Michigan Math. J. 61 (2012), 123–131

A Relation between Height, Area, and Volume for Compact Constant Mean Curvature Surfaces in $\mathbb{M}^2 \times \mathbb{R}$

CLAUDEMIR LEANDRO & HAROLD ROSENBERG

1. Introduction

Let Σ be a compact CMC-H surface in $\mathbb{M}^2 \times \mathbb{R}$ with $\Gamma = \partial\Sigma \subset \mathbb{M}^2 \times \{0\}$, where $\mathbb{M}^2$ is a Hadamard surface with curvature $K_{\mathbb{M}^2} \leq -\kappa \leq 0$. Let Σ_1 be the connected component of the part of Σ above the plane $Q = \mathbb{M}^2 \times \{0\}$, and let h be the height of Σ_1 above Q. We will determine a volume V_1 bounded by Σ_1 and prove that

$$h \leq \frac{H|\Sigma_1|}{2\pi} - \frac{\kappa V_1}{4\pi};$$

here $|\Sigma_1|$ is the area of Σ_1. We also state conditions under which equality occurs.

We then let $\mathbb{M}^2 = \mathbb{H}^2$ be the hyperbolic plane of curvature -1, with $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ a compact CMC-H surface as just described. Finally, we give a condition that guarantees Σ lies in a half-space determined by Q.

We introduce some definitions and notation as follows. Let $\gamma \subset Q$ be a complete geodesic. We call $P = \gamma \times \mathbb{R}$ a *vertical plane* of $\mathbb{M}^2 \times \mathbb{R}$. Let $\beta(t)$ be a complete geodesic of Q, with $\beta(0)$ in the vertical plane P and $\beta'(0)$ orthogonal to P. Let $P_\beta(t)$ be the vertical plane of $\mathbb{M}^2 \times \mathbb{R}$ that passes through $\beta(t)$ and is orthogonal to β at $\beta(t)$. We call $P_\beta(t)$ the vertical plane *foliation* determined by P and β.

2. The Main Result

Let $\Sigma \subset \mathbb{M}^2 \times \mathbb{R}$ be a CMC-H surface as before and suppose that Σ meets Q transversally along $\Gamma = \partial\Sigma \subset Q$. We put $\Sigma^+ = \Sigma \cap (\mathbb{M}^2 \times \mathbb{R}_+)$ and $\Sigma^- = \Sigma \cap (\mathbb{M}^2 \times \mathbb{R}_-)$. There is a connected component of Σ^+ or Σ^- that contains Γ. We can assume, without loss of generality, that $\Gamma \subset \partial\Sigma^+$. We use Σ_1 to denote the connected component of Σ^+ that contains Γ.

Let $\hat{\Sigma}_1$ be the symmetry of Σ_1 through the plane Q. Then $\hat{\Sigma}_1 \cup \Sigma_1$ is a compact embedded surface with no boundary, and with corners along $\partial\Sigma_1$, that bounds a domain U in $\mathbb{M}^2 \times \mathbb{R}$. Let U_1 be the intersection of U with the half-space above Q. Thus U_1 is a bounded domain in $\mathbb{M}^2 \times \mathbb{R}$ whose boundary, ∂U_1, consists of the smooth connected surface Σ_1 and the union Ω of finitely smooth, compact and connected surfaces in Q. We define A^+ to be the area of Σ_1.

Received May 26, 2010. Revision received September 21, 2011.

THEOREM 2.1. *Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \leq -\kappa \leq 0$. Let Σ be a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$, with boundary belonging to $Q = \mathbb{M}^2 \times \{0\}$ and transverse to Q. If h denotes the height of Σ with respect to Q, then*

$$h \leq \frac{HA^+}{2\pi} - \frac{\kappa \operatorname{Vol}(U_1)}{4\pi} \tag{1}$$

for A^+ and U_1 defined as before. There is equality if and only if $K \equiv -\kappa$ inside U_1 and Σ is a rotational spherical cap.

