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# On the Arithmetic Nature of the Values of the Gamma Function, Euler's Constant, and Gompertz's Constant

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## 1. Introduction

In this paper, we prove some results concerning the arithmetic nature of the values of the Gamma function $\Gamma$ at rational or algebraic points and for Euler's constant $\gamma$. A (completely open) conjecture of Rohrlich and Lang predicts that all polynomial relations between Gamma values over $\mathbb{Q}$ come from the functional equations satisfied by the Gamma function. This conjecture implies the transcendence over $\mathbb{Q}$ of $\Gamma(\alpha)$ at all algebraic nonintegral numbers. But at present, the only known results are the transcendance of $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1/3)$, and $\Gamma(1/4)$ (the last two are algebraically independent of $\pi$; see [5]). Using the well-known functional equations satisfied by $\Gamma$, we deduce the transcendence of other Gamma values, such as $\Gamma(1/6)$, but not of $\Gamma(1/5)$. Nonetheless, in [7, Thm. 3.3.5] it is proved that the set $\{\pi, \Gamma(1/5), \Gamma(2/5)\}$ contains at least two algebraically independent numbers. In positive characteristic, all polynomial relations between values of the analogue of the Gamma function are known to come from the analogue of the Rohrlich–Lang conjecture (see [1]).

The results proved here are steps in the direction of transcendence results for the Gamma function. We start with a specific quantitative theorem and then prove more general results of a qualitative nature. We define $\log(z)$ and $z^\alpha$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ with the principal value of the argument $-\pi < \arg(z) < \pi$. An important function in the paper is the function

$$\mathcal{G}_\alpha(z) := z^{-\alpha} \int_0^\infty (t+z)^{\alpha-1} e^{-t}\, \mathrm{d}t.$$

For any $\alpha \in \mathbb{C}$, it is an analytic function of $z$ in $\mathbb{C} \setminus (-\infty, 0]$. When $\alpha = 0$ and $z = 1$, $\mathcal{G}_0(1)$ is known as Gompertz's constant (see [6]).

The main result of the paper is the following.

Theorem 1. (i) *For any rational number $\alpha \notin \mathbb{Z}$, any rational number $z > 0$, and any $\varepsilon > 0$, there exists a constant $c(\alpha, \varepsilon, z) > 0$ such that for any $p, q, r \in \mathbb{Z}$, $q \neq 0$, we have*

$$\left|\frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q}\right| + \left|\mathcal{G}_\alpha(z) - \frac{r}{q}\right| \geq \frac{c(\alpha, \varepsilon, z)}{H^{3+\varepsilon}}, \tag{1.1}$$

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*where* $H = \max(|p|, |q|, |r|)$. *In particular, at least one of* $\Gamma(\alpha)/z^\alpha$ *and* $\mathcal{G}_\alpha(z)$ *is an irrational number.*

(ii) *For any rational number* $z > 0$ *and for any* $\varepsilon > 0$, *there exists a constant* $d(\varepsilon, z) > 0$ *such that for any* $p, q, r \in \mathbb{Z}$, $q \neq 0$, *we have*

$$\left|\gamma + \log(z) - \frac{p}{q}\right| + \left|\mathcal{G}_0(z) - \frac{r}{q}\right| \geq \frac{d(\varepsilon, z)}{H^{3+\varepsilon}}. \tag{1.2}$$

*In particular, at least one of* $\gamma + \log(z)$ *and* $\mathcal{G}_0(z)$ *is an irrational number.*

REMARKS. The constants $c(\alpha, \varepsilon, z)$ and $d(\varepsilon, z)$ could be made explicit, but this is not necessary here.

Aptekarev [2] was apparently the first to state explicitly that at least one of $\gamma$ and $\mathcal{G}_0(1)$ is irrational. He constructed and studied precisely a sequence of linear forms in 1, $\gamma$, and $\mathcal{G}_0(1)$ with integer coefficients and tending to zero. The technique presented here is different, but we show in Section 6 how to construct such linear forms using our approach. For other constructions of rational approximations for Gamma values, see [11; 12].

The proof of Theorem 1 is a consequence of the construction of Hermite–Padé-type approximants to 1, exp, and a specific $E$-function (in Siegel's sense; for the definition, see [13]). As almost always with Hermite–Padé approximants, they provide precise Diophantine estimates but at the cost of less generality. In fact, using the much more general theorems of Shidlovskii on the algebraic independence of values of $E$-functions, we can obtain better qualitative results that we now explain. (Some of them are variations of results due to Mahler [8].) However, it is not clear to us that the precise irrationality measures in Theorem 1 could be obtained by Shidlovskii's methods.

THEOREM 2. (i) *For any algebraic number* $z \notin (-\infty, 0]$ *and any algebraic number* $\alpha \notin \mathbb{Z}$, *the transcendence degree of the field generated by* $e^z$, $\Gamma(\alpha)/z^\alpha$, *and* $\mathcal{G}_\alpha(z)$ *is at least 2. In particular, at least one of the numbers* $\Gamma(\alpha)/z^\alpha$ *and* $\mathcal{G}_\alpha(z)$ *is transcendental.*

(ii) *For any algebraic number* $z \notin (-\infty, 0]$, *the transcendence degree of the field generated by* $\gamma + \log(z)$, $e^z$, *and* $\mathcal{G}_0(z)$ *is at least 2. In particular, at least one of* $\gamma + \log(z)$ *and* $\mathcal{G}_0(z)$ *is transcendental.*

Since $\Gamma(1/2) = \sqrt{\pi}$, we have the following corollaries to Theorem 2(i), which are appealing because of the simultaneous occurences of the numbers $\pi$ and $e$, whose algebraic independence over $\mathbb{Q}$ is still conjectural.

COROLLARY 1. *For any algebraic number* $z \notin (-\infty, 0]$, *the transcendence degree of the field generated by* $\pi$, $e^z$, *and* $\mathcal{G}_{1/2}(z)$ *is at least 2.*

In particular, for $z = 1$ we have the following result.

