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Grimm's Conjecture and Smooth Numbers

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1. Introduction

In 1969, C. A. Grimm [8] proposed a seemingly innocent conjecture regarding prime factors of consecutive composite numbers. We begin by stating this conjecture.

Let $n \geq 1$ and $k \geq 1$ be integers. *Suppose* $n+1, \dots, n+k$ *are all composite numbers*; *then there are distinct primes* P_i *such that* $P_i \mid (n+i)$ *for* $1 \leq i \leq k$. That this is a difficult conjecture having several interesting consequences was first pointed out by Erdős and Selfridge [6]. For example, the conjecture implies there is a prime between two consecutive square numbers, something that is out of bounds even for the Riemann hypothesis. In this paper, we will pursue that theme. We will relate several results and conjectures regarding smooth numbers (defined in what follows) to Grimm's conjecture.

To begin, we say that Grimm's conjecture holds for n and k if there are distinct primes P_i such that $P_i \mid (n+i)$ for $1 \leq i \leq k$ whenever $n+1, \dots, n+k$ are all composites. For positive integers $n > 1$ and k, we say that (n,k) *has a prime representation* if there are distinct primes $P_1, P_2, \dots, P_k$ with $P_j \mid (n+j)$ for $1 \leq j \leq k$. We define $g(n)$ to be the maximum positive integer k such that (n,k) has a prime representation. It is an interesting problem to find the best possible upper and lower bounds for $g(n)$. If n' is the smallest prime greater than n, then Grimm's conjecture implies that $g(n) > n' - n$. However, it is clear that $g(2^m) < 2^m$ for $m > 3$.

The question of obtaining lower bounds for $g(n)$ was attacked using methods from transcendental number theory by Ramachandra, Shorey, and Tijdeman [15], who derived

$$g(n) \geq c\left(\frac{\log n}{\log\log n}\right)^3$$

for $n > 3$ and an absolute constant $c > 0$. In other words, for any sufficiently large natural number n, (n,k) has a prime representation if $k \ll (\log n/\log\log n)^3$.

We prove the following theorem.

Theorem 1. (i) *There exists an* $\alpha < \frac{1}{2}$ *such that* $g(n) < n^\alpha$ *for sufficiently large* n.

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(ii) *For* $\varepsilon > 0$, *we have* $|\{n \le X : g(n) \ge n^{\varepsilon}\}| \ll X \exp(-(\log X)^{1/3-\varepsilon})$, *where the implied constant depends only on* ε.

We show in Section 3 that $0.45 < \alpha < 0.46$ is permissible in Theorem 1(i).

For real x and y, let $\Psi(x, y)$ denote the number of positive integers $\le x$ all of whose prime factors do not exceed y. These are *y-smooth numbers,* which have been well studied. In 1930 Dickman [3] proved that, for any $\alpha \le 1$,

$$\lim_{x\to\infty} \frac{\Psi(x, x^{\alpha})}{x}$$

exists and equals $\rho(1/\alpha)$, where $\rho(t)$ is defined for $t \ge 0$ as the continuous solution of the equations $\rho(t) = 1$ for $0 \le t \le 1$ and $-t\rho'(t) = \rho(t-1)$ for $t \ge 1$. Later authors derived refined results; see [11] for an excellent survey on smooth numbers. An important conjecture on smooth numbers in short intervals is as follows.

CONJECTURE 1.1. *Let* $\varepsilon > 0$. *For sufficiently large* x, *we have*

$$\Psi(x + x^{\varepsilon}, x^{\varepsilon}) - \Psi(x, x^{\varepsilon}) \gg x^{\varepsilon}.$$

This remains an open question, but it leads to the following statement.

THEOREM 2. *Assume Conjecture 1.1 holds, and let* $\varepsilon > 0$. *Then* $g(n) < n^{\varepsilon}$ *for large* n.

Let p_i denote the ith prime. As a consequence of Theorem 2, we obtain the following result.

