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The ∞-Poincaré Inequality in Metric Measure Spaces

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1. Introduction

A useful feature of the Euclidean n-space, $n \geq 2$, is that every pair of points x and y can be joined not only by the line segment $[x, y]$ but also by a large family of curves whose length is comparable to the distance between the points. Once one has found such a "thick" family of curves, the deduction of important Sobolev and Poincaré inequalities is an abstract procedure in which the Euclidean structure no longer plays a role.

The classical Poincaré inequality allows one to obtain integral bounds on the oscillation of a function using integral bounds on its derivatives. In this type of inequality, the derivative itself is not needed and only the size of the function's gradient is used; a nice discussion of this can be found in [17]. This is the idea behind generalizations of Poincaré inequalities in spaces where we may not have a linear structure. Heinonen and Koskela [8; 9] introduced a notion of "upper gradients", which serve the role of derivatives in a metric space X. A nonnegative Borel function g on X is said to be an *upper gradient* for an extended real-valued function u on X if $|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g$ for every rectifiable curve $\gamma\colon [a, b] \to X$. The following Poincaré inequality is now standard in literature on analysis in metric measure spaces.

Definition 1.1. Let $1 \leq p < \infty$. We shall say that (X, d, μ) supports a *weak p-Poincaré inequality* if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that, for every Borel measurable function $u\colon X \to \mathbb{R} \cup \{\infty\}$ and every upper gradient $g\colon X \to [0, \infty]$ of u, the pair (u, g) satisfies the inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_p r \left(\fint_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p}$$

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for each ball $B(x,r) \subset X$. The modifier *weak* refers to the possibility that λ may be strictly greater than 1.

Here $B(x,r)$ is an open ball with center at x and radius $r > 0$. For arbitrary $A \subset X$ with $0 < \mu(A) < \infty$, we write

$$u_A = \fint_A u = \frac{1}{\mu(A)} \int_A u \, d\mu.$$

There is a long list of metric spaces supporting a Poincaré inequality, including some standard examples such as $\mathbb{R}^n$, Riemannian manifolds with nonnegative Ricci curvature, and Carnot groups (in particular, the Heisenberg group) as well as other, non-Riemannian metric measure spaces of fractional Hausdorff dimension (see e.g. [7; 14] and the references therein). Metric spaces equipped with a p-Poincaré inequality support a nontrivial potential theory and geometric theory even without a priori smoothness structure of the metric space. Metric spaces with doubling measure and p-Poincaré inequality admit a first-order differential calculus theory akin to that in Euclidean spaces. One surprising fact is that some geometric consequences of this condition seem to be independent of the parameter p, and the picture is not yet clear.

It follows from Hölder's inequality that, if a space admits a p-Poincaré inequality, then it admits a q-Poincaré inequality for each $q \ge p$. Keith and Zhong [11] proved a self-improving property for Poincaré inequalities—namely, if X is a complete metric space equipped with a doubling measure satisfying a p-Poincaré inequality for some $1 < p < \infty$, then there exists an $\varepsilon > 0$ such that X supports a q-Poincaré inequality for all $q > p - \varepsilon$. The strongest of all these inequalities would be a 1-Poincaré inequality, and it is well known that the 1-Poincaré inequality is equivalent to the relative isoperimetric property [1; 16]. On the other hand, even for $p > 1$ the p-Poincaré inequality has strong links with the geometry of the underlying metric measure space. For instance, the Poincaré inequality implies that any pair of points in the space can be connected by curves that are not too long; this property is called *quasiconvexity*. Hence a natural question is: What is the weakest version of p-Poincaré inequality that still gives reasonable information on the geometry of the metric space? One of the goals of this paper is to answer that question by studying the following version of ∞-Poincaré inequality.

DEFINITION 1.2. We say that (X,d,μ) supports a *weak ∞-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \ge 1$ such that, for every Borel measurable function $u \colon X \to \mathbb{R} \cup \{\infty\}$ and every upper gradient $g \colon X \to [0,\infty]$ of u, the pair (u,g) satisfies the inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le Cr \|g\|_{L^\infty(B(x,\lambda r))}$$

for each ball $B(x,r) \subset X$.

The main result of this paper is a characterization of spaces supporting an ∞-Poincaré inequality; this is given in Theorem 4.7. A metric measure space is said to be *thick quasiconvex* if, loosely speaking, every pair of sets of positive measure that are a positive distance apart can be connected by a "thick" family of quasiconvex curves in the sense that the ∞-modulus of this family of curves is positive. The first aim of this paper is to show that a connected complete doubling metric measure space supports a weak ∞-Poincaré inequality if and only if it is thick quasiconvex, which is a purely geometric condition. We will also prove that this condition is equivalent to the purely analytic condition that $\mathrm{LIP}^{\infty}(X) = N^{1,\infty}(X)$, with comparable energy seminorms, in the sense described before Example 4.5.

The paper is organized as follows. In Section 2 we recall some standard notation and relevant notions regarding metric spaces supporting a doubling measure, ∞-modulus of curves, and Newtonian–Sobolev spaces $N^{1,\infty}(X)$. In Section 3 we introduce ∞-Poincaré inequality and present an example (Example 3.3) of a non-doubling metric space that supports an ∞-Poincaré inequality but does not support any p-Poincaré inequality for $p < \infty$. We do not know whether there is a metric space with a doubling measure that supports an ∞-Poincaré inequality but does not support any p-Poincaré inequality for $p < \infty$. Furthermore, we give some geometric implications of the ∞-Poincaré inequality—in particular, that the space is quasiconvex. However, as one can appreciate in Corollary 4.15, quasiconvexity is not a sufficient condition for a space to support an ∞-Poincaré inequality. In Section 4 we introduce the stronger notion of thick quasiconvexity (Definition 4.1), which leads us in Theorem 4.7 to obtain the desired analytic and geometric characterization of ∞-Poincaré inequality.

Unless otherwise stated, the letter C denotes various positive constants whose exact values are not important, and the value might change even from line to line.

2. Notation and Preliminaries

We assume throughout the paper that (X, d, μ) is a metric measure space, that is, a metric space equipped with a metric d and a Borel measure μ such that $0 < \mu(B) < \infty$ for each open ball $B \subset X$.

A measure μ is *doubling* if there is a constant $C_\mu > 0$ such that, for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \le C_\mu \mu(B(x, r)).$$

Here $B(x,r) := \{y \in X : d(x, y) < r\}$. Also $\bar{B}(x,r) := \{y \in X : d(x, y) \le r\}$ and $\lambda B(x,r) := \{y \in X : d(x, y) < \lambda r\}$. We point out here that, in the abstract metric setting, $\bar{B}(x,r)$ contains the closure of $B(x,r)$ yet might be larger.

An iteration of the preceding inequality shows that μ is also s_1-*homogeneous* for some $s_1 > 0$. In other words, there are constants C and s depending only on C_μ such that—whenever B is a ball in X, $x \in B$, and $r > 0$ with $B(x,r) \subset B$—we have

$$\frac{\mu(B(x,r))}{\mu(B)} \ge \frac{1}{C}\left(\frac{r}{\mathrm{rad}(B)}\right)^{s_1}. \tag{1}$$

If in addition X is connected and has at least two points, then the doubling property also implies the existence of a constant $s_2 > 0$ such that, for all balls $B \subset X$ and $B(x,r) \subset B$,

$$\frac{\mu(B(x,r))}{\mu(B)} \leq \frac{1}{C}\left(\frac{r}{\mathrm{rad}(B)}\right)^{s_2}. \qquad (2)$$

By this inequality, letting $r \to 0$ we see that for all $x \in X$ we have $\mu(\{x\}) = 0$; that is, μ has no atoms.

In a complete metric space X, the existence of a doubling measure that is finite on balls and not trivial implies that X is separable and *proper*. The latter means that closed bounded subsets of X are compact. In particular, X is locally compact.

Some of the classical theorems in analysis in the Euclidean setting can be extended to doubling metric measure spaces. The Lebesgue differentiation theorem is such an example: if u is a locally integrable function on a doubling metric space X, then

$$u(x) = \lim_{r\to 0} \fint_{B(x,r)} u \, d\mu$$

for μ-a.e. point in X. In other words, almost every point in X is a *Lebesgue point* for u (see e.g. [7]).

REMARK 2.1. The hypothesis of completeness is not so restrictive. The completion $(\hat{X}, \hat{d})$ of a metric space (X, d) is unique up to isometry. Note that (X, d) is a subspace of $(\hat{X}, \hat{d})$ and that X is dense in $\hat{X}$. For our purposes, the crucial observation is that the essential features of X are inherited by $\hat{X}$. Indeed, if X is locally complete and there is a doubling Borel measure μ that is nontrivial and finite on balls, we may extend this measure to $\hat{X}$ so that $\hat{X} \setminus X$ has zero measure and the extended measure has the same properties as the original one. Also, if X supports a weak p-Poincaré inequality for some $1 \leq p \leq \infty$, then so does $\hat{X}$. See also [10] for further discussions on this topic.

