 = 0$.

If we compose this map with the automorphism of $\mathbb{P}^1$ that sends t to $1/t$, then the new map is branched at infinity and at points satisfying $t^3 + at + b = 0$. The elliptic curve that appears in the stable limit is consequently isomorphic to an elliptic curve, in the (x, y)-plane, that is given by $y^2 = x^3 + ax + b$. The j-invariant of such a curve is easily calculated as

$$j = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Since every elliptic curve is isomorphic to one of the form $y^2 = x^3 + Ax + B$ in the plane, we see that the elliptic curve appearing in our stable limit is a general elliptic curve.

If C is any smooth curve of genus 2 and Q is a point of C that is not a hyperelliptic Weierstrass point, then $|K + 2Q|$ determines a map from C to $\mathbb{P}^2$. This maps C to a cuspidal quartic that is smooth away from the cusp. Furthermore, after an automorphism of $\mathbb{P}^2$, we can assume that the cuspidal quartic is given by $F(X, Y, Z) = 0$ for some choice of $\alpha_{i,j}$ and $\beta_{i,j}$. Thus we see that (i) an open dense subset of the genus-2 curves can appear in the limit and (ii) the point $E \cap C$ can be any point of C other than one of the six hyperelliptic Weierstrass points. This shows that there exists an open dense subset of Δ_1 any points of which can appear as the stable limit of our family.

We now use this family to prove the following theorem.

THEOREM 3.5. *With notation as in the proof of Proposition 2.2, let* $\phi\colon E \to \mathbb{P}^1$ *be the double cover of* $\mathbb{P}^1$ *determined* (*up to automorphism of* $\mathbb{P}^1$) *by* $|2P_1|$. *Let* S_1, S_2, *and* S_3 *be the points of* E, *other than* P_1, *that are ramified over* $\mathbb{P}^1$. *Then, as a scheme,* $D_1(\sigma')$ *is reduced except at* S_1, S_2, S_3, *and* P_2, *where the ideal locally defining* $D_1(\sigma')$ *is the product of the maximal ideal at* S_i, $i = 1, 2, 3$, *and the ideal defining* E (*or of the maximal ideal at* P_2 *and the ideal defining* E_0).

Proof. We use the family $\pi : X \to \mathbb{A}^1_u$ constructed before. It is clear that we can apply Lemmas 3.3 and 3.4 to our situation, where D_X, D_Y, and D_Z are simply the proper transforms of $\{X = 0\}$, $\{Y = 0\}$, and $\{Z = 0\}$. We first find local equations for s_X, s_Y, and s_Z on each of the open sets U_0, U_1, and V. We have already identified local equations for E and C in the descriptions of each of these open sets given previously. Moreover, these open sets are disjoint from D_Z and so, on each patch, a local equation for s_Z is the same as a local equation for C. Thus, if we can determine local equations for D_X and D_Y then we can, in turn, determine local equations for s_X and s_Y.

A careful schematic drawing of the sequence of blow-ups, base changes, and blow-downs in the foregoing stable reduction process will reveal that D_X intersects the central fiber in those points of E that are the images of E_4' and E_4'' after the blow-down. Therefore, a local equation for D_X can be found on each patch by determining a local equation for $E_4' \cup E_4''$ prior to the blow-down. Similarly, D_Y intersects the central fiber in those points of E that are the images of E_2', E_2'', and E_2''', and so a local equation for D_Y can be found on each patch by determining a local equation for $E_2' \cup E_2'' \cup E_2'''$ prior to the blow-down. As a result, on U_0

we have $s_X = x$, $s_Y = x^2 v$, and $s_Z = t$; on U_1 we have $s_X = x$, $s_Y = x^2 y$, and $s_Z = v$; and on V we have $s_X = yx$, $s_Y = y^2$, and $s_Z = 1$.

Now we can explicitly compute the map at each point of $D_1(\sigma')$. Note that all points of E are contained in U_0 except for those that are the result of blowing down E_4' and E_4'' (contained in V) and the point where E meets C (contained in U_1).

