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Irregularity of the Bergman Projection on Worm Domains in $\mathbb{C}^n$

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1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ denote the Bergman space of square-integrable holomorphic functions on Ω. The Bergman projection on Ω is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$.

The Bergman projection is known to be regular in the sense that it maps W^s to W^s for all $s \geq 0$, where W^s denotes the Sobolev space of order s, on a large class of smooth bounded pseudoconvex domains (throughout this paper a domain is smooth if its boundary is a smooth manifold). Regularity is usually established through the $\bar{\partial}$-Neumann problem, the solution operator for the complex Laplacian $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on square-integrable $(0,1)$-forms. For more information on this matter we refer the reader to [BS3; S] and the references therein.

Irregularity of the Bergman projection is not understood nearly as well as regularity. The story of irregularity goes back to the discovery of the worm domains in $\mathbb{C}^2$ by Diederich and Fornæss [DF]. Worm domains were constructed to show that the closure of some smooth bounded pseudoconvex domains may not have Stein neighborhood bases (a compact set $K \subset \mathbb{C}^n$ is said to have a Stein neighborhood basis if for every open set U containing K there exists a pseudoconvex domain V such that $K \subset V \subset U$). Indeed, Diederich and Fornæss showed that the closure of a worm domain does not have a Stein neighborhood basis if the total winding is no less than π. It turned out that worm domains are also counterexamples for regularity of the Bergman projection. In 1991, Kiselman [Ki] showed that the Bergman projection does not satisfy Bell's condition R on nonsmooth worm domains (a domain Ω satisfies Bell's condition R if the Bergman projection maps $C^\infty(\bar{\Omega})$ to $C^\infty(\bar{\Omega})$). In 1992, Barrett [Ba] showed that the Bergman projection on a smooth worm domain does not map W^s into W^s if $s \geq \pi/(\text{total winding})$. On the other hand, Boas and Straube [BS2] showed that the Bergman projection maps W^k into W^k if $k \leq \pi/(2 \times \text{total winding})$ and k is a positive integer or $k = 1/2$. Finally, in 1996 Christ [Ch] showed that the Bergman projections on smooth worm domains with any positive winding do not satisfy Bell's condition R. More recently, Krantz and Peloso [KP1; KP2] studied the asymptotics for the Bergman

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kernel on the model domains in $\mathbb{C}^2$ and derived L^p (ir)regularity for the Bergman projection on worm domains in $\mathbb{C}^2$.

In this paper we (i) construct smooth bounded pseudoconvex domains $\Omega_{\alpha\beta} \subset \mathbb{C}^n$ that are higher-dimensional generalizations of the worm domains in $\mathbb{C}^2$ and (ii) study the irregularity of the Bergman projection on these domains on L^p Sobolev spaces for $1 \leq p < \infty$. We will use the method developed in [Ba] to show that irregularity on L^2 Sobolev spaces depends only on the total winding whereas the irregularity on L^p spaces with $p \neq 2$ depends not only on the total winding but also on the dimension n.

The two parameters α and β in $\Omega_{\alpha\beta}$ represent the speed of the winding and the thickness of the annulus, respectively. Both parameters play a role in the proof of Theorem 1, but we find it interesting that the actual results depend only on the total winding—whether this is achieved by fast winding along a thin annulus or slow winding along a thick annulus.

The domains $\Omega_{\alpha\beta} \subset \mathbb{C}^n$, $n \geq 3$, are defined by

$$\Omega_{\alpha\beta} = \{(z_1, z', z_n) \in \mathbb{C}^n : r(z_1, z', z_n) < 0\}$$

with

$$r(z_1, z', z_n) = |z_1 - e^{2i\alpha \ln|z_n|}|^2 + |z'|^2 - 1 + \sigma(|z_n|^2 - \beta^2) + \sigma(1 - |z_n|^2);$$

here $z' = (z_2, \ldots, z_{n-1})$, $|z'|^2 = |z_2|^2 + \cdots + |z_{n-1}|^2$, the constants $\alpha > 0$ and $\beta > 1$, and

$$\sigma(t) = \begin{cases} Me^{-1/t}, & t > 0, \\ 0, & t \leq 0 \end{cases}$$

for some $M > 0$.

In Section 2 we show that $\Omega_{\alpha\beta}$ is smoothly bounded and pseudoconvex when M is sufficiently large. The main result of this paper is the following theorem.

THEOREM 1. *The Bergman projection for $\Omega_{\alpha\beta}$ does not map $W^{p,s}(\Omega_{\alpha\beta})$ into $W^{p,s}(\Omega_{\alpha\beta})$ when $1 \leq p < \infty$ and $s \geq \frac{\pi}{2\alpha \ln \beta} + n\left(\frac{1}{p} - \frac{1}{2}\right)$.*

Here $W^{p,s}(\Omega_{\alpha\beta})$ is the Sobolev space of order s with exponent p, and we write $W^{p,s}(\Omega_{\alpha\beta}) \not\subset L^2(\Omega_{\alpha\beta})$ to indicate that the $W^{p,s}$ bounds do not hold for the Bergman projection on $W^{p,s}(\Omega_{\alpha\beta}) \cap L^2(\Omega_{\alpha\beta})$. The denominator $2\alpha \ln \beta$ appearing in the theorem may be interpreted as the total amount of winding along the annulus $1 < |z_n| < \beta$ (see equation (1) in Section 3).