By a *curve* γ we mean a continuous mapping $\gamma\colon [a,b] \to X$. Recall that the *length* of a continuous curve $\gamma : [a,b] \to X$ in a metric space (X,d) is defined as

$$\ell(\gamma) = \sup\left\{\sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))\right\},$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a,b]$. We will say that a curve γ is *rectifiable* if $\ell(\gamma) < \infty$. The integral of a Borel function g over a rectifiable path γ is usually defined via the path-length parameterization γ_0 of γ in the following way:

$$\int_\gamma \rho \, ds = \int_0^{\ell(\gamma)} g \circ \gamma_0(t) \, dt.$$

Recall that every rectifiable curve γ admits a parameterization by the arc length; that is, with $\gamma_0\colon [0, \ell(\gamma)] \to X$, for all t_1, t_2 with $t_1 \leq t_2$ we have $\ell(\gamma_0|_{[t_1,t_2]}) = t_2 - t_1$. Hence, we consider only curves that are arc-length parameterized.

We denote by $\mathrm{LIP}^\infty(X)$ the space of bounded Lipschitz functions on X. In what follows, $\|\cdot\|_{L^\infty}$ denotes the essential supremum norm (provided we have a measure on X). In addition, $\mathrm{LIP}(\cdot)$ will denote the Lipschitz constant:

$$\mathrm{LIP}(u) := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|u(y)-u(x)|}{d(y,x)}.$$

The norm on $\mathrm{LIP}^\infty(X)$ is given by

$$\|u\|_{\mathrm{LIP}^\infty(X)} := \sup_{x\in X}|u(x)| + \mathrm{LIP}(u).$$

We recall the definition of ∞-*modulus*: an outer measure on the collection of all paths in X. Let $\Upsilon \equiv \Upsilon(X)$ denote the family of all nonconstant rectifiable curves in X. It may be that Υ is empty, but we are mainly interested in finding out when metric spaces have large enough Υ.

DEFINITION 2.2. For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho\colon X \to [0,\infty]$ such that

$$\int_\gamma \rho \geq 1 \quad \text{for all } \ \gamma \in \Gamma.$$

We define the ∞-*modulus of* Γ by

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho\in F(\Gamma)}\{\|\rho\|_{L^\infty}\}.$$

If some property holds for all curves $\gamma \notin \Gamma$ for some $\Gamma \subset \Upsilon$ that satisfies $\mathrm{Mod}_\infty\Gamma = 0$, then we say that the property holds for ∞-*a.e. curve*.

It can be easily checked that Mod_∞ is an outer measure as it is for $1 \leq p < \infty$; see for example [5, Thm. 5.2].

REMARK 2.3. Note that if we have two measures μ and λ defined on X with the same zero measure sets, then the ∞-modulus of Γ is the same regardless of the measure we use to compute it.

DEFINITION 2.4. Let $E \subset X$. Then Γ_E^+ is the family of curves γ such that $\mathscr{L}^1(\gamma^{-1}(\gamma\cap E)) > 0$, where $\mathscr{L}^1$ denotes the 1-dimensional Lebesgue measure.

Recall that we consider only curves that are arc-length parameterized.

LEMMA 2.5. *Let $E \subset X$. If $\mu(E) = 0$, then* $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$.

Proof. Since μ is a Borel measure, by enlarging E if necessary we may assume that E is a Borel set. Let $g = \infty\cdot\chi_E$. For $\gamma \in \Gamma_E^+$, we have that $\mathscr{L}^1(\gamma^{-1}(\gamma\cap E)) > 0$ and so

$$\int_\gamma g\, ds = \int_{\gamma\cap E} g\, ds = \infty.$$

Hence, by the definition of modulus,

$$\mathrm{Mod}_\infty(\Gamma_E^+) \leq \|g\|_{L^\infty(X)} = 0. \qquad \square$$

A related generalization of Sobolev spaces to general metric spaces are the so-called *Newtonian spaces* $N^{1,p}$ introduced in [18; 19]. Its definition is based on the notion of upper gradients of Heinonen and Koskela. In this paper, we will focus on the case $p = \infty$ studied in [3].

DEFINITION 2.6. A nonnegative Borel function g on X is an *∞-weak upper gradient* of an extended real-valued function u on X if, for ∞-a.e. curve $\gamma \in \Upsilon$,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g$$

when both $u(\gamma(a))$ and $u(\gamma(b))$ are finite and $\int_\gamma g = \infty$ otherwise. If the family of curves for which this requirement is not satisfied is an empty family, then we say that g is an *upper gradient* of u.

Let $\tilde{N}^{1,\infty}(X,d,\mu) = \tilde{N}^{1,\infty}(X)$ be the class of all Borel functions $u \in L^\infty(X)$ for which there exists an ∞-weak upper gradient g in $L^\infty(X)$. For $u \in \tilde{N}^{1,\infty}(X,d,\mu)$ we set

$$\|u\|_{\tilde{N}^{1,\infty}} = \|u\|_{L^\infty} + \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞-weak upper gradients g of u.

DEFINITION 2.7. We define an equivalence relation in $\tilde{N}^{1,\infty}(X)$ by $u \sim v$ if and only if $\|u - v\|_{\tilde{N}^{1,\infty}} = 0$. The space $N^{1,\infty}(X,d,\mu) = N^{1,\infty}(X)$ denotes the quotient $\tilde{N}^{1,\infty}(X,d,\mu)/\sim$ and is equipped with the norm

$$\|u\|_{N^{1,\infty}} = \|u\|_{\tilde{N}^{1,\infty}}.$$

It was shown in [3] that $N^{1,\infty}(X)$ is a Banach space. Note that if $u \in \tilde{N}^{1,\infty}(X)$ and $v = u$ μ-a.e., then it is not necessarily true that $v \in \tilde{N}^{1,\infty}$. Nevertheless, the following lemma shows that if $u, v \in \tilde{N}^{1,\infty}$ and $v = u$ μ-a.e., then $\|u - v\|_{\tilde{N}^{1,\infty}} = 0$.

LEMMA 2.8 [3, 5.13]. *Let $u_1, u_2 \in \tilde{N}^{1,\infty}(X,d,\mu)$ be such that $u_1 = u_2$ μ-a.e. Then $u_1 \sim u_2$; in other words, both functions define exactly the same element in $N^{1,\infty}(X,d,\mu)$.*

If g is an ∞-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \to g$ in $L^\infty(X)$. It follows from the Lebesgue differentiation theorem that, if μ is doubling, then μ-a.e. $x \in X$ is a Lebesgue point of $N^{1,\infty}(X,d,\mu)$. Observe also that if $u \in \mathrm{LIP}^\infty(X)$ then the Lipschitz constant $\mathrm{LIP}(u)$ is an upper gradient for u. Therefore, $\|\cdot\|_{N^{1,\infty}} \leq \|\cdot\|_{\mathrm{LIP}^\infty}$ for every $u \in \mathrm{LIP}^\infty(X)$.

3. ∞-Poincaré Inequality in Metric Measure Spaces

We recall here the definition of ∞-Poincaré inequality referred to in Section 1.

DEFINITION 3.1. We say that (X,d,μ) supports a *weak ∞-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that, for every Borel measurable function $u\colon X \to \mathbb{R} \cup \{\infty\}$ and every ∞-weak upper gradient $g\colon X \to [0,\infty]$ of u, the pair (u,g) satisfies the inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le Cr\|g\|_{L^\infty(B(x,\lambda r))}$$

for each ball $B(x,r) \subset X$.

REMARK 3.2. Observe that

$$\begin{aligned}\fint_B |u(x) - u_B|\, d\mu(x) &= \fint_B \left| \fint_B (u(x) - u(y))\, d\mu(y) \right| d\mu(x) \\ &\le \fint_B \fint_B |u(x) - u(y)|\, d\mu(y)\, d\mu(x)\end{aligned}$$

and so, when we want to check whether (X, d, μ) supports a weak ∞-Poincaré inequality, it is enough to prove that each pair (u, g) satisfies

$$\fint_B \fint_B |u(x) - u(y)|\, d\mu(y)\, d\mu(x) \le Cr\|g\|_{L^\infty(\lambda B)} \tag{3}$$

for each ball $B \subset X$ with radius r. On the other hand, inequality (3) is necessary to verify ∞-Poincaré inequality also. To see this, note that

$$\begin{aligned}\fint_B \fint_B |u(x) - u(y)|\, d\mu(y)\, d\mu(x) &= \fint_B \fint_B |u(x) - u_B + u_B - u(y)|\, d\mu(y)\, d\mu(x) \\ &\le 2 \fint_B |u(x) - u_B|\, d\mu(x).\end{aligned}$$

The following example shows that there exist spaces with a weak ∞-Poincaré inequality that do not admit a weak p-Poincaré inequality for any finite p.