Proof. From the surface Σ we obtain the surface Σ_1, the bounded domain $U_1 \subset \mathbb{M}^2 \times \mathbb{R}$, and the union Ω of finitely smooth, compact and connected surfaces in Q as just described. Let $\vec{H}$ denote the mean curvature vector of Σ_1, and take the unit normal N of Σ_1 to point inside U_1. Let $\pi_1 \colon \mathbb{M}^2 \times \mathbb{R} \to \mathbb{M}^2$ and $\pi_2 \colon \mathbb{M}^2 \times \mathbb{R} \to \mathbb{R}$ be the usual projections. If we denote by $h_1 \colon \Sigma_1 \to \mathbb{R}$ the height function of Σ_1—that is, $h_1(p) = \pi_2(p)$ and $\nu = \langle N, \frac{\partial}{\partial t}\rangle$—then we can write

$$\frac{\partial}{\partial t} = T + \nu N, \tag{2}$$

where T is a tangent vector field on Σ_1. Since $\frac{\partial}{\partial t}$ is the gradient in $\mathbb{M}^2 \times \mathbb{R}$ of the function t, it follows that T is the gradient of h_1 on Σ_1.

If $H = 0$ then h (the height of Σ) is a harmonic function and therefore, by the maximum principle, $\Sigma \subset \mathbb{M}^2 \times \{0\}$. So we suppose that $H > 0$.

Let $A(t)$ be the area of $\Sigma_t = \{p \in \Sigma_1;\ h_1(p) \geq t\}$ and let $\Gamma(t) = \{p \in \Sigma_1;\ h_1(p) = t\}$. By [6, Thm. 5.8] we have

$$A'(t) = -\int_{\Gamma(t)} \frac{1}{\|\nabla h_1\|}\, ds_t, \quad t \in \mathcal{O},$$

where $\mathcal{O}$ is the set of all regular values of h_1.

If $L(t)$ denotes the length of the planar curve $\Gamma(t)$, then the Schwartz inequality yields

$$L^2(t) \leq \int_{\Gamma(t)} \|\nabla h_1\|\, ds_t \int_{\Gamma(t)} \frac{1}{\|\nabla h_1\|}\, ds_t = -A'(t) \int_{\Gamma(t)} \|\nabla h_1\|\, ds_t, \quad t \in \mathcal{O}. \tag{3}$$

But from (2) we have that, along the curve $\Gamma(t)$,

$$\|\nabla h_1\|^2 = 1 - \nu^2 = \left\langle \eta^t, \frac{\partial}{\partial t} \right\rangle^2;$$

here η^t is the inner conormal of Σ_t along $\partial\Sigma_t$. Since Σ_t is above the plane $Q(t)$, we know that $\langle \eta^t, \frac{\partial}{\partial t}\rangle \geq 0$. Hence

$$\|\nabla h_1\| = \left\langle \eta^t, \frac{\partial}{\partial t} \right\rangle.$$

Therefore, (3) may be rewritten as

$$L^2(t) \leq -A'(t) \int_{\Gamma(t)} \left\langle \eta^t, \frac{\partial}{\partial t} \right\rangle ds_t. \tag{4}$$

Now we recall the flux formula. Let Σ_t and $\Omega(t)$ be two compact, smooth, and embedded but not necessarily connected surfaces in $\mathbb{M}^2 \times \mathbb{R}$ such that their boundaries coincide. Assume that there exists a compact domain $U(t)$ in $\mathbb{M}^2 \times \mathbb{R}$ such that the boundary of $U(t)$ is $\partial U(t) = \Sigma_t \cup \Omega(t)$ and is orientable. Notice that the boundary of $U(t)$ is smooth except perhaps along $\partial\Sigma_t = \partial\Omega(t)$.