COROLLARY 1.2. *Assume Grimm's conjecture and Conjecture 1.1. Then, for any* $\varepsilon > 0$,

$$p_{i+1} - p_i < p_i^{\varepsilon} \tag{1}$$

for sufficiently large i.

If we assume Grimm's conjecture alone, then Erdős and Selfridge [6] have shown that

$$p_{i+1} - p_i \ll (p_i/\log p_i)^{1/2},$$

a statement well beyond what the Riemann hypothesis would imply about gaps between consecutive primes. Indeed, the Riemann hypothesis implies an upper bound of $O(p_i^{1/2}(\log p_i))$. In 1936, Cramér [2] conjectured that

$$p_{i+1} - p_i \ll (\log p_i)^2.$$

If Cramér's conjecture is true, then the result of [15] implies Grimm's conjecture—at least for sufficiently large numbers. In [12], Laishram and Shorey verified Grimm's conjecture for all $n < 1.9 \times 10^{10}$. They also checked that $p_{i+1} - p_i < 1 + (\log p_i)^2$ for $i \le 8.5 \times 10^8$.

It is worth mentioning the existence of several weaker versions of Grimm's conjecture that have also been attacked using methods of transcendental number theory. For an integer $\nu > 1$, we denote by $\omega(\nu)$ the number of distinct prime divisors of ν and let $\omega(1) = 0$. A weaker version of Grimm's conjecture runs as follows: *If* $n+1, n+2, \ldots, n+k$ *are all composite numbers, then* $\omega\big(\prod_{i=1}^{k}(n+i)\big) \geq k$. This conjecture, too, remains open, though much progress toward proving it has been made by Ramachandra, Shorey, and Tijdeman [16].

We define $g_1(n)$ to be the maximum positive integer k such that

$$\omega\left(\prod_{i=1}^{l}(n+i)\right) \geq l$$

for all $1 \leq l \leq k$. Observe that $g_1(n) \geq g(n)$. In Section 5 we prove the following.

THEOREM 3. *There exists a* γ *with* $0 < \gamma < \frac{1}{2}$ *such that*

$$g(n) \leq g_1(n) < n^{\gamma} \tag{2}$$

for large values of n.

We show in Section 5 that $\gamma = \frac{1}{2} - \frac{1}{390}$ is permissible. This result will be proved as a consequence of our next theorem, which is of independent interest.

THEOREM 4. *Suppose there exist* $0 < \alpha < \frac{1}{2}$ *and* $\delta > 0$ *such that*

$$\sum_{j \leq m^{\alpha}} \left\{ \pi\left(\frac{m+m^{\alpha}}{j}\right) - \pi\left(\frac{m}{j}\right) \right\} \geq \delta m^{\alpha} \tag{3}$$

for large m. *Then* $g_1(n) < n^{\gamma}$ *with*

$$\gamma = \max\left(\alpha, \frac{1-\delta(1-\alpha)}{2-\delta}\right) < \frac{1}{2}$$

for large n.

A conjecture arising from primes in short intervals states that

$$\pi(x + x^{\alpha}) - \pi(x) \sim \frac{x^{\alpha}}{\log x} \quad \text{as } x \to \infty$$

(see e.g. [13]). Assuming this conjecture, for $m \to \infty$ we obtain

$$\sum_{j \leq m^{\alpha}} \left\{ \pi\left(\frac{m+m^{\alpha}}{j}\right) - \pi\left(\frac{m}{j}\right) \right\} \sim \sum_{j \leq m^{\alpha}} \frac{\frac{m^{\alpha}}{j}}{\log \frac{m}{j}} = \frac{m^{\alpha}}{\log m} \sum_{j \leq m^{\alpha}} \frac{1}{j\big(1 - \frac{\log j}{\log m}\big)}$$
$$\sim \frac{m^{\alpha}}{\log m} \int_{1}^{m^{\alpha}} \frac{dt}{t\big(1 - \frac{\log t}{\log m}\big)}.$$