EXAMPLE 3.3. Let T be a nondegenerate triangular region in $\mathbb{R}^2$, and let T' be an identical copy of T. Let X be the metric space obtained by identifying a vertex V of T with a vertex V' of T' ($V = V' = \{0\}$) and the metric defined by

$$d(x,y) = \begin{cases} |x - y| & \text{if } x, y \in T \text{ or } x, y \in T', \\ |x - V| + |V' - y| & \text{if } x \in T \text{ and } y \in T'. \end{cases}$$

This space X is equipped with the weighted measure μ given by $d\mu(x) = \omega(x)\, d\mathscr{L}^2(x)$, where $\omega(x) = \exp(-1/|x|^2)$. Note that μ and the Lebesgue measure $\mathscr{L}^2$ have the same zero measure sets. It is already known that this space equipped with the Lebesgue measure $\mathscr{L}^2$ admits a p-Poincaré inequality for $p > 2$ (see e.g. [18]). We now show that (X, d, μ) does not admit a weak p-Poincaré inequality for any finite p but does admit a weak ∞-Poincaré inequality.

First observe that, given a measurable function u in X,

$$\fint_B |u - u_B|\, d\mu \le 2 \inf_{c \in \mathbb{R}} \fint_B |u - c|\, d\mu, \tag{4}$$

where $u_B = \fint_B u\, d\mu$. Indeed, let $c \in \mathbb{R}$ and suppose $c \ge u_B$ (the case $c < u_B$ is analogous). Then

$$\fint_B |c - u_B|\, d\mu = c - u_B = \fint_B c - \fint_B u = \fint_B (c - u) \le \fint_B |c - u|\, d\mu.$$

Since $|u(x)-u_B| \le |u(x)-c| + |c-u_B|$ for each $x \in X$, we have that

$$\fint_B |u-u_B|\,d\mu \le \fint_B |u-c|\,d\mu + \fint_B |c-u_B|\,d\mu \le 2\fint_B |u-c|\,d\mu.$$

If we take the infimum over c on the right-hand of this inequality, the result is inequality (4). Now let us consider an upper gradient g of u. We obtain the following chain of inequalities by using Hölder's inequality for $2 < p < q$:

$$\begin{aligned}\fint_B |u-u_B|\,d\mu &\overset{(4)}{\le} 2\inf_{c\in\mathbb{R}} \fint_B |u-c|\,d\mu \le 2\fint_B |u-u_{B,\mathscr{L}^2}|\,d\mu\\ &\le 2\|u-u_{B,\mathscr{L}^2}\|_{L^\infty(\mu)} = 2\|u-u_{B,\mathscr{L}^2}\|_{L^\infty(\mathscr{L}^2)}\\ &\le C_p r\left(\fint_{5\lambda B} g^p\,d\mathscr{L}^2\right)^{1/p} \le C_p r\left(\fint_{5\lambda B} g^q\,d\mathscr{L}^2\right)^{1/q},\end{aligned}$$

where $u_{B,\mathscr{L}^2} = \fint_B u\,d\mathscr{L}^2$. In the third line of this chain of inequalities we have applied [6, Thm. 5.1]. Letting q tend to infinity yields

$$\fint_B |u-u_B|\,d\mu \le C_p r\|g\|_{L^\infty(\mathscr{L}^2,5\lambda B)} = C_p r\|g\|_{L^\infty(\mu,5\lambda B)},$$

whence (X,d,μ) admits a weak ∞-Poincaré inequality.

Next we show that (X,d,μ) does not admit a p-Poincaré inequality for any finite p. Indeed, consider the function $u=1$ in T, $u=0$ in T' and in the vertex. It is not difficult to check that the function $g_\alpha(x) = \alpha/|x|$ is an upper gradient for u for each $\alpha > 0$. Taking the ball $B = X$, we have that $u_X > 0$ and therefore $\fint_X |u-u_X|\,d\mu > 0$. Nevertheless, $\int_X g_\alpha^p\,d\mu$ tends to zero when α tends to zero for $1 < p < \infty$, so X does not admit a weak p-Poincaré inequality for any finite p.

Observe that the measure μ in the previous example is *not* doubling.

One of the most useful geometric implications of the p-Poincaré inequality for finite p is that, if a complete doubling metric measure space supports a p-Poincaré inequality, then there exists a constant such that each pair of points can be connected with a curve whose length is at most the constant times the distance between the points (see [6; 17]); that is, the space is *quasiconvex*. If X is known only to support an ∞-Poincaré inequality then the same conclusion holds, as demonstrated by the following proposition.

PROPOSITION 3.4. *Suppose that (X,d,μ) is a complete metric measure space with μ a doubling measure. If X supports a weak ∞-Poincaré inequality, then X is quasiconvex with a constant depending only on the constants of the Poincaré inequality and the doubling constant.*

Proof. Let $\varepsilon > 0$. We say that $x, z \in X$ lie in the same ε-component of X if there exists an ε-chain joining x with z—that is, there exists a finite chain $z_0, z_1, \dots, z_n$ such that $z_0 = x$, $z_n = z$, and $d(z_i, z_{i+1}) \le \varepsilon$ for all $i = 0, \dots, n-1$. If x and y lie in different ε-components, then it is obvious that there exists no rectifiable curve joining x and y. Thus, the function $g \equiv 0$ is an upper gradient for the characteristic function of any of the components. Note that, for every x in one of the components, the ball $B(x, \varepsilon/2)$ is a subset of that component; in other words, each component is open and hence is a measurable set. By applying the weak ∞-Poincaré inequality to the characteristic function of any component, it follows that all the points of X lie in the same ε-component.

Now let us fix $x, y \in X$ and prove that there exists a curve γ joining x and y such that $\ell(\gamma) \le C\, d(x, y)$, where C is a constant that depends only on the doubling constant and the constants involved in the Poincaré inequality. We define the ε-distance from x to z as

$$\rho_{x,\varepsilon}(z) := \inf \sum_{i=0}^{N-1} d(z_i, z_{i+1}),$$

where the infimum is taken over all finite ε-chains $\{z_i\}$. Observe that $\rho_{x,\varepsilon}(z) < \infty$ for all $z \in X$. In addition, if $d(z, w) \le \varepsilon$ then $|\rho_{x,\varepsilon}(z) - \rho_{x,\varepsilon}(w)| \le d(z, w)$. Hence $\rho_{x,\varepsilon}$ is a locally 1-Lipschitz function; in particular, every point is a Lebesgue point of $\rho_{x,\varepsilon}$ and, moreover, for all $\varepsilon > 0$ the function $g \equiv 1$ is an upper gradient of $\rho_{x,\varepsilon}$. For each $i \in \mathbb{Z}$, define $B_i = B(x, 2^{1-i}\, d(x, y))$ if $i \ge 0$ and $B_i = B(y, 2^{1+i}\, d(x, y))$ if $i \le -1$. Then a telescopic argument together with weak ∞-Poincaré inequality yields the following chain of expressions:

$$\begin{aligned}
|\rho_{x,\varepsilon}(y)| &= |\rho_{x,\varepsilon}(x) - \rho_{x,\varepsilon}(y)| \\
&\le \sum_{i\in\mathbb{Z}} \left| \fint_{B_i} \rho_{x,\varepsilon}\, d\mu - \fint_{B_{i+1}} \rho_{x,\varepsilon}\, d\mu \right| \\
&\le C \sum_{i\in\mathbb{Z}} \frac{1}{\mu(B_i)} \fint_{B_i} \left| \rho_{x,\varepsilon} - \fint_{B_{i+1}} \rho_{x,\varepsilon}\, d\mu \right| d\mu \\
&\le CC\, d(x, y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(\lambda B_i)} \\
&\le C\, d(x, y);
\end{aligned} \tag{5}$$

here C is a constant that depends only on X.

Because X is complete, the existence of a nontrivial doubling measure implies that closed balls are compact. Using a standard limiting argument that involves Arzela–Ascoli's theorem and inequality (5), we can construct a 1-Lipschitz rectifiable curve connecting x and y with length at most $C\, d(x, y)$. Since x and y were arbitrary, this completes the proof. For further details about the construction of the curve, see [13, Thm. 3.1]. $\square$

The following technical lemma will be useful in the sequel.

LEMMA 3.5. *Let X be a complete separable metric space equipped with a σ-finite Borel measure μ, and let $g\colon X \to [0, \infty]$ be a Borel function. Then, for each $x_0 \in X$, the function*

$$u(z) = \inf_{\gamma \text{ connects } z \text{ to } B(x_0, r)} \int_\gamma g\, ds$$

is μ-measurable. Moreover, if $k \in \mathbb{R}$ then the function g is an upper gradient for $v = \min\{u, k\}$.

Proof. Following the lines of [10, Cor. 1.10], one can prove that u is μ-measurable.

To see that g is an upper gradient of v on X, we argue as follows. Fix $z_1, z_2 \in X$ and let β be a rectifiable curve in X connecting z_1 to z_2. There are three possible cases:

1. $v(z_1) = u(z_1)$ and $v(z_2) = u(z_2)$;
2. $v(z_1) = u(z_1)$ and $v(z_2) = k$;
3. $v(z_1) = k = v(z_2)$.