We consider U_0 first. Recall that on this open set the total space of the family is given by

$$\left\{\frac{t^3}{x^6} f(x^2 t^{-1}, x^3 v t^{-2}) - a t^2 - b t^3 = 0\right\} \cap \{t \neq 0\} \subseteq \mathbb{A}^3_{(x,t,v)}$$

with a fiber of the family given by $x = u$. Thus we have

$$dx = 0$$

and

$$d\left(\frac{t^3}{x^6} f(x^2 t^{-1}, x^3 v t^{-2}) - a t^2 - b t^3\right) = 0;$$

this allows us to write

$$0 = p(x,t,v)\, dt + q(x,t,v)\, dv,$$

where

$$\begin{aligned} p(x,t,v) &= \frac{\partial}{\partial t}\left(\frac{t^3}{x^6} f(x^2 t^{-1}, x^3 v t^{-2}) - a t^4 - b t^3\right) \\ &= -v^2 t^{-2} + \sum_{i+j=3} -j\alpha_{i,j} x^j v^j t^{-j-1} \\ &\quad + \sum_{i+j=4} (-1-j)\beta_{i,j} x^{2+j} v^j t^{-2-j} - 2at - 3bt^2, \\ q(x,t,v) &= \frac{\partial}{\partial v}\left(\frac{t^3}{x^6} f(x^2 t^{-1}, x^3 v t^{-2}) - a t^2 - b t^3\right) \\ &= 2vt^{-1} + \sum_{i+j=3} j\alpha_{i,j} x^j v^{j-1} t^{-j} + \sum_{i+j=4} j\beta_{i,j} x^{2+j} v^{j-1} t^{-1-j}. \end{aligned}$$

Suppose $(0,\gamma,\zeta)$ is a point on E with $\zeta \neq 0$. Since $t \neq 0$ on all of U_0, it follows that $2\zeta\gamma^{-1} \neq 0$ and hence $q(0,\gamma,\zeta) \neq 0$. Thus, in a neighborhood of $(0,\gamma,\zeta)$, we have

$$dv = \frac{-p(x,t,v)}{q(x,t,v)}\, dt.$$

Let (x_0, t_0, v_0) be a point in such neighborhood. The linearizations of x, $x^2 v$, and t at this point are

$$\begin{aligned} x &= x_0 + dx \\ &= x_0, \\ x^2 v &= x_0^2 v_0 + 2x_0 v_0\, dx + x_0^2\, dv \\ &= x_0^2 v_0 - x_0^2 \left(\frac{-p(x_0,t_0,v_0)}{q(x_0,t_0,v_0)}\right) dt, \\ t &= t_0 + dt. \end{aligned}$$

Locally, then, the map σ' can be given by the matrix

$$\begin{bmatrix} x & x^2v & t \\ 0 & -x^2\left(\dfrac{-p(x,t,v)}{q(x,t,v)}\right) & 1 \end{bmatrix}.$$

The ideal of

$$\frac{\mathbb{C}[x,t,v,t^{-1}]}{\left(\frac{t^3}{x^6}f(x^2t^{-1},x^3vt^{-2}) - at^2 - bt^3\right)}$$

generated by the 2×2 minor determinants of this matrix is (x); in particular, $D_1(\sigma')$ is reduced at such points.

Next we consider the points $(0,\gamma,0)$ of E. Notice that if $x = v = 0$ then we have $1+a\gamma^2+b\gamma^3$, and if $-2a\gamma - 3b\gamma^2 = 0$ we have $\gamma = -2a/3b$. Combining these yields

$$\begin{aligned} 0 &= 1 + a\left(\frac{4a^2}{9b^2}\right) - b\left(\frac{8a^3}{27b^3}\right), \\ -1 &= \frac{4a^3}{27b^2}. \end{aligned}$$

Hence we must have $-2a\gamma - 3b\gamma^2 \neq 0$ and so $p(0,\gamma,0) \neq 0$. Thus, in a neighborhood of $(0,\gamma,0)$, we have

$$dt = \frac{-q(x,t,v)}{p(x,t,v)}\,dv.$$

Again, let (x_0,t_0,v_0) be a point in such a neighborhood. The linearizations of x, x^2v, and t at this point are

$$\begin{aligned} x &= x_0 + dx \\ &= x_0, \\ x^2v &= x_0^2v_0 + 2x_0v_0\,dx + x_0^2\,dv \\ &= x_0^2v_0 + x_0^2\,dv, \\ t &= t_0 + dt \\ &= t_0 + \frac{-q(x_0,t_0,v_0)}{p(x_0,t_0,v_0)}\,dv. \end{aligned}$$