Taking $u = \frac{\log t}{\log m}$, we get

$$\sum_{j \leq m^{\alpha}} \left\{ \pi\left(\frac{m+m^{\alpha}}{j}\right) - \pi\left(\frac{m}{j}\right) \right\}$$
$$\sim m^{\alpha} \int_{0}^{\alpha} \frac{du}{1-u} = m^{\alpha}[-\log(1-u)]_{0}^{\alpha} = -m^{\alpha}\log(1-\alpha)$$

as $m \to \infty$. Continuing as in the proof of Theorem 4, we obtain $g_1(n) < n^{\alpha_1}$ with

$$\alpha_1 = \max\left(\alpha, \frac{1 + (1-\alpha)\log(1-\alpha)}{2 + \log(1-\alpha)}\right).$$

Since $\log(1-\alpha) \approx -\alpha$ for $0 < \alpha < 1$, it follows that

$$\frac{1 + (1-\alpha)\log(1-\alpha)}{2 + \log(1-\alpha)} \approx \frac{1 - \alpha(1-\alpha)}{2 - \alpha} = \frac{1}{2}(1 - \alpha + \alpha^2)\left(1 - \frac{\alpha}{2}\right)^{-1}$$
$$\approx \frac{1}{2}(1 - \alpha + \alpha^2)\left(1 + \frac{\alpha}{2}\right) = \frac{1}{4}(2 - \alpha + \alpha^2 + \alpha^3)$$

and the function $\frac{1}{4}(2 - \alpha + \alpha^2 + \alpha^3)$ attains its maximum at $\alpha = \frac{1}{3}$, where the value of $\alpha_1 \approx 0.4567$. We are therefore unlikely to get a result with $g_1(n) < n^\gamma$ and $\gamma < 0.4567$ by these methods. This value $g_1(n) = O(n^\alpha)$ does seem to agree with the permissible value of $0.45 < \alpha < 0.46$ in $g(n) = O(n^\alpha)$.

It was noted by Erdős and Selfridge in [6] that "the assertion $\gamma < \frac{1}{2}$ seems to follow from a recent result of Ramachandra [14] but we do not give the details here." In [5], Erdős and Pomerance noted again that, "from the proof in [14], it follows that there is an $\alpha > 0$ such that for all large n a positive proportion of the integers in $(n, n + n^\alpha]$ are divisible by a prime which exceeds $n^{15/26}$. Using this result with the method in [6] gives $g(n) < n^{1/2-c}$ for some fixed $c > 0$ and all large n." However, there is no proof anywhere in the literature about this fact. We give a complete proof here by generalizing the result of [14] in Lemma 2.5.

2. Preliminaries and Lemmas

We introduce some notation. We shall always write p for a prime number. Let $\Lambda(n)$ be the von Mangoldt function, which is defined as $\Lambda(n) = \log p$ if $n = p^r$ for some positive integer r and as 0 otherwise. We write $\theta(x) = \sum_{p \le x} \log p$. For real x and y, let $\Psi(x, y)$ denote the number of positive integers $\le x$ all of whose prime factors do not exceed y. We also write $\log_2 x$ for $\log\log x$. We begin with some results from prime number theory.

LEMMA 2.1. *Let $k, t \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then:*

(i) $\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right)$ *for* $x > 1$;
(ii) $p_t > t(\log t + \log_2 t - c_1)$ *for some* $c_1 > 0$ *and for large* t;
(iii) $\theta(x) \le 1.00008x$ *for* $x > 0$;
(iv) $\theta(p_t) > t(\log t + \log_2 t - c_2)$ *for some* $c_2 > 0$ *and for large* t; *and*
(v) $k! > \sqrt{2\pi k}\, e^{-k} k^k e^{1/(12k+1)}$ *for* $k > 1$.

The estimate (ii) is due to Rosser and Schoenfeld [18]. Inequalities (i), (iii), and (iv) are due to Dusart [4]. The estimate (v) is Stirling's formula; see [17].