Let N_{Σ_t} and $N_{\Omega(t)}$ be the unit normal fields to Σ_t and $\Omega(t)$, respectively, that point inside $U(t)$. Finally, assume that Σ_t is a compact surface with constant mean curvature $H = \langle \vec{H}, N_{\Sigma_t}\rangle > 0$. Let Y be a Killing vector field in $\mathbb{M}^2 \times \mathbb{R}$. Then the flux formula (i.e., [4, Prop. 3]) yields

$$\int_{\partial\Sigma_t} \langle Y, \eta^t\rangle = 2H \int_{\Omega(t)} \langle Y, N_{Q(t)}\rangle. \tag{5}$$

Using (5), we can take $Y = \frac{\partial}{\partial t}$ to obtain

$$\int_{\Gamma(t)} \left\langle \frac{\partial}{\partial t}, \eta^t \right\rangle = 2H\|\Omega(t)\|,$$

where $\|\Omega(t)\|$ is the area of the planar region $\Omega(t)$. Hence substituting into (4) results in

$$L^2(t) \leq -2HA'(t)\|\Omega(t)\| \quad \text{for almost every } \; t \geq 0,\, t \in \mathcal{O}. \tag{6}$$

Next we will show that

$$L^2(t) \geq 4\pi\|\Omega(t)\| + \kappa\|\Omega(t)\|^2. \tag{7}$$

We put $\Omega(t) = \bigcup_{i=1}^{n_t} \Omega_i(t)$, where $\Omega_1(t), \ldots, \Omega_{n_t}(t)$ are bounded domains determined in the plane $Q(t)$ by the closed curve $\Gamma(t)$ and where $\|\Omega_i(t)\|$ (with $i = 0, \ldots, n_t$) is the area of the corresponding $\Omega_i(t)$. Then $\|\Omega(t)\| = \sum_{i=1}^{n_t}\|\Omega_i(t)\|$. We know by [2] that equation (7) holds if $n_t = 1$. Supposing the result is true for $n_t = m$, we will prove it to be true also for $m + 1$.

Let $\tilde{L}(t)$ be the length of $\tilde{\Omega}(t) = \bigcup_{i=1}^{m} \Omega_i(t)$. We know that

$$\tilde{L}^2(t) \geq 4\pi\|\tilde{\Omega}(t)\| + \kappa\|\tilde{\Omega}(t)\|^2 \quad \text{(by hypothesis of induction)}, \tag{8}$$

$$L^2_{m+1}(t) \geq 4\pi\|\Omega_{m+1}(t)\| + \kappa\|\Omega_{m+1}(t)\|^2 \quad \text{(by [2])}. \tag{9}$$

Inequalities (8) and (9) imply, respectively,

$$\tilde{L}(t) \geq \sqrt{\kappa}\|\tilde{\Omega}(t)\|,$$

$$L_{m+1}(t) \geq \sqrt{\kappa}\|\Omega_{m+1}(t)\|.$$

Therefore,

$$\tilde{L}(t)L_{m+1}(t) \geq \kappa\|\tilde{\Omega}(t)\|\|\Omega_{m+1}(t)\|$$
$$\Longrightarrow\; 2\tilde{L}(t)L_{m+1}(t) \geq 2\kappa\|\tilde{\Omega}(t)\|\|\Omega_{m+1}(t)\|. \tag{10}$$

Combining (8), (9), and (10) yields

$$(\tilde{L}(t) + L_{m+1}(t))^2 \geq 4\pi(\|\tilde{\Omega}(t)\| + \|\Omega_{m+1}(t)\|) + \kappa(\|\tilde{\Omega}(t)\| + \|\Omega_{m+1}(t)\|)^2,$$

and this proves (7).