So locally, the map σ' can be given by the matrix

$$\begin{bmatrix} x & x^2v & t \\ 0 & x^2 & \dfrac{-q(x,t,v)}{p(x,t,v)} \end{bmatrix},$$

and the ideal of

$$\frac{\mathbb{C}[x,t,v,t^{-1}]}{\left(\frac{t^3}{x^6}f(x^2t^{-1},x^3vt^{-2}) - at^2 - bt^3\right)}$$

generated by the 2×2 minor determinants of this matrix is

$$\left(x^3, \frac{xq(x,t,v)}{p(x,t,v)}, x^2\left(\frac{vq(x,t,v)}{p(x,t,v)} + t\right)\right).$$

Passing to the complete local ring at $(0, \gamma, 0)$ we see that, since $\frac{vq(x,t,v)}{p(x,t,v)} + t \neq 0$ and $p(x,t,v) \neq 0$, it follows that the ideal is given by

$$(xq(x,t,v), x^2) = (xvt^4, x^2) = (xv, x^2).$$

In particular, $D_1(\sigma')$ is nonreduced at such points.

Next we consider the points of V that are not contained in $U_0 \cup U_1$. There are only two such points: those obtained from blowing down E_4' and E_4''. Recall that, on V, the total space of the family is given by

$$\left\{\frac{1}{y^6v^4} f(xy^2v^2, y^3v^2) - axv^2 - bv^2 = 0\right\} \subseteq \mathbb{A}^3_{(x,y,v)}$$

with a fiber given by $yv = u$. Thus we have

$$d(yv) = y\,dv + v\,dy = 0.$$

Since $v \neq 0$, this gives

$$dy = \frac{-y}{v}\,dv;$$

also,

$$d\left(\frac{1}{y^6v^4} f(xy^2v^2, y^3v^2) - axv^2 - bv^2\right) = 0.$$

These relations allow us to write

$$0 = p(x,y,v)\,dx + q(x,y,v)\,dv,$$

where

$$\begin{aligned}
p(x,y,v) &= \frac{\partial}{\partial x}\left(\frac{1}{y^6v^4} f(xy^2v^2, y^3v^2) - axv^2 - bv^2\right)\\
&= \sum_{i+j=3} i\alpha_{i,j}x^{i-1}y^jv^2 + \sum_{i+j=4} i\beta_{i,j}x^{i-1}y^{2+j}v^4 - av^2\\
&= v^2\left(\sum_{i+j=3} i\alpha_{i,j}x^{i-1}y^j + \sum_{i+j=4} i\beta_{i,j}x^{i-1}y^{2+j}v^2 - a\right),\\
q(x,y,v) &= \frac{\partial}{\partial v}\left(\frac{1}{y^6v^4} f(xy^2v^2, y^3v^2) - axv^2 - bv^2\right)\\
&\quad - \frac{y}{v}\frac{\partial}{\partial y}\left(\frac{1}{y^6v^4} f(xy^2v^2, y^3v^2) - axv^2 - bv^2\right)\\
&= \sum_{i+j=3} 2\alpha_{i,j}x^iy^jv + \sum_{i+j=4} 4\beta_{i,j}x^iy^{2+j}v^3 - 2axv - 2bv\\
&\quad - \sum_{i+j=3} j\alpha_{i,j}x^iy^jv - \sum_{i+j=4} (2+j)\beta_{i,j}x^iy^{2+j}v^3\\
&= \sum_{i+j=3} (2-j)\alpha_{i,j}x^iy^jv + \sum_{i+j=4} (2-j)\beta_{i,j}x^iy^{2+j}v^3 - 2axv - 2bv.
\end{aligned}$$

The points of V that we wish to consider are $(0,0,\zeta)$, where $1-b\zeta^2=0$. At such a point we have

$$p(0,0,\zeta)=-a\zeta^2\neq 0.$$

Thus, in a neighborhood of such points,

$$dx=\frac{-q(x,y,v)}{p(x,y,v)}\,dv.$$

Let (x_0,y_0,v_0) be a point in such a neighborhood. The linearizations of xy, y^2, and 1 at this point are

$$\begin{aligned}
xy &= x_0y_0+y_0\,dx+x_0\,dy\\
&= x_0y_0-\left(y_0\frac{q(x_0,y_0,v_0)}{p(x_0,y_0,v_0)}+\frac{x_0y_0}{v_0}\right)dv,\\
y^2 &= y_0^2+2y_0\,dy\\
&= y_0^2-2\frac{y_0^2}{v_0}\,dv,\\
1 &= 1+0\,dv.
\end{aligned}$$

