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Extensions of Two Chow Stability Criteria to Positive Characteristics

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1. Introduction

We work over an algebraically closed field k of arbitrary characteristic. Let $X \subset \mathbb{P}_k^n$ be an effective cycle of dimension r and degree d in a projective space of dimension n. Analysis of the Chow (semi)stability of X is one of the basic problems in geometric invariant theory (GIT). Contrary to the case of asymptotic Chow (semi)stability, the precise classification of Chow (semi)stable cycles is quite a subtle problem and is known for only a few cases, even for projective hypersurfaces. For example, Shah [Sh] studied the case of plane sextics and Laza [La] studied the case of cubic 4-folds—both in relation to period maps.

On the other hand, there are two sufficient conditions for Chow (semi)stability in terms of the singularity of X or that of the Chow divisor $Z(X) \subset \mathbb{G} = \mathrm{Grass}_k(n-r, n+1)$, which deal with general situations. Both have been proved in characteristic 0, and the purpose of this paper is to extend them to arbitrary characteristics. Namely, we prove the following two theorems.

Theorem 1.1 (= Theorem 3.1). *If $d \geq 3$, then any nonsingular projective hypersurface of degree d is Chow stable.*

Theorem 1.2 (= Theorem 4.1). *Let X be an effective cycle of dimension r and degree d in $\mathbb{P}_k^n$. Let $(\mathbb{G}, Z(X))$ be the log pair defined by the Chow divisor $Z(X)$ of X. If $\mathrm{lct}(\mathbb{G}, Z(X))$ is greater than (respectively, is greater than or equal to) $\frac{n+1}{d}$, then X is Chow stable (respectively, Chow semistable).*

In the statement of Theorem 1.2, $\mathrm{lct}(\mathbb{G}, Z(X))$ denotes the log canonical threshold of $(\mathbb{G}, Z(X))$, which measures how good the singularity of $Z(X)$ is (see Section 2.2 for details). The characteristic-0 case of Theorem 1.1 is due to Mumford [GIT, Chap. 4, Sec. 2], and that of Theorem 1.2 is due to Lee [Le].

The original proof of Theorem 1.1 works only when the characteristic of the base field does not divide d (see Section 3). To prove the general case, we depend on the corresponding result in characteristic 0.

We sketch the proof of Theorem 1.1 in positive characteristic. First we take a suitable lift of the equation of a given hypersurface over the ring of Witt vectors. This defines a family of projective hypersurfaces over the ring. We are assuming that the closed fiber is nonsingular; hence the geometric generic fiber is also

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nonsingular. Since we know that Theorem 1.1 holds in characteristic 0, we obtain some inequalities for the Hilbert–Mumford numerical functions of the lift. By the choice of the lift, those numerical functions coincide with those of the original hypersurface. Thus we obtain the inequalities for the numerical functions of the original one, concluding the proof.

The point is that the singularity of the hypersurface over the generic point is better than that of the special fiber, so we can use the corresponding stability criterion in characteristic 0. This method seems to be applicable to other stability problems (see the remark at the beginning of Section 4).

In Section 3.2 we show that the complement of the locus of a nonsingular hypersurface is an irreducible divisor, even when p divides d. In general, some multiple of the defining equation of this divisor lifts to the usual discriminant in characteristic 0.

Theorem 1.2 will be proved along the same lines as the proof given in [Le], but we must modify several points. This is because some properties of log canonicity that hold in characteristic 0 fail in positive characteristic owing to the existence of wild ramifications and inseparable morphisms.

We can prove that our required property of log canonicity still holds for finite separable morphisms. It turns out that this is enough for our purpose, since we can use a perturbation technique and thereby avoid dealing with the inseparable morphisms (see Section 4). In Section 4 we also discuss some other properties of log canonicity and provide a few (counter)examples.

In the Appendix we prove the following result.

PROPOSITION 1.3 (= Proposition A.1). *Let Y and Z be Chow semistable cycles of the same dimension in a projective space $\mathbb{P}^n_k$. Then $Y + Z$ is Chow semistable. Furthermore, if Y is Chow stable then so is $Y + Z$.*

This proposition may be well known to experts, but the author could not find it in the literature. The proof is a simple application of the fact that the stability can be checked via 1-parameter subgroups, an approach that is essentially the same as employing the numerical criterion. Yet the conclusion itself seems to be rather surprising: if we have two Chow stable cycles, the sum of them is always Chow stable regardless of the way they touch each other.

Proposition 1.3 will be used to give a family of stable projective hypersurfaces whose stability cannot be detected by Theorem 1.2 (see Example A.5).