From (6) and (7) it follows that

$$4\pi\|\Omega(t)\| + \kappa\|\Omega(t)\|^2 \le -2HA'(t)\|\Omega(t)\|,$$

$$4\pi\|\Omega(t)\| + \kappa\|\Omega(t)\|^2 + 2HA'(t)\|\Omega(t)\| \le 0,$$

$$(4\pi + 2HA'(t) + \kappa\|\Omega(t)\|)\|\Omega(t)\| \le 0,$$

$$4\pi + 2HA'(t) + \kappa\|\Omega_i(t)\| \le 0.$$

After integrating the last inequality from 0 to $h = \max_{p\in\Sigma} h_1(p) \ge 0$, we have

$$4\pi h + 2H(A(h) - A(0)) + \kappa\,\mathrm{Vol}(U_1) \le 0;$$

then

$$A^+ = A(0) \ge \frac{2\pi h}{H} + \frac{\kappa\,\mathrm{Vol}(U_1)}{2H},$$

which is the inequality that we were seeking.

If equality holds, then all the preceding inequalities become equalities. In particular, by [2] it will follow that $\Gamma(t)$ is the boundary of a geodesic disk in $\mathbb{M}^2 \times \{t\}$ for every $t \ge 0$ and that $K_{\mathbb{M}^2}(p) \equiv -\kappa$ for all $p \in U$.

Let $D \subset \mathbb{M}^2 \times \{0\}$ be the geodesic disk such that $\partial D = \partial\Sigma$, and let $p \in D$ be the center of D. Let γ be a horizontal, complete, oriented geodesic passing through the point p with $\gamma(0) = p$, and let $P_\gamma(t)$ be the oriented foliation of vertical planes along the γ. Let $P_\gamma(t_1)$ be a vertical plane in this horizontal foliation that does not touch Σ. Now, performing Alexandrov reflection with the planes $P_\gamma(t)$, starting at $t = t_1$ and then decreasing t, we obtain—by the symmetries of ∂D—that Σ is symmetric with respect to $P_\gamma(0)$. Since γ is an arbitrary horizontal complete geodesic passing through the point p, it follows that Σ is a rotational spherical cap.

COROLLARY 2.1. *Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \le -\kappa \le 0$. Let Σ be a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$ without boundary but with area A, and let U be the compact domain bounded by Σ. Then*

$$2HA \ge \kappa\,\mathrm{Vol}(U) + 4\pi h.$$

Equality holds if and only if Σ is a sphere of revolution.

COROLLARY 2.2. *Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \le -\kappa \le 0$. If Σ is a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$ with boundary in a plane Q and transverse to Q, then*

$$\kappa\,\mathrm{Vol}(U_1) < 2\pi HA^+$$

for A^+ and U_1 defined as before Theorem 2.1.

3. Horizontal H-cylinders in $\mathbb{H}^2 \times \mathbb{R}$

Now we use a translation-invariant H-hypersurface given by P. Bérard and R. Sa Earp in [3] to give some conditions implying that Σ lies above $Q = \mathbb{H}^2 \times \{0\}$ when $\partial\Sigma \subset Q$. We recall some ideas here.

Let γ_1 be a geodesic passing through $0 \in \mathbb{H}^2 \times \{0\}$ in $Q = \mathbb{H}^2 \times \{0\}$ and let $P_1 = \gamma_1 \times \mathbb{R} = \{(\gamma_1(s), t);\ (s,t) \in \mathbb{R}^2\}$ be the vertical plane, where s is the signed hyperbolic distance to 0 on γ_1.

Take a geodesic γ_2 such that $\gamma_2(0) = \gamma_1(0)$ and $\gamma_2'(0) \perp \gamma_1'(0)$. We consider the hyperbolic translation with respect to the geodesic γ_2. In the vertical plane P_1 we take the curve $\alpha(s) = (s, f(s))$, where f is a real function.