As a result, locally the map σ' can be given by the matrix

$$\begin{bmatrix} xy & y^2 & 1\\ -y\dfrac{q(x,y,v)}{p(x,y,v)}-\dfrac{xy}{v} & \dfrac{-2y^2}{v} & 0\end{bmatrix}.$$

The ideal of

$$\frac{\mathbb{C}[x,y,v]}{\left(\frac{1}{y^6v^4}f(xy^2v^2,y^3v^2)-axv^2-bv^2\right)}$$

generated by the 2×2 minor determinants of this matrix is

$$\left(y\left(\frac{q(x,y,v)}{p(x,y,v)}+\frac{xy}{v}\right),y^2\right),$$

but

$$\frac{q(0,0,\zeta)}{p(0,0,\zeta)}=\frac{2b\zeta}{a}\neq 0.$$

Therefore, in the complete local ring at $(0,0,\zeta)$, the ideal is given simply by (y); in particular, $D_1(\sigma')$ is reduced at the points $(0,0,\zeta)$.

Finally, we consider the point $P=(0,1,0)$ of U_1. Recall that the total space of the family on U_1 is given by

$$\left\{y^4+y^3\sum_{i+j=3}\alpha_{i,j}x^j+y^2\sum_{i+j=4}\beta_{i,j}x^{j+2}-ayv^4-bv^6=0\right\}\cap\{y\neq 0\}\subseteq\mathbb{A}^3_{(x,y,v)}$$

with a fiber given by $xv=u$. So in a neighborhood of P we have

$$\begin{aligned}
0 &= d\left(y^4+y^3\sum_{i+j=3}\alpha_{i,j}x^j+y^2\sum_{i+j=4}\beta_{i,j}x^{j+2}-ayv^4-bv^6\right)\\
&= p(x,y,v)\,dx+q(x,y,v)\,dy+r(x,y,v)\,dv,
\end{aligned}$$

where

$$p(x,y,v) = y^3 \sum_{i+j=3} j\alpha_{i,j} x^{j-1} + y^2 \sum_{i+j=4} (j+2)\beta_{i,j} x^{j+1},$$

$$q(x,y,v) = 4y^3 + 3y^2 \sum_{i+j=3} \alpha_{i,j} x^j + 2y \sum_{i+j=4} \beta_{i,j} x^{j+2} - av^4,$$

$$r(x,y,v) = -4ayv^3 - 6bv^5.$$

However, $q(0,1,0) = 4 + 3\alpha_{3,0} = 1$ (recall that $\alpha_{3,0}$ was assumed to be -1). Thus, in a neighborhood of P we have

$$dy = \frac{-p(x,y,v)}{q(x,y,v)}\,dx - \frac{r(x,y,v)}{q(x,y,v)}\,dv;$$

we also have

$$0 = d(xv) = v\,dx + x\,dv.$$

Hence $\mathcal{F}$ is locally generated by 1, dx, and dv but with a nontrivial relation at P.

We now consider the map from $\mathcal{E}(U_1)$ to the free module generated by 1, dx, and dv. Let (x_0, y_0, v_0) be a point in a neighborhood of P. The linearizations of x, x^2y, and v at this point are

$$\begin{aligned} x &= x_0 + dx,\\ x^2y &= x_0^2y_0 + 2x_0y_0\,dx + x_0^2\,dy\\ &= x_0^2y_0 + 2x_0y_0\,dx + x_0^2\frac{-p(x_0,y_0,v_0)}{q(x_0,y_0,v_0)}\,dx - x_0^2\frac{r(x_0,y_0,v_0)}{q(x_0,y_0,v_0)}\,dv\\ &= x_0^2y_0 + \left(2x_0y_0 - x_0^2\frac{p(x_0,y_0,v_0)}{q(x_0,y_0,v_0)}\right)dx - x_0^2\frac{r(x_0,y_0,v_0)}{q(x_0,y_0,v_0)}\,dv,\\ v &= v_0 + dv. \end{aligned}$$

Locally this map is given by

$$\begin{bmatrix} x & x^2y & v\\ 1 & 2xy - x^2\dfrac{p(x,y,v)}{q(x,y,v)} & 0\\ 0 & -x^2\dfrac{r(x,y,v)}{q(x,y,v)} & 1 \end{bmatrix}. \tag{3.1}$$

Since $\mathcal{F}$ is not free at P, we apply the process of [D] (described previously). As mentioned before, the smallest nonzero Fitting ideal of $\mathcal{F}$ is the maximal ideal of P. We therefore blow up along the maximal ideal of P, pull back $\mathcal{E}$, $\mathcal{F}$, and σ, and then take their double duals. This gives the map $\sigma' \colon \mathcal{E}' \to \mathcal{F}'$. We have two patches to consider.