If we choose $p = 2$ then the amount of irregularity provided by a fixed amount of winding is independent of the dimension.

COROLLARY 1. *The Bergman projection for $\Omega_{\alpha\beta}$ does not map $W^{2,s}(\Omega_{\alpha\beta})$ to $W^{2,s}(\Omega_{\alpha\beta})$ when $s \geq \frac{\pi}{2\alpha \ln \beta}$.*

REMARK 1. Suppose we have the Bergman projection P_U of a domain U bounded on $L^p(U)$, where $p > 2$. Then the duality and self-adjointness of the Bergman projection imply that P_U is also bounded on $L^q(U)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, interpolation implies that P_U is bounded on L^r for all $r \in [q, p]$.

Thus, when $s = 0$ and $n\alpha \ln \beta > \pi$, Remark 1 and Theorem 1 together imply the following corollary.

COROLLARY 2. *The Bergman projection for $\Omega_{\alpha\beta}$ does not map $L^p(\Omega_{\alpha\beta})$ to $L^p(\Omega_{\alpha\beta})$ when* $0 < \frac{1}{p} \le \frac{1}{2} - \frac{\pi}{2n\alpha \ln \beta}$ *or* $\frac{1}{2} + \frac{\pi}{2n\alpha \ln \beta} \le \frac{1}{p} < 1$.

Theorem 1 is proved in Section 4. The proof is based on model domain asymptotics developed in Section 3.

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2. Geometry of Worm Domains

PROPOSITION 1. *The domain $\Omega_{\alpha\beta}$ is smoothly bounded and pseudoconvex whenever M is sufficiently large.*

Proof. We start by requiring $M > e^2$. Then $\Omega \subset \left\{z \in \mathbb{C}^n : |z_1| < 3, |z'| < 2, 1/2 < |z_n| < \sqrt{\beta^2 + 1/2}\right\}$ and so Ω is bounded. Now, by considering z_1-, z'-, and z_n-derivatives in order, it is easy to check that the gradient of $r(z)$ does not vanish on $\{z \in \mathbb{C}^n : r(z) = 0\}$; hence Ω has smooth boundary.

It remains to show that $\Omega_{\alpha\beta}$ is pseudoconvex. It suffices to check this locally. We focus on the case $|z_n| \ge (1+\beta)/2$; the case $|z_n| \le (1+\beta)/2$ is similar.

Multiplying $r(z)$ by $\exp\{\operatorname{Arg}(z_n^{2\alpha})\}$, we obtain the new defining function

$$r_1(z) = r_2(z) - 2\operatorname{Re}(z_1 z_n^{-2\alpha i}),$$

where

$$r_2(z) = (|z_1|^2 + |z'|^2 + \lambda(z_n)) \exp\{\operatorname{Arg}(z_n^{2\alpha})\} \quad \text{and} \quad \lambda(z_n) = \sigma(|z_n|^2 - \beta^2).$$

Since $2\operatorname{Re}(z_1 z_n^{-2\alpha i})$ is pluriharmonic it will suffice to show that r_2 is plurisubharmonic. To simplify the notation, let $A(z) = |z_1|^2 + |z'|^2 + \lambda(z_n)$ and $B(z) = \operatorname{Arg}(z_n^{2\alpha})$. Let $W = \sum_{j=1}^n w_j \frac{\partial}{\partial z_j}$ with w_j constant. (In the following calculations, $H_f(W)$ denotes the complex Hessian of f in the direction W.) Then $W(r_2) = e^B(W(A) + AW(B))$ and so the Cauchy–Schwarz inequality implies that

$$-2\operatorname{Re}\left(\bar{w}_n B_{\bar{z}_n} \sum_{j=1}^{n-1} w_j \bar{z}_j\right) \le \sum_{j=1}^{n-1} |w_j|^2 + |\bar{w}_n B_{\bar{z}_n}|^2 \sum_{j=1}^{n-1} |z_j|^2.$$

Using this inequality in the second line below yields

$$\begin{aligned} H_{r_2}(W) &= e^B\big(H_A(W) + 2\operatorname{Re}(W(A)\overline{W}(B)) + A|W(B)|^2 + AH_B(W)\big) \\ &\ge |w_n|^2 e^B(\lambda_{z_n\bar{z}_n} + 2\operatorname{Re}(\lambda_{z_n} B_{\bar{z}_n}) + \lambda |B_{\bar{z}_n}|^2). \end{aligned}$$

