Michigan Math. J. 61 (2012), 279–296

Quadratic Involutions on Binary Forms

Abdelmalek Abdesselam
& Jaydeep Chipalkatti

1. Introduction

Given a smooth conic C in the projective plane $\mathbb{P}^2$, a point in $\mathbb{P}^2 \setminus C$ will define an involution (i.e., a degree-2 automorphism) on C. Several familiar objects in the invariant theory of binary forms (such as the quartic catalecticant, or the Hermite invariant) can be defined as sets of divisors that are fixed by such an involution. In this paper we study a wide class of such involutions and the corresponding fixed loci.

We begin with an elementary introduction to the subject. The main results are described in Section 2.4 after the required notation is available. Many of the proofs involve elaborate calculations using the graphical or symbolic method, as in [2; 3]. We have omitted them from this paper for the sake of brevity. Instead, the reader is referred to [5], which is the online archival version of this paper, for complete proofs of all the results announced here. We have made an attempt to ensure that such a division of the text does not lead to a breach of continuity.

The reader may consult [9; 11; 16] for classical introductions to the invariant theory of binary forms and [7; 14; 15; 17] for more modern accounts.

1.1. Representations of SL_2

Throughout, the base field will be $\mathbb{C}$ (complex numbers). Let V denote a two-dimensional $\mathbb{C}$-vector space. For a nonnegative integer m, let $S_m = \mathrm{Sym}^m V$ denote the mth symmetric power. If $x = \{x_1, x_2\}$ is a basis of V, then S_m can be identified with the space of binary forms of order m in the variables x. The $\{S_m : m \geq 0\}$ are a complete set of finite-dimensional irreducible representations of $\mathrm{SL}(V)$ (see e.g. [13, Sec. I.9]).

Following a notation introduced by Cayley, we will write the binary form $\sum_{i=0}^{d} a_i \binom{d}{i} x_1^{d-i} x_2^i$ as $(a_0, \dots, a_d \between x_1, x_2)^d$.

1.2. Transvectants

Given integers $m, n \geq 0$ and $0 \leq r \leq \min(m, n)$, we have a transvectant morphism (see [4])

$$S_m \otimes S_n \to S_{m+n-2r}.$$

Received October 11, 2010. Revision received February 7, 2011.

If A, B are binary forms of orders m, n respectively, the image of $A \otimes B$ via this morphism is called their rth transvectant; it will be denoted by $(A, B)_r$. We have an explicit formula

$$(A, B)_r = \frac{(m-r)!\,(n-r)!}{m!\,n!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r A}{\partial x_1^{r-i} \partial x_2^i} \frac{\partial^r B}{\partial x_2^{r-i} \partial x_1^i}.$$

In the notation of symbolic calculus, if $A = \alpha_x^m$ and $B = \beta_x^n$, then $(A, B)_r = (\alpha\beta)^r \alpha_x^{m-r} \beta_x^{n-r}$.

1.3

Consider the Veronese embedding

$$v \colon \mathbb{P}V \to \mathbb{P}S_2, \quad [\ell] \to [\ell^2];$$

whose image C is a smooth conic in $\mathbb{P}^2$. If $(a_0, a_1, a_2 \between x_1, x_2)^2$ is identified with $[a_0, a_1, a_2] \in \mathbb{P}^2$, then C is defined by the equation $a_1^2 = a_0 a_2$.

Fix a point $q \in \mathbb{P}^2 \setminus C$. Given $t \in C$, draw the line $\overline{qt}$, and let $\sigma_q(t)$ denote the other point where the line intersects C. This defines an order-2 automorphism

$$\sigma_q \colon C \to C, \quad t \to \sigma_q(t).$$

The two intersection points of C with the polar line of q are the fixed points of this automorphism.

Assume that q corresponds to $Q = (q_0, q_1, q_2 \between x_1, x_2)^2 \in S_2$. Define

$$\Delta_Q = -2(Q, Q)_2 = 4(q_1^2 - q_0 q_2).$$

(The normalization is chosen such that $\Delta_{x_1 x_2} = 1$.) We have $\Delta_Q \neq 0$, since $[Q] \notin C$. We will merely write Δ for Δ_Q if no confusion is likely.

LEMMA 1.1. *If the point $t \in C$ corresponds via v to $\ell \in V$, then $\sigma_q(t)$ corresponds to $(Q, \ell)_1$.*

Proof. It is enough to show that the three forms $Q, \ell^2, ((Q, \ell)_1)^2$ are linearly dependent, and hence the corresponding points in $\mathbb{P}^2$ are collinear. By SL_2-equivariance, we may take $\ell = x_1$. Then $(Q, x_1)_1 = -(q_1 x_1 + q_2 x_2)$, and it is immediate that $((Q, x_1)_1)^2 - q_2 Q = (q_1^2 - q_0 q_2) x_1^2$, which proves the claim. $\square$

1.4

If $\ell \in V$, then a simple calculation shows the identity

$$(Q, (Q, \ell)_1)_1 = \frac{1}{4} \Delta \ell. \tag{1}$$

Let $F \in S_d$ be a nonzero binary d-ic that factors into linear forms as $F = \prod_{i=1}^{d} \ell_i$. Define

$$\sigma_Q(F) = 2^d \prod_{i=1}^{d} (Q, \ell_i)_1. \tag{2}$$

By formula (1),

$$\sigma_Q^2(F) = \Delta^d F. \tag{3}$$

Now F corresponds to the divisor $A = \sum v([\ell_i])$ on C and $\sigma_Q(F)$ to its image $\sigma_q(A) = \sum \sigma_q([\ell_i^2])$. The divisor A is said to be *in involution* with respect to q if $\sigma_q(A) = A$. Figure 1 illustrates typical divisors in involution for orders 6 and 7.

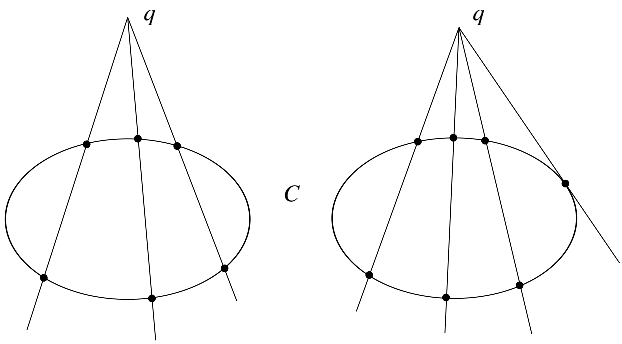

Figure 1

1.5

Define

$$\mathcal{X}_d^\circ = \{A \in \mathbb{P}S_d : A \text{ is in involution with respect to some } q \notin C\}, \tag{4}$$

and let $\mathcal{X}_d \subseteq \mathbb{P}S_d$ denote its Zariski closure. It is an SL_2-invariant irreducible projective subvariety of $\mathbb{P}^d$.