On the first patch we have the relation $v = xv$. The result of pulling back (3.1) is

$$\begin{bmatrix} x & x^2y & xv\\ 1 & 2xy - x^2\dfrac{p(x,y,xv)}{q(x,y,xv)} & 0\\ 0 & -x^2\dfrac{r(x,y,xv)}{q(x,y,xv)} & 1 \end{bmatrix}.$$

Yet on this patch the relation $0 = v\,dx + x\,dv$, after taking the double dual of the pull-back of $\mathcal{F}$, becomes $0 = v\,dx + dv$. Thus the map σ' is given locally by

$$\begin{bmatrix} x & x^2y & xv \\ 1 & 2xy - x^2\dfrac{p(x,y,xv)}{q(x,y,xv)} + x^2v\dfrac{r(x,y,xv)}{q(x,y,xv)} & -v \end{bmatrix}. \tag{3.2}$$

The ideal generated by the 2×2 minor determinants in

$$\frac{\mathbb{C}[x,y,v,y^{-1}]}{\left(y^4 + y^3\sum_{i+j=3}\alpha_{i,j}x^j + y^2\sum_{i+j=4}\beta_{i,j}x^{j+2} - ax^4yv^4 - bx^6v^6\right)}$$

is

$$\left(xv, x^2\left(y - x\frac{p(x,y,xv)}{q(x,y,xv)}\right)\right).$$

In the complete local ring at any point of this patch along the exceptional divisor, this ideal becomes

$$(xv, x^2).$$

If $v \neq 0$, then this ideal is simply (x), and we see that $D_1(\sigma')$ is reduced.

On the second patch we have the relation $x = xv$. The result of pulling back (3.1) is

$$\begin{bmatrix} xv & x^2yv^2 & v \\ 1 & 2xyv - x^2v^2\dfrac{p(xv,y,v)}{q(xv,y,v)} & 0 \\ 0 & -x^2v^2\dfrac{r(xv,y,v)}{q(xv,y,v)} & 1 \end{bmatrix}.$$

But on this patch the relation $0 = v\,dx + x\,dv$, after taking the double dual of the pull-back of $\mathcal{F}$, becomes $0 = dx + x\,dv$. Thus the map σ' is given locally by

$$\begin{bmatrix} xv & x^2yv^2 & v \\ -x & -x^2v^2\dfrac{r(xv,y,v)}{q(xv,y,v)} - 2x^2yv + x^3v^2\dfrac{p(xv,y,v)}{q(xv,y,v)} & 1 \end{bmatrix}. \tag{3.3}$$

The ideal generated by the 2×2 minor determinants in

$$\frac{\mathbb{C}[x,y,v,y^{-1}]}{\left(y^4 + y^3\sum_{i+j=3}\alpha_{i,j}x^jv^j + y^2\sum_{i+j=4}\beta_{i,j}x^{j+2}v^{j+2} - ayv^4 - bv^6\right)}$$

is (xv). This shows that, at $(0,1,0)$, $D_1(\sigma')$ is simply the union of the exceptional divisor and E.

Combining this with (3.2) completes the proof of Theorem 3.5. □

4. Excess Porteous

Let $\pi : X \to B$ be a generic 1-parameter family of smooth, nonhyperelliptic curves of genus 3 degenerating to a general element of Δ_1; let E and C be the elliptic and genus-2 curves, respectively, meeting transversely at P. Let $\sigma : \mathcal{E} \to \mathcal{F}$ be the map of coherent sheaves described in Section 2.

Let $g\colon X' \to X$ be the blow-up of X at the maximal ideal of P, with E_0 the exceptional divisor, and let $\sigma'\colon \mathcal{E}' \to \mathcal{F}'$ be the map of vector bundles on X' described in Section 2.