COROLLARY 2. *The transcendence degree of the field generated by* $\pi$, $e$, *and* $\int_0^\infty e^{-t}/\sqrt{1+t}\,\mathrm{d}t$ *is at least 2.*

It is easy to see that the asymptotic expansion

$$\mathcal{G}_\alpha(z) \sim \sum_{m=0}^{\infty} (-1)^m \frac{(1-\alpha)_m}{z^{m+1}}$$

holds as $|z| \to \infty$ in any open angular sector that does not contain $(-\infty, 0]$. Here, $(x)_m := x(x+1)\cdots(x+m-1)$ is the Pochhammer symbol. The divergent asymptotic series on the right-hand side is a Gevrey series of exact order 1. (A formal power series $\sum_{n\geq 0} a_n z^n$ with $a_n \in \mathbb{C}$ is a Gevrey series of order $s$, $s \in \mathbb{R}$, if the associated power series $\sum_{n\geq 0}(a_n/n!^s)z^n$ has a nonzero radius of convergence; it is of exact order $s$ if the radius of convergence is finite nonzero.) The Taylor series for exp is a Gevrey series of exact order $-1$ and an $E$-function in the sense of Siegel; $\pi$ is the sum of the series $4\sum_{m=0}^{\infty} \frac{z^m}{2m+1}$ at $z = -1$, which is a Gevrey series of exact order 0 and a $G$-function in the sense of Siegel; and the asymptotic expansion of $\mathcal{G}_{1/2}$ is a Gevrey series of exact order 1. Hence Corollary 1 deals with three numbers at different levels in the hierarchy of Gevrey series. However, this is rather accidental because the proof remains purely at the level of $E$-functions. It is still a difficult open problem to find transcendence methods that would enable one to construct "good" auxiliary functions mixing $E$-functions and $G$-functions for example.

In Section 2, we prove the relation between $\Gamma(\alpha)/z^\alpha, \mathcal{G}_\alpha(z)$, respectively $\gamma + \log(z), \mathcal{G}_0(z)$, and the $E$-functions just mentioned. In Section 3, we construct certain Hermite–Padé-type approximants to these $E$-functions, which are needed for the proof of Theorem 1 in Section 4. In Section 5 we give the proof of Theorem 2, and in Section 6 we construct a sequence of linear forms. Finally, in Section 7 we explain why Theorem 2 is implicit in a paper of Mahler [8].

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## 2. Some Useful Functional Relations

In this section, we discuss the relations at the origin of Theorems 1 and 2. We define the function

$$\mathcal{E}_\alpha(z) := \sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+\alpha+1)}$$

for any $z \in \mathbb{C}$ and $\alpha \in \mathbb{C}$, $\alpha \neq -1, -2, \ldots,$ and we let

$$\mathcal{E}(z) := \sum_{m=1}^{\infty} \frac{z^m}{m!\,m}$$

for any $z \in \mathbb{C}$. Both functions are $E$-functions discussed in Shidlovskii's book [13].

Proposition 1. (i) *For any $z \in \mathbb{C} \setminus (-\infty, 0]$ and any $\alpha \in \mathbb{C}$, $\alpha \neq -1, -2, \ldots,$ we have*

$$\frac{\Gamma(\alpha+1)}{z^{\alpha+1}} = \mathcal{E}_\alpha(-z) + e^{-z}\mathcal{G}_{\alpha+1}(z). \tag{2.1}$$

(ii) *For any* $z \in \mathbb{C} \setminus (-\infty, 0]$, *we have*

$$\gamma + \log(z) = -\mathcal{E}(-z) - e^{-z}\mathcal{G}_0(z). \tag{2.2}$$

*Proof.* (i) We fix $z > 0$ and $\alpha$ such that $\Re(\alpha) > -1$; then

$$\begin{aligned}
\Gamma(\alpha+1) &= \int_0^z e^{-t}t^\alpha \,\mathrm{d}t + \int_z^\infty e^{-t}t^\alpha \,\mathrm{d}t \\
&= z^{\alpha+1}\int_0^1 e^{-tz}t^\alpha \,\mathrm{d}t + \int_0^\infty e^{-(t+z)}(t+z)^\alpha \,\mathrm{d}t \\
&= z^{\alpha+1}\int_0^1 e^{-tz}t^\alpha \,\mathrm{d}t + e^{-z}z^{\alpha+1}\mathcal{G}_{\alpha+1}(z).
\end{aligned}$$

This identity can be analytically continued to any $z$ such that $z \in \mathbb{C} \setminus (-\infty, 0]$ and any $\alpha \in \mathbb{C}$, $\alpha \neq -1, -2, \ldots$. This is nothing but (2.1) because

$$\int_0^1 e^{tz}t^\alpha \,\mathrm{d}t = \mathcal{E}_\alpha(z).$$

(ii) We use the same strategy as before. It is well known that $\gamma = -\Gamma'(1)$. Hence, for any $z > 0$,

$$\begin{aligned}
-\gamma &= \int_0^\infty e^{-t}\log(t)\,\mathrm{d}t = \int_0^z e^{-t}\log(t)\,\mathrm{d}t + \int_z^\infty e^{-t}\log(t)\,\mathrm{d}t \\
&= z\int_0^1 e^{-tz}\log(tz)\,\mathrm{d}t + \int_0^\infty e^{-(t+z)}\log(t+z)\,\mathrm{d}t \qquad (2.3)\\
&= \log(z) + z\int_0^1 e^{-tz}\log(t)\,\mathrm{d}t + e^{-z}\int_0^\infty \frac{e^{-t}}{t+z}\,\mathrm{d}t
\end{aligned}$$

(after an integration by parts in the last integral of (2.3)). By analytic continuation this holds for any $z \in \mathbb{C} \setminus (-\infty, 0]$, giving (2.2) because

$$z\int_0^1 e^{-tz}\log(t)\,\mathrm{d}t = \mathcal{E}(-z). \qquad \square$$

We conclude this section with an identity that is irrelevant for the questions considered in this paper but is still interesting because it expresses $\mathcal{G}_\alpha(z)$ in terms of a more natural integral of Stieltjes type.

PROPOSITION 2. *For any complex number* $\alpha$ *such that* $\Re(\alpha) < 1$ *and any* $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$\mathcal{G}_\alpha(z) = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{t^{-\alpha}e^{-t}}{t+z}\,\mathrm{d}t. \tag{2.4}$$

*Proof.* With $x = 1/z > 0$ and $\alpha < 1$, it is enough to prove that

$$\Gamma(1-\alpha)\int_0^\infty \frac{e^{-t}}{(1+xt)^{1-\alpha}}\,\mathrm{d}t = \int_0^\infty \frac{t^{-\alpha}e^{-t}}{1+xt}\,\mathrm{d}t;$$

the complete result then follows by analytic continuation in $x$ and $\alpha$.