If $d \leq 4$, then given a general set of d points on C, one can always complete the diagram to find a q; hence $\mathcal{X}_d = \mathbb{P}^d$. Assume $d \geq 5$. If $d = 2n$, then a typical point in $\mathcal{X}_d^\circ$ can be constructed as follows: choose an arbitrary point q away from C and a degree-n divisor B, and let $A = B + \sigma_q(B)$. If $d = 2n + 1$, then choose q, B as before, together with any one point $t \in C$ such that $\overline{qt}$ is tangent to C, and let $A = B + \sigma_q(B) + t$. In either case, a parameter count shows that $\dim \mathcal{X}_d = n + 2$.

The hypersurfaces $\mathcal{X}_5$ and $\mathcal{X}_6$ are respectively of degrees 18 and 15. Their defining equations are given by well-known skew invariants of binary quintics and sextics (see Sections 4.10 and 4.11). No such equations seem to be known for higher values of d.

1.6. Canonical Form

Let us write $\square$ for an arbitrary scalar that need not be precisely specified. Suppose that the divisor A is in involution with respect to $q \in \mathbb{P}^2 \setminus C$. Then A is a sum of pairs of the form $t + \sigma_q(t)$ together with some points w such that $w = \sigma_q(w)$.

By a change of variables, we may assume that $q = [Q]$ for $Q = x_1 x_2$. If $\ell = l_1 x_1 + l_2 x_2$, then $(Q, \ell)_1 = -\frac{1}{2}(l_1 x_1 - l_2 x_2)$. If $\ell = \Box (Q, \ell)_1$, then $\ell = \Box x_1$ or $\Box x_2$. If $\ell \neq \Box (Q, \ell)_1$, then $\ell (Q, \ell)_1$ is a form in x_1^2 and x_2^2. Alternately, $\Box x_1^2 + \Box x_2^2$ can be factored as $\ell (Q, \ell)_1$. We have proved the following.

PROPOSITION 1.2. *A divisor A is in involution if and only if, up to a linear change of variables, it corresponds to a binary form that can be written as*

$$x_1^r x_2^s \times \text{a form in } x_1^2 \text{ and } x_2^2$$

for some r, s.

Later we will state a similar result for a more general class of involutions.

2. The System $\mathfrak{S}(d)$

2.1

We begin by generalizing the map σ_Q and subsequently the notion of an involution. The first step is to rewrite the expression (2) for $\sigma_Q(F)$ in terms of transvectants of only Q and F without involving the factors of F. This is done in [5, Sec. 6]; here we merely state the result.

Let $n = \lfloor d/2 \rfloor$. Then we have an expansion

$$\sigma_Q(F) = \sum_{i=0}^{n} g_i \Delta^i (Q^{d-2i}, F)_{d-2i}, \tag{5}$$

where

$$g_i = \frac{2^{d-2i} \cdot d!\,(d-i)!\,(2d-4i+1)!}{i!\,(d-2i)!^2\,(2d-2i+1)!}.$$

Since the construction of $\sigma_Q(F)$ is covariant in Q and F, a result of Gordan (see [11, Sec. 103]) implies that it should be expressible in terms of compound transvectants of the two arguments. Thus, one knows a priori that an identity such as (5) should exist for *some* rational numbers g_i.

2.2

All of this suggests the following construction. Let $z = (z_0, \dots, z_n)$ be a sequence of complex numbers, and consider the function

$$\sigma_{Q,z} \colon S_d \to S_d, \quad F \to \sum_{i=0}^{n} z_i \Delta^i (Q^{d-2i}, F)_{d-2i}.$$

One should like to write a set of equations in z_i that encodes the condition that $\sigma_{Q,z}$ be involutive. Now,

$$\begin{aligned}\sigma_{Q,z}^2(F) &= \sum_{i=0}^{n} z_i \Delta^i \sum_{j=0}^{n} z_j \Delta^j (Q^{d-2j}, (Q^{d-2i}, F)_{d-2i})_{d-2j} \\ &= \sum_{0 \le i, j \le n} z_i z_j \Delta^{i+j} \underbrace{(Q^{d-2j}, (Q^{d-2i}, F)_{d-2i})_{d-2j}}_{(\star)}.\end{aligned}$$

It is possible to rewrite $(\star)$ by expanding the compound transvectant expression $(Q^\bullet, (Q^\bullet, F))$ into a sum of terms of the form $\Delta^\bullet(Q^\bullet, F)$. Such a general formula is given in Section 6, where the reader will find the definition of the rational numbers ω that are needed below. The end result is as follows. There is an expansion

$$(\star) = \sum_t \alpha_{i,j}^{(t)} \Delta^{d-i-j-t/2}(Q^t, F)_t, \tag{6}$$

where the sum is quantified over all even t in the range

$$2|i-j| \leq t \leq \min\{d, 2(d-i-j)\} \tag{7}$$

and we have written

$$\alpha_{i,j}^{(t)} = \omega(d-2j, d-2i; d-2i, d-2j; t)$$

for brevity. If t does not lie in the range (7), then define $\alpha_{i,j}^{(t)}$ to be zero. In particular, $\alpha_{i,j}^{(0)} = 0$ unless $i = j$. Thus

$$\sigma_{Q,z}^2(F) = \sum_{\substack{t=0\\ t \text{ even}}}^{d} \left\{ \Delta^{d-t/2} \left(\sum_{0 \leq i,j \leq n} \alpha_{i,j}^{(t)} z_i z_j \right) (Q^t, F)_t \right\}.$$

Now we require that the coefficient of $\Delta^{d-t/2}$ be 1 for $t = 0$ and vanish for $t \neq 0$, which would force

$$\sigma_{Q,z}^2(F) = \Delta^d F. \tag{8}$$

This gives the system of n homogeneous quadratic equations

$$\sum_{0 \leq i,j \leq n} \alpha_{i,j}^{(t)} z_i z_j = 0, \quad t = 2, 4, \ldots, 2n, \tag{9}$$

together with the condition

$$\sum_{i=0}^{n} \alpha_{i,i}^{(0)} z_i^2 = 1. \tag{10}$$

The combined set (9) and (10) will be denoted by $\mathfrak{S}(d)$. For instance, the system $\mathfrak{S}(6)$ consists of

$$\left.\begin{array}{c} -\dfrac{25}{20328} z_0^2 + \dfrac{5}{3234} z_0 z_1 - \dfrac{1}{2058} z_1^2 + \dfrac{22}{735} z_1 z_2 + \dfrac{11}{210} z_2^2 + 2 z_2 z_3 \\ \dfrac{5}{1331} z_0^2 - \dfrac{15}{847} z_0 z_1 + \dfrac{5}{121} z_0 z_2 - \dfrac{69}{5390} z_1^2 - \dfrac{2}{77} z_1 z_2 + 2 z_1 z_3 + \dfrac{2}{5} z_2^2 \\ -\dfrac{5}{2541} z_0^2 + \dfrac{4}{165} z_0 z_1 - \dfrac{7}{33} z_0 z_2 + 2 z_0 z_3 - \dfrac{1}{35} z_1^2 + \dfrac{2}{15} z_1 z_2 \end{array}\right\} = 0$$

together with

$$\frac{1}{6468} z_0^2 + \frac{11}{22050} z_1^2 + \frac{1}{75} z_2^2 + z_3^2 = 1.$$