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2. Preliminaries

2.1. Notation from Scheme Theory

We need some notation from [Ha]. Let R be an $\mathbb{N}$-graded ring. For a homogeneous ideal I of R we denote by $V(I)$ the corresponding closed subscheme $\operatorname{Proj}(R/I)$ of $\operatorname{Proj} R$.

For a homogeneous element $f \in R$, we use $D_+(f)$ to denote $\operatorname{Proj} R \setminus V(f)$. This open subscheme is known to be affine, with coordinate ring

$$R_{(f)} = \left\{ \frac{r}{f^n} \;\middle|\; r \in R,\ \deg(r) = n \cdot \deg(f) \right\}.$$

2.2. Notions of Singularities

Here we summarize the notions of singularities of pairs that will be needed later.

Definition 2.1 (discrepancy, log canonical). Let X be a normal variety over k, and let Δ be an effective $\mathbb{R}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.

Let $\pi : Y \to X$ be a birational morphism from another normal variety Y over k and let $E \subset Y$ be a prime divisor. Then, in a neighborhood of the generic point of E, the following canonical bundle formula holds:

$$K_Y = \pi^*(K_X + \Delta) + aE.$$

The real number a in this equation is called the *discrepancy* of E with respect to (X, Δ) and is denoted by $a(E; X, \Delta)$. It is independent of the choice of Y and π and depends only on the valuation of $k(X)$ that corresponds to E.

We say that the log pair (X, Δ) is *log canonical* ("lc" for short) if $a(E; X, \Delta) \geq -1$ holds for all the divisors E as above.

A finer version of Definition 2.1 is as follows.

Definition 2.2. Let $x \in X$ be a point. We say that the log pair (X, Δ) is *log canonical at* x if the restriction of (X, Δ) to an open neighborhood of x is log canonical.

Definition 2.3 (log canonical threshold). Let (X, Δ) be a log canonical pair and let D be an effective $\mathbb{R}$-Cartier divisor on X. The *log canonical threshold* of D with respect to (X, Δ) is defined as

$$\operatorname{lct}(X, \Delta; D) = \sup\{t \in \mathbb{R} \mid (X, \Delta + tD) \text{ is log canonical}\}.$$

For a point $x \in X$, we set

$$\operatorname{lct}_x(X, \Delta; D) = \sup\{t \in \mathbb{R} \mid (X, \Delta + tD) \text{ is log canonical at } x\}.$$

If the pair $(X, \Delta + D)$ has a log resolution, then it is easy to see that "sup" in this definition is actually "max".

When considering the case $\Delta = 0$, we write $\operatorname{lct}(X, \Delta; D) = \operatorname{lct}(X, D)$ for short (and $\operatorname{lct}_x(X, \Delta; D) = \operatorname{lct}_x(X, D)$).

2.3. Chow Stability and the Numerical Criterion

Let $X \subset \mathbb{P}_k^n$ be an effective r-dimensional cycle of degree d. We associate to X its *Chow divisor* $Z(X)$, which is a hypersurface of degree d of the Grassmannian $\mathbb{G} = \mathrm{Grass}_k(n-r, n+1)$, as follows (see [GKZ; Ko1] for details). If X itself is a variety, set $Z(X) = \{L \in \mathbb{G} \mid L \cap X \neq \emptyset\}$. For a general cycle X, define $Z(X)$ additively. The defining equation of $Z(X)$ is called the *Chow form* of X (Chow form is determined by X only up to scalar multiplication). The homogeneous coordinate ring of $\mathbb{G}$ with respect to the Plücker embedding is denoted by $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$. This is the subring of the polynomial ring of $(n+1)(n-r)$ indeterminants $U_i^{(j)}$, where (i, j) runs through the range $i = 0, \ldots, n$ and $j = 1, \ldots, n-r$, which are generated by all the $(n-r) \times (n-r)$ minors of the matrix $(U_i^{(j)})$. The Chow form of a cycle X is an element of $\mathcal{B}_d$ (up to scalar multiplication), so the Chow divisor $Z(X)$ of X can be regarded as an element of the projective space $\mathbb{P}_* \mathcal{B}_d$. The canonical action of $\mathrm{SL}(n+1, k)$ on $\mathbb{P}_k^n$ naturally induces a linear action on $\mathcal{B}_d$; hence we can discuss the GIT (semi)stability of an element of $\mathbb{P}_* \mathcal{B}_d$. (Here we are using "stable" in the sense of "properly stable" in [GIT], which requires finiteness of the stabilizer subgroup; we use this terminology because we rely heavily on the numerical criterion (the author would like to thank Dr. S. Ma for this remark). Chow (semi)stability of X is defined to be the (semi)stability of $Z(X)$ in the sense just described.