One can check that $\lambda_{z_n}(z_n) = \bar{z}_n \sigma'(|z_n|^2 - \beta^2)$, $|B_{\bar{z}_n}| = \frac{\alpha}{|z_n|}$, and

$$\lambda_{z_n\bar{z}_n}(z_n) = |z_n|^2 \sigma''(|z_n|^2 - \beta^2) + \sigma'(|z_n|^2 - \beta^2).$$

We remark that, because $\lambda(z_n) = \lambda_{z_n}(z_n) = \lambda_{z_n\bar{z}_n}(z_n) = 0$ for $|z_n| \le \beta$, we can assume without loss of generality that $|z_n| > \beta$. Using the fact that $\beta < |z_n| < \sqrt{\beta^2 + 1/2}$ and $t = |z_n|^2 - \beta^2$ in the third line of the following display, we obtain

$$\begin{aligned}
\lambda_{z_n\bar{z}_n} &+ 2\,\mathrm{Re}(\lambda_{z_n} B_{\bar{z}_n}) + \lambda|B_{\bar{z}_n}|^2 \\
&\geq \lambda_{z_n\bar{z}_n} - \frac{2\alpha|\lambda_{z_n}|}{|z_n|} \\
&\geq |z_n|^2\sigma''(|z_n|^2-\beta^2) + (1-2\alpha)\sigma'(|z_n|^2-\beta^2) \\
&= Me^{-1/t}\left(\frac{\beta^2+t}{t^4} - \frac{2(\beta^2+t)}{t^3} + \frac{1-2\alpha}{t^2}\right) \\
&= \frac{M(\beta^2+t)e^{-1/t}}{t^4}\left(1-2t+\frac{(1-2\alpha)t^2}{\beta^2+t}\right).
\end{aligned}$$

We can choose M sufficiently large that $z \in \Omega_{\alpha\beta} \cap \{z \in \mathbb{C}^n : |z_n| \geq \beta\}$ implies t is sufficiently small. That, in turn, implies

$$1 - 2t + \frac{(1-2\alpha)t^2}{\beta^2+t} > 0.$$

This inequality implies that $\lambda_{z_n\bar{z}_n} + 2\,\mathrm{Re}(\lambda_{z_n} B_{\bar{z}_n}) + \lambda|B_{\bar{z}_n}|^2 \geq 0$ for $z \in \Omega_{\alpha\beta}$ such that $|z_n| \geq (1+\beta)/2$. Therefore, the domain $\Omega_{\alpha\beta}$ is pseudoconvex for sufficiently large M. □

REMARK 2. A similar calculation shows that the set of weakly pseudoconvex points in the boundary is the set $\{(0,\dots,0,z_n) \in \mathbb{C}^n : 1 \leq |z_n| \leq \beta\}$.

REMARK 3. We note that regularity of the $\bar{\partial}$-Neumann operator is closely connected to regularity of the Bergman projection [BS1]. In particular, if the $\bar{\partial}$-Neumann operator of a smooth bounded pseudoconvex domain is globally regular then the Bergman projection satisfies Bell's condition R. One can show that, on the set $\{(0,\dots,0,z_n) \in \mathbb{C}^n : 1 \leq |z_n| \leq \beta\}$, the Levi form of r has only one vanishing eigenvalue because the form has positive eigenvalues in the direction transversal to the z_n-axis. In this case, [ŞS, Thm. 1] implies that the $\bar{\partial}$-Neumann operator is not compact on $(0,1)$-forms (recall that compactness of the $\bar{\partial}$-Neumann operator implies that it is globally regular by [KoN]). However, showing irregularity of the Bergman projection in Sobolev scale requires more work.

3. Model Domains

In this section we define a family of simplified model domains and calculate the asymptotics for the Bergman kernels of these model domains. We use a modified version of the method developed in [Ba].

For $\lambda > 0$, let

$$\tau_\lambda(z_1, z', z_n) = (2\lambda^2 z_1, \lambda z', z_n),$$
$$r_\lambda = \lambda^2 r \circ \tau_\lambda^{-1},$$
$$D_\lambda = \tau_\lambda(\Omega_{\alpha\beta}).$$

Then for $1 \leq |z_n| \leq \beta$ we have $r_\lambda \searrow r_\infty$ as $\lambda \to \infty$, where

$$r_\infty(z_1, z', z_n) = |z'|^2 - \mathrm{Re}(z_1 e^{-2\alpha i \ln|z_n|});$$

for $|z_n|$ outside this range we have $r_\lambda \to \infty$. It follows that the D_λ converge in an appropriate sense to the limit domain

$$D = D_{\alpha\beta} = \{(z_1, z', z_n) \in \mathbb{C}^n : \operatorname{Re}(z_1 e^{-2\alpha i \ln|z_n|}) > |z'|^2,\ 1 < |z_n| < \beta\}, \quad (1)$$

where the limit is increasing over the annulus $1 \le |z_n| \le \beta$.