2.3

A sequence $z = (z_0, \dots, z_n)$ will be called an involutor if it satisfies $\mathfrak{S}(d)$. In particular, $g = (g_0, \dots, g_n)$ will be called the *geometric* involutor. It is clear that if z is an involutor, then so is $-z = (-z_0, \dots, -z_n)$. If d is even, then $(0, \dots, 0, \pm 1)$ will be called the *improper* involutors. (In these cases $\sigma_{Q,z}$ merely multiplies F by a scalar; i.e., it is the identity map at the level of divisors on C.)

2.4. Summary of Results

Since (9) is a system of n homogeneous quadratic equations in $n+1$ variables, Bézout's theorem implies that the number of homogeneous solutions, if finite, should be at most 2^n. After imposing condition (10), one expects at most 2^{n+1} affine solutions. We programmed the system $\mathfrak{S}(d)$ in Maple and found that, for the first few values of d, there are always precisely 2^{n+1} involutors. This prompted us to ask whether it should be possible to write down all the solutions to $\mathfrak{S}(d)$. We will carry this out in Section 3. It turns out that an involutor arises naturally from a *sign sequence* (see Definition 3.1), and moreover there is an explicit bijection between sign sequences and involutors. This is formally stated in Theorem 3.2.

Given an involutor z, one can define a subvariety $\mathcal{Y}(z) \subseteq \mathbb{P}S_d$ in analogy with Section 1.4. We initiate a study of these varieties in Sections 4 and 5. In particular, Theorem 4.1 will show that a point in $\mathcal{Y}(z)$ admits a canonical form that can be read off from the corresponding sign sequence. It is evident from the examples in Sections 4.4–4.7 and Section 5 that these varieties display a wide range of geometries depending on the sign sequence, and they furnish a great deal of matter for further study.

3. Sign Sequences and Involutors

3.1

In this section we will describe a complete classification of all involutors. The following definition and formulas are justified by Theorem 3.2. Recall that $n = \lfloor d/2 \rfloor$.

Definition 3.1. A sign sequence for d is of the form $s = (s_0, s_1, \dots, s_d)$, where $s_i = \pm 1$ and $s_{d-i} = (-1)^d s_i$.

It is clear that the segment $(s_0, \dots, s_n)$ can be made up arbitrarily, and then it determines the rest. Thus there are 2^{n+1} sign sequences for d. We will write $\pm$ for ± 1 if no confusion is likely. Define γ to be the alternating sign sequence $(-, +, \dots, -, +)$ if d is odd and $(+, -, +, \dots, -, +)$ if d is even.

3.2

Given a sign sequence s and an index i such that $0 \le i \le n$, let

$$E_{1,i} = \frac{d!\,(2d-4i+1)!}{2^{2i-1}(d-2i)!^2}$$

and
$$E_{2,i} = \sum \{s_\ell m_\ell (-1)^q \xi(e, \ell, p, q)\}, \tag{11}$$
where
$$\xi(e, \ell, p, q) = \frac{(d-2e)!\,(d-i-e)!}{(2d-2i-2e+1)!\,(i-e)!\,p!\,q!\,(\ell-p)!\,(d-\ell-q)!}.$$
The sum in (11) is quantified over all integer quadruples (e, ℓ, p, q) such that
$$0 \le e \le i, \quad 0 \le \ell \le n, \quad 0 \le p \le \ell, \quad 0 \le q \le d-\ell, \quad p+q = d-2e.$$
Here m_ℓ is defined to be $\frac{1}{2}$ if $d = 2\ell$, and 1 otherwise. Now define
$$z_i(s) = E_{1,i} E_{2,i}. \tag{12}$$

THEOREM 3.2. *With notation as before, $z(s) = (z_0(s), \ldots, z_n(s))$ is an involutor. Moreover, every involutor arises in this way from a unique sign sequence.*

It follows that each involutor is defined over the rationals. The proof of the theorem is given in [5, Sec. 8]. The following proposition lists some properties of this correspondence.

PROPOSITION 3.3. *With notation as before*:

(i) *if $z = z(s)$ and s' is the sign sequence such that $s'_i = -s_i$, then $z(s') = -z$;*
(ii) *the geometric involutor corresponds to γ;*
(iii) *when d is even, the improper involutors correspond to the sequences $(+, +, \ldots, +)$ and $(-, -, \ldots, -)$.*

Part (i) is obvious from the definition of $E_{2,i}$. The rest is proved in [5, Sec. 9.1].

For instance, if $d = 4$, then $(+, -, -, -, +)$ corresponds to the involutor $(4, 48/7, -1/5)$ and $(+, -, +, -, +)$ corresponds to the geometric involutor $(16, 24/7, 1/5)$.

4. Varieties of Forms in Involution

4.1

Consider the product $\mathbb{P}S_2 \times \mathbb{P}S_d$ with projections π_1 and π_2 onto $\mathbb{P}S_2$ and $\mathbb{P}S_d$, respectively. Given an involutor z, define the locus $\widetilde{\mathcal{Y}}^\circ(z) \subseteq \mathbb{P}S_2 \times \mathbb{P}S_d$ as the set of pairs $\langle Q, F \rangle$ satisfying $\Delta_Q \neq 0$, and
$$\begin{aligned} \sigma_{Q,z}(F) &= \Delta^{d/2} F \quad \text{if } d \text{ is even}, \\ [\sigma_{Q,z}(F)]^2 &= \Delta^d F^2 \quad \text{if } d \text{ is odd}. \end{aligned} \tag{13}$$

If (13) holds, then we will say that $[Q] \in \mathbb{P}^2$ is a center of involution for $[F]$ with respect to z. For instance, in Figure 1, the point q is such a center with respect to γ for the divisors shown. However, if $z \neq \gamma$, then it is not clear to us whether there is any hidden "geometry" in the relationship defined by (13).