By definition of $\Gamma(1-\alpha)$, we have

$$\begin{aligned}
\Gamma(1-\alpha)&\int_0^\infty \frac{e^{-t}}{(1+xt)^{1-\alpha}}\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \frac{e^{-(s+t)}s^{-\alpha}}{(1+xt)^{\alpha+1}}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty e^{-u}\left(\int_0^u v^{1-\alpha}\frac{1+xu}{(1+xv)^2}\,\frac{1+xv}{v(1+xu)}\,\mathrm{d}v\right)\mathrm{d}u \\
&= \int_0^\infty e^{-u}\left(\int_0^u \frac{v^{-\alpha}}{1+xv}\,\mathrm{d}v\right)\mathrm{d}u \\
&= \int_0^\infty \frac{v^{-\alpha}}{1+xv}\left(\int_v^\infty e^{-u}\,\mathrm{d}u\right)\mathrm{d}v = \int_0^\infty e^{-v}\frac{v^{-\alpha}}{1+xv}\,\mathrm{d}v,
\end{aligned}$$

which proves the expected identity. Here we used the change of variables

$$s = v\frac{1+xu}{1+xv}, \qquad t = \frac{u-v}{1+xv},$$

and the application of Fubini's theorem is licit by positivity. □

## 3. Hermite–Padé-type Approximants of $E$-Functions

In this section, we present the constructions of explicit Hermite–Padé-type approximants of the functions $1, \exp, \mathcal{E}_\alpha$ on the one hand (Section 3.1), and $1, \exp, \mathcal{E}$ on the other hand (Section 3.2). In the latter case, the construction is an adaptation of the techniques in [14]. Propositions 3 and 4 are crucial ingredients in the proof of Theorem 1. Both are generalizations of a classical construction of diagonal Padé approximants of exp, based on the study of the integral

$$\frac{z^{2n+1}}{n!}\int_0^1 e^{tz}t^n(1-t)^n\,\mathrm{d}t \in \mathbb{Z}[z] + \mathbb{Z}[z]\exp(z)$$

(for details, see e.g. [3]).

### *3.1. Approximations to the Functions* 1, exp, *and* $\mathcal{E}_\alpha$

PROPOSITION 3. *Let us fix $\alpha$ such that $\Re(\alpha) > -1$ and $\alpha \notin \mathbb{Z}$. For any integer $n \geq 0$, there exist some polynomials $A_n, C_n$ (of degree $\leq n$) and $B_n$ (of degree $\leq n+1$) with coefficients in $\mathbb{Q}(\alpha)$ and such that*

$$\begin{aligned}
R_{n,\alpha}(z) &:= \frac{z^{3n+1}}{n!^2}\int_0^1\int_0^1 e^{zuv}u^{2n+\alpha}(1-u)^n v^{2n}(1-v)^n\,\mathrm{d}u\,\mathrm{d}v \\
&= A_n(z)e^z + B_n(z)\mathcal{E}_\alpha(z) + C_n(z). \qquad (3.1)
\end{aligned}$$

*The order of $R_n(z)$ at $z=0$ is $3n+1$.*

Explicit expressions for the polynomials are provided by the proof. The condition that $\alpha \notin \mathbb{Z}$ is not necessary to define $R_{n,\alpha}(z)$, but the polynomials cannot be defined for $\alpha \in \mathbb{Z}$ in the explicit expressions. This is fixed in Section 3.2 in the case $\alpha = 0$.

Proposition 3 fails to give a solution to the problem of finding the simultaneous Hermite–Padé approximants $[n; n+1; n]$ to the functions 1, exp, and $\mathcal{E}_\alpha$. But that failure is by a small margin because this would have been the case if the order at $z = 0$ of $R_n(z)$ were $3n + 3$.

To prove this proposition, we need the following lemma.

LEMMA 1. *For any integers $k, j \geq 0$, any $z \in \mathbb{C}$, and any $\alpha \notin \mathbb{Z}$, $\Re(\alpha) > -1$, we have*

$$\int_0^1 \int_0^1 e^{zuv} u^{k+\alpha} v^j \, \mathrm{d}u \, \mathrm{d}v = \frac{1}{j-k+\alpha}\left(\frac{1}{z} M_{j,k,\alpha}\left(\frac{1}{z}\right) e^z + (-1)^k \frac{(\alpha+1)_k}{z^k} \mathcal{E}_\alpha(z) + (-1)^j \frac{j!}{z^{j+1}}\right),$$

*where*

$$M_{j,k,\alpha}(z) = \sum_{\ell=0}^{k-1} (k-\ell+\alpha+1)_\ell (-z)^\ell - \sum_{\ell=0}^{j} (j-\ell+1)_\ell (-z)^\ell.$$

REMARK. The lemma does not hold when $\alpha \in \mathbb{Z}$, in which case it must be replaced by Lemma 2.

*Proof of Lemma 1.* Expanding $\exp(zuv)$ in series of powers of $zuv$, we get

$$\begin{aligned}
\int_0^1 \int_0^1 e^{zuv} u^{k+\alpha} v^j \, \mathrm{d}u \, \mathrm{d}v
&= \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \frac{1}{(m+k+\alpha+1)(m+j+1)} \\
&= \frac{1}{j-k-\alpha}\left(\sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+k+\alpha+1)} - \sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+j+1)}\right).
\end{aligned}$$

To evaluate both series, we remark that

$$\sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+j+1)} = \int_0^1 e^{zt} t^j \, \mathrm{d}t,$$

$$\sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+k+\alpha+1)} = \int_0^1 e^{zt} t^{k+\alpha} \, \mathrm{d}t$$

and that, by repeated integration by parts, we have

$$\int_0^1 e^{zt} t^j \, \mathrm{d}t = e^z \sum_{\ell=0}^{j} (-1)^\ell \frac{(j-\ell+1)_\ell}{z^{\ell+1}} + (-1)^{j+1} \frac{j!}{z^{j+1}} \tag{3.2}$$

and

$$\int_0^1 e^{zt} t^{k+\alpha}\,\mathrm{d}t = e^z \sum_{\ell=0}^{k-1} (-1)^\ell \frac{(k-\ell+\alpha+1)_\ell}{z^{\ell+1}} + (-1)^k \frac{(\alpha+1)_k}{z^k} \mathcal{E}_\alpha(z).$$