In the first case, both $u(z_1)$ and $u(z_2)$ are finite. Fix $\varepsilon > 0$. Then we can find a rectifiable curve connecting z_1 to $B(x, \varepsilon)$ such that $u(z_1) \geq \int_\gamma g\, ds - \varepsilon$; hence

$$u(z_2) - u(z_1) \leq \int_{\gamma \cup \beta} g\, ds - \int_\gamma g\, ds + \varepsilon = \int_\beta g\, ds + \varepsilon,$$

where we can cancel $\int_\gamma g\, ds$ because it is a finite value. A similar argument gives

$$u(z_1) - u(z_2) \leq \int_\beta g\, ds + \varepsilon.$$

Combining the previous two inequalities and letting $\varepsilon \to 0$ now yields

$$|v(z_1) - v(z_2)| = |u(z_1) - u(z_2)| \leq \int_\beta g\, ds.$$

In the second case, $u(z_1) = v(z_1) \leq v(z_2) \leq u(z_2)$. In this case again, $u(z_1)$ is finite. For $\varepsilon > 0$ we can find a rectifiable curve γ connecting z_1 to $B(x, \varepsilon)$ such that $u(z_1) \geq \int_\gamma g\, ds - \varepsilon$, so

$$\begin{aligned} |v(z_1) - v(z_2)| = v(z_2) - v(z_1) \leq u(z_2) - u(z_1) &\leq \int_{\gamma \cup \beta} g\, ds - \int_\gamma g\, ds + \varepsilon \\ &= \int_\beta g\, ds + \varepsilon; \end{aligned}$$

once again, we cancel the term $\int_\gamma g\, ds \leq u(z_1) + \varepsilon$ because it is finite. Letting $\varepsilon \to 0$, we again obtain

$$|v(z_1) - v(z_2)| \leq \int_\beta g\, ds.$$

In the third case we easily obtain the preceding inequality because, in this case, $v(z_1) - v(z_2) = 0$. □

The next example shows one of the difficulties in working with $p = \infty$ as opposed to finite values of p.

EXAMPLE 3.6. Let X be a complete metric space that supports a doubling Borel measure μ that is nontrivial and finite on balls, and suppose that X supports a weak ∞-Poincaré inequality. Denote by $\Gamma_{x_0,r,R}$ the family of curves that connect $B(x_0,r)$ to the complement of the ball $B(x_0,R)$ with $0<r<R/2<\operatorname{diam}(X)/4$.

We will prove that there is a constant $C>0$, independent of R, r, and x_0, such that

$$\operatorname{Mod}_\infty(\Gamma_{x_0,r,R})\geq \frac{C}{R}.$$

To see this, let g be a nonnegative Borel measurable function on X such that, for all $\gamma\in\Gamma_{x_0,r,R}$, the integral $\int_\gamma g\,ds\geq 1$. Notice here that X is quasiconvex by Proposition 3.4. We then set

$$\tilde{u}(z)=\inf_{\gamma \text{ connects } z \text{ to } B(x_0,r)}\int_\gamma g\,ds$$

and consider $u=\min\{\tilde{u},2\}$. It then follows that $u=0$ on $B(x_0,r)$ and, by the choice of g, that $u\geq 1$ on $X\setminus B(x_0,R)$. By [10, Cor. 1.10], u is measurable; by Lemma 3.5, g is an upper gradient of u. In short, $u\in N^{1,\infty}(X)$.

Given $x\in B(x_0,r)$ and $y\in B(x_0,R+r)\setminus B(x_0,R)$, for each $i\in\mathbb{Z}$ define $B_i=B(x,2^{1-i}\,d(x,y))$ if $i\geq 0$ and $B_i=B(y,2^{1+i}\,d(x,y))$ if $i\leq -1$. By the weak ∞-Poincaré inequality and the doubling property of μ, we get for Lebesgue points $x\in B(x_0,r)$ and $y\in X\setminus B(x_0,R)$ that

$$\begin{aligned}1\leq |u(x)-u(y)| &\leq \sum_{i\in\mathbb{Z}}\left|\fint_{B_i}u\,d\mu-\fint_{B_{i+1}}u\,d\mu\right|\\ &\leq C\sum_{i\in\mathbb{Z}}\fint_{B_i}\left|u-\fint_{B_i}u\,d\mu\right|d\mu\\ &\leq CC\,d(x,y)\sum_{i\in\mathbb{Z}}2^{-|i|}\|g\|_{L^\infty(\lambda B_i)}\\ &\leq C\,d(x,y)\|g\|_{L^\infty(X)}.\end{aligned}$$

Therefore,

$$\|g\|_{L^\infty(X)}\geq\frac{1}{C\,d(x,y)}\geq\frac{1}{C(R+r)}\geq\frac{1}{2CR}.$$

Taking the infimum over all such g yields the desired inequality for the ∞-modulus. An analogous statement holds for $\operatorname{Mod}_p(\Gamma_{x_0,r,R})$ if X supports a weak p-Poincaré inequality for sufficiently large finite p (i.e., with p larger than the lower mass bound exponent s_1 obtained from the doubling property of the measure μ). For such finite p, we can approximate test functions g from above in $L^p(X)$ by lower semicontinuous functions (as follows from the Vitali–Caratheodory theorem [4, pp. 209–213]), so we could show (as in [8]) that the p-modulus of the collection of all curves that connect x_0 itself to $X\setminus B(x_0,R)$ is positive. Unfortunately, such an approximation by lower semicontinuous functions in the L^∞-norm is not valid in general, so we cannot conclude from the foregoing computation that the ∞-modulus of the collection of all curves connecting x_0 to $X\setminus B(x_0,R)$ is positive *if X is known only to support a weak ∞-Poincaré inequality.*

The previous example highlights the difficulties when working with the L^∞-norm. Namely, the L^∞-norm is insensitive to local changes, and the Vitali–Caratheodory theorem does not apply.

4. Geometric Characterization of Weak ∞-Poincaré Inequality

The connection between isoperimetric and Sobolev-type inequalities in the Euclidean setting is well understood (see [1; 16]). In the context of metric spaces supporting a doubling measure, Miranda proved in [16] that a 1-weak Poincaré inequality implies a relative isoperimetric inequality for sets of finite perimeter. More recently, Kinnunen and Korte [12] gave further characterizations of Poincaré-type inequalities (in the context of Newtonian spaces) in terms of isoperimetric and isocapacitary inequalities.

In what follows we shall prove that ∞-Poincaré inequality also has a geometric characterization—namely, it is equivalent to *thick quasiconvexity.*

DEFINITION 4.1. (X, d, μ) is a *thick quasiconvex* space if there exists a $C \geq 1$ such that—for all $x, y \in X$, $0 < \varepsilon < d(x, y)/4$, and all measurable sets $E \subset B(x, \varepsilon)$ and $F \subset B(y, \varepsilon)$ satisfying $\mu(E)\mu(F) > 0$—we have

$$\mathrm{Mod}_\infty(\Gamma(E, F, C)) > 0,$$

where $\Gamma(E, F, C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \leq C\, d(p, q)$. Here we do not require quantitative control on the modulus of the curve family.

REMARK 4.2. Note that every complete thick quasiconvex space X supporting a doubling measure is quasiconvex. Indeed, let $x, y \in X$ and choose a sequence ε_j that tends to zero. Since X is thick quasiconvex, there is a constant $C \geq 1$ such that, for every ε_j, there exist $x_j \in B(x, \varepsilon_j)$, $y_j \in B(y, \varepsilon_j)$, and a curve γ_j connecting x_j to y_j with $\ell(\gamma_j) \leq C\, d(x_j, y_j)$. Thus we obtain a sequence $\{\gamma_j\}$ of curves such that

$$\ell(\gamma_j) \leq C\, d(x_j, y_j) \leq 2C\, d(x, y),$$

that is, a sequence of curves with uniformly bounded length. Because X is a complete doubling metric space and therefore proper, we may use the Arzela–Ascoli theorem to obtain a subsequence, also denoted $\{\gamma_j\}$, that converges uniformly to a curve γ connecting x and y with

$$\ell(\gamma) = \lim_{j\to\infty} \ell(\gamma_j) \leq C \lim_{j\to\infty} d(x_j, y_j) = C\, d(x, y).$$

However, the converse is not true. In Example 4.14 we will describe a quasiconvex space endowed with a doubling measure that is not thick quasiconvex.

In what follows, we assume that X is a connected complete metric space supporting a doubling Borel measure μ that is nontrivial and finite on balls.

We have already proved in Proposition 3.4 that weak ∞-Poincaré inequality for Lipschitz functions implies quasiconvexity. However, in the following proposition we prove that weak ∞-Poincaré inequality for Newtonian functions implies the stronger property of thick quasiconvexity.