In $\mathbb{H}^2 \times \{f(s)\}$ we translate the point $\alpha(s)$ by the translations with respect to $\gamma_2 \times \{f(s)\}$, which yields the equidistant curves $(\gamma_2)_{\alpha(s)}$ passing through $\alpha(s)$ at a distance s from $\gamma_2 \times \{f(s)\}$. The curve α then generates a translation surface $C = \bigcup_s (\gamma_2)_{\alpha(s)}$ in $\mathbb{H}^2 \times \mathbb{R}$.

Principal Curvatures. The principal directions of curvature of C are tangent to the curve α in P_1 and the directions tangent to $(\gamma_2)_{\alpha(s)}$. The corresponding principal curvatures with respect to the unit normal pointing downward are given by

$$k_{P_1} = -f''(s)(1 + (f'(s))^2)^{-3/2} \quad \text{and}$$
$$k_{(\gamma_2)_{\alpha(s)}} = -f'(s)(1 + (f'(s))^2)^{-1/2} \tanh(s).$$

The first equality holds because P_1 is totally geodesic and flat. The second equality follows because $(\gamma_2)_{\alpha(s)}$ is at a distance s from $\gamma_2 \times \{f(s)\}$ in $\mathbb{H}^2 \times \{f(s)\}$.

Mean Curvature. The mean curvature of the translation surface C associated with f is given by

$$\begin{aligned} 2H(s) &= -f''(s)(1 + (f'(s))^2)^{-3/2} - f'(s)(1 + (f'(s))^2)^{-1/2} \tanh(s), \\ 2H(s)\cosh(s) &= -f''(s)(1 + (f'(s))^2)^{-3/2} \cosh(s) \\ &\quad - f'(s)(1 + (f'(s))^2)^{-1/2} \sinh(s), \\ 2H(s)\cosh(s) &= -\frac{d}{ds}\big(f'(s)(1 + (f'(s))^2)^{-1/2} \cosh(s)\big). \end{aligned}$$

We assume that $H =$ constant. Observe that in our case $H > 0$. The generating curves of translation surfaces with mean curvature H are given by the differential equation

$$-f'(s)(1 + (f'(s))^2)^{-1/2} \cosh(s) = 2H \sinh(s) + d_1,$$

where d_1 is a constant.

We want that $f'(0) = 0$, so we take $d_1 = 0$. Therefore,

$$\begin{aligned} -f'(s)(1 + (f'(s))^2)^{-1/2} &= 2H \tanh(s), \\ -f'(s) &= 2H \tanh(s)(1 + (f'(s))^2)^{1/2}, \\ (f'(s))^2 &= 4H^2 \tanh^2(s)(1 + (f'(s))^2) \\ &= 4H^2 \tanh^2(s) + (f'(s))^2 4H^2 \tanh^2(s) \\ &= \frac{4H^2 \tanh^2(s)}{1 - 4H^2 \tanh^2(s)}. \end{aligned}$$

We have two first-order, linear ordinary differential equations given by

$$f_+'(s) = -\frac{2H\tanh(s)}{\sqrt{1-4H^2\tanh^2(s)}} \quad \text{and} \quad f_-'(s) = \frac{2H\tanh(s)}{\sqrt{1-4H^2\tanh^2(s)}}$$

with $s \in (-s_H, s_H)$, where $s_H = \operatorname{arctanh}(1/2H)$.

We assume that $H > 1/2$. After resolving the previous equations, we get

$$f_+(s) = -\frac{2H}{\sqrt{4H^2-1}} \arctan\left(\frac{\sqrt{4H^2-1}}{\sqrt{1-4H^2\tanh^2(s)}}\right) + d_2$$

and

$$f_-(s) = \frac{2H}{\sqrt{4H^2-1}} \arctan\left(\frac{\sqrt{4H^2-1}}{\sqrt{1-4H^2\tanh^2(s)}}\right) + d_3,$$

respectively, where d_2 and d_3 are constant.