The lemma follows immediately. □

*Proof of Proposition 3.* We fix $\alpha$ such that $\Re(\alpha) > -1$. Set

$$P_n(t) = \frac{1}{n!}(t^n(1-t)^n)^{(n)} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{n} t^k \in \mathbb{Z}[t],$$

$$\begin{aligned} Q_{n,\alpha}(t) &= \frac{1}{n!\,t^{n+\alpha}}(t^{2n+\alpha}(1-t)^n)^{(n)} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2n+k+\alpha}{n} t^k \in \mathbb{Z}[\alpha][t], \end{aligned}$$

which are of degree $n$ in $t$. Here, $\binom{2n+k+\alpha}{n} := \frac{(n+k+\alpha+1)_n}{n!}$ and it is standard that if $\alpha = a/b \in \mathbb{Q}$, with $a, b \in \mathbb{Z}$, then $b^{2n}\binom{2n+k+\alpha}{n} \in \mathbb{Z}$, so that $b^{2n} Q_{n,\alpha}(t) \in \mathbb{Z}[t]$ in this case.

Let us define

$$I_{n,\alpha}(z) = \int_0^1 \int_0^1 e^{zuv} u^\alpha Q_{n,\alpha}(u) P_n(v)\,\mathrm{d}u\,\mathrm{d}v \tag{3.3}$$

for any $z \in \mathbb{C}$. For simplicity, we write

$$P_n(t) = \sum_{j=0}^{n} p_{j,n} t^j, \qquad Q_{n,\alpha}(t) = \sum_{k=0}^{n} q_{k,n,\alpha} t^k.$$

Hence,

$$\begin{aligned} I_{n,\alpha}(z) &= \sum_{k=0}^{n}\sum_{j=0}^{n} q_{k,n,\alpha} p_{j,n} \int_0^1 \int_0^1 e^{zuv} u^{k+\alpha} v^j\,\mathrm{d}u\,\mathrm{d}v \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n} \frac{q_{k,n,\alpha} p_{j,n}}{j-k+\alpha} \left( \frac{1}{z} M_{j,k,\alpha}\left(\frac{1}{z}\right) e^z + \frac{(\alpha+1)_k}{z^k} \mathcal{E}_\alpha(z) - \frac{j!}{z^{j+1}} \right) \end{aligned} \tag{3.4}$$

by Lemma 1. Clearly, it follows that

$$z^{n+1} I_{n,\alpha}(z) = A_n(z) e^z + B_n(z) \mathcal{E}_\alpha(z) + C_n(z)$$

for some polynomials $A_n$, $B_n$, and $C_n$ as described in Proposition 3.

To conclude, it remains to prove that

$$z^{n+1} I_{n,\alpha}(z) = R_{n,\alpha}(z).$$

This is easily done as follows: in $z^{n+1} I_{n,\alpha}(z)$, we integrate $n$ times by parts in $v$, and then $n$ times by parts in $u$, which gives $R_{n,\alpha}(z)$.

### *3.2. Approximations to the Functions* 1, exp, *and* $\mathcal{E}$

In Proposition 3, the integral $R_{n,\alpha}(z)$ is well-defined for $\alpha = 0$, but its expansion as a linear form in 1, $\exp(z)$, and $\mathcal{E}_0(z) = (\exp(z) - 1)/z$ does not hold because the polynomials $A_n, B_n, C_n$ are not defined for $\alpha = 0$ (more precisely, because of the factor $1/(j - k - \alpha)$). However, this can be corrected.

PROPOSITION 4. *For any integer $n \geq 0$, there exist some polynomials $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ (all of degree $\leq n$) with coefficients in $\mathbb{Q}$ and such that*

$$\begin{aligned} R_{n,0}(z) &:= \frac{z^{3n+1}}{n!^2} \int_0^1 \int_0^1 e^{zuv} u^{2n}(1-u)^n v^{2n}(1-v)^n \, \mathrm{d}u \, \mathrm{d}v \\ &= \mathcal{A}_n(z) e^z + \mathcal{B}_n(z)\mathcal{E}(z) + \mathcal{C}_n(z). \end{aligned} \tag{3.5}$$

*The order at $z = 0$ of $R_{n,0}(z)$ is $3n + 1$.*

To prove the proposition, we need an analogue of Lemma 1 in the case when $\alpha = 0$.

LEMMA 2. *Fix any integers $k, j \geq 0$ and any $z \in \mathbb{C}$.*
*If $k \neq j$, then*

$$\int_0^1 \int_0^1 e^{zuv} u^k v^j \, \mathrm{d}u \, \mathrm{d}v = \frac{1}{j-k} \left( \frac{1}{z} \mathcal{M}_{j,k}\left(\frac{1}{z}\right) e^z + (-1)^{k+1} \frac{k!}{z^{k+1}} + (-1)^j \frac{j!}{z^{j+1}} \right),$$

*where*

$$\mathcal{M}_{j,k}(z) = \sum_{\ell=0}^{k} (k - \ell + 1)_\ell (-z)^\ell - \sum_{\ell=0}^{j} (j - \ell + 1)_\ell (-z)^\ell.$$

*If $k = j$, then*

$$\int_0^1 \int_0^1 e^{zuv} u^k v^k \, \mathrm{d}u \, \mathrm{d}v = \frac{1}{z} \mathcal{M}_k\left(\frac{1}{z}\right) e^z + (-1)^{k+1} \frac{k!}{z^{k+1}} \mathcal{E}(z) + \frac{(-1)^k k!}{z^{k+1}} \sum_{j=1}^{k} \frac{1}{j},$$

*where*

$$\mathcal{M}_k(z) = \sum_{\ell=1}^{k} \sum_{m=0}^{k-\ell} (-1)^{\ell+m+1} \frac{(k-\ell-m+1)_m k!}{(k-\ell+1)!} z^{\ell+m}.$$

*Proof.* If $k \neq j$, we expand $\exp(zuv)$ in powers of $zuv$ to get

$$\begin{aligned} \int_0^1 \int_0^1 e^{zuv} u^k v^j \, \mathrm{d}u \, \mathrm{d}v &= \sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+k+1)(m+j+1)} \\ &= \frac{1}{j-k} \left( \sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+k+1)} - \sum_{m=0}^{\infty} \frac{z^m}{m!\,(m+j+1)} \right) \\ &= \frac{1}{j-k} (I_k - I_j), \end{aligned}$$

where

$$I_k := \int_0^1 e^{zt} t^k \,\mathrm{d}t.$$

To conclude this case we then use identity (3.2), which enables us to evaluate $I_k$ and $I_j$.