Define $\mathcal{Y}^\circ(z) \subseteq \mathbb{P}S_d$ to be the image $\pi_2(\widetilde{\mathcal{Y}}^\circ(z))$, and let $\mathcal{Y}(z) \subseteq \mathbb{P}S_d$ denote its Zariski closure. We may equally well denote these loci by $\mathcal{Y}(s), \ldots$ by referring to the sign sequence and further shorten them to $\widetilde{\mathcal{Y}}^\circ, \mathcal{Y}, \ldots$ if no confusion is likely.

It is clear that if d is odd, then $\mathcal{Y}(z) = \mathcal{Y}(-z)$. This may or may not hold for even d (see Section 4.7).

4.2. The Relation between $\mathcal{X}$ and $\mathcal{Y}$

Assume that A is a divisor on C in involution with respect to $q = [Q] \in \mathbb{P}^2 \setminus C$. If $F \in S_d$ represents A, then $\sigma_{Q,\gamma}(F) = cF$ for some constant c. Formula (3) in Section 1.4 implies that $c^2 = \Delta_Q^d$; that is, $c = \pm\Delta_Q^{d/2}$. Hence,

$$\mathcal{X}_d^\circ = \mathcal{Y}^\circ(\gamma) \cup \mathcal{Y}^\circ(-\gamma).$$

For odd d, this reduces to $\mathcal{X}_d^\circ = \mathcal{Y}^\circ(\gamma)$.

4.3. Canonical Forms

It turns out that an element in $\mathcal{Y}^\circ$ admits a canonical form analogous to Proposition 1.2. Let s be a sign sequence for d, and let $z = z(s)$. Assume $Q = x_1x_2$ after a change of variables, and consider the condition

$$\langle x_1x_2, F\rangle \in \widetilde{\mathcal{Y}}^\circ(z). \tag{14}$$

THEOREM 4.1. (i) *Assume d to be even. Then* (14) *holds if and only if F is a linear combination of terms in the set*

$$\{x_1^{d-i}x_2^i : s_i = 1\}.$$

(ii) *Assume d to be odd. Then* (14) *holds if and only if F is a linear combination of terms in either one of the sets*

$$\{x_1^{d-i}x_2^i : s_i = 1\} \quad \textit{or} \quad \{x_1^{d-i}x_2^i : s_i = -1\}.$$

For instance, if $s = (+,-,-,+,-,-,+)$, then such an F is of the form $\Box x_1^6 + \Box x_1^3x_2^3 + \Box x_2^6$. If $s = (+,-,-,+,+,-)$, then it is of the form

$$\Box x_1^5 + \Box x_1^2x_2^3 + \Box x_1x_2^4 \quad \text{or} \quad \Box x_2^5 + \Box x_2^2x_1^3 + \Box x_2x_1^4.$$

In general, $\mathcal{Y}^\circ(z)$ is a union of SL_2-orbits of such forms.

Proof of Theorem 4.1. See [5, Sec. 9.2]. □

4.4

Let $p(s)$ be the number of $+$ signs in s. If d is odd, then $p(s) = \frac{1}{2}(d+1)$. By Theorem 4.1, each fiber of the projection $\widetilde{\mathcal{Y}}^\circ \to \mathbb{P}S_2 \setminus C$ has dimension $p(s) - 1$. Hence

$$\dim \mathcal{Y} \le \min\{p(s)+1, d\}. \tag{15}$$

This inequality may be strict. For instance, let $d = 3$ and $s = (+,+,-,-)$. A typical element in $\mathcal{Y}$ can be written as $x_1^2\ell$ up to a change of variables, hence $\mathcal{Y} \subseteq \mathbb{P}^3$ is the discriminant surface of degree 4.

In general, if d is odd and

$$s = (\underbrace{+,\dots,+}_{\frac{d+1}{2}\text{ times}},-,\dots,-),$$

then the canonical form shows that a typical element in $\mathcal{Y}$ is a binary d-ic with a root of multiplicity $\frac{d+1}{2}$. Hence $\dim \mathcal{Y} = \frac{d+1}{2}$, and (15) is strict.

However, if $z = \gamma$, then (15) is an equality for $d \geq 3$. Indeed, $p(\gamma) = \frac{d+2}{2}$ or $\frac{d+1}{2}$ according to whether d is even or odd, and the right-hand side reduces to $\lfloor d/2 \rfloor + 2$.

4.5

Let $d = 4$. A binary quartic F has covariants

$$A_F = (F, F)_4, \qquad B_F = (F, (F, F)_2)_4.$$

(See [11, Sec. 89]; however, our notation differs from theirs.) Usually, B_F is called the catalecticant. If F has distinct roots, its j-invariant is defined to be (cf. [11, Sec. 171])

$$j(F) = \frac{A_F^3}{A_F^3 - 6B_F^2}.$$

Recall that two binary quartics (with distinct roots) are in the same SL_2-orbit exactly when they have the same j-invariant (see e.g. [12, Exm. 10.12]). Now let

$$s = (+, -, -, -, +), \qquad t = (-, +, -, +, -),$$

so the canonical forms are respectively

$$G_s = \Box x_1^4 + \Box x_2^4, \qquad G_t = x_1 x_2(\Box x_1^2 + \Box x_2^2).$$

A straightforward calculation shows that $j(G_s) = j(G_t) = 1$, hence $\mathcal{Y}(s) = \mathcal{Y}(t)$ is the cubic hypersurface defined by the equation $B_F = 0$. (Classically, these were called the harmonic binary quartics.)

In general, for even d and $s = (+, -, \ldots, -, +)$, the variety $\mathcal{Y}(s)$ is the chordal threefold (union of secant lines) of the rational normal d-ic curve.

4.6

Let d be even, and let $s = (-, \ldots, -, +, -, \ldots, -)$. The canonical form is $(x_1 x_2)^{d/2}$; in other words, $\mathcal{Y}$ is the variety of d-ics that are expressible as powers of quadratic forms. It is shown in [3] that its ideal is generated by (an explicitly given) list of cubic polynomials.

4.7

Let $s = (+, -, -, -, +)$ and $s' = (-, +, +, +, -)$. A general quartic can be written as $x_1 x_2(\Box x_1^2 + \Box x_1 x_2 + \Box x_2^2)$ up to a change of variables, hence $\mathcal{Y}(s') = \mathbb{P}^4$. Since $\mathcal{Y}(s)$ is a threefold, $\mathcal{Y}(s) \neq \mathcal{Y}(s')$.

By contrast, let $d = 2$ and $u = (+, -, +)$, $u' = (-, +, -)$. Then $\mathcal{Y}(u) = \mathcal{Y}(u') = \mathbb{P}^2$.

4.8

In general, one should like to know formulas for the dimension and degree of $\mathcal{Y}$ as a function of s; moreover, it would be of interest to be able to write down a set of SL_2-equivariant defining equations for $\mathcal{Y}$. Our next example shows (if nothing else) that such questions can be rather involved.