The Bergman projection P of D is defined by $Pf(z) = \int_D K(z,w) f(w)\, dV(w)$, where $f \in L^2(D)$ and $K\colon D \times D \to \mathbb{C}$ is the Bergman kernel characterized by the following conditions:

(i) $K(z,w) \in A^2(D)$ for fixed $w \in D$;
(ii) $K(w,z) = \overline{K(z,w)}$;
(iii) $\int_D K(z,w) f(w)\, dV(w) = f(z)$ for $f \in A^2(D)$.

If $f_1, f_2, \ldots$ is an orthonormal basis for $A^2(D)$, then $K(z,w) = \sum_j f_j(z)\overline{f_j(w)}$.

To study the Bergman kernel of D, we begin by performing a Fourier decomposition. We define

$$(P_{Jk} f)(z_1, z', z_n) = \frac{1}{2^{n-1}\pi^{n-1}} \int_{[-\pi,\pi]^{n-1}} f(z_1, e^{iS} z', e^{it} z_n) e^{-iJS} e^{-ikt}\, dS\, dt, \quad (2)$$

where $k \in \mathbb{Z}$ and

$$\begin{aligned} e^{iS} &= (e^{is_1}, \ldots, e^{is_{n-2}}), \\ S &= (s_1, \ldots, s_{n-2}) \in [-\pi,\pi]^{n-2}, \\ J &= (j_1, \ldots, j_{n-2}) \in \mathbb{N}^{n-2}, \\ JS &= j_1 s_1 + \cdots + j_{n-2} s_{n-2}, \\ dS &= ds_1 \cdots ds_{n-2}. \end{aligned}$$

Let us define the mapping $\rho_{St}(z_1, z', z_n) = (z_1, e^{iS} z', e^{it} z_n)$. Then P_{Jk} is the orthogonal projection from $A^2(D)$ onto

$$A^2_{Jk}(D) = \{f \in A^2(D) : f \circ \rho_{St} = e^{iJS} e^{ikt} f \text{ for all } S, t\}.$$

Hence the Bergman space $A^2(D)$ can be written as the orthogonal sum

$$A^2(D) = \bigoplus_{J \in \mathbb{N}^{n-2},\, k \in \mathbb{Z}} A^2_{Jk}(D)$$

and the Bergman kernel $K(z,w)$ for D satisfies

$$K(z,w) = \sum_{J \in \mathbb{N}^{n-2},\, k \in \mathbb{Z}} K_{Jk}(z,w),$$

where $K_{Jk}(z,w)$ is the kernel for $A^2_{Jk}(D)$.

One can show that for $f \in A^2_{Jk}(D)$ the function $f(z_1, z', z_n) z_2^{-j_1} \cdots z_{n-1}^{-j_{n-2}} z_n^{-k}$ is locally independent of (z', z_n). We notate such functions as functions of z_1, where it is understood that z_1 ranges over the Riemann domain described by $-\pi/2 < \operatorname{Arg} z_1 < 2\alpha \ln \beta + \pi/2$.

Let $|J| = j_1 + \cdots + j_{n-2}$. Then a square-integrable holomorphic function f on D can be written as

$$f(z) = \sum_{J \in \mathbb{N}^{n-2},\, k \in \mathbb{Z}} F_{Jk}(z),$$

where

$$F_{Jk}(z_1, z', z_n) = z_1^{-(|J|+n)/2} f_{Jk}(z_1) z'^J z_n^k$$

and the sum converges locally uniformly.

Now we will calculate the L^2-norm of F_{Jk} on D. Let $z_1 = r_1 e^{i\theta_1}$, $r_j = |z_j|$ for $j = 1, \dots, n$, $r' = \sqrt{r_2^2 + \cdots + r_{n-1}^2}$, and $s = \ln|z_n|^2$. Then D is described by the following inequalities:

$$\begin{aligned} 0 &< r_1 < \infty, \\ 0 &< s < 2\ln\beta, \\ |\theta_1 - \alpha s| &< \pi/2, \\ 0 &\le r' < \sqrt{r_1 \cos(\theta_1 - \alpha s)}. \end{aligned}$$

We have

$$\begin{aligned} &\|F_{Jk}\|_D^2 \\ &\quad = \int_D |f_{Jk}(r_1 e^{i\theta_1})|^2 r_1^{-|J|-n+1} r_2^{2j_2+1} \cdots r_{n-1}^{2j_{n-2}+1} r_n^{2k+1}\, d\theta_1 \cdots d\theta_n\, dr_1 \cdots dr_n \\ &\quad = C_{nJ} \int_{\substack{0<r_1<\infty \\ |\theta_1 - \alpha s|<\pi/2 \\ 0<s<2\ln\beta}} |f_{Jk}(r_1 e^{i\theta_1})|^2 \cos^{|J|+n-2}(\theta_1 - \alpha s) e^{s(k+1)} r_1^{-1}\, d\theta_1\, dr_1\, ds \\ &\quad = \int_{\substack{0<|z_1|<\infty \\ -\pi/2<\arg(z_1)<2\alpha\ln\beta+\pi/2}} |f_{Jk}(z_1)|^2 W_{Jk}(\theta_1) |z_1|^{-2}\, dV(z_1); \end{aligned} \tag{3}$$