If $k = j$, then

$$\int_0^1 \int_0^1 e^{zuv} u^k v^k \,\mathrm{d}u \,\mathrm{d}v = -\int_0^1 e^{zt} t^k \log(t) \,\mathrm{d}t =: -J_k.$$

We have $J_0 = z^{-1}\mathcal{E}(z)$. For $k \geq 1$, using integration by parts yields

$$J_k = -\frac{1}{z} I_{k-1} - \frac{k}{z} J_{k-1},$$

which we iterate to obtain

$$\begin{aligned} J_k &= \sum_{\ell=1}^{k} \frac{(-1)^\ell}{z^\ell} \cdot \frac{k!}{(k-\ell+1)!} I_{k-\ell} + (-1)^k \frac{k!}{z^k} J_0 \\ &= -\frac{1}{z} \mathcal{M}_k\left(\frac{1}{z}\right) e^z + (-1)^k \frac{k!}{z^{k+1}} \mathcal{E}(z) + \frac{(-1)^{k+1} k!}{z^{k+1}} \sum_{j=1}^{k} \frac{1}{j}. \end{aligned}$$

This concludes the proof of the lemma. □

*Proof of Proposition 4.* We start from the integral

$$I_{n,0}(z) := \int_0^1 \int_0^1 e^{zuv} Q_{n,0}(u) P_n(v) \,\mathrm{d}u \,\mathrm{d}v.$$

Expanding the polynomials $Q_{n,0}$ and $P_n$ and using Lemma 2, we see that

$$\begin{aligned} I_{n,0}(z) &= \sum_{k=0}^{n} \sum_{j=0}^{n} q_{k,n,0} p_{j,n} \int_0^1 \int_0^1 e^{zuv} u^k v^j \,\mathrm{d}u \,\mathrm{d}v \\ &= \sum_{\substack{j,k=0 \\ j \neq k}}^{n} \frac{q_{k,n,0} p_{j,n}}{j-k} \left( \frac{1}{z} \mathcal{M}_{j,k}\left(\frac{1}{z}\right) e^z + (-1)^{k+1} \frac{k!}{z^{k+1}} + (-1)^j \frac{j!}{z^{j+1}} \right) \\ &\quad + \sum_{k=0}^{n} q_{k,n,0} p_{k,n} \left( \frac{1}{z} \mathcal{M}_k\left(\frac{1}{z}\right) e^z + (-1)^{k+1} \frac{k!}{z^{k+1}} \mathcal{E}(z) + \frac{(-1)^k k!}{z^{k+1}} \sum_{j=1}^{k} \frac{1}{j} \right). \end{aligned}$$

It follows that $z^{n+1} I_{n,0}(z) = \mathcal{A}_n(z) e^z + \mathcal{B}_n(z) \mathcal{E}(z) + \mathcal{C}_n(z)$, where the polynomials $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{C}_n$ are as described in Proposition 4. To prove that $z^{n+1} I_{n,0}(z) = R_{n,0}(z)$, we integrate $n$ times by parts in $v$ and then $n$ times by parts in $u$. □

## 4. Proof of Theorem 1

The Hermite–Padé approximants constructed in Section 3.1 provide good *functional* simultaneous approximations to the functions $\exp(z)$ and $\mathcal{E}_\alpha(z)$, and as usual it is natural to expect that they also provide good *numerical* simultaneous approximations to the values of both functions. In our situation, the transfer is operated by means of Nesterenko's [9] criterion for linear independence of real numbers, which we recall next.

PROPOSITION 5 (Nesterenko's criterion). *Let $\xi_1, \dots, \xi_N$ denote $N$ real numbers such that there exist $N$ sequences of integers $(p_{j,n})_{n\geq 0}$, $j = 1, \dots, N$, four positive real numbers $\tau_1, \tau_2, c_1, c_2$, and a monotonically increasing function $\sigma$ (defined on $\mathbb{R}$) that satisfy the following properties*:

$$\text{(i)} \qquad \lim_{t\to+\infty} \sigma(t) = +\infty \quad \textit{and} \quad \limsup_{t\to+\infty} \frac{\sigma(t+1)}{\sigma(t)} = 1;$$

$$\text{(ii)} \qquad \max_{j=1,\dots,N} |p_{j,n}| \leq e^{\sigma(n)};$$

$$\text{(iii)} \qquad c_1 e^{-\tau_1\sigma(n)} \leq \left|\sum_{j=1}^{N} p_{j,N}\xi_j\right| \leq c_2 e^{-\tau_2\sigma(n)}.$$

*Then the dimension of the vector space spanned over $\mathbb{Q}$ by $\xi_1, \dots, \xi_N$ is at least $\frac{\tau_1+1}{1+\tau_1-\tau_2}$.*

We will also use a quantitative version of the criterion when $\tau_1 = \tau_2 = N-1$. In that case the dimension is maximal and equal to $N$ and, for any $\varepsilon > 0$, there exists a constant $\eta_\varepsilon > 0$ such that for any $(a_1, \dots, a_N) \in \mathbb{Z}^N \setminus \{(0, \dots, 0)\}$ we have

$$\left|\sum_{j=1}^{N} a_j\xi_j\right| \geq \frac{\eta_\varepsilon}{\max_{j=1,\dots,N}|a_j|^{N-1+\varepsilon}}. \tag{4.1}$$

This is a consequence of the theorem stated in [9, p. 72], which in fact encompasses Proposition 5.

To apply the proposition and (4.1), we need two lemmas. The first one is used for case (i) of Theorem 1 whereas the second is used for case (ii). Set $d_n := \mathrm{lcm}(1, 2, \dots, n)$.