Let $d = 6$ and $s = (-,-,+,-,+,-,-)$. Up to a change of variables, a form in $\mathcal{Y}^\circ(s)$ can be written as $G_s = x_1^2 x_2^2(x_1^2 + x_2^2)$. This corresponds to the divisor $2[\ell_1] + 2[\ell_2] + [\ell_3] + [\ell_4]$, where

$$\ell_1 = x_1, \quad \ell_2 = x_2, \quad \ell_3 = x_1 + \sqrt{-1}x_2, \quad \ell_4 = x_1 - \sqrt{-1}x_2.$$

Thus we can describe $\mathcal{Y}^\circ$ as the set of divisors $2p_1 + 2p_2 + p_3 + p_4$ on the conic C such that the intersection point $[Q]$ of tangents at p_1, p_2 falls on the line $\overline{p_3 p_4}$. The entire configuration is determined by Q and an arbitrary line through it, hence $\dim \mathcal{Y} = 3$. Let

$$B = \mathrm{Sym}^\bullet(S_6) = \bigoplus_{r \geq 0} \mathrm{Sym}^r(S_6)$$

denote the coordinate ring of $\mathbb{P}S_6$. (Since each S_m is canonically isomorphic to its dual, here it is unnecessary to distinguish between the two.) One can computationally determine the defining ideal $J \subseteq B$ of $\mathcal{Y} \subseteq \mathbb{P}^6$ as follows. Introduce indeterminates $\alpha, \beta, \gamma, \delta$, and let $\hat{G}(x_1, x_2)$ denote the form obtained by making simultaneous substitutions

$$x_1 \to \alpha x_1 + \beta x_2, \qquad x_2 \to \gamma x_1 + \delta x_2$$

into G_s. Now write $(a_0, \ldots, a_6 \between x_1, x_2)^6 = \hat{G}(x_1, x_2)$ and equate the coefficients of $x_1^{6-i} x_2^i$ on both sides. This expresses each a_i in terms of $\alpha, \ldots, \delta$ and defines a ring morphism

$$f : \mathbb{C}[a_0, \ldots, a_6] \to \mathbb{C}[\alpha, \beta, \gamma, \delta].$$

Then $J = \ker f$. (This captures the idea that J is the set of forms that vanish on the orbit of G_s.) We carried out this computation in Macaulay-2. One finds that the degree of $\mathcal{Y}$ as a variety is 18. Furthermore, J is generated by one form in degree 4 and 36 forms in degree 5. More precisely, its minimal resolution begins as

$$0 \leftarrow B/J \leftarrow B \leftarrow B(-4) \otimes M_1 \oplus B(-5) \otimes M_{36} \leftarrow \cdots,$$

where M_i is an SL_2-representation of dimension i.

Clearly $M_1 \simeq S_0$. Since the complete minimal system of binary sextics is known (see [11, Secs. 132–134]), it is a mechanical task to find the irreducible decomposition of M_{36}. Indeed, $M_{36} \subseteq \mathrm{Sym}^5(S_6)$, and the latter can be decomposed into irreducibles by the Cayley–Sylvester formula (see [17, Cor. 4.2.8]). Thus we only need to identify those degree-5 covariants of sextics that vanish when specialized to G_s. The decomposition turns out to be

$$M_{36} \simeq S_{14} \oplus S_{10} \oplus S_6 \oplus S_2.$$

Let θ_{rn} stand for the covariant of degree-order (r, n) as given in the table on [11, p. 156]; for instance, $\theta_{38} = (F, (F, F)_4)_1$. We will explicitly write down the covariants corresponding to these representations. They are, in degree 4,

$$\text{order } 0 \rightsquigarrow 7\theta_{20}^2 - 50\theta_{40}, \tag{16}$$

and in degree 5,

$$
\begin{aligned}
\text{order } 14 &\rightsquigarrow 20\theta_{16}\theta_{24}^2 - 21\theta_{16}\theta_{20}\theta_{28} + 10\theta_{16}^2\theta_{32} + 90\theta_{28}\theta_{36}, \\
\text{order } 10 &\rightsquigarrow 2\theta_{16}\theta_{20}\theta_{24} - 6\theta_{16}\theta_{44} - 27\theta_{28}\theta_{32}, \\
\text{order } 6 &\rightsquigarrow 50\theta_{16}\theta_{40} - 42\theta_{20}\theta_{36} - 105\theta_{24}\theta_{32}, \\
\text{order } 2 &\rightsquigarrow \theta_{20}\theta_{32} - 10\theta_{52}.
\end{aligned} \tag{17}
$$

In conclusion, a binary sextic F belongs to $\mathcal{Y}(s)$ if and only if all the covariants in (16) and (17) vanish on F. It is not clear whether a simpler set of equations may be found.

4.9

If $d = 5$ or 6, then $\mathcal{Y}(\gamma)$ is a hypersurface in $\mathbb{P}^d$. In either case, there are classical methods to deduce its equation (cf. [16, Sec. 260]); we briefly explain them in what follows. One needs to recall some preliminaries on the weights of covariants of binary forms (see [9, Chap. 2]).

Let $F = (a_0, \dots, a_d \between x_1, x_2)^d$, and let $\Psi = (\psi_0, \dots, \psi_n \between x_1, x_2)^n$ be a covariant of degree-order (r, n). Each ψ_k is homogeneous of degree r in the a_i. Moreover, ψ_k is isobaric of weight $w_k = \frac{1}{2}(rd - n) + k$; that is, for every monomial $\prod_{i=0}^d a_i^{m_i}$ appearing in ψ_k, we have $\sum_i i m_i = w_k$.

For instance, if $d = 4$, then $\Psi = (F, (F, F)_2)_1$ is a covariant of degree-order $(3, 6)$. The coefficient ψ_k is isobaric of weight $3 + k$; for example, $\psi_2 = \frac{1}{3} a_0 a_1 a_4 + \frac{2}{3} a_1^2 a_3 - a_0 a_2 a_3$.

Lemma 4.2. *Assume that $F = (a_0, \dots, a_d \between x_1, x_1)^d$, where $a_i = 0$ for odd indices i. Then $\psi_k = 0$ whenever w_k is odd.*

Proof. Since a set of even numbers cannot add up to an odd number, each monomial in ψ_k must vanish. □

4.10

Assume $d = 6$ and let $\beta = (F, F)_4$. Consider the following three covariants of respective degree-orders $(3, 2)$, $(5, 2)$, and $(7, 2)$ (cf. [11, p. 156]):

$$\mu_{32} = (F, \beta)_4, \quad \mu_{52} = (\beta, \mu_{32})_2, \quad \mu_{72} = (F, \mu_{32}^2)_4. \tag{18}$$

Consider their Wronskian $\mathcal{U}$, defined to be the determinant of the 3×3 matrix

$$(i, j) \to \frac{\partial^2 \mu_{2i+3,2}}{\partial x_1^{2-j} \partial x_2^j} \quad \text{for } 0 \le i, j \le 2.$$

By construction, $\mathcal{U}$ is an invariant of binary sextics of degree $3 + 5 + 7 = 15$.