Let $D = D_1(\sigma') = \{x \in X' \mid \operatorname{rank}(\sigma'_x) \le 1\}$. The expected codimension of D is $(3-1)(2-1) = 2$, but the contribution to D of the central fiber is the union of E_0, E, and the six hyperelliptic Weierstrass points of C (i.e., $Q_1, \dots, Q_6$). We'd like to compute the class $\mathbb{D}_1(\sigma')$ of D in $A_0(D)$; since the codimension of D is less than 2, we use the excess Porteous formula [F, Exm. 14.4.7]. Specifically, since $D_0(\sigma') = \emptyset$, there exist vector bundles $\mathcal{K}$ and $\mathcal{C}$ (of ranks 2 and 1, respectively) on D as well as an exact sequence

$$0 \to \mathcal{K} \to \mathcal{E}'_D \to \mathcal{F}'_D \to \mathcal{C} \to 0,$$

where $\mathcal{E}'_D$ and $\mathcal{F}'_D$ are the restrictions of $\mathcal{E}'$ and $\mathcal{F}'$ to D. Then

$$\mathbb{D}_1(\sigma') = \{c(\mathcal{K}^\vee \otimes \mathcal{C}) \cap s(D, X')\}_0.$$

Proposition 4.1. *Let P_1 and P_2 be as in the proof of Proposition 2.2; let S_1, S_2, and S_3 be as in Theorem 3.2; and let Q_i, $i = 1, \dots, 6$, be the hyperelliptic Weierstrass points of C. Then*

$$s(D, X') = [E_0] + [E] + \sum_{i=1}^{6} [Q_i] + \sum_{i=1}^{3} [S_i] + [P_1] + [P_2].$$

In particular, we have

$$\int_D s(D, X') = 11.$$

Proof. Let $f\colon \tilde{X}' \to X'$ be the blow-up of X' along $\mathcal{I}_D$ (the ideal sheaf of D). By [Ha, Exm. II.7.11(b)] we can identify this blow-up with the blow-up along $Q_1 \cup \dots \cup Q_6 \cup S_1 \cup S_2 \cup S_3 \cup P_2$. Let E' be the exceptional divisor of this blow-up and let Q'_i, S'_i, and P'_2 be the respective components lying above Q_i, S_i, and P_2. Then $D' := f^{-1}(D) = f^*(E_0 + E) + E'$. Let $h\colon D' \to D$ be the projection. By [F, Cor. 4.2.2] we have

$$\begin{aligned}
s(D, X') &\\
&= h_*[D'] - h_*(D' \cdot [D'])\\
&= [E_0] + [E]\\
&\quad - h_*(f^*(E_0 + E) \cdot [f^*(E_0 + E)] + 2f^*(E_0 + E) \cdot [E'] + E' \cdot [E'])\\
&= [E_0] + [E] - (E_0 + E) \cdot [E_0 + E]\\
&\quad - h_*(2f^*(E_0 + E) \cdot [E'] + E' \cdot [E'])\\
&= [E_0] + [E] - (E_0 + E) \cdot [E_0 + E]\\
&\quad + \sum_{i=1}^{6} [Q_i] + \sum_{i=1}^{3} [S_i] + [P_2] - h_*(2f^*(E_0 + E) \cdot [E'])\\
&= [E_0] + [E] - (E_0 + E) \cdot [E_0 + E] + \sum_{i=1}^{6} [Q_i] + \sum_{i=1}^{3} [S_i] + [P_2].
\end{aligned}$$

The second and third equalities follow from [F, Prop. 2.3(c)]. The fourth equality holds because each component of E' is a rational curve with self-intersection -1 and disjoint from the other components. The last equality is shown as follows:

$$\begin{aligned} f^*(E_0+E)\cdot[E'] &= (E_0+E+S'_1+S'_2+S'_3+P'_2) \\ &\qquad \cdot [Q'_1+\cdots+Q'_6+S'_1+S'_2+S'_3+P'_2] \\ &= E_0\cdot[P'_2]+\sum_{i=1}^{3} E\cdot[S'_i]+\sum_{i=1}^{3} S'_i\cdot[S'_i]+P'_2\cdot[P'_2] \\ &= (E_0+P'_2)\cdot[P'_2]+\sum_{i=1}^{3}(E+S'_i)\cdot[S'_i] \\ &= 0, \end{aligned}$$

where by abuse of notation we identify E_0 and E with their proper transforms in D'. Furthermore, we have

$$\begin{aligned} E_0\cdot[E_0] &= -[P_1], \\ E_0\cdot[E] &= [P_1], \\ E\cdot[E] &= -2[P_1]. \end{aligned}$$

Consequently,

$$(E_0+E)\cdot([E_0+E]) = E_0\cdot[E_0]+2E\cdot[E_0]+E\cdot[E] = -[P_1]. \qquad \square$$