here C_{nJ} is a positive constant,

$$W_{Jk}(\theta_1) = C_{nJ} \int_{-\infty}^{\infty} \cos^{|J|+n-2}(\theta_1 - \alpha t) \chi_{\pi/2}(\theta_1 - \alpha t) e^{t(k+1)} \chi_{\ln\beta}(t - \ln\beta)\, dt,$$

and $\chi_a(t)$ is the characteristic function of the interval $[-a, a]$ for $a > 0$. (The positivity of C_{nJ} follows because we are integrating only over positive values of r_j.)

Now we use a change of coordinates $z = \ln z_1$ in the last integral to obtain

$$\begin{aligned} \|F_{Jk}\|_D^2 &= \int_{\substack{-\infty<x<\infty \\ -\pi/2<y<2\alpha\ln\beta+\pi/2}} |f_{Jk}(e^z)|^2 W_{Jk}(y)\, dV(z) \\ &= \int_{\substack{-\infty<x<\infty \\ -\pi/2<y<2\alpha\ln\beta+\pi/2}} |\tilde{f}_{Jk}(z)|^2 W_{Jk}(y)\, dV(z), \end{aligned} \tag{4}$$

where $z = x + iy$ and $\tilde{f}_{Jk}(z) = f_{Jk}(e^z)$. Then $\tilde{f}_{Jk}$ is a square-integrable holomorphic function on $S_{\alpha\beta} = \{z \in \mathbb{C} : -\pi/2 < \operatorname{Im}(z) < \pi/2 + 2\alpha\ln\beta\}$ with weight W_{Jk}. Furthermore, the Bergman kernel K_{Jk} for $A^2_{Jk}(D)$ can be calculated as

$$K_{Jk}(z, w) = K_{Jk}^{\alpha\beta}(\ln z_1, \ln w_1) \frac{z'^J z_n^k \bar{w}'^J \bar{w}_n^k}{z_1^{(|J|+n)/2} \bar{w}_1^{(|J|+n)/2}}, \tag{5}$$

where $K_{Jk}^{\alpha\beta}$ is the Bergman kernel on $S_{\alpha\beta}$ with weight W_{Jk}. (One way of seeing this is to observe that (4) allows us to convert an orthonormal basis for the Bergman space on $S_{\alpha\beta}$ with weight W_{Jk} to an orthonormal basis for A^2_{Jk}.)

Let $\mathcal{F}(f)$ denote the Fourier transform of f; thus

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\xi t}\, dt,$$

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{i\xi t}\, d\xi.$$

PROPOSITION 2. *$K_{Jk}^{\alpha\beta}$ is given by the integral*

$$K_{Jk}^{\alpha\beta}(z,w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{i(z-\bar{w})\xi}}{\mathcal{F}(W_{Jk})(-2i\xi)}\, d\xi. \tag{6}$$

Proof. See [Ba] and [CSh, Lemma 6.5.1].

Note also that $-\pi < \operatorname{Im}(z-\bar{w}) < \pi + 4\alpha \ln \beta$ for $z, w \in S_{\alpha\beta}$.

PROPOSITION 3. *The Fourier transform of W_{Jk} is given by*

$$\mathcal{F}(W_{Jk})(\xi) = D_{nJ} e^{-i\xi\pi/2} \frac{E_{Jk}(\xi)}{(\xi + |J| + n - 2)(\xi + |J| + n - 4) \cdots (\xi - |J| - n + 2)}, \tag{7}$$

where

$$E_{Jk}(\xi) = (e^{i\xi\pi} - (-1)^{|J|+n}) \left(\frac{e^{2(k+1-i\alpha\xi)\ln\beta} - 1}{k + 1 - i\alpha\xi} \right).$$

We postpone the proof of Proposition 3 until later in this section.

In order to apply residue methods to equation (6), we must find the zeros of $\mathcal{F}(W_{Jk})(-2i\xi)$. Let us denote the set $\{s \in \mathbb{Z} : -m \le s \le m\}$ by $\mathbb{I}(m)$. From Proposition 3 we see that if $|J| + n$ is even then the zeros of $\mathcal{F}(W_{Jk})(-2i\xi)$ are located at

$$\left\{ mi : m \in \mathbb{Z} \setminus \mathbb{I}\left(\frac{|J| + n - 2}{2} \right) \right\} \cup \left\{ \frac{m\pi i}{2\alpha \ln \beta} + \frac{k+1}{2\alpha} : m \in \mathbb{Z} \setminus \{0\} \right\}$$

whereas if $|J| + n$ is odd then they are located at

$$\left\{ mi + \frac{i}{2} : m \in \mathbb{Z} \setminus \left(\mathbb{I}\left(\frac{|J| + n - 3}{2} \right) \cup \left\{ -\frac{|J| + n - 1}{2} \right\} \right) \right\} \cup \left\{ \frac{m\pi i}{2\alpha \ln \beta} + \frac{k+1}{2\alpha} : m \in \mathbb{Z} \setminus \{0\} \right\}.$$

For simplicity we focus on the case $J = 0$, $k = -2$; in so doing, we guarantee that the zeros just enumerated are simple (see Remark 4 to follow).