The following results are due to Friedlander and Lagarias [7].

LEMMA 2.2. *Let $0 < \varepsilon < 1$ be fixed. Then there are positive constants c_0 and c_1, depending only on ε, such that there are at most $c_1 X \exp(-(\log X)^{1/3-\varepsilon})$ many n with $1 \le n \le X$ that do not satisfy*

$$\Psi(n + n^{\varepsilon}, n^{\varepsilon}) - \Psi(n, n^{\varepsilon}) \ge c_0 n^{\varepsilon}. \tag{4}$$

LEMMA 2.3. *There exist positive absolute constants α and c_1, with $\frac{3}{8} < \alpha < \frac{1}{2}$, such that*

$$\Psi(n + n^{\alpha}, n^{\alpha}) - \Psi(n, n^{\alpha}) > c_1 n^{\alpha} \tag{5}$$

for sufficiently large n.

Lemma 2.2 is obtained by taking $\alpha = \beta = \varepsilon$ in [7, Thm. 5], and Lemma 2.3 is obtained by taking $x = n$ and $y = z = n^{\alpha}$ with $\alpha = \frac{1}{2} - \frac{\eta}{2}$ in [7, Thm. 2.4]. From [10, Thm. 2] and the remarks after that, a permissible value of α in Lemma 2.3 is given by α in the range $0.45 < \alpha < 0.46$.

Our next lemma is the key result following from the definition of $g(n)$; it relates the study of $g(n)$ to smooth numbers.

LEMMA 2.4. *Let $x, y, z \in \mathbb{R}$ be such that $\Psi(x + z, y) - \Psi(x, y) > \pi(y)$. Then $g(\lfloor x \rfloor) < z$.*

Proof. Let $x \le n_1 < n_2 < \cdots < n_t \le x + z$ be all y-smooth numbers with $t > \pi(y)$. Then $(n_1, n_t - n_1)$ does not have a prime representation. In particular, $(\lfloor x \rfloor, \lfloor z \rfloor)$ has no prime representation. Thus $g(\lfloor x \rfloor) < z$. □

The next lemma generalizes a result of Ramachandra [14].

LEMMA 2.5. *Let $\frac{1}{33} < \lambda < \frac{1}{29}$. For $\alpha = \frac{1-\lambda}{2}$ and for sufficiently large x, we have*

$$\sum_{n \le x^{\alpha}} \left\{ \pi\left(\frac{x + x^{\alpha}}{n}\right) - \pi\left(\frac{x}{n}\right) \right\} \ge \left(\frac{1}{4} + \frac{\lambda}{2} - \varepsilon'\right) x^{\alpha}, \tag{6}$$

where $\varepsilon' > 0$ is arbitrary small.

We postpone the proof of Lemma 2.5 to Section 4.

3. Proof of Theorem 1 and Theorem 2

Proof of Theorem 1. (i) Let α be given by Lemma 2.3. We apply Lemma 2.4 by taking $x = n$ and $z = y = n^{\alpha}$. Since $\pi(y) = \pi(n^{\alpha}) < 2\frac{n^{\alpha}}{\alpha \log n} < c_1 n^{\alpha}$ for sufficiently large n, the assertion follows from Lemma 2.4 and Lemma 2.3. As remarked after Lemma 2.3, a permissible value of α is given by $0.45 < \alpha < 0.46$.

(ii) Let $\varepsilon > 0$ be given. By part (i) we may assume that $\varepsilon < \frac{1}{2}$. Since $\pi(n^{\varepsilon}) < 2\frac{n^{\varepsilon}}{\varepsilon \log n} < c_0 n^{\varepsilon}$ for sufficiently large n, with c_0 as in Lemma 2.2, the assertion now follows from Lemma 2.4 (taking $x = n$ and $z = y = n^{\varepsilon}$) and Lemma 2.2. □