PROPOSITION 4.3. *If X supports a weak ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$ with upper gradients in $L^\infty(X)$, then X is thick quasiconvex.*

We remark that $N^{1,\infty}(X)$ consists precisely of the functions in $L^\infty(X)$ that have an upper gradient in $L^\infty(X)$.

Proof of Proposition 4.3. Let $x, y \in X$ be such that $x \neq y$, and let $0 < \varepsilon < d(x,y)/4$. Fix $n \in \mathbb{N}$ and let $\Gamma_n = \Gamma(B(x,\varepsilon), B(y,\varepsilon), n)$ be the collection of all rectifiable curves connecting $B(x,\varepsilon)$ to $B(y,\varepsilon)$ such that $\ell(\gamma) \leq n\, d(x,y)$. Observe that, by our choice of ε, if p, q are the endpoints of γ then $d(p,q)/4 \leq d(x,y) \leq 4\, d(p,q)$.

Suppose that $\mathrm{Mod}_\infty(\Gamma_n) = 0$. By [3, Lemma 5.7] there exists a nonnegative Borel measurable function $g \in L^\infty(X)$ such that $\|g\|_{L^\infty(X)} = 0$ and, for all $\gamma \in \Gamma_n$, the path integral $\int_\gamma g\, ds = \infty$. In this case we define

$$u(z) = \inf_{\gamma \text{ connects } z \text{ to } B(x,\varepsilon)} \int_\gamma (1+g)\, ds.$$

Observe that $\|1+g\|_{L^\infty(X)} = 1$ and $u = 0$ on $B(x,\varepsilon)$. If $z \in B(y,\varepsilon)$ and γ is a rectifiable curve connecting z to $B(x,\varepsilon)$, then either (i) $\gamma \in \Gamma_n$, in which case $\int_\gamma (1+g)\, ds \geq \int_\gamma g\, ds = \infty$, or (ii) $\gamma \notin \Gamma_n$, in which case $\ell(\gamma) > n\, d(x,y)$ and so $\int_\gamma (1+g)\, ds \geq \int_\gamma 1\, ds > n\, d(x,y)$; therefore, $u(z) \geq n\, d(x,y)$. It follows that the function $v = \min\{u, 2n\, d(x,y)\}$ has the properties that

1. $v = 0$ on $B(x,\varepsilon)$,
2. $v \geq n\, d(x,y)$ on $B(y,\varepsilon)$,
3. $v \in N^{1,\infty}(X)$, and
4. $1+g$ is an upper gradient of v on X (see Lemma 3.5) with $\|g\|_{L^\infty(X)} = 0$.

Let $y_0 \in B(y,\varepsilon/2)$ be a Lebesgue point of v. Then, by using the weak ∞-Poincaré inequality and the chain of balls $B_i = B(x, 2^{1-i}\, d(x,y))$ if $i \geq 0$ and $B_i = B(y_0, 2^{1+i}\, d(x,y))$ if $i \leq -1$, we obtain

$$\begin{aligned} n\, d(x,y) \leq v(y_0) = |v(x) - v(y_0)| &\leq \sum_{i\in\mathbb{Z}} |v_{B_i} - v_{B_{i+1}}| \\ &\leq C \sum_{i\in\mathbb{Z}} \fint_{2B_i} |v - v_{B_i}|\, d\mu \\ &\leq C \sum_{i\in\mathbb{Z}} 2^{-|i|}\, d(x,y) \|1+g\|_{L^\infty(\lambda B_i)} \\ &= C\, d(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \leq C\, d(x,y). \end{aligned}$$

Observe that x is a Lebesgue point of v since $v = 0$ on $B(x,\varepsilon)$. We must therefore have $n \le C$, where C depends only on the doubling constant and the constant of the Poincaré inequality. Hence if $n > C$ then the curve family $\Gamma_n = \Gamma(B(x,\varepsilon), B(y,\varepsilon), n)$ must have positive ∞-modulus, completing the proof in the simple case where $E = B(x,\varepsilon)$ and $F = B(y,\varepsilon)$. The proof for more general E, F is similar; we modify the definition of u by looking at curves that connect z to E and then observing that almost every point in E and almost every point in F are Lebesgue points for the modified function v, where $v = 0$ on E and $v \ge n\, d(x,y)$ on F. This completes the proof of the proposition. $\square$

The following result describes one advantage of a thick quasiconvex space.

LEMMA 4.4. *Let X be a thick quasiconvex space. If u is a measurable function (finite μ-a.e.) on X, if g is an upper gradient of u, and if B is a ball in X such that $\|g\|_{L^\infty(2CB)} < \infty$, then there is a set $F \subset B$ with $\mu(F) = 0$ such that u is $C\|g\|_{L^\infty(2CB)}$-Lipschitz continuous on $B \setminus F$. Here C is the constant appearing in the definition of thick quasiconvexity.*

Proof. Since u is measurable (and finite μ-a.e.), by Lusin's theorem [4, p. 61] for every $n \in \mathbb{N}$ there is a measurable set $E_n \subset X$ such that $\mu(E_n) < 1/n$ and $u|_{B\setminus E_n}$ is continuous. Moreover, for each $n \ge 1$ we can choose G_n be an open set such that $E_n \subset G_n$, $\mu(G_n) < 1/n$ (see [15, Thm. 1.10]), and $u|_{X\setminus G_n}$ is continuous. Now $V_n = G_1 \cap G_2 \cap \cdots \cap G_n$ is an open set with $\mu(V_n) < 1/n$. Observe that $B \setminus V_n = (B \setminus G_1) \cup \cdots \cup (B \setminus G_n)$ and that $u|_{B\setminus V_n}$ is continuous.

We will show that u is $C\|g\|_{L^\infty(2CB)}$-Lipschitz continuous on $B \setminus V_n$. Let $P = \{x \in 2CB : g(x) > \|g\|_{L^\infty(2CB)}\}$; then, by assumption, $\mu(P) = 0$ and so it follows from Lemma 2.5 that $\mathrm{Mod}_\infty(\Gamma_P^+) = 0$. To prove that u is $C\|g\|_{L^\infty(2CB)}$-Lipschitz continuous on $B \setminus V_n$, we fix $x, y \in B \setminus V_n$ as points of density for $B \setminus V_n$. Let $0 < \delta < d(x,y)/4$. By thick quasiconvexity applied to the sets $E_\delta := B(x,\delta) \setminus V_n$ and $F_\delta := B(y,\delta) \setminus V_n$, there is a curve γ connecting the points $x_\delta \in E_\delta$ and $y_\delta \in F_\delta$ with $\ell(\gamma) \le C\, d(x_\delta, y_\delta)$ and $\mathscr{L}^1(\gamma^{-1}(\gamma \cap P)) = 0$. Notice that, since x is a point of density for $B \setminus V_n$, we have

$$\lim_{\rho\to 0} \frac{\mu(B(x,\rho) \cap (B \setminus V_n)}{\mu(B(x,\rho))} = 1$$

and so $\mu(E_\delta) > 0$. Analogously we obtain that, since y is a point of density for $B \setminus V_n$, we have $\mu(F_\delta) > 0$. Hence we can apply the thick quasiconvexity property to E_δ and F_δ. Therefore,

$$|u(x_\delta) - u(y_\delta)| \le \int_\gamma g\, ds \le \|g\|_{L^\infty(2CB)} \ell(\gamma) \le C\|g\|_{L^\infty(2CB)}\, d(x_\delta, y_\delta). \tag{6}$$

Since u is continuous on $B \setminus V_n$, we can let $\delta \to 0$ in (6) to show that

$$|u(x) - u(y)| \le C\|g\|_{L^\infty(2CB)}\, d(x,y)$$

as desired.

Next we put $F = \bigcap_n V_n$. Note that, since $\{V_n\}_n$ is a decreasing sequence of sets, $\mu(F) = \lim_{n\to\infty} \mu(V_n) = 0$. To conclude, let $x, y \in B \setminus F$. Since $B \setminus V_n$ is an increasing sequence of sets, there exists an $n \in \mathbb{N}$ such that $x, y \in B \setminus V_n$; hence $u|_{B\setminus F}$ is $C\|g\|_{L^\infty(2CB)}$-Lipschitz. □

In what follows we say that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms* if there is a constant $C > 0$ such that, for all $u \in N^{1,\infty}(X)$, there exists a $u_0 \in \mathrm{LIP}^\infty(X)$ with $u = u_0$ μ-a.e. and

$$\mathrm{LIP}(u_0) \le C \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞-weak upper gradients g of u.

Our next example shows that the requirement that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ as Banach spaces does not in itself imply that these two Banach spaces have comparable energy seminorms. But if the two seminorms are comparable, then the two Banach space norms are equivalent.