Next we recall the *Hilbert–Mumford numerical criterion* ("numerical criterion" for short) for stability and obtain an explicit description of the numerical function μ following [GIT, Prop. 2.3]. We start with some preparations.

For a nonnegative integer n, set

$$[n] = \{0, 1, \ldots, n\}. \tag{1}$$

For a subset $I \subset [n]$ with $\#I = n-r$, let Δ_I be the $(n-r) \times (n-r)$ minor of the matrix $(U_i^{(j)})$ obtained by picking out the $n-r$ rows according to I. Recall that $\mathcal{B}_d$ is a k-vector space generated by the set $\{\Delta_{I_1} \cdots \Delta_{I_d} \mid I_\ell \subset [n], \#I_\ell = n-r$ for all $\ell = 1, \ldots, d\}$.

Now fix X. Take any $g \in \mathrm{SL}(n+1, k)$ and let F be the Chow form of g^*X (i.e., the defining equation of $g^*Z(X)$). Set

$$\mathcal{R} = \left\{ \mathbf{r} = (r_0, \ldots, r_n) \in \mathbb{Z}^{n+1} \setminus \{0\} \;\middle|\; \sum_{i=0}^{n} r_i = 0,\ r_0 \leq r_1 \leq \cdots \leq r_n \right\}. \tag{2}$$

An element $\mathbf{r}$ of $\mathcal{R}$ corresponds to a nontrivial 1-parameter subgroup (1-PS) $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(n+1, k)$ of $\mathrm{SL}(n+1, k)$ that is defined by $\lambda(t) = \mathrm{diag}(t^{r_0}, \ldots, t^{r_n})$. If we regard $\mathcal{B}_d$ as a representation of $\mathbb{G}_m$ via λ, then the 1-dimensional subspace of $\mathcal{B}_d$ spanned by $\Delta_{I_1} \cdots \Delta_{I_d}$ is an eigenspace of weight

$$\mathrm{wt}(I_1, \ldots, I_d) = \sum_{\ell=1}^{d} \sum_{i \in I_\ell} r_i = \sum_{i=0}^{n} r_i \cdot \#\{\ell \mid i \in I_\ell\}.$$

Using this notation, the numerical function of X with respect to $g \in \mathrm{SL}(n+1, k)$ and $\mathbf{r} \in \mathcal{R}$ is defined as follows.

DEFINITION 2.4 (numerical function). Let $\mathcal{I}(F)$ be the set of d-tuples $(I_1, \ldots, I_d)$ such that the coefficient of $\Delta_{I_1} \cdots \Delta_{I_d}$ in F is not zero. Then set

$$\mu(Z(X), g, \mathbf{r}) = \mu(V(F), \mathrm{id}, \mathbf{r}) = \min_{(I_1, \ldots, I_d) \in \mathcal{I}(F)} \mathrm{wt}(I_1, \ldots, I_d).$$

REMARK 2.5. $\mu(Z(X), g, \mathbf{r})$ depends only on g, $\mathbf{r}$, and the set $\mathcal{I}(F)$.

Now the numerical criterion is as follows.

PROPOSITION 2.6. *The cycle X is Chow stable (resp., semistable) if and only if $\mu(Z(X), g, \mathbf{r}) < 0$ (resp., ≤ 0) holds for any $g \in \mathrm{SL}(n+1, k)$ and $\mathbf{r} \in \mathcal{R}$.*

We will recast Proposition 2.6 in such a way as to prove Theorem 4.1. This reinterpretation is simply a generalization of [Le, Lemma 2.1]. However, first we need to make some preparations. Take an arbitrary $g \in \mathrm{SL}(n+1, k)$ and let F be the Chow form of g^*X.

Let f be the local equation of F on $D_+(\Delta_{[n-r-1]}) \simeq \operatorname{Spec} \mathcal{B}_{(\Delta_{[n-r-1]})}$. Recall that $\mathcal{B}_{(\Delta_{[n-r-1]})}$ is the polynomial ring over k with indeterminants $x_I = \frac{\Delta_I}{\Delta_{[n-r-1]}}$, where I runs through those subsets of $[n]$ (see (1)) that satisfy the following two conditions:

$$\begin{aligned} \#I &= n - r; \\ \#(I \cap [n-r-1]) &= n - r - 1. \end{aligned} \tag{3}$$

Therefore, f is a polynomial in the x_I. Now assign nontrivial integral weights $\mathbf{r} = (r_0, \ldots, r_n) \in \mathcal{R}$ to $X_0, \ldots, X_n$ so that the induced weight $w(x_I)$ on x_I satisfies

$$w(x_I) = \sum_{i \in I} r_i - \sum_{i=0}^{n-r-1} r_i, \tag{4}$$

which is nonnegative by the assumption $r_0 \leq r_1 \leq \cdots \leq r_n$.