Now assume that F is a general sextic in geometric involution with respect to $x_1 x_2$, so that $F = (a_0, 0, a_2, 0, a_4, 0, a_6 \between x_1, x_2)^6$. Each μ reduces to the form $\Box x_1^2 + \Box x_2^2$ by the previous lemma. Hence the three forms in (18) are linearly dependent, and $\mathcal{U}$ must vanish. By invariance, $\mathcal{U}$ vanishes on each point of $\mathcal{Y}(\gamma)$; in other words, we have an inclusion of hypersurfaces $\mathcal{Y}(\gamma) \subseteq \{\mathcal{U} = 0\}$. It can be shown that the latter is irreducible (see [11, p. 156]), hence the two must in fact coincide.

In [11, Secs. 132ff], the following expression for $\mathcal{U}$ is given:

$$\mathcal{U} = ((F, \beta)_1, \mu_{32}^4)_8 \quad \text{(up to a scalar).}$$

4.11

Now assume $d = 5$. Let $\eta = (F, F)_4$ and $\tau = (F, (F, F)_2)_1$, and define (cf. [11, p. 131])

$$\nu_{51} = (\eta^2, F)_4, \qquad \nu_{13,1} = (\eta^5, \tau)_9,$$

which are covariants of respective degree-orders $(5, 1)$ and $(13, 1)$. Let $\mathbb{H}$ denote their Jacobian $(\nu_{51}, \nu_{13,1})_1$, which is an invariant of degree 18 (usually called the Hermite invariant of binary quintics).

If F is a general quintic in geometric involution with respect to $x_1 x_2$, then (after possibly interchanging x_1 and x_2) we have $F = (a_0, 0, a_2, 0, a_4, 0 \between x_1, x_2)^5$. Now both ν reduce to the form $\square x_1$, hence $\mathbb{H}$ vanishes. Then an argument similar to the foregoing shows that $\mathcal{Y}(\gamma) = \{\mathbb{H} = 0\}$. Other expressions for $\mathbb{H}$ are given in [1, p. 787] and [6, Sec. 1].

5. The Locus of Centers of Involution

Fix an involutor z, and let F be a d-ic. The locus of centers of involution for F often has interesting geometric structure, especially when $\mathcal{Y}^\circ$ is dense in $\mathbb{P}S_d$. We adduce a few such examples.

Write $Q = (q_0, q_1, q_2 \between x_1, x_2)^2$, and let $R = \mathbb{C}[q_0, q_1, q_2]$ denote the coordinate ring of $\mathbb{P}S_2$. For $F \in S_d$, define

$$\vartheta(F) = \{[Q] \in \mathbb{P}^2 : \Delta_Q \neq 0 \text{ and (13) holds}\}.$$

Suppose that ℓ is a linear factor in F, and let $Q = \ell^2$. Then it is easy to see that $(Q^d, F)_d$ vanishes. Indeed, by equivariance we may assume $\ell = x_1$, and then

$$(Q^d, F)_d = \text{constant} \times x_1^d \frac{\partial^d F}{\partial x_2^d} = 0.$$

Since $\sigma_{Q,z}(F)$ and Δ_Q are zero, (13) is satisfied. It follows that when $\vartheta(F)$ is nonempty, its Zariski closure will always contain the points $[\ell^2] \in C$ corresponding to the linear factors of F. Recall that by our definition, such points do not count as centers of involution.

5.1

Let $d = 4$ and $s = (+, +, -, +, +)$; then $z = (-12, 24/7, 3/5)$ by the formulas in Section 3.

PROPOSITION 5.1. *We have $\mathcal{Y}^\circ(z) = \mathbb{P}^4$. Moreover, for a general binary quartic F, the locus $\overline{\vartheta(F)}$ is a smooth conic in $\mathbb{P}^2$.*

Proof. For arbitrary Q and F, there is an identity (see Section 5.2)

$$\sigma_{Q,z}(F) - \Delta^2 F = -12 Q^2 (Q^2, F)_4. \tag{19}$$

As a result, $[Q] \in \vartheta(F)$ if and only if $[Q] \notin C$ and $(Q^2, F)_4 = 0$. The latter is a quadratic equation in the q_i whose coefficients depend upon F. Hence $\vartheta(F) \neq \emptyset$ for any F, and thus $\mathcal{Y}^\circ = \mathbb{P}^4$. The discriminant of this quadratic is $B_F = (F, (F, F)_2)_4$ (see Section 5.3). Hence $\overline{\vartheta(F)}$ is a smooth conic when F is not harmonic. □

By what we have already said, this conic passes through the four points on C corresponding to the linear factors of F.

5.2

Identity (19) is an easy consequence of Proposition 6.1 in Section 6. Indeed, the left-hand side is

$$-12(Q^2, F)_4 + \frac{24}{7}\Delta(Q^2, F)_2 - \frac{2}{5}\Delta^2 F,$$

which is a rescaled expansion of $(Q^2, (Q^2, F)_4)_0$. Identities (20)–(25) can all be proved using a similar technique; this is left as an exercise for the reader.

5.3

The claim about B_F can be easily established by writing down the determinantal expression for the discriminant of a ternary quadratic followed by a straightforward expansion. A more elegant way to do this calculation is to use the macroscopic-to-microscopic rewriting as in [2, eq. (2.6)]. The embedding $\mathrm{SL}(V) \hookrightarrow \mathrm{SL}(S_{n-1}V) \simeq \mathrm{SL}_n$ allows us to reformulate the invariant theory of SL_n entirely in terms of that of SL_2. In the symbolic formalism, that amounts to replacing n-ary brackets by homogenized binary Vandermonde determinants—that is, products of $\frac{n(n-1)}{2}$ binary brackets. If we carry out the procedure on the example at hand, the outcome is the well-known symbolic expression $B_F = (ab)^2(ac)^2(bc)^2$ for the catalecticant.

5.4

Let $d = 6$ and $s = (+, +, +, -, +, +, +)$; then $z = (40, -180/11, 20/7, 5/7)$. This example is similar to the previous one.