We want that $\lim_{s\to\pm s_H} f_+(s) = \lim_{s\to\pm s_H} f_-(s) = 0$, so we take $d_2 = -d_3 = H\pi/\sqrt{4H^2-1}$. Hence

$$f_+(s) = -\frac{2H}{\sqrt{4H^2-1}} \left(\arctan\left(\frac{\sqrt{4H^2-1}}{\sqrt{1-4H^2\tanh^2(s)}}\right) - \frac{\pi}{2}\right)$$

and

$$f_-(s) = \frac{2H}{\sqrt{4H^2-1}} \left(\arctan\left(\frac{\sqrt{4H^2-1}}{\sqrt{1-4H^2\tanh^2(s)}}\right) - \frac{\pi}{2}\right).$$

We have two curves, $\alpha_+(s) = (s, f_+(s))$ and $\alpha_-(s) = (s, f_-(s))$. The curve $\alpha = \alpha_+ \cup \alpha_-$ generates a complete embedded translation invariant H-surface, C_H, which we call an H-cylinder.

Observe that the height of C_H is given by

$$h_{C_H} = -\frac{4H}{\sqrt{4H^2-1}} \left(\arctan\left(\sqrt{4H^2-1}\right) - \tfrac{\pi}{2}\right).$$

Since $\arctan(1/x) = \pi/2 - \arctan x$ for $x > 0$, it follows that

$$h_{C_H} = \frac{4H}{\sqrt{4H^2-1}} \arctan\left(\frac{1}{\sqrt{4H^2-1}}\right).$$

But $\arctan x = \arcsin\left(x/\sqrt{1+x^2}\right)$, so

$$h_{C_H} = \frac{4H}{\sqrt{4H^2-1}} \arcsin\left(\frac{1}{2H}\right).$$

By Aledo, Espinar, and Gálvez [1] we have that the height of the rotational H-sphere, S_H, is equal to

$$\frac{8H}{\sqrt{4H^2-1}} \arcsin\left(\frac{1}{2H}\right);$$

therefore,

$$h_{C_H} = \frac{h_{S_H}}{2}.$$

We can use these C_H-cylinders to prove the theorem that follows.

REMARK. In the rest of this paper, the height of a compact H-surface Σ embedded into $\mathbb{H}^2 \times \mathbb{R}$ is the height difference between its upper point and lower point.

THEOREM 3.1. *Let Σ be a compact H-surface* ($H > 1/2$), *embedded into* $\mathbb{H}^2 \times \mathbb{R}$, *whose boundary is a convex planar curve contained in the plane $Q = \mathbb{H}^2 \times \{0\}$. Assume that $2h_\Sigma < h_{S_H}$, where h_Σ and h_{S_H} denote* (*respectively*) *the height of the surface Σ and that of the H-sphere. Then Σ stays in a half-space determined by Q and is transverse to Q along the boundary. Moreover, Σ inherits the symmetries of its boundary.*

To prove this, we need the following lemma.

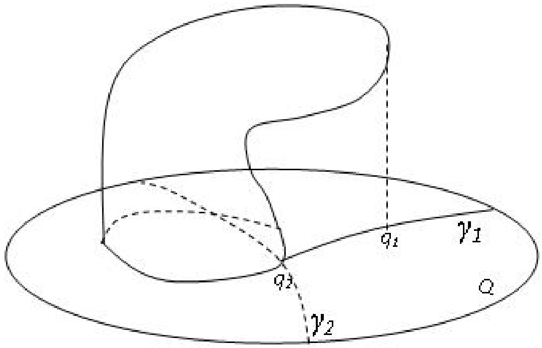

Figure 1

LEMMA 3.1. *Let Σ be a compact H-surface* ($H > 1/2$) *embedded in* $\mathbb{H}^2 \times \mathbb{R}$ *and with planar boundary. If $2h_\Sigma < h_{S_H}$* (*where h denotes height as before*), *then the surface Σ lies inside the right vertical cylinder determined by the convex hull of its boundary.*

Proof (see Figure 1). Suppose there is a point of Σ projecting on a point $q_1 \in Q$ outside the convex hull V of the boundary of Σ, and choose $q_2 \in V$ to minimize the distance to q_1. Denote by γ_1 the geodesic of Q passing through q_1 and q_2; we have $\gamma_1(0) = q_2$ and $\gamma_1(a) = q_1$ for $a > 0$. Let $\gamma_2 \subset \mathbb{H}^2 \times \{0\}$ be a complete geodesic with $\gamma_2(0) = \gamma_1(0)$ and $\gamma_2'(0) \perp \gamma_1'(0)$.