Proof of Theorem 2. Let $\varepsilon > 0$ be given. We apply Lemma 2.4 by taking $x = n$ and $z = y = n^{\varepsilon}$. Because $\pi(y) = \pi(n^{\varepsilon}) < 2\frac{n^{\varepsilon}}{\varepsilon \log n} \ll n^{\varepsilon}$ for sufficiently large n, the assertion follows from Lemma 2.4 and Conjecture 1.1. □

4. Proof of Lemma 2.5

We follow the proof of Ramachandra in [14] and fill in the details as we go along. Let $\alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. By taking $\varepsilon = x^{\alpha-1}$ in [14, Lemma 1], we obtain

$$\sum_{n \le x^{1-\alpha}} \left\{\pi\left(\frac{x + x^{\alpha}}{n}\right) - \pi\left(\frac{x}{n}\right)\right\} \log \frac{x}{n} = (1-\alpha)x^{\alpha} \log x + O(x^{\alpha}). \tag{7}$$

We segment the interval $[\beta, 1-\alpha]$ as $0 < \beta = \beta_0 < \beta_1 < \cdots < \beta_m = 1-\alpha$ for some m. For $0 < r < s < 1$, let

$$S(r,s) = \sum_{x^r \le n \le x^s} \left\{\pi\left(\frac{x + x^{\alpha}}{n}\right) - \pi\left(\frac{x}{n}\right)\right\} \log \frac{x}{n}. \tag{8}$$

We would like to obtain an upper bound for $S(\beta, 1-\alpha) = \sum_{i=0}^{m-1} S(\beta_i, \beta_{i+1})$. We first prove the following lemma, which is minor refinement of [14, Lemma 3].

LEMMA 4.1. *Let $x \ge 1$ and $1 \le R \le S \le x^{1-\alpha}$. For an integer $d \ge 1$, let*

$$R_d = \sum_{R \le n \le S} \left\{\left[\frac{x + x^{\alpha}}{nd}\right] - \left[\frac{x}{nd}\right]\right\}. \tag{9}$$

Then

$$\sum_{R \le n \le S} \left\{\pi\left(\frac{x + x^{\alpha}}{n}\right) - \pi\left(\frac{x}{n}\right)\right\}$$
$$\le \frac{(2-\varepsilon)x^{\alpha}}{\log z} \log\left(\frac{S}{R} + 2\right)\left(1 + O\left(\frac{1}{R} + \frac{1}{\log z}\right)\right) + O\left(z \max_{d \le z} |R_d|\right), \tag{10}$$

where $z \ge 3$ is an arbitrary real number and $\varepsilon > 0$ is arbitrary small.

Proof. Let

$$T = \bigcup_{R \le n \le S} \left(\left(\frac{x}{n}, \frac{x + x^{\alpha}}{n}\right] \cap \mathbb{Z}\right).$$

From T we remove those elements that are divisible by primes $\le \sqrt{z}$ and let T_1 denote the remaining set. We remark that, for each d, the number of integers in T that are divisible by d is equal to

$$\frac{x^{\alpha}}{d} \sum_{R \le n \le S} \frac{1}{n} + R_d.$$

Using Selberg's sieve as in [14], we obtain the assertion of lemma. □

Let $\phi(u) = u - [u] - \frac{1}{2}$. Then we can write

$$\left[\frac{x + x^\alpha}{nd}\right] - \left[\frac{x}{nd}\right] = \frac{x^\alpha}{nd} - \phi\left(\frac{x + x^\alpha}{nd}\right) + \phi\left(\frac{x}{nd}\right).$$

The next lemma is a restatement of [14, Lemma 2], which follows from a result of van der Corput (see [14]).