PROPOSITION 5.2. *We have $\mathcal{Y}^\circ = \mathbb{P}^6$. Moreover, for a general binary sextic F, the locus $\overline{\vartheta(F)}$ is a smooth cubic in $\mathbb{P}^2$.*

Proof. The argument is parallel to the previous proposition. We have an identity

$$\sigma_{Q,z}(F) - \Delta^3 F = 40 Q^3 (Q^3, F)_6. \tag{20}$$

The equation $(Q^3, F)_6 = 0$ defines a planar cubic curve whose discriminant is a degree-12 invariant of F. It is not necessary to calculate it explicitly; for our purposes it would suffice to check that it is not identically zero. Specialize to $F = x_1^6 + x_2^6 + x_1^2 x_2^4$ when

$$(Q^3, F)_6 = q_0^3 + \frac{4}{5} q_0 q_1^2 + \frac{1}{5} q_0^2 q_2 + q_2^3.$$

This is easily seen to be a nonsingular curve. □

The following lemma will be used in the next section.

LEMMA 5.3. *Assume A and B to be nonzero binary forms of the same order. If $(A,B)_1 = 0$, then the forms are equal up to a multiplicative constant.*

Proof. See [10, Lemma 2.2]. □

5.5

In the next two examples in this section, $\vartheta(F)$ is a finite set of points. The simplest such case is that of the geometric involution for $d = 4$. Let $F \in S_4$ correspond to the divisor $A = p_1 + p_2 + p_3 + p_4$ consisting of four distinct points on C. Now A has three centers of involution—namely, the pairwise intersections of lines

$$\overline{p_1p_2} \cap \overline{p_3p_4}, \quad \overline{p_2p_3} \cap \overline{p_1p_4}, \quad \overline{p_1p_3} \cap \overline{p_2p_4}.$$

One knows that the ideal of three noncollinear planar points is generated by a net of conics; we will see that this ideal can be written down in terms of F. (The reader may refer to [8, Chap. 3] for generalities on ideals of finite point-sets in $\mathbb{P}^2$.) There is an identity

$$\sigma_{Q,\gamma}(F) - \Delta^2 F = Q\big[\underbrace{16(Q^3,F)_4 + \tfrac{24}{5}(Q,F)_2\Delta}_{\alpha}\big]. \tag{21}$$

Hence, $[Q] \in \vartheta(F)$ implies that $\alpha = 0$. The problem, as usual, is that α also vanishes if $Q = \ell^2$ with $\ell | F$. We need an expression that would force α to be zero without itself vanishing on such points. The useful identity is

$$\alpha = -32(\underbrace{(Q^2,F)_3}_{\beta}, Q)_1. \tag{22}$$

Once F is fixed, β is of the form $(\varphi_0, \varphi_1, \varphi_2 \between x_1, x_2)^2$, where $\varphi_i(q_0,q_1,q_2)$ are homogeneous degree-2 expressions in the q_i. Let $I = (\varphi_0,\varphi_1,\varphi_2) \subseteq R$ be the ideal generated by the coefficients of β.

PROPOSITION 5.4. *Assume that F has no repeated linear factors. Then the ideal of the three points $\vartheta(F)$ is I.*

Proof. Let Θ denote the ideal of $\vartheta(F)$. Assume that $[Q] \in \vartheta(F)$; then $(\beta, Q)_1$ vanishes at $[Q]$, and Lemma 5.3 implies that $\beta = cQ$ for some constant c. Since $[Q] \notin C$, we may assume $Q = x_1x_2$ after a change of variables. If

$$F = (a_0, \dots, a_4 \between x_1, x_2)^4, \tag{23}$$

then a direct calculation shows that $\beta = (Q^2, F)_3 = \frac{1}{2}(a_1x_1^2 - a_3x_2^2)$, which forces $c = 0$. Hence $I \subseteq \Theta$.

Alternately, assume $\beta = 0$ at $[Q]$. Then the left-hand side of (21) is zero. We claim that $[Q] \notin C$, and hence $[Q] \in \vartheta(F)$. Indeed, if $[Q] \in C$, then we may assume $Q = x_1^2$ and F is as in (23). Then $(Q^2,F)_3 = a_3x_1^2 + a_4x_1x_2 = 0$. Hence $a_3 = a_4 = 0$, which would force F to have a repeated linear factor.

Thus the zero locus of I is $\vartheta(F)$. The forms $\{\varphi_0, \varphi_1, \varphi_2\}$ must be linearly independent (otherwise I would either define a conic or a scheme of length 4), which implies that $I = \Theta$. □

The first syzygies of I can also be written down using transvectants. They correspond to the identities

$$(\beta, Q)_2 = (\beta, (Q, F)_2)_2 = 0. \tag{24}$$

Altogether, this accounts for the minimal free resolution

$$0 \leftarrow R/I \leftarrow R \leftarrow R(-2)^3 \leftarrow R(-3)^2 \leftarrow 0.$$

The correspondence between transvectant identities and syzygies is explained in [6, Secs. 1.7, 4.1].

5.6

This example is very similar to the previous one, hence we will keep the arguments brief. Let $d = 6$ and $s = (+,+,-,+,-,+,+)$; then $z = (-60, -60/11, 30/7, 3/7)$. It turns out that the set $\vartheta(F)$ consists of seven points, and its ideal can be written down as follows.

We have an identity

$$\sigma_{Q,z}(F) - \Delta^3 F = -60\,Q^2\big[\underbrace{(Q^4, F)_6 + \tfrac{2}{7}\Delta(Q^2, F)_4}_{\mu}\big],$$

and furthermore

$$\mu = -2(\lambda, Q)_1, \quad \text{where } \lambda = (Q^3, F)_5.$$

Write $\lambda = (\psi_0, \psi_1, \psi_2 \between x_1, x_2)^2$ and $I = (\psi_0, \psi_1, \psi_2) \subseteq R$.

PROPOSITION 5.5. *Let F be a general binary sextic. Then the ideal of $\vartheta(F)$ is I.*

Proof. The fact that the zero locus of I is $\vartheta(F)$ is proved exactly as in the previous proposition. Now specialize F to $x_1^6 + x_2^6$ when

$$\lambda = (-q_1 q_2^2, \frac{q_0^3 - q_2^3}{2}, q_0^2 q_1 \between x_1, x_2)^2.$$

It is easy to see that the coefficients of λ are linearly independent and that I is a saturated radical ideal. (We verified this in Macaulay-2.) Thus the assertions remain true for a general F. □

We have identities

$$(\lambda, Q)_2 = (\lambda, (Q^2, F)_4)_2 = 0, \tag{25}$$

and using the techniques of [11, Chap. VI] one verifies that each syzygy satisfied by λ is a polynomial in these two. Hence I has two first syzygies, and we have a minimal resolution

$$0 \leftarrow R/I \leftarrow R \leftarrow R(-3)^3 \leftarrow R(-4) \oplus R(-5) \leftarrow 0.$$

This allows us to calculate the Hilbert function of R/I, which shows that $\deg \vartheta(F) = 7$.