Consider the horizontal CMC cylinder C_H generated by $\alpha \subset P_1 = \gamma_1 \times \mathbb{R}$, as described previously, with curvature H. We consider a half-cylinder C_{γ_1} generated by $\alpha(s)$, where $s \in [0, s_H]$ or $s \in [-s_H, 0]$. We move C_{γ_1} (by horizontal translation along γ_1) far enough so that it does not touch the surface Σ, and we place its concave side in front of Σ.

The surface Σ is inside a slab B parallel to Q with height less than $h_{S_H}/2$. This slab is not necessarily symmetric with respect to Q. However, we may utilize half-cylinders with axes in the central plane of B; then, making a vertical translation if necessary, we can suppose that B is symmetric with respect to Q. See Figure 2.

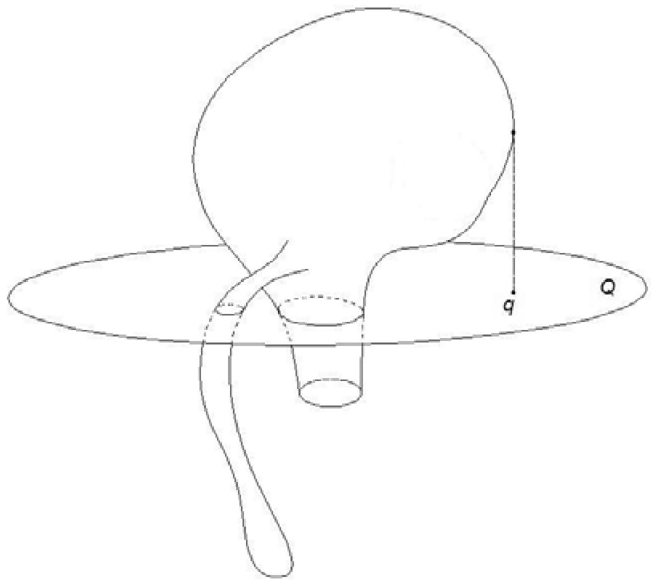

Figure 2

Now we proceed to approach the half-cylinder C_{γ_1} to Σ via the horizontal translation along γ_1, thereby obtaining a first (and so tangential) contact point between the two surfaces.

Since γ_2 lies inside Q and since there is a point of Σ projecting on the point q_1 outside the convex hull of the boundary, it follows that the contact point so obtained is a nonboundary point of the surface Σ. It is also an interior point of the half-cylinder C_{γ_1} because Σ is inside the slab $B \subset \mathbb{H}^2 \times (-h_{S_H}/2, h_{S_H}/2)$. On the other hand, this half-cylinder has constant mean curvature H with respect to the normal field pointing to its concave part. We already know that Σ is in that concave part, so by elementary comparison we have that the same choice of normal at the contact point gives the mean curvature H for Σ. Yet because this contradicts the maximum principle, all the points of the surface Σ must project on the convex hull of its boundary. □

Proof of Theorem 3.1. By the lemma just proved, if Ω is a compact convex domain in Q with $\partial\Omega = \partial\Sigma$ then $\Sigma \cap \text{ext}(\Omega) = \emptyset$. Hence we can consider a hemisphere S under the plane Q whose boundary disc D is contained in Q and is large enough that $\Omega \subset \text{int}(D)$ and $S \cap \Sigma = \emptyset$. Therefore, $\Sigma \cup (D - \Omega) \cup (S - D)$ is a compact embedded surface in $\mathbb{H}^2 \times \mathbb{R}$ and so determines an interior domain, which we call U. Choose a unit normal N for Σ in such a way that N points into U at each point. If there are points of the surface Σ in both half-spaces determined by Q, then N takes the same value at the points where the height function attains its respective maximum and minimum. Reversing N if necessary, we conclude that the normal of Σ (for which $H > 0$) takes the same value at the highest and the lowest points of the surface.