LEMMA 3. *Let $\alpha = a/b \in \mathbb{Q}\setminus\mathbb{Z}$, $\alpha > -1$, $b \geq 1$, and $z = u/v \in \mathbb{Q}^*$.*

(i) *The numbers*

$$b^{3n}v^n d_{bn+|a|}A_n(z),\ b^{3n}v^{n+1}d_{bn+|a|}B_n(z),\ b^{3n}v^n d_{bn+|a|}C_n(z)$$

*are integers.*

(ii) *For all large enough $n$, we have $\max(|A_n(z)|, |B_n(z)|, |C_n(z)|) \leq c_3^n n!$ for some $c_3 > 0$ that depends on $\alpha$ and $z$.*

(iii) *We have $R_{n,\alpha}(z) = c_4^{n(1+o(1))}/n!^2$, where $c_4 := 16z^3/81$.*

LEMMA 4. *Set $z = u/v \in \mathbb{Q}^*$.*

(i) *The numbers*

$$v^n d_n \mathcal{A}_n(z),\ v^{n+1}\mathcal{B}_n(z),\ v^n d_n \mathcal{C}_n(z)$$

*are integers.*

(ii) *For all large enough* $n$, *we have* $\max(|\mathcal{A}_n(z)|, |\mathcal{B}_n(z)|, |\mathcal{C}_n(z)|) \le c_5^n n!$ *for some* $c_5 > 0$ *that depends on* $z$.

(iii) *We have* $R_{n,0}(z) = c_4^{n(1+o(1))}/n!^2$, *where* $c_4 := 16z^3/81$.

We prove only Lemma 3, since the proof of Lemma 4 is analogous.

*Proof of Lemma 3.* (i) This is immediate from (3.4) for $I_{n,\alpha}(z)$ and since $p_{k,n} \in \mathbb{Z}$ and $v^{2n} q_{k,n} \in \mathbb{Z}$ $\big($the latter because $v^{2n}\binom{2n+k+u/v}{n} \in \mathbb{Z}\big)$.

(ii) This part is also immediate from expression (3.4). Indeed, the coefficients $p_{k,n}$ and $q_{k,n,\alpha}$ of the polynomials $P_n$ and $Q_n$ are uniformly bounded (for $k = 0, \dots, n$) by $c_6^n$ for some constant $c_6$ that depends only on $\alpha$.

(iii) An application of Laplace's method to the integral expression (3.1) for $R_n(z)$ shows that

$$\lim_{n \to +\infty} (n!^2 R_n(z))^{1/n} = z^3 \max_{(u,v) \in [0,1]^2} (u^2(1-u)v^2(1-v)) = \frac{16z^3}{81}.$$

(The fact that $\alpha$ and $z$ are real is used here.) □

*Proof of Theorem 1.* We prove only (i), since (ii) is proved in a similar fashion. First, we remark that the restriction that $\alpha > -1$ in Lemma 3 is not essential: we can remove it provided we assume $n$ is large enough, say $n \ge N(\alpha)$, which of course is possible in the lemma and in Proposition 5.

For $n \ge N(\alpha)$, we construct a sequence of linear forms

$$\ell_n = a_n e^z + b_n \mathcal{E}_\alpha(z) + c_n$$

with $a_n, b_n, c_n \in \mathbb{Z}$ by setting

$$\ell_n = b^{3n} v^{n+1} d_{bn+|a|} R_n(z), \qquad a_n = b^{3n} v^{n+1} d_{bn+|a|} A_n(z),$$
$$b_n = b^{3n} v^{n+1} d_{bn+|a|} B_n(z), \qquad c_n = b^{3n} v^{n+1} d_{bn+|a|} C_n(z).$$

Since $d_n = e^{n(1+o(1))}$, the various estimates in Lemma 3 show that we can apply Proposition 5 with $\sigma(n) = \log(n!) = n\log(n)(1+o(1))$ and $\tau_1 = \tau_2 = 2$. (The exact values of $c_1, c_2 > 0$ are not important.) It follows that the dimension of the vector space spanned over $\mathbb{Q}$ by 1, $e^z$, and $\mathcal{E}_\alpha(z)$ is exactly 3.

Recall (2.1)—that is, that

$$\frac{\Gamma(\alpha+1)}{z^{\alpha+1}} = \mathcal{E}_\alpha(-z) + e^{-z}\mathcal{G}_{\alpha+1}(z).$$

Since $\mathcal{E}_\alpha(-z)$ and $e^{-z}$ are $\mathbb{Q}$-linearly independent, at least one of $\Gamma(\alpha+1)/z^{\alpha+1}$ and $\mathcal{G}_{\alpha+1}(z)$ is irrational for any $z \in \mathbb{Q}^*$, $z > 0$ and any $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$. We now prove a quantitative version of this statement. (We change $\alpha$ to $\alpha - 1$ for simplicity.) Indeed, we are in a situation where we can use the linear independence measure (4.1): for any integers $p, q, r$ not all zero and any $\varepsilon > 0$, we have

$$|p + qe^{-z} + r\mathcal{E}_{\alpha-1}(-z)| \ge \frac{c_7}{H^{2+\varepsilon}}, \tag{4.2}$$

where $H = \max(|p|, |q|, |r|)$ and $c_7$ depends on $\alpha$, $\varepsilon$, and $z$.

We claim this implies that, for any integers $p, q, r$ not all zero and any $\varepsilon > 0$,

$$\left|\frac{q\Gamma(\alpha)}{z^\alpha} - p\right| + |q\mathcal{G}_\alpha(z) - r| \geq \frac{c_8}{H^{2+\varepsilon}}, \tag{4.3}$$

where $c_8 = c_7/(1+e^{-z})$. To get a contradiction, let us assume we can find some integers $p', q', r'$ not all zero and an $\varepsilon > 0$ such that

$$\left|\frac{q'\Gamma(\alpha)}{z^\alpha} - p'\right| + |q'\mathcal{G}_\alpha(z) - r'| < \frac{c_8}{\tilde{H}^{2+\varepsilon}},$$

where $\tilde{H} = \max(|p'|, |q'|, |r'|)$. Hence

$$\left|\frac{q'\Gamma(\alpha)}{z^\alpha} - p'\right| < \frac{c_8}{\tilde{H}^{2+\varepsilon}}$$

and

$$|q'e^{-z}\mathcal{G}_\alpha(z) - r'e^{-z}| < \frac{c_8 e^{-z}}{\tilde{H}^{2+\varepsilon}}.$$

On the other hand, by (4.2),

$$\begin{aligned}\frac{c_7}{\tilde{H}^{2+\varepsilon}} &\leq |-p' + r'e^{-z} + q'\mathcal{E}_{\alpha-1}(-z)| = \left|-p' + r'e^{-z} + q'\left(\frac{\Gamma(\alpha)}{z^\alpha} - \mathcal{G}_\alpha(z)\right)\right| \\ &\leq \left|\frac{q'\Gamma(\alpha)}{z^\alpha} - p'\right| + e^{-z}|q'\mathcal{G}_\alpha(z) - r'| < \frac{c_8(1+e^{-z})}{\tilde{H}^{2+\varepsilon}} = \frac{c_7}{\tilde{H}^{2+\varepsilon}}.\end{aligned}$$

This is a contradiction, and thus (4.3) holds, which is the inequality (1.1) in disguise with $c(\alpha, \varepsilon, z) = c_8$.