The pairs of examples in Sections 5.1 and 5.4 and Sections 5.5 and 5.6 suggest that there is an infinite class of results of each type. The reader has probably noticed that all the considerations in Section 4 exclusively use the sign sequence, whereas those in Section 5 exclusively use the involutor. The connection between the two is mediated by a rather complicated formula, and each of them seems to contain a kind of algebraic information on its surface that is not easy to extract from the other.

6. The Recoupling Coefficients

In this section, we give a general formula for a compound transvectant of the form $(Q^{\bullet}, (Q^{\bullet}, F))$, where Q is a quadratic and F a d-ic. This result is used to build the system $\mathfrak{S}(d)$ in Section 2.2. Broadly speaking, each term of the expansion is obtained by first taking a transvectant of the two powers of Q and then taking a transvectant of the result with F. In the terminology of mathematical physics, such a move is called "recoupling".

In general, if A, B, C are binary forms of arbitrary orders, then a compound transvectant of the form $(A, (B, C))$ can be written as a sum of terms $((A, B), C)$ (with rational coefficients). Such a general recoupling formula is proved in [5, Sec. 7].

Let a, b, r, s denote nonnegative integers such that $r \leq \min\{d, 2b\}$ and $s \leq \min\{2a, 2b + d - 2r\}$. Let $\Delta = -2(Q, Q)_2$. Given an integer t in the range

$$\max\{|a + b - r - s|, |a - b|\} \leq t \leq \min\{a + b + d - r - s, a + b\}, \quad t \equiv a + b \pmod 2; \tag{26}$$

define rational numbers P_1, P_2, P_3 as follows:

$$P_1 = \{a!\, b!\, r!\, s!\, (d - r)!\, (2b - r)!\, (2a - s)!\, (2b + d - 2r - s)!\} \times \{(2a)!\, (2b)!\, (2b + d - 2r)!\}^{-1},$$

$$P_2 = \left\{(2t + 1)! \left(\frac{a + b + t}{2}\right)! \, (a + b - t)!\, (a - b + t)!\, (b - a + t)!\right\} \times \left\{t!\, (a + b + t + 1)!\, (a + b + d - r - s + t + 1)! \left(\frac{a + b - t}{2}\right)! \times \left(\frac{a - b + t}{2}\right)! \left(\frac{b - a + t}{2}\right)!\right\}^{-1},$$

and

$$P_3 = \sum_z (-1)^z (z + 1)! \times \{(z - a - b - t)!\, (z - 2b - d + r)!\, (z - 2a - 2b - d + 2r + s)! \times (z - a - b - d + r + s - t)!\, (a + b + d - r + t - z)! \times (2a + 2b + d - r - s - z)!\, (a + 3b + d - 2r - s + t - z)!\}^{-1},$$

where the sum is quantified over all integers z such that the arguments of all factorials in the denominator are nonnegative. Now, let

$$\omega(a,b;r,s;t) = (-1)^{d+r+s+(1/2)(a+b-t)} P_1 P_2 P_3. \tag{27}$$

THEOREM 6.1. *With notation as before, we have a formula*

$$(Q^a, (Q^b, F)_r)_s = \sum_t \omega(a,b;r,s;t) \Delta^{(a+b-t)/2} (Q^t, F)_{r+s-a-b+t},$$

where the sum is quantified over (26).

For instance, if $d = 5$, then $(Q^5, (Q^6, F)_2)_4$ can be expanded into

$$\frac{95}{286286} \Delta^3 (Q^5, F)_0 + \frac{575}{1123122} \Delta^2 (Q^7, F)_2 + \frac{95}{9438} \Delta (Q^9, F)_4.$$

Proof of Theorem 6.1. Specialize the general recoupling formula in [5, Thm. 7.2] to the case where A and B are powers of the same quadratic form. Now a transvectant of the type $(Q^\bullet, Q^\bullet)$ can be evaluated using [3, Prop. 6.1], and the result follows. □

ACKNOWLEDGMENTS. The second author has received financial support from NSERC, Canada. We are grateful to Daniel Grayson and Michael Stillman, the authors of Macaulay-2.

References

[1] A. Abdesselam, *A computational solution to a question by Beauville on the invariants of the binary quintic,* J. Algebra 303 (2006), 771–788.

[2] ———, *On the volume conjecture for classical spin networks,* J. Knot Theory Ramifications 21 (2012), 1250022 (62 pp.).

[3] A. Abdesselam and J. Chipalkatti, *Brill–Gordan loci, transvectants and an analogue of the Foulkes conjecture,* Adv. Math. 208 (2007), 491–520.

[4] ———, *The higher transvectants are redundant,* Ann. Inst. Fourier (Grenoble) 59 (2009), 1671–1713.

[5] ———, *Quadratic involutions on binary forms,* preprint, 2010, arXiv:1008.3117v2.

[6] J. Chipalkatti, *On Hermite's invariant for binary quintics,* J. Algebra 317 (2007), 324–353.

[7] I. Dolgachev, *Lectures on invariant theory,* London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2003.

[8] D. Eisenbud, *The geometry of syzygies,* Grad. Texts in Math., 229, Springer-Verlag, New York, 2005.

[9] O. Glenn, *The theory of invariants,* Ginn, Boston, 1915.

[10] L. Goldberg, *Catalan numbers and branched coverings by the Riemann sphere,* Adv. Math. 85 (1991), 129–144.

[11] J. H. Grace and A. Young, *The algebra of invariants,* reprint edition, Chelsea, New York, 1962.

[12] J. Harris, *Algebraic geometry. A first course,* Grad. Texts in Math., 133, Springer-Verlag, New York, 1992.

[13] A. Knapp, *Lie groups beyond an introduction,* 2nd ed., Progr. Math., 140, Birkhäuser, Boston, 2002.

[14] P. Olver, *Classical invariant theory,* London Math. Soc. Stud. Texts, 44, Cambridge Univ. Press, Cambridge, 1999.
[15] C. Procesi, *Lie groups. An approach through invariants and representations,* Springer-Verlag, New York, 2007.
[16] G. Salmon, *Lessons introductory to higher algebra,* reprint edition, Chelsea, New York, 1965.
[17] B. Sturmfels, *Algorithms in invariant theory,* Texts Monogr. Symbol. Comput., Springer-Verlag, Wien, 1993.

A. Abdesselam
Department of Mathematics
University of Virginia
Charlottesville, VA 22904

malek@virginia.edu

J. Chipalkatti
Department of Mathematics
University of Manitoba
Winnipeg, MB R3T 2N2
Canada

chipalka@cc.umanitoba.ca