LEMMA 4.2. *Let $u \geq 1$, let V and V_1 be real numbers satisfying $3 \leq V < V_1 \leq 2V$ and $V_1 \geq V + 1$, and let $u \leq \eta \leq 2u$. Then*

$$\sum_{V \leq n \leq V_1} \phi\left(\frac{\eta}{n}\right) = O(V^{1/2} \log V + V^{3/2} u^{-1/2} + u^{1/3}). \tag{11}$$

To obtain an upper bound for $S(\beta_i, \beta_{i+1})$, we take $R = x^{\beta_i}$ and $S = x^{\beta_{i+1}}$ in Lemma 4.1. Recall that $\beta_{i+1} \leq 1 - \alpha$. We subdivide $(R, S]$ into intervals of type $(V, 2V]$ and at most one interval of type $(V, V_1]$ with $V_1 \leq 2V$. We apply Lemma 4.2 twice, taking $\eta = \frac{x}{d}$ and $\eta = \frac{x + x^\alpha}{d}$, to get

$$\begin{aligned} R_d &= O\left(\left(x^{\beta_{i+1}/2} + x^{(3\beta_{i+1}-1)/2} d^{1/2} + \left(\frac{x}{d}\right)^{1/3}\right)(\log x)^2\right) \\ &= O\big((x^{1-3\alpha/2} d^{1/2} + x^{1/3})(\log x)^2\big), \end{aligned}$$

since $\beta_{i+1} \leq 1 - \alpha$. Let $3\alpha - \frac{4}{3} < \delta < \frac{5\alpha - 2}{3}$ and take $z = x^\delta$. Then

$$z \max_{d \leq z} |R_d| = O(x^{1-3\alpha/2+3\delta/2} (\log x)^2)$$

and $1 - \frac{3}{2}\alpha + \frac{3}{2}\delta < \alpha$. From (10) we obtain

$$\sum_{x^{\beta_i} \leq n \leq x^{\beta_{i+1}}} \left\{\pi\left(\frac{x + x^\alpha}{n}\right) - \pi\left(\frac{x}{n}\right)\right\} \leq \frac{2x^\alpha}{\delta}(\beta_{i+1} - \beta_i);$$

hence an upper bound for

$$\sum_{x^\beta \leq n \leq x^{1-\alpha}} \left\{\pi\left(\frac{x + x^\alpha}{n}\right) - \pi\left(\frac{x}{n}\right)\right\} \log \frac{x}{n}$$

is

$$\frac{2x^\alpha \log x}{\delta}\{(\beta_1 - \beta_0)(1 - \beta_0) + (\beta_2 - \beta_1)(1 - \beta_1) + \cdots + (\beta_m - \beta_{m-1})(1 - \beta_{m-1})\}.$$

We take the β_i to be equally spaced and take m sufficiently large. Since

$$\frac{2x^\alpha \log x}{\delta} \int_\beta^{1-\alpha} (1 - t)\, dt = \frac{x^\alpha \log x}{\delta}(1 - \alpha^2 - \beta(2 - \beta)),$$

it follows from (7) that

$$\sum_{n \leq x^\beta} \left\{\pi\left(\frac{x + x^\alpha}{n}\right) - \pi\left(\frac{x}{n}\right)\right\} \geq \left(1 - \alpha - \varepsilon' - \frac{1 - \alpha^2 - \beta(2 - \beta)}{\delta}\right) x^\alpha, \tag{12}$$

where $1 - \frac{3}{2}\alpha + \frac{3}{2}\delta < \alpha$ and $\varepsilon' > 0$ is arbitrarily small.

Let $\frac{1}{33} < \lambda < \frac{1}{29}$, and put $\alpha = \beta = \frac{1-\lambda}{2}$ and $\delta = 4\lambda$. Then $1 - \frac{3}{2}\alpha + \frac{3}{2}\delta < \alpha$ and hence, from (12), we obtain (6). This completes the proof of Lemma 2.5.