Inequality (4.3) quantifies the assertion "at least one of $\Gamma(\alpha)/z^\alpha$ and $\mathcal{G}_\alpha(z)$ is irrational". Indeed, if $\Gamma(\alpha)/z^\alpha$ or $\mathcal{G}_\alpha(z)/z$ is rational (say, $\Gamma(\alpha)/z^\alpha = p_0/q_0 \in \mathbb{Q}^*$ for simplicity), we set $p = p_0 q'$, $q = q_0 q'$, and $r = q_0 p'$ in inequality (4.3) for any integers $p', q' \neq 0, r'$. In particular,

$$\left|\frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q}\right| = 0.$$

Consequently,

$$\frac{c_8}{H^{3+\varepsilon}} \leq \left|\frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q}\right| + \left|\mathcal{G}_\alpha(z) - \frac{r}{q}\right| = \left|\mathcal{G}_\alpha(z) - \frac{p'}{q'}\right|$$

with $H := \max(|p|, |q|, |r|) = \max(|p_0 q'|, |q_0 q'|, |q_0 p'|) \leq \max(|p_0|, |q_0|) \cdot \tilde{H}$. Let $\hat{H} := \max(|p'|, |q'|)$. Then, setting $c_9 = c_8 \max(|p_0|, |q_0|)^{3+\varepsilon}$, for any integers $p', q' \neq 0$ we have

$$\left|\mathcal{G}_\alpha(z) - \frac{p'}{q'}\right| \geq \frac{c_9}{\hat{H}^{3+\varepsilon}},$$

which shows that $\mathcal{G}_\alpha(z)$ is an irrational number (and even a non-Liouville number). □

## 5. Proof of Theorem 2

(i) For any $\alpha \notin \mathbb{Z}$, the functions $\exp(z)$ and $\mathcal{E}_\alpha(z)$ are algebraically independent over $\mathbb{C}(z)$ [13, Lemma 7], and both functions satisfy the homogeneous linear differential system

$$\begin{aligned} y_1' &= y_1, \\ y_2' &= \frac{1}{z}y_1 - \frac{\alpha+1}{z}y_2. \end{aligned} \tag{5.1}$$

If $\alpha$ is an algebraic noninteger, then Shidlovskii's classical theorem on $E$-functions [13, Thm. 3] yields that for any algebraic number $z \neq 0$, the numbers $\mathcal{E}_\alpha(-z)$ and $\exp(-z)$ are algebraically independent over $\mathbb{Q}$.

We now use identity (2.1) to deduce that, for any $\alpha \in \bar{\mathbb{Q}} \setminus \mathbb{Z}$ and any $z \in \bar{\mathbb{Q}}^*$, $z \notin (-\infty, 0]$, the field generated over $\mathbb{Q}$ by the numbers $\Gamma(\alpha)/z^\alpha$, $e^z$, and $\mathcal{G}_\alpha(z)$ has transcendence degree at least 2. This is the content of Theorem 2(i).

(ii) Although this is not proved in [13], the functions $\exp(z)$ and $\mathcal{E}(z)$ are algebraically independent over $\mathbb{Q}(z)$. Because they satisfy the inhomogeneous linear differential system

$$\begin{aligned} y_1' &= y_1, \\ y_2' &= \frac{1}{z}y_1 - \frac{1}{z}, \end{aligned}$$

we can apply Shidlovskii's second fundamental theorem [13, p. 123] to deduce that for any $z \in \bar{\mathbb{Q}}^*$, the numbers $\mathcal{E}(-z)$ and $\exp(-z)$ are algebraically independent over $\mathbb{Q}$. Together with identity (2.2), this immediately implies Theorem 2(ii).

## 6. A Sequence of Linear Forms in 1, $\Gamma(\alpha)/z^\alpha$, and $\mathcal{G}_\alpha(z)$

In this section, we construct an explicit sequence of linear forms

$$L_n(\alpha, z) \in \mathbb{Z} + \mathbb{Z}\frac{\Gamma(\alpha)}{z^\alpha} + \mathbb{Z}\mathcal{G}_\alpha(z)$$

that tends to 0 as $n \to +\infty$ under the assumptions that $z \in \mathbb{Q}^*$, $z > 0$, and $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$.

The principle of the construction is simple and was already used in [10; 11] (for a different purpose, however). We consider $R_{n,\alpha}(-z)$ and $R_{n+1,\alpha}(-z)$ simultaneously and define the five determinants

$$S_n(z) = \begin{vmatrix} A_n(-z) & R_{n,\alpha}(-z) \\ A_{n+1}(-z) & R_{n+1,\alpha}(-z) \end{vmatrix}, \qquad T_n(z) = \begin{vmatrix} R_{n,\alpha}(-z) & B_n(-z) \\ R_{n+1,\alpha}(-z) & B_{n+1}(-z) \end{vmatrix},$$

$$U_n(z) = \begin{vmatrix} A_n(-z) & C_n(-z) \\ A_{n+1}(-z) & C_{n+1}(-z) \end{vmatrix}, \qquad V_n(z) = \begin{vmatrix} A_n(-z) & B_n(-z) \\ A_{n+1}(-z) & B_{n+1}(-z) \end{vmatrix},$$

and

$$W_n(z) = \begin{vmatrix} C_n(-z) & B_n(-z) \\ C_{n+1}(-z) & B_{n+1}(-z) \end{vmatrix}.$$

Clearly, $U_n, V_n, W_n$ are polynomials in $z$ of degree at most $2n+2$, with coefficients in $\mathbb{Q}(\alpha)$. Furthermore, we have the relations

$$\begin{aligned} V_n(z)\mathcal{E}_\alpha(-z) + U_n(z) &= S_n(z) = \mathcal{O}(z^{3n+1}), \\ V_n(z)e^{-z} + W_n(z) &= T_n(z) = \mathcal{O}(z^{3n+1}). \end{aligned}$$

(These functional approximations almost provide the diagonal simultaneous Padé approximants of type II for the functions $\exp(z)$ and $\mathcal{E}_\alpha(z)$.)