Lowering a sphere S_H^2 to the highest point or pushing it up to the lowest one, we obtain a contradiction via the interior maximum principle. Thus the surface lies in one of the half-spaces determined by the plane Q and rises in it by less than $h_{S_H}/2$. Again using half-cylinders C_H with axes in a plane parallel to Q and height $h_{S_H}/2$, we see that the boundary maximum principle implies that the surface is transversal along its boundary.

Let γ be a horizontal, complete, oriented geodesic passing through the origin $O \in \mathbb{H}^2 \times \mathbb{R}$, and let $P_\gamma(t_1)$ be a vertical plane such that $P_\gamma(t_1) \cap \Sigma = \emptyset$. We take the oriented foliation of vertical planes along γ with $P = P_\gamma(0)$. Finally, we apply Alexandrov reflection with these planes—starting at $t = t_1$ and then decreasing t—to obtain that Σ has all the symmetries of its boundary. $\square$

COROLLARY 3.1. *Let Σ be a compact H-surface ($H > 1/2$) embedded in $\mathbb{H}^2 \times \mathbb{R}$ and with convex planar boundary. Then Σ is a graph if and only if $h_\Sigma < h_{S_H}/2$, where again h_Σ and h_{S_H} are the height of the surface Σ and of the H-sphere, respectively.*

Proof. If Σ is a graph then the proof follows by [1, Thm. 2.1]. Suppose now that $h_\Sigma < h_{S_H}/2$. By Theorem 3.1 we have that Σ must be contained in one of the half-spaces determined by the boundary plane; and by Lemma 3.1, Σ is inside the right vertical cylinder determined by the convex hull of its boundary. Using Alexandrov reflection with horizontal planes, we deduce that Σ is a graph. $\square$

References

[1] J. A. Aledo, J. Espinar, and J. Gálvez, *Height estimates for surfaces with positive constant mean curvature in $M^2 \times \mathbb{R}$*, Illinois J. Math. 52 (2008), 203–211.

[2] L. Barbosa and M. do Carmo, *A proof of a general isoperimetric inequality for surfaces,* Math. Z. 162 (1978), 245–261.

[3] P. Bérard and R. Sa Earp, *Examples of H-hypersufaces in $\mathbb{H}^n \times \mathbb{R}$ and geometric applications,* Mat. Contemp. 34 (2008), 19–51.

[4] D. Hoffman, J. de Lira, and H. Rosenberg, *Constant mean curvature surfaces in* $\mathbb{M}^2 \times \mathbb{R}$, Trans. Amer. Math. Soc. 358 (2006), 491–507.

[5] R. López and S. Montiel, *Constant mean curvature with planar boundary,* Duke Math. J. 85 (1996), 583–604.

[6] T. Sakai, *Riemannian geometry,* Transl. Math. Monogr., 149, Amer. Math. Soc., Providence, RI, 1992.

[7] J. Serrin, *On surfaces of constant mean curvature which span a given space curve,* Math. Z. 112 (1969), 77–88.

C. Leandro
Instituto Nacional de Matemática
Pura e Aplicada
Rio de Janeiro 22460-320
Brazil

claudemi@impa.br

H. Rosenberg
Instituto Nacional de Matemática
Pura e Aplicada
Rio de Janeiro 22460-320
Brazil

rosen@impa.br