In order to determine the equivalence of $\mathbb{D}_1(\sigma')$, we need only look at the intersection $c_1(\mathcal{K}^\vee\otimes\mathcal{C})\cap([E_0]+[E_1])$. By [F, Exm. 3.2.2] we have

$$c_1(\mathcal{K}^\vee\otimes\mathcal{C}) = 2c_1(\mathcal{C})+c_1(\mathcal{K}^\vee) = 2c_1(\mathcal{C})-c_1(\mathcal{K}).$$

Also, from the exact sequence

$$0\to\mathcal{K}\to\mathcal{E}'_D\to\mathcal{F}'_D\to\mathcal{C}\to 0$$

we obtain the relation

$$c_1(\mathcal{K})+c_1(\mathcal{F}'_D) = c_1(\mathcal{E}'_D)+c_1(\mathcal{C}).$$

Combining then yields

$$c_1(\mathcal{K}^\vee\otimes\mathcal{C}) = c_1(\mathcal{C})+c_1(\mathcal{F}'_D)-c_1(\mathcal{E}'_D).$$

Moreover, $\mathcal{E}'_D$ is the trivial bundle on both E_0 and E, so $c_1(\mathcal{E}'_D)=0$.

Since $\mathcal{F}'_D\to\mathcal{C}$ is surjective, its kernel is a vector bundle (of rank 1). But the kernel of this map is the image of $\mathcal{E}'_D\to\mathcal{F}'_D$, which we will call $\mathcal{A}$. Thus we have

$$c_1(\mathcal{C})+c_1(\mathcal{A}) = c_1(\mathcal{F}'_D).$$

If we combine this equality with our previous results, then

$$c_1(\mathcal{K}^\vee\otimes\mathcal{C}) = 2c_1(\mathcal{F}'_D)-c_1(\mathcal{A}).$$

The proof of [D, Lemma 5] immediately generalizes to our situation to show that

$$c_1(\mathcal{F}'_D) = 3\gamma_D-E_0,$$

where $\gamma_D = c_1(\omega_{X/B}) \cdot D$. The restriction of $\omega_{X/B}$ to E is $K_1(P_1)$, where K_1 is the canonical bundle on E, so γ_D has degree 1 on this curve; the restriction of $\omega_{X/B}$ to C is $K_2(P_2)$, where K_2 is the canonical bundle on C, so γ_D has degree 3 on this curve. Because the degree of γ_D on any member of the family is 4, the degree on E_0 must be 0. Also, we see that $\#(E_0 \cdot [E_0]) = -1$ and $\#(E_0 \cdot [E]) = 1$. Thus we have the following statement.

Proposition 4.2.

$$\int_D 2c_1(\mathcal{F}'_D) \cap ([E_0] + [E]) = 2(3\gamma_D - E_0) \cdot ([E_0] + [E]) = 6.$$

It remains to determine $c_1(\mathcal{A}) \cap ([E_0] + [E])$. For this we use the following two propositions.

Proposition 4.3. *We have*

$$\mathcal{A}_{E_0} \cong \mathcal{O}(1).$$

Proof. We consider the map $\mathcal{E}' \to \mathcal{F}'$ on E_0 given by restricting the matrices (3.2) and (3.3) to E_0. On one affine patch of E_0 we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -v \end{bmatrix}$$

and on the other

$$\begin{bmatrix} 0 & 0 & 0 \\ -x & 0 & 1 \end{bmatrix},$$

where v and x are affine parameters on their respective patches. On the first patch $v = 0$ is a local equation for C, and on the second $x = 0$ is a local equation for E. Since C and E both meet E_0 transversely, we see that s_1 and s_3 map to sections with a simple zero on E_0. □

Let $0 \to \mathcal{F}'_2 \to \mathcal{F}' \to \mathcal{F}'_1 \to 0$ be the exact sequence given in [D, Lemma 5]. The proof of that lemma generalizes to show that this remains a filtration of $\mathcal{F}'$ in our case.