Let $\nu_{\alpha\beta} = \frac{\pi}{2\alpha \ln \beta}$ and $\mu_\alpha = \frac{1}{2\alpha} > 0$.

PROPOSITION 4. *The kernels $K_{0,-2}$ satisfy*

$$K_{0,-2}(z,w) = \sum_{\ell=0}^{[\nu_{\alpha\beta} - n/2]} C_\ell z_1^\ell \bar{w}_1^{-\ell-n} z_n^{-2} \bar{w}_n^{-2} + C z_1^{\nu_{\alpha\beta} - n/2 - i\mu_\alpha} \bar{w}_1^{-\nu_{\alpha\beta} - n/2 + i\mu_\alpha} z_n^{-2} \bar{w}_n^{-2} + \mathcal{R}(z,w), \tag{8}$$

where $\varepsilon > 0$, the constants C and C_ℓ are nonzero, and the remainder term $\mathcal{R}(z,w)$ satisfies

$$\left(\frac{\partial}{\partial z_1}\right)^m \mathcal{R}(z,w) = O(z_1^{\nu_{\alpha\beta}-n/2+\varepsilon-m}\,\bar{w}_1^{-\nu_{\alpha\beta}-n/2-\varepsilon})$$

uniformly on closed subannuli of $1 < |z_n| < \beta$.

Proof. We apply the residue theorem to the integral in (6) along the strip $-\nu_{\alpha\beta}-\varepsilon \leq \operatorname{Im}\xi \leq 0$ to obtain

$$K^{\alpha\beta}_{0,-2}(z,w) = \sum_{\ell=0}^{[\nu_{\alpha\beta}-n/2]} C_\ell e^{(\ell+n/2)(z-\bar{w})} + Ce^{(\nu_{\alpha\beta}-i\mu_\alpha)(z-\bar{w})} + \widetilde{\mathcal{R}}(z,w)$$

for nonzero C and C_ℓ, where $\widetilde{\mathcal{R}}(z,w)$ and all of its derivatives are $O(e^{(\nu_{\alpha\beta}+\varepsilon)(z-\bar{w})})$ on closed substrips of $S_{\alpha\beta}$. If we plug the preceding equality into (5), the result is (8). □

REMARK 4. We have focused on the case $J = 0$, $k = -2$ because this is the simplest choice and avoids possible problems with double poles. Analogous formulas hold for other values of k in the absence of double poles. When double poles do occur, they contribute factors of $\ln(z_1 - \bar{w}_1)$.

LEMMA 1.

$$\sum_{s=0}^{j}\binom{j}{s}\frac{(-1)^s}{\xi+\alpha(j-2s)} = \frac{(-2\alpha)^j j!}{(\xi+\alpha j)(\xi+\alpha(j-2))\cdots(\xi-\alpha j)}.$$

Proof. The statement is true for $j = 0$.

Working inductively and recalling that $\binom{j}{s} = \binom{j-1}{s-1} + \binom{j-1}{s}$, we have

$$\begin{aligned}
&\sum_{s=0}^{j}\binom{j}{s}\frac{(-1)^s}{\xi+\alpha(j-2s)}\\
&\quad= \sum_{s=0}^{j-1}\binom{j-1}{s}\frac{(-1)^s}{\xi+\alpha(j-2s)} + \sum_{s=1}^{j}\binom{j-1}{s-1}\frac{(-1)^s}{\xi+\alpha(j-2s)}\\
&\quad= \frac{(-2\alpha)^{j-1}(j-1)!}{(\xi+\alpha j)(\xi+\alpha(j-2))\cdots(\xi+\alpha(-j+2))}\\
&\qquad - \frac{(-2\alpha)^{j-1}(j-1)!}{(\xi+\alpha(j-2))(\xi+\alpha(j-4))\cdots(\xi-\alpha j)}\\
&\quad= \frac{(-2\alpha)^{j-1}(j-1)!}{(\xi+\alpha(j-2))\cdots(\xi+\alpha(-j+2))}\left(\frac{1}{\xi+\alpha j}-\frac{1}{\xi-\alpha j}\right)\\
&\quad= \frac{(-2\alpha)^j j!}{(\xi+\alpha j)(\xi+\alpha(j-2))\cdots(\xi-\alpha j)}.
\end{aligned}$$