We now use (2.1) in the form

$$\mathcal{E}_\alpha(-z) = \frac{\Gamma(\alpha+1)}{z^{\alpha+1}} - e^{-z}\mathcal{G}_{\alpha+1}(z),$$

so that

$$\begin{aligned} S_n(z) &= V_n(z)\frac{\Gamma(\alpha+1)}{z^{\alpha+1}} - V_n(z)e^{-z}\mathcal{G}_{\alpha+1}(z) + U_n(z), \\ T_n(z) &= V_n(z)e^{-z} + W_n(z), \end{aligned}$$

from which we finally obtain that

$$S_n(z) + \mathcal{G}_{\alpha+1}(z)T_n(z) = V_n(z)\frac{\Gamma(\alpha+1)}{z^{\alpha+1}} + W_n(z)\mathcal{G}_{\alpha+1}(z) + U_n(z). \tag{6.1}$$

The estimates given in Lemma 3 show that there exist some constants $c_{10}$ and $c_{11}$ (depending on $\alpha$ and $z$) such that

$$|S_n(z) + \mathcal{G}_{\alpha+1}(z)T_n(z)| \leq \frac{c_{10}^n}{n!}$$

and, when $z > 0$ and $\alpha \notin \mathbb{Z}$ are rational numbers, the common denominator $D_n$ of the coefficients of $V_n(z)$, $W_n(z)$, and $U_n(z)$ is bounded by $c_{11}^n$. Hence

$$L_n(\alpha+1, z) := D_n(S_n(z) + \mathcal{G}_{\alpha+1}(z)T_n(z)) \in \mathbb{Z} + \mathbb{Z}\frac{\Gamma(\alpha+1)}{z^{\alpha+1}} + \mathbb{Z}\mathcal{G}_{\alpha+1}(z)$$

tends to 0 essentially as fast as $1/n!$ (up to some factor with exponential growth in $n$). To conclude that at least one of $\Gamma(\alpha+1)/z^{\alpha+1}$ and $\mathcal{G}_{\alpha+1}(z)$ is irrational, it remains to prove that $L_n(\alpha+1, z) \neq 0$ for infinitely many $n$. As seen in Section 4, this is a consequence of the linear independence of the numbers $\exp(z)$ and $\mathcal{E}_\alpha(z)$ over $\mathbb{Q}$. This is not an easy task if we don't want to remember this fact. In principal, we could explicitly compute the recurrence satisfied by $A_n, B_n, C_n, R_n$, deduce that it is satisfied by $S_n, T_n, U_n, V_n, W_n$, and then find the exact asymptotic behavior of $zS_n(-z) + \mathcal{G}_{\alpha+1}(z)T_n(-z)$ by means of Birkhoff–Trjitzinski theory. A similar construction of sequences of linear forms in $\gamma + \log(z)$ and $\mathcal{G}_0(z)$ is possible.

## 7. Connection with Mahler's Paper

In Section 1, we mentioned that Theorem 2 is related to Mahler's article [8], where he says: "the results proved in this paper are quite trivial consequences of Shidlovski's work, and they do not even imply the irrationality of $\gamma$ or of $\zeta(3)$. However, they deserve perhaps a little interest because, up to now, nothing was known about the arithmetic of these constants." Mahler's comment refers to his remark that the number $\frac{\pi Y_0(2)}{2J_0(2)} - \gamma$ and other similar numbers are transcendental, but it could certainly be applied to our Theorem 2. Note that [8] was published in 1968, many years before Apéry's proof of the irrationality of $\zeta(3)$.

In the last five lines of [8], he mentions without proof the following theorem: *For* $z \in \bar{\mathbb{Q}}^*$, *integer* $k \geq 0$, *and rational number* $\alpha > -1$, *any finite number of integrals*

$$\int_0^1 t^\alpha \log(t)^k e^{-zt}\,\mathrm{d}t \tag{7.1}$$

*are algebraically independent over* $\mathbb{Q}$. Clearly, this contains as a particular case the algebraic independence over $\mathbb{Q}$ of the numbers $\exp(z)$ and $\mathcal{E}_\alpha(z)$, respectively of the numbers $\exp(z)$ and $\mathcal{E}(z)$ in the previous conditions. Although Mahler did not give a proof, it is clear that it was based on the observation that the integral in (7.1) is an $E$-function (of the variable $z$) strongly similar to $\mathcal{E}_\alpha$ and $\mathcal{E}$.

As an application of Mahler's result, we mention a generalization of Theorem 2(ii): *For any* $z \in \bar{\mathbb{Q}}$, $z \notin (-\infty, 0]$, *and any integer* $s \geq 1$, *the transcendence degree of the field generated over* $\mathbb{Q}$ *by*

$$\Gamma^{(s)}(1),\ \log(z),\ e^z,\ \textit{and}\ \int_0^\infty \log(t+z)^s e^{-t}\,\mathrm{d}t$$

*is at least 2. In particular, at least one of* $\Gamma^{(s)}(1) = \int_0^\infty \log(t)^s e^{-t}\,\mathrm{d}t$ *and* $\int_0^\infty \log(1+t)^s e^{-t}\,\mathrm{d}t$ *is transcendental.*

The proof amounts to the observation that

$$\begin{aligned}\Gamma^{(s)}(1) &= z\int_0^1 \log(tz)^s e^{-zt}\,\mathrm{d}t + \int_z^\infty \log(t)^s e^{-t}\,\mathrm{d}t\\ &= z\sum_{j=0}^{s}\binom{s}{j}\log(z)^{s-j}\int_0^1 \log(t)^j e^{-tz}\,\mathrm{d}t + e^{-z}\int_0^\infty \log(t+z)^s e^{-t}\,\mathrm{d}t\end{aligned}$$

for any $z \in \mathbb{C} \setminus (-\infty, 0]$, at which point we can use Mahler's result. We conclude by mentioning that, for any integer $s \geq 1$, $\Gamma^{(s)}(1)$ can be expressed as a polynomial in $\gamma, \zeta(2), \zeta(3), \ldots, \zeta(s)$ with rational coefficients (see [10, eq. (3.1)] for a precise statement). For example, $\Gamma'(1) = -\gamma$, $\Gamma''(1) = \zeta(2) + \gamma^2$, and $\Gamma'''(1) = -2\zeta(3) - 3\gamma\zeta(2) - \gamma^3$.

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