Proposition 4.4. *We have $\mathcal{A}_E \cong \mathcal{O}_E$. In particular, $c_1(\mathcal{A}) \cap E = 0$.*

Proof. We consider the composition of maps $\mathcal{E}' \to \mathcal{F}' \to \mathcal{F}'_1$. For a point $x \in E$ this map takes sections of the relative dualizing sheaf, expands them about a local coordinate at x, first maps them to the constant and linear term, and then maps them to the constant term. We can choose a basis for the sections of the dualizing sheaf such that two of them (when expanded about $x \in E$) are 0 and the other is a section of $K_1(P_1)$, where K_1 is the canonical bundle of E. Thus $\mathcal{E}'_x \to (\mathcal{F}'_1)_x$ is surjective if and only if the section on E fails to vanish at x. This is the case for all $x \neq P_1$. Considering the map $\mathcal{E}' \to \mathcal{F}'_1(-P_1)$ shows that, away from P_1, this is the same as before. But at P_1 this map will take the linear term of the section of the dualizing sheaf, instead of the constant term, and so is surjective at all points of E_1. Hence $\mathcal{A}_E \to (\mathcal{F}'_1)_E(-P_1)$ is a surjective map of vector bundles of the same rank and thus is an isomorphism. But $\mathcal{F}_1'$ is simply the relative dualizing sheaf of the family, so $\mathcal{F}_1'$ is $K_1(P_1)$ on E. Therefore, $\mathcal{A}_E \cong K_1 = \mathcal{O}_E$. □

From these two propositions, we see that

$$c_1(\mathcal{A}) \cap ([E_0] + [E]) = c_1(\mathcal{O}(1)) \cap [E_0],$$
$$\#(c_1(A) \cap ([E_0] + [E])) = 1.$$

These equalities yield the following result.

THEOREM 4.5.
$$\int_D \mathbb{D}_1(\sigma') = 16.$$

Proof. From the foregoing it follows that

$$\int_D s(D, X') = 11,$$
$$\int_D 2c_1(\mathcal{F}_D') \cap ([E_0] + [E]) = 6,$$
$$\int_D c_1(\mathcal{A}) \cap ([E_0] + [E]) = 1.$$

Thus

$$\int_D \mathbb{D}_1(\sigma') = 11 + 6 - 1 = 16.$$ □

5. The Hyperelliptic Locus in $\overline{\mathfrak{M}}_3$

Let H be the hyperelliptic locus in $\mathfrak{M}_3$ and $\bar{H}$ its closure in $\overline{\mathfrak{M}}_3$. Let $\bar{h} \in \mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3)$ be the rational divisor class on the moduli stack associated to $\bar{H}$ by [HM, Prop. 3.88]. We wish to combine Theorem 4.5 with [D, Lemma 5] to obtain an expression for $\bar{h}$ in terms of the generators λ, δ_0, and δ_1 of $\mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3)$.

Let $\pi : X \to B$ be a generic 1-parameter family of stable curves of genus 3. Let $\sigma' \colon \mathcal{E}' \to \mathcal{F}'$ be the map described before on X', where $g \colon X' \to X$ is the blow-up along the nodes of singular fibers of π. Applying the standard Thom–Porteous formula now gives

$$[D_1(\sigma')] = c_2(\mathcal{E}'^* - \mathcal{F}'^*).$$

From the proof of [D, Lemma 5], we have

$$c(\mathcal{E}'^*) = 1 - \lambda \quad \text{and}$$
$$c(\mathcal{F}') = 1 + 3\gamma - E_0 + 2\gamma^2,$$

where E_0 is the exceptional divisor of the blow-up $g \colon X' \to X$. This gives

$$[D_1(\sigma')] = 7\omega^2 - 3\omega\lambda + E_0^2.$$

But there is one component of E_0 for each fiber of π from δ_0 as well as one component for each fiber from δ_1, and each component has square -1. Hence

$$(\pi g)_*([D_1(\sigma')]) = 7\kappa - 12\lambda - \delta_0 - \delta_1.$$

We use $\lambda = (\kappa + \delta_0 + \delta_1)/12$ from [HM, 3.110] to obtain

$$(\pi g)_*([D_1(\sigma')]) = 72\lambda - 8\delta_0 - 8\delta_1.$$

We observe that generic members of Δ_1 are not contained in $\bar{H}$, but by Theorem 4.5 we know that the standard Thom–Porteous formula will count generic members of Δ_1 16 times each. Hence we need to subtract $16\delta_1$ from the previous result, which gives

$$72\lambda - 8\delta_0 - 24\delta_1.$$

Since each smooth hyperelliptic curve contains eight hyperelliptic Weierstrass points, we divide through by 8 to obtain

$$\bar{h} = 9\lambda - \delta_0 - 3\delta_1.$$

This agrees with the Harris–Morrison result [HM, p. 188].

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