□

Proof of Proposition 3. Write

$$W_{Jk}(y) = C_{nJ}(W_{Jk1} * W_{Jk2})(y/\alpha)$$

for $-\pi/2 < y < \pi/2 + 2\alpha \ln \beta$, where $f * g$ denotes the convolution of f and g and where

$$W_{Jk1}(t) = \cos^{|J|+n-2}(\alpha t)\chi_{\pi/2}(\alpha t),$$
$$W_{Jk2}(t) = e^{t(k+1)}\chi_{\ln\beta}(t - \ln\beta).$$

To calculate the Fourier transform of W_{Jk} we first calculate

$$\cos^j(t) = \frac{1}{2^j}\sum_{s=0}^{j}\binom{j}{s}e^{i(2s-j)t}.$$

One can then calculate that

$$\mathcal{F}(\cos^j(t)\chi_{\pi/2}(t))(\xi) = \frac{1}{i\sqrt{2\pi}\,2^{j-1}}\sum_{s=0}^{j}\binom{j}{s}\frac{(e^{i(\xi+j-2s)\pi/2} - e^{-i(\xi+j-2s)\pi/2})}{2(\xi + j - 2s)}.$$

Lemma 1 now implies that

$$\begin{aligned}
&\mathcal{F}(\cos^j(\alpha t)\chi_{\pi/2}(\alpha t))(\xi)\\
&\quad= \frac{1}{\alpha}\mathcal{F}(\cos^j(t)\chi_{\pi/2}(t))\left(\frac{\xi}{\alpha}\right)\\
&\quad= \frac{i^{j-1}(e^{i\xi\pi/2\alpha} - (-1)^j e^{-i\xi\pi/2\alpha})}{\sqrt{2\pi}\,2^j}\sum_{s=0}^{j}\binom{j}{s}\frac{(-1)^s}{\xi + \alpha(j-2s)}\\
&\quad= \frac{(-\alpha i)^j j!\,(e^{i\xi\pi/2\alpha} - (-1)^j e^{-i\xi\pi/2\alpha})}{i\sqrt{2\pi}(\xi+\alpha j)(\xi+\alpha(j-2))\cdots(\xi-\alpha j)}.
\end{aligned}$$

We also need to find the Fourier transform of $e^{kt}\chi_a(t-a)$:

$$\mathcal{F}(e^{kt}\chi_a(t-a))(\xi) = \frac{1}{\sqrt{2\pi}}\frac{e^{2a(k-i\xi)} - 1}{k - i\xi}.$$

Using $\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g)$, we find that the Fourier transform of W_{Jk} is given by (7). □

4. Proof of Theorem 1

The proof of Theorem 1 follows immediately from Lemmas 3 and 4.

LEMMA 2. *If P is continuous on $W^{p,s}(\Omega_{\alpha\beta})$ then*

$$\left\| |r_\lambda|^t \left(\frac{\partial}{\partial z_1}\right)^m P_\lambda f \right\|_{L^p(D_\lambda)} \le C\|f\|_{W^{p,s}(D_\lambda)}, \tag{9}$$

where m is a nonnegative integer, $0 \le t < 1$ such that $m = s + t$, and the constant C is independent of λ and f.

Proof. Assume that P maps $W^{p,s}(\Omega_{\alpha\beta})$ onto itself continuously and let $T_\lambda f = f \circ \tau_\lambda$. Then one can check that

$$\left\| \left(\frac{\partial}{\partial z}\right)^{P} \left(\frac{\partial}{\partial \bar{z}}\right)^{Q} T_\lambda f \right\|_{L^p(\Omega_{\alpha\beta})} = 2^{p_1+q_1-2/p} \lambda^{2p_1+2q_1+|P'|+|Q'|-2n/p} \left\| \left(\frac{\partial}{\partial z}\right)^{P} \left(\frac{\partial}{\partial \bar{z}}\right)^{Q} f \right\|_{L^p(D_\lambda)};$$

in this equation, $P = (p_1, \dots, p_n)$, $Q = (q_1, \dots, q_n)$, $P' = (p_2, \dots, p_{n-1})$, $Q' = (q_2, \dots, q_{n-1})$, $|P'| = p_1 + \cdots + p_{n-1}$, and $|Q'| = q_1 + \cdots + q_{n-1}$. Therefore,

$$\|T_\lambda f\|_{W^{p,k}(\Omega_{\alpha\beta})} \le 2^{k-2/p} \lambda^{2k-2n/p} \|f\|_{W^{p,k}(D_\lambda)}.$$

By interpolation we also have $\|T_\lambda f\|_{W^{p,s}(\Omega_{\alpha\beta})} \le 2^{s-2/p} \lambda^{2s-2n/p} \|f\|_{W^{p,s}(D_\lambda)}$ for all $s > 0$.

Let $s = m - t$, where m is a nonnegative integer and $0 \le t < 1$. Then

$$\left\| |r|^t \left(\frac{\partial}{\partial z_1}\right)^m f \right\|_{L^p(\Omega_{\alpha\beta})} \le C_1 \|f\|_{W^{p,s}(\Omega_{\alpha\beta})} \tag{10}$$

for f holomorphic on $\Omega_{\alpha\beta}$ (see e.g. [L]).