Example 4.5. Consider the set $X = \mathbb{R}^2 \setminus \bigcup_{n=1}^\infty R_n$, where R_n is the open rectangle $R_n = (2n, 2n+1) \times (0, n)$. We endow X with the Euclidean distance and the 2-dimensional Lebesgue measure. It is clear that X is not quasiconvex. Nevertheless, X is uniformly locally thick quasiconvex; in other words, for every $p \in X$, the ball $B(p, 1)$ in X with center p and radius $1/2$ is thick quasiconvex with quasiconvexity constant 2. Indeed, if the ball does not contain any corner of the rectangles R_n, $n \in \mathbb{N}$, then it is thick quasiconvex with quasiconvexity constant 1; if it contains a corner of one of the rectangles R_n then the ball is thick quasiconvex with quasiconvexity constant 2. Now we will show that each $u \in N^{1,\infty}(X)$ coincides a.e. with a function in $\mathrm{LIP}^\infty(X)$. The set $E = \{x \in X : u(x) > \|u\|_{L^\infty}\}$ has measure 0. If $x, y \in X \setminus E$ with $d(x, y) \ge 1/8$, then $|u(x) - u(y)| \le 2\|u\|_{L^\infty(X)} \le 16\|u\|_{L^\infty(X)}\, d(x, y)$.

Fix an upper gradient $g \in L^\infty(X)$ of u. Let (p_j) be an enumeration of the points in X having rational coordinates, and for each j consider the ball $B(p_j, 1/2)$. By Lemma 4.4, for each j there is a set F_j of measure 0 such that u is $2\|g\|_{L^\infty(B(p_j,1))}$-Lipschitz on $B(p_j, 1/2) \setminus F_j$ and hence is $2\|g\|_{L^\infty(X)}$-Lipschitz continuous on $B(p_j, 1/2) \setminus F_j$. The set $F = \bigcup_{j=1}^\infty F_j \cup E$ is of measure 0. If $x, y \in X \setminus F$ is such that $d(x, y) < 1/8$, then there is some j with $x, y \in B(p_j, 1/2)$ and so $|u(x) - u(y)| \le 2\|g\|_{L^\infty(X)}\, d(x, y)$. Thus for all $x, y \in X \setminus F$,

$$|u(x) - u(y)| \le 2[\|u\|_{L^\infty(X)} + 8\|g\|_{L^\infty(X)}]\, d(x, y).$$

Now the restriction $u|_{X\setminus F}$ can be extended to a Lipschitz function on X (for example, via McShane extension; see e.g. [7, Thm. 6.2]). In this way we obtain the equality $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$. Finally, because X is not quasiconvex, it follows from Theorem 4.7 (to follow) that we do not have comparable energy seminorms for this case.

Proposition 4.6. *If X is a thick quasiconvex space, then* $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms.*

Proof. We know that, given a Lipschitz function u on X, the constant function $\rho(x) = \mathrm{LIP}(u)$ is an upper gradient of u; hence we have a continuous embedding

$\mathrm{LIP}^\infty(X) \subset N^{1,\infty}(X)$. It therefore suffices to check that we have a continuous embedding $N^{1,\infty}(X) \subset \mathrm{LIP}^\infty(X)$. This follows from Lemma 4.4 once we exhaust X by balls of large radii and then modify $f \in N^{1,\infty}(X)$ on the exceptional set of measure 0 via McShane extension (see e.g. [7, Thm. 6.2]). □

We are now ready to state the main result of this paper.

THEOREM 4.7. *Suppose that X is a connected complete metric space supporting a doubling Borel measure μ that is nontrivial and finite on balls. Then the following conditions are equivalent.*

(a) *X supports a weak ∞-Poincaré inequality.*
(b) *X is thick quasiconvex.*
(c) $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms.*
(d) *X supports a weak ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$.*

The equivalence of condition (c) with the other three conditions requires the additional assumption of connectedness of X, since the union of two disjoint planar discs satisfies (c) but fails the other three conditions. Conditions (a), (b), and (d) directly imply that X is connected.

The result (a) $\Rightarrow$ (d) is immediate, so we split the proof of Theorem 4.7 into three parts as follows:

- (d) $\Rightarrow$ (b) has been proved as Proposition 4.3;
- (b) $\Rightarrow$ (c) has been proved as Proposition 4.6;
- (c) $\Rightarrow$ (a) will be proved in Proposition 4.11.

REMARK 4.8. We point out that if X is complete, connected, and equipped with a nontrivial doubling measure, then the following statements are equivalent.

(i) X is quasiconvex.
(ii) X supports an ∞-Poincaré inequality for locally Lipschitz continuous functions with continuous upper gradients.
(iii) $\mathrm{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms.

Recall that $D^\infty(X)$ is the class of all bounded functions $u\colon X \to \mathbb{R}$ for which the local Lipschitz constant function $\mathrm{Lip}\, u$ is uniformly bounded; see [3]. The norm on $D^\infty(X)$ is given by

$$\|u\|_{D^\infty(X)} := \sup_{x\in X}|u(x)| + \sup_{x\in X} \mathrm{Lip}\, u(x),$$

where

$$\mathrm{Lip}\, u(x) := \limsup_{\substack{y\to x\\ y\neq x}} \frac{|u(x)-u(y)|}{d(x,y)}.$$

So by $\mathrm{LIP}^\infty(X) = D^\infty(X)$ *with comparable energy seminorms* we mean that the two sets coincide and there is a constant $C > 0$ such that, for all $u \in \mathrm{LIP}^\infty(X)$,

$$\mathrm{LIP}(u) \le C \sup_{x\in X} \mathrm{Lip}\, u(x).$$

It is well known that $\mathrm{LIP}^\infty(X)$ is a Banach space. However, $D^\infty(X)$ is not, in general, a Banach space (as shown in [3]). But $\mathrm{LIP}^\infty(X) \subset D^\infty(X)$ is an isometric embedding because, if u is a Lipschitz function, then

$$\operatorname{Lip} u(x) \le \mathrm{LIP}(u) \quad \text{for every } \ x \in X.$$

When condition (iii) is satisfied, $D^\infty(X)$ will also be a Banach space.

The implication (ii) $\Rightarrow$ (i) is given by the proof of Proposition 3.4; we need only apply the Poincaré inequality to the locally Lipschitz continuous function $\rho_{x,\varepsilon}$ and its continuous upper gradient 1. The implication (i) $\Rightarrow$ (ii) follows from the argument that, if g is a continuous upper gradient of a locally Lipschitz continuous function u, then for $x, y \in X$ we can choose a quasiconvex path γ connecting x to y and obtain

$$|u(x) - u(y)| \le \int_\gamma g\, ds \le C\, d(x,y) \sup_{z \in B(x, C\, d(x,y))} g(z).$$

So if B is a ball in X and if x, y are points in B, then

$$\fint_B \fint_B |u(x) - u(y)|\, d\mu(x)\, d\mu(y) \le C\, \mathrm{rad}(B) \sup_{z \in CB} g(z) = C\, \mathrm{rad}(B) \|g\|_{L^\infty(CB)}.$$

That condition (i) implies condition (iii) is shown in [3, Lemma 2.3, Cor. 2.4].

Now suppose that condition (iii) holds. Then, as in the proof of Proposition 3.4, for each $x \in X$ and $\varepsilon > 0$ we consider the function $\rho_{x,\varepsilon}$ and, since X is connected, we see that $\rho_{x,\varepsilon}$ is finite valued everywhere and that $|\rho_{x,\varepsilon}(z) - \rho_{x,\varepsilon}(w)| \le d(z,w)$ when $d(z,w) < \varepsilon$. Thus, for all $w \in X$, we have $\operatorname{Lip} \rho_{x,\varepsilon}(w) \le 1$ and hence $\rho_{x,\varepsilon}$ belongs to $D^\infty(X)$. Because (iii) holds, there is a constant $C > 0$ such that $\mathrm{LIP}(\rho_{x,\varepsilon}) \le C$ with C independent of x and ε. It follows that, for all $y \in X$ and all $\varepsilon > 0$,

$$|\rho_{x,\varepsilon}(y)| = |\rho_{x,\varepsilon}(y) - \rho_{x,\varepsilon}(x)| \le \mathrm{LIP}(\rho_{x,\varepsilon})\, d(x,y) \le C\, d(x,y).$$

Consequently, as in the proof of Proposition 3.4, there is a curve γ connecting x to y with length $\ell(\gamma) \le C\, d(x,y)$; that is, X is quasiconvex. These two arguments prove that conditions (i) and (iii) are equivalent.

Now we continue with our proof of Theorem 4.7 as outlined before Remark 4.8.

The following two technical lemmas will be useful in the sequel.

Lemma 4.9. *Suppose $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ with comparable energy seminorms. Then there exists a constant $C \ge 1$ such that, for every $E \subset X$ with $\mu(E) = 0$ and for every $x \in X$ and $r > 0$, there is a set $F \subset X$ with $\mu(F) = 0$ such that, whenever $y \in X \setminus (B(x,2r) \cup F)$, there is a rectifiable curve γ_y connecting y to $\bar{B}(x,r)$ such that $\ell(\gamma_y) \le C\, d(x,y)$ and $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$.*

Proof. Let $E \subset X$ be such that $\mu(E) = 0$; since μ is a Borel measure, we may assume (by enlarging E if necessary) that E is a Borel set. Then $\rho = \infty \cdot \chi_E \in L^\infty(X)$ is a nonnegative Borel measurable function. Let Γ_E^+ be the collection of all rectifiable curves γ for which $\mathscr{L}^1(\gamma^{-1}((\gamma \cap E))) > 0$. Then for such curves γ

we clearly have $\int_\gamma \rho\, ds = \infty$ and so $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. As before, we define (for $r > 0$)

$$u(z) = \inf_{\gamma \text{ connects } z \text{ to } B(x,r)} \int_\gamma (1+\rho)\, ds,$$

where $\|1+\rho\|_{L^\infty(X)} = 1$. For positive integers k we set $u_k = \min\{k, u\}$. Then $u_k \in N^{1,\infty}(X)$ with $1+\rho$ as an upper gradient (see Lemma 3.5) and $u = 0$ on $B(x,r)$. Let F_k be the exceptional set on which u_k must be modified in order to be Lipschitz continuous; then $\mu(F_k) = 0$. Observe that, since $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms, we have

$$\mathrm{LIP}(u_k) \le C \inf_g \|g\|_{L^\infty} \le C\|1+\rho\|_{L^\infty(X)} = C;$$

here the infimum is taken over all ∞-weak upper gradients g of u_k.

Let $F = \bigcup_{k\in\mathbb{N}} F_k$. Then, for $y \in X \setminus (F \cup B(x,2r))$, there exists a positive integer k such that $d(x,y) < k/2C$. In addition, we have that

$$|u_k(y)| = |u_k(y) - u_k(x_1)| \le C\, d(x_1,y) \le C(d(x_1,x) + d(x,y)) \le 2C\, d(x,y)$$

for any $x_1 \in B(x,r) \setminus F_k$ and that $u_k(y) = \tilde{u}(y)$ is finite. Thus, there exists a rectifiable curve γ_y such that

$$\ell(\gamma_y) + \int_{\gamma_y} \rho\, ds = \int_{\gamma_y} (1+\rho)\, ds \le C\, d(x,y).$$

Hence we have

$$\ell(\gamma_y) \le C\, d(x,y) \quad \text{and} \quad \int_{\gamma_y} \rho < +\infty,$$

so $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$ as desired. □

LEMMA 4.10. *Let $u \in N^{1,\infty}(X)$ and let $g \in L^\infty(X)$ be an upper gradient of u. If v is a Lipschitz continuous function on X such that $u = v$ μ-a.e., then g is an ∞-weak upper gradient of v and so there is a Borel measurable function $0 \le \rho \in L^\infty(X)$ with $\rho = g$ μ-a.e. such that ρ is an upper gradient of v.*

Proof. Let $E = \{x \in X : u(x) \ne v(x)\}$; then $\mu(E) = 0$ and so $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. If $x, y \in X \setminus E$ and if β is a rectifiable curve connecting x to y in X, then

$$|u(x) - u(y)| = |v(x) - v(y)| \le \int_\beta g\, ds.$$

Let γ be a nonconstant rectifiable compact curve with endpoints x and y and such that $\gamma \notin \Gamma_E^+$. Then we can find two sequences of points $\{z_i\}$ and $\{w_i\}$ from the trajectory of γ such that for i we have $z_i, w_i \in \gamma \setminus E$ as well as $z_i \to x$ and $w_i \to y$ as $i \to \infty$. Let γ_i be a subcurve of γ with endpoints z_i and w_i; then, by the foregoing discussion,

$$|v(z_i) - v(w_i)| \le \int_{\gamma_i} g\, ds \le \int_\gamma g\, ds.$$

Since v is Lipschitz continuous, we can let $i \to \infty$ in this expression and obtain

$$|v(x) - v(y)| \le \int_\gamma g\, ds.$$

It follows that g is an ∞-weak upper gradient of v. Since $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$, by [3, Lemma 5.7] there is a nonnegative Borel measurable function ρ_0 such that $\|\rho_0\|_{L^\infty(X)} = 0$ even though the integral $\int_\gamma \rho_0\, ds = \infty$ for all $\gamma \in \Gamma_E^+$. It follows that $\rho = g + \rho_0$ is an upper gradient of v with the desired property. $\square$

PROPOSITION 4.11. *Suppose that X is connected and that $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ with comparable energy seminorms. In this case, X supports a weak ∞-Poincaré inequality.*

Proof. Let $u \in N^{1,\infty}(X)$, let $g \in L^\infty(X)$ be an upper gradient of u, and fix a ball $B \subset X$. By Lemma 4.10 and our assumption that $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$, we may assume that u is itself Lipschitz continuous on X. Let $E = \{w \in 2CB : g(w) > \|g\|_{L^\infty(2CB)}\}$, where C is the constant from Lemma 4.9. Then $\mu(E) = 0$. Fix $\varepsilon > 0$.

Observe that, since μ is doubling and X is connected, we may deduce that $\mu(\{x\}) = 0$ for all $x \in X$ (see inequality (2)). So for $x \in B$ we can choose $r > 0$ sufficiently small such that:

1. $B(x, 2r) \subset B$;
2. $\mu(B(x, 2r)) < \mu(B)/2$;
3. for all $w \in \bar{B}(x, r)$ we have $|u(w) - u(x)| < \varepsilon$ (this is possible because u is Lipschitz continuous); and
4. $\int_{\bar{B}(x,2r)} |u - u(x)|\, d\mu \le \frac{1}{2} \int_B |u - u(x)|\, d\mu$.

Then

$$\begin{aligned} \fint_B |u - u(x)|\, d\mu &\le \frac{2}{\mu(B)} \int_{B \setminus B(x,2r)} |u - u(x)|\, d\mu \\ &\le 2 \fint_{B \setminus B(x,2r)} |u(y) - u(x)|\, d\mu(y). \end{aligned}$$

Let $F \subset X$ be the set given by Lemma 4.9 with respect to x and r, and for $y \in B \setminus (F \cup B(x, 2r))$ let γ_y be the corresponding curve connecting y to $B(x, r)$. We denote the other endpoint of γ_y as $w_y \in \bar{B}(x, r)$. By the choice of r, we see that $|u(y) - u(x)| \le |u(y) - u(w_y)| + |u(w_y) - u(x)| < |u(y) - u(w_y)| + \varepsilon$. Hence $|u(y) - u(x)| \le \varepsilon + \int_{\gamma_y} g\, ds \le \varepsilon + C\|g\|_{L^\infty(2CB)}\, d(x, y)$, where we have used that $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$. Therefore,

$$\begin{aligned} \fint_B |u - u(x)|\, d\mu &\le 2 \fint_{B \setminus (F \cup B(x,2r))} (\varepsilon + C\|g\|_{L^\infty(2CB)}\, d(x, y))\, d\mu(y) \\ &\le 4 \fint_{B \setminus (F \cup B(x,2r))} (\varepsilon + C\|g\|_{L^\infty(2CB)}\, \mathrm{rad}(B))\, d\mu(y) \\ &= 4(\varepsilon + C\|g\|_{L^\infty(2CB)}\, \mathrm{rad}(B)). \end{aligned}$$

Now integrating over x, we obtain

$$\fint_B \fint_B |u(y) - u(x)|\, d\mu(y)\, d\mu(x) \le 4(\varepsilon + C\|g\|_{L^\infty(2CB)}\, \mathrm{rad}(B)).$$

Letting $\varepsilon \to 0$ yields the inequality

$$\fint_B \fint_B |u(y) - u(x)|\, d\mu(y)\, d\mu(x) \le 2C\, \mathrm{rad}(B)\|g\|_{L^\infty(2CB)},$$

which implies (by Remark 3.2) the weak ∞-Poincaré inequality for the pair (u, g). Since the constants are independent of u, g, B, it follows that (X, d, μ) supports a weak ∞-Poincaré inequality for Newtonian functions. Then, by Proposition 4.3, X is thick quasiconvex.

To complete the proof, we have to check that (X, d, μ) admits a weak ∞-Poincaré inequality for every Borel measurable function $u\colon X \to \mathbb{R}$ and every upper gradient. Let u be a measurable function and let g be a measurable upper gradient for u. Fix B. If $\|g\|_{L^\infty(2CB)} = \infty$ then we are done, so assume that $\|g\|_{L^\infty(2CB)} < \infty$. We have shown that X is thick quasiconvex, so we can invoke Lemma 4.4 to see that u is Lipschitz in $B \subset X$ up to a set of measure 0. By Lemma 4.10, we can assume that u is Lipschitz in all of B and that g is an upper gradient of u in B. Hence we can repeat the foregoing proof for the pair u and g, completing the proof of Proposition 4.11. □

EXAMPLE 4.12. The space (X, d, μ) considered in Example 3.3 with a measure that rapidly decays to zero at the origin (the point where the two triangular regions are glued) is thick quasiconvex. We can prove this with the aid of Theorem 4.7 even though μ is not doubling. Indeed, since $(X, d, \mathscr{L}^2)$ supports a p-Poincaré inequality for $p > 2$ [18, 4.3.1], it also supports an ∞-Poincaré inequality. By Theorem 4.7, it is also thick quasiconvex (we can apply this theorem because $\mathscr{L}^2$ is a doubling measure). Using the idea described in Remark 2.3, we conclude that (X, d, μ) is also thick quasiconvex.

The rest of this section is devoted to showing that, in Theorem 4.7, thick quasiconvexity cannot be replaced with the weaker notion of quasiconvexity.

The next lemma is useful for verifying that a metric space does not support any Poincaré inequality. Its proof is an adaptation of [2, Lemma 4.3] for the case $p = \infty$.

LEMMA 4.13. *Let (X, d, μ) be a bounded doubling metric measure space that admits a weak ∞-Poincaré inequality, and let $f\colon X \to I$ be a surjective Lipschitz function from X onto an interval $I \subset \mathbb{R}$. Then $\mathscr{L}^1|_I \ll f_\# \mu$, where $f_\# \mu$ denotes the pushforward measure of μ under f.*

Proof. Suppose the contrary, and denote $L = \mathrm{LIP}(f)$. Then there exists a Borel set N in I such that $\mathscr{L}^1(N) > 0$ and $\mu(f^{-1}(N)) = f_\# \mu(N) = 0$. On X we consider the function

$$u(x) = \int_0^{f(x)} \chi_N(t)\, d\mathscr{L}^1(t).$$

This function is L-Lipschitz because, for $x, y \in X$, we have

$$\begin{aligned}|u(y) - u(x)| &= \left| \int_{f(x)}^{f(y)} \chi_N \, d\mathscr{L}^1 \right| \\ &= \mathscr{L}^1([f(x), f(y)] \cap N) \le |f(y) - f(x)| \le L\, d(y, x).\end{aligned}$$

Moreover, $g = L(\chi_N \circ f)$ is an upper gradient of u. Indeed, for each rectifiable curve $\gamma\colon [a,b] \to X$ one has (we assume w.l.o.g. that $f(\gamma(a)) < f(\gamma(b))$)

$$\begin{aligned}|u(\gamma(a)) - u(\gamma(b))| &= \left| \int_{f(\gamma(a))}^{f(\gamma(b))} \chi_N(t)\, d\mathscr{L}^1(t) \right| \\ &= \mathscr{L}^1([f(\gamma(a)), f(\gamma(b))] \cap N)\end{aligned}$$

and

$$\int_\gamma g = \int_a^b L \cdot (\chi_N \circ f(\gamma(t)))\, d\mathscr{L}^1(t) = L\mathscr{L}^1([a,b] \cap (f \circ \gamma)^{-1}(N)).$$

Because γ is arc-length parameterized, $f \circ \gamma$ is L-Lipschitz. It follows that

$$\mathscr{L}^1([a,b] \cap (f \circ \gamma)^{-1}(N)) \ge L^{-1}\mathscr{L}^1([f(\gamma(a)), f(\gamma(b))] \cap N)$$

and so

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g \, d\mathscr{L}^1(t)$$

for each rectifiable curve γ in X. However, $\mu\{x \in X : f(x) \in N\} = f_\#\mu(N) = 0$ by hypothesis, so $\chi_N \circ f(x) = 0$ μ-a.e. Therefore, by the weak ∞-Poincaré inequality, $\int_X |u - u_X| \, d\mu = 0$; this means that u is constant μ-a.e. on X. Because u is Lipschitz continuous on X, it follows that u is constant on X—contradicting the fact that u is nonconstant on the set $f^{-1}(N)$ (this set is nonempty because f is surjective, and u is not constant here because $\mathscr{L}^1(N) > 0$). □

Example 4.14. Let $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square. Divide Q into nine equal squares of side length $1/3$ and remove the central one. In this way, we obtain a set Q_1 that is the union of eight squares of side length $1/3$. Repeating this procedure on each square yields a sequence of sets Q_j consisting of 8^j squares of side length $1/3^j$. We define the *Sierpinski carpet* to be $S = \bigcap Q_j$. If d is the distance in $\mathbb{R}^2$ given by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

then (S, d) is a complete geodesic metric space. Let μ be the Hausdorff measure on (S, d) of dimension s, where s is given by the formula $3^s = 8$. It can be checked that μ is a doubling measure and that the metric d just defined is bi-Lipschitz equivalent to the restriction of the Euclidean metric.

The Sierpinski carpet (S, d, μ) is clearly quasiconvex. Hence the following corollary demonstrates that the quasiconvexity property is not sufficient to guarantee ∞-Poincaré inequality.

COROLLARY 4.15. *The Sierpinski carpet (S, d, μ) does not admit an ∞-Poincaré inequality.*

Proof. Let f be the projection on the horizontal axis. It can be shown that $f_{\#}\mu \perp \mathscr{L}^1$ (see [2, 4.5]). Indeed, as shown in [2], if given a point $0 < x < 1$ we can see, by way of the ternary expansion of x, that the interval I_n centered at x and of radius 3^{-n} has Lebesgue measure $\mathscr{L}^1(I_n) \approx 3^{-n}$. However, $f_{\#}\mu(I_n) \approx e^{-\psi(x,n)}$ for an appropriately chosen function ψ with the property that

$$\lim_{n\to\infty} \frac{f_{\#}\mu(I_n)}{\mathscr{L}^1(I_n)} \approx \limsup_{n\to\infty} \frac{e^{-\psi(x,n)}}{3^{-n}},$$

which for $\mathscr{L}^1$-a.e. x is either 0 or ∞. In conjunction with the Radon–Nikodym theorem, this implies that $f_{\#}\mu$ is singular with respect to the Lebesgue measure $\mathscr{L}^1$.

The result now follows from Lemma 4.13. □

Added in proof. After this paper was accepted for publication, the first and third authors, together with Alex Williams, constructed a metric space equipped with a doubling measure that supports an ∞-Poincaré inequality but no p-Poincaré inequality for each finite p; see E. Durand-Cartagena, N. Shanmugalingam, and A. Williams, *p-Poincaré inequality vs. ∞-Poincaré inequality*; *some counterexamples,* Math. Z. (to appear).

References

[1] S. G. Bobkov and C. Houdré, *Some connections between isoperimetric and Sobolev-type inequalities,* Mem. Amer. Math. Soc. 129 (1997).

[2] M. Bourdon and H. Pajot, *Quasi-conformal geometry and hyperbolic geometry,* Rigidity in dynamics and geometry (Cambridge, 2000), pp. 1–17, Springer-Verlag, Berlin, 2002.

[3] E. Durand-Cartagena and J. A. Jaramillo, *Pointwise Lipschitz functions on metric spaces,* J. Math. Anal. Appl. 363 (2010), 525–548.

[4] G. B. Folland, *Real analysis. Modern techniques and their applications,* Pure Appl. Math. (N.Y.), Wiley, New York, 1999.

[5] P. Hajłasz, *Sobolev spaces on metric-measure spaces,* Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), Contemp. Math., 338, pp. 173–218, Amer. Math. Soc, Providence, RI, 2003.

[6] P. Hajlasz and P. Koskela, *Sobolev met Poincaré,* Mem. Amer. Math. Soc. 145 (2000).

[7] J. Heinonen, *Lectures on analysis on metric spaces,* Springer-Verlag, New York, 2001.

[8] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry,* Acta Math. 181 (1998), 1–61.

[9] ———, *A note on Lipschitz functions, upper gradients, and the Poincaré inequality,* New Zealand J. Math. 28 (1999), 37–42.

[10] E. Järvenpää, M. Järvenpää, N. Shanmugalingam, K. Rogovin, and S. Rogovin, *Measurability of equivalence classes and MEC$_p$-property in metric spaces,* Rev. Mat. Iberoamericana 23 (2007), 811–830.

[11] S. Keith and X. Zhong, *The Poincaré inequality is an open ended condition,* Ann. of Math. (2) 167 (2008), 575–599.
[12] J. Kinnunen and R. Korte, *Characterizations of Sobolev inequalities on metric spaces,* J. Math. Anal. Appl. 344 (2008), 1093–1104.
[13] R. Korte, *Geometric implications of the Poincaré inequality,* Licentiate thesis, Helsinki University of Technology, 2006.
[14] T. J. Laakso, *Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality,* Geom. Funct. Anal. 10 (2000), 111–123.
[15] P. Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability,* Cambridge Stud. Adv. Math., 44, Cambridge Univ. Press, 1995.
[16] M. Miranda, *Functions of bounded variation on "good" metric spaces,* J. Math. Pures Appl. (9) 82 (2003), 975–1004.
[17] S. Semmes, *Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities,* Selecta Math. (N.S.) 2 (1996), 155–295.
[18] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces,* Ph.D. thesis, University of Michigan, 1999, ⟨http:math.uc.edu/~nages/papers.html⟩.
[19] ———, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces,* Rev. Mat. Iberoamericana 16 (2000), 243–279.

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