Now Proposition 2.6 is equivalent to the following statement.

LEMMA 2.7. *A cycle X is Chow stable (resp. semistable) if and only if*

$$\frac{w(f)}{\sum_I w(x_I)} < \frac{d}{n+1} \tag{5}$$

(*resp.* $\leq \frac{d}{n+1}$) *holds for all $g \in \mathrm{SL}(n+1, k)$ and $\mathbf{r} \in \mathcal{R}$ (see (2) for the definition of $\mathcal{R}$).*

Here f is the local equation on $D_+(\Delta_{[n-r-1]})$ of the Chow form of g^*X as before. In the left-hand side of (5), $w(f)$ denotes the weighted multiplicity of f (i.e., the lowest weight of the monomials occurring in f) with respect to the weight $(w(x_I))_I$.

Proof of Lemma 2.7. We discuss only the stable case. The semistable case can be proven similarly.

The inequality (5) is equivalent to

$$d\sum_I w(x_I) - (n+1)w(f) > 0. \tag{6}$$

Combining the calculation of $w(x_I)$ (see (4)) with the definition of $w(f)$, we see that the left-hand side of (6) is equal to

$$d\Bigg(\sum_I \sum_{i\in I} r_i - (n-r)(r+1)\sum_{i=0}^{n-r-1} r_i\Bigg) - (n+1)\Bigg(\mu(X,g,\mathbf{r}) - d\sum_{i=0}^{n-r-1} r_i\Bigg).$$

Recalling the conditions (3) imposed on I, we see that

$$\sum_I \sum_{i\in I} r_i = (n-r-1)(r+1)\sum_{i=0}^{n-r-1} r_i + (n-r)\sum_{i=n-r}^{n+1} r_i.$$

A little calculation shows that the left-hand side of (6) boils down to

$$d(n-r)\sum_{i=0}^{n} r_i - (n+1)\mu(X,g,\mathbf{r}) = -(n+1)\mu(X,g,\mathbf{r}),$$

since we assumed that $\sum_{i=0}^n r_i = 0$.

Therefore, (5) is equivalent to the condition $\mu(X,g,\mathbf{r}) < 0$. □

2.4. Chow Stability in Characteristic p from Characteristic 0

Let k be a field of characteristic $p > 0$ and let X be a cycle in $\mathbb{P}_k^n$. In this section we propose a method to deduce the Chow (semi)stability of X from the corresponding results in characteristic 0.

From now on, we denote by $W = W(k)$ the ring of Witt vectors. This is a discrete valuation ring (DVR) of characteristic 0 whose residue field is isomorphic to k (see [S, Chap. 2, Sec. 5, Thm. 5]). These are, in fact, the only properties of W that we need in this paper. We denote by K the field of fractions of W and by m_W the unique maximal ideal of W.

Take $g \in \mathrm{SL}(n+1,k)$ and let F be the Chow form of g^*X as in Section 2.3. Let F_W be a lift of F over W such that a monomial not appearing in F never appears in F_W, which is equivalent to the assumption $\mathcal{I}(F) = \mathcal{I}(F_W)$ (see Definition 2.4 for the definition of $\mathcal{I}$). Note that F_W defines a hypersurface $V(F_W) \subset \mathrm{Grass}_{\bar{K}}(n-r, n+1)$ of degree d, where $\bar{K}$ is the algebraic closure of K.

THEOREM 2.8. *Assume that, for any* $g \in \mathrm{SL}(n+1,k)$, *we can take* F_W *such that* $\mathcal{I}(F) = \mathcal{I}(F_W)$ *holds and* $V(F_W)$ *is stable* (*resp. semistable*) *with respect to the induced action of* $\mathrm{SL}(n+1,\bar{K})$. *Then* X *is Chow stable* (*resp. Chow semistable*).

Proof. Since F_W is (semi)stable, $\mu(V(F_W), \mathrm{id}, \mathbf{r}) < 0$ (resp. ≤ 0) holds for any $\mathbf{r} \in \mathcal{R}$ (see (2) for the definition of $\mathcal{R}$). Yet $\mu(V(F_W), \mathrm{id}, \mathbf{r}) = \mu(Z(X), g, \mathbf{r})$ because $\mathcal{I}(F) = \mathcal{I}(F_W)$ (see Remark 2.5). Therefore, $\mu(Z(X), g, \mathbf{r}) < 0$ (resp. ≤ 0) holds for all $g \in \mathrm{SL}(n+1,k)$ and $\mathbf{r} \in \mathcal{R}$; hence the Chow (semi)stability of X follows from Proposition 2.6. □

REMARK 2.9. By a result of Seshadri ([Se, Prop. 