Let P_λ be the Bergman projection for D_λ. Then $P_\lambda = T_\lambda^{-1} P T_\lambda$ and

$$\begin{aligned}
\left\| |r_\lambda|^t \left(\frac{\partial}{\partial z_1}\right)^m P_\lambda f \right\|_{L^p(D_\lambda)} &= \left\| |r_\lambda|^t \left(\frac{\partial}{\partial z_1}\right)^m T_\lambda^{-1} P T_\lambda f \right\|_{L^p(D_\lambda)} \\
&= 2^{2/p-m} \lambda^{2t+2n/p-2m} \left\| |r|^t \left(\frac{\partial}{\partial z_1}\right)^m P T_\lambda f \right\|_{L^p(\Omega_{\alpha\beta})} \\
&\le C_2 \lambda^{2t+2n/p-2m} \|P T_\lambda f\|_{W^{p,s}(\Omega_{\alpha\beta})} \\
&\le C_3 \lambda^{2n/p-2s} \|T_\lambda f\|_{W^{p,s}(\Omega_{\alpha\beta})} \\
&\le C_4 \|f\|_{W^{p,s}(D_\lambda)},
\end{aligned}$$

where the constants are independent of λ. □

LEMMA 3. *If the estimate* (9) *holds on D_λ then*

$$\left\| |r_\infty|^t \left(\frac{\partial}{\partial z_1}\right)^m P_\infty f \right\|_{L^p(D)} \le C \|f\|_{W^{p,s}(D)},$$

where P_∞ is the Bergman projection on D and the constant C is independent of f.

Proof. The proof follows that of [Ba, Lemma 1]. □

LEMMA 4. *Let $s \ge \nu_{\alpha\beta} + n\left(\frac{1}{p} - \frac{1}{2}\right)$, where $\nu_{\alpha\beta} = \frac{\pi}{2\alpha \ln \beta}$ and $s = m - t$ as before. Then there exists an $f \in C_0^\infty(D)$ such that $|r_\infty|^t \left(\frac{\partial}{\partial z_1}\right)^m P_\infty f$ is not in $L^p(D)$.*

Proof. Because P_{Jk} maps $W^{p,\delta}(D) \cap A^p(D)$ onto $W^{p,\delta}(D) \cap A^p_{Jk}(D)$ for all $\delta \ge 0$, it is sufficient to prove that there exists an $f \in C_0^\infty(D)$ such that $P_{Jk} P_\infty f \notin W^{p,s}(D)$. Fix $w \in D$, $J = 0$, and $k = -2$. Let f be a nonnegative smooth function with compact support in D such that f depends on $|z - w|$ and $\int_D f = 1$. Then $K_{0,-2}(\cdot, w) = P_{0,-2} P_\infty f$. We can write $s = m - t$ for m a nonnegative integer and $0 \le t < 1$. In view of [10] adapted to D, it suffices to show that $|r_\infty(z)|^t \frac{\partial^m}{\partial z_1^m} K_{0,-2}(z,w) \notin L^p(D)$ for fixed w. Proposition 4 implies that

$$\frac{\partial^m}{\partial z_1^m} K_{0,-2}(z,w) = C z_1^{\nu_{\alpha\beta}-n/2-i\mu_\alpha-m} + O(z_1^{\nu_{\alpha\beta}-n/2+\varepsilon-m}).$$

Let

$$D' = \big\{(z_1,z',z_n) \in \mathbb{C}^n : \mathrm{Re}(z_1 e^{-2\alpha i \ln|z_n|}) > |z'|^2,\ 1+\delta < |z_n| < \beta - \delta,\\ |z_1| < \delta,\ |\theta_1 - 2\alpha \ln|z_n|| < \pi/4\big\}$$

for suitably small $\delta > 0$. Then $|r_\infty|$ is comparable to $|z_1|$ on D' and

$$\int_D |r_\infty(z)|^{pt} \left|\frac{\partial^m}{\partial z_1^m} K_{0,-2}(z,w)\right|^p dV(z) \ge \int_{D'} |r_\infty(z)|^{pt} \left|\frac{\partial^m}{\partial z_1^m} K_{0,-2}(z,w)\right|^p dV(z)\\ \ge c \int_0^\delta r_1^{p\nu_{\alpha\beta}+pt-pm+n-1-pn/2}\, dr_1,$$

where c is a positive constant. The last integral in this expression is divergent if $s \ge \nu_{\alpha\beta} + n\big(\frac{1}{p} - \frac{1}{2}\big)$. As a result,

$$|r_\infty(z)|^t \frac{\partial^m}{\partial z_1^m} P_{0,-2} P_\infty f = |r_\infty(z)|^t \frac{\partial^m}{\partial z_1^m} K_{0,-2}(z,w) \notin L^p(D)$$

for $s \ge \nu_{\alpha\beta} + n\big(\frac{1}{p} - \frac{1}{2}\big)$. □

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