5. Proof of Theorem 4 and Theorem 3

Proof of Theorem 4. Recall that $g_1(n)$ is the largest integer k such that

$$\omega\left(\prod_{i=1}^{l}(n+i)\right) \geq l$$

for $1 \leq l \leq k$. Now suppose that $g_1(n) > n^\gamma$, in which case $g_1(n) > n^\alpha$. Let $k = [n^\alpha]$. Then

$$\omega(P) \geq k, \quad P = \prod_{i=1}^{k}(n+i).$$

By (3), we have

$$\sum_{j \leq k} \pi\left(\frac{n+k}{j}\right) - \pi\left(\frac{n}{j}\right) \geq \delta k.$$

Now the intervals $[n, n+k]$, $[n/2, (n+k)/2], \ldots$ are disjoint intervals. In fact, if we write $I_j = [n/j, (n+k)/j] = [a_j, b_j]$ (say), then it is easy to see $b_1 > a_1 > b_2 > a_2 > b_3 > a_3 \cdots$ by virtue of the condition that $k < n^\alpha$ with $\alpha < 1/2$. A prime q_i (say) lying in the interval I_j satisfies $n < jq_i < n+k$ and consequently is a prime dividing P. Since these primes q_i are all distinct and since all of them are greater than $n/k \geq n^{1-\alpha}$, we deduce that there are at least δk distinct primes greater than $n^{1-\alpha}$ dividing P. Let $\delta' \geq \delta$ be such that $\delta' k = \lceil \delta k \rceil$. Since $\omega(P) \geq k$, there are at least $(1-\delta')k$ other primes dividing P and we have $(1-\delta')k \in \mathbb{Z}$. Also, $k!|P$ because P is a product of k consecutive numbers. All the prime factors of $k!$ are less than or equal to $k < n^\alpha < n^{1-\alpha}$ since $\alpha < 1/2$. Therefore,

$$P \geq k!\left(\prod_{k<p<p_{(1-\delta')k}} p\right)(n^{1-\alpha})^{\delta' k}.$$

Now we apply the bounds provided by Lemma 2.1. By Lemma 2.1(iii) and (iv), we obtain

$$\begin{aligned}\log\left(\prod_{k<p\leq p_{(1-\delta')k}} p\right) &= \theta(p_{(1-\delta')k}) - \theta(k)\\ &\geq (1-\delta')k\log(1-\delta')k + (1-\delta')k\{\log_2(1-\delta')k - c_2\}\\ &\quad - 1.00008k\\ &> (1-\delta')k\log(1-\delta')k + k(c_3\log_2 c_3 k - c_4),\end{aligned}$$

where c_3 and c_4 are positive constants. This, together with $k! > (k/e)^k$ (by Lemma 2.1(v)) and $P < (2n)^k$, implies that

$$\begin{aligned}2n &> \frac{k}{e}(1-\delta')^{1-\delta'}k^{1-\delta'}c_5(\log c_3 k)^{c_3}n^{\delta'(1-\alpha)}\\ &= \frac{1}{e}(1-\delta')^{1-\delta'}c_5(\log c_3 k)^{c_3}\left(\frac{k}{n^\gamma}\right)^{2-\delta'} n^{\gamma(2-\delta')}n^{\delta'(1-\alpha)}\\ &> 2n^{\gamma(2-\delta')+\delta'(1-\alpha)} \geq 2n^{\gamma(2-\delta)+\delta(1-\alpha)}2n\end{aligned}$$

for large n, since $\delta' > \delta$ and $1-\alpha > \frac{1}{2} > \gamma$. This is a contradiction, so $g_1(n) < k \leq n^\alpha \leq n^\gamma$. □

Proof of Theorem 3. From Lemma 2.5, we obtain (3) with $\alpha = \frac{1-\alpha}{2}$ and $\delta = \frac{1}{4} + \frac{\lambda}{2} - es'$ for some $\frac{1}{33} < \lambda < \frac{1}{29}$. Now the assertion follows from (4). Taking $\lambda = \frac{1}{30} + 2\varepsilon'$, for instance, we get $\gamma \leq \frac{1}{2} - \frac{1}{390}$. □

REMARK. It is possible to improve the result obtained here, but the improvement is not substantial. Indeed, the result of van der Corput has been improved, and a small refinement is possible using methods of Baker and Harman [1]. The details are rather technical and will be discussed in a future paper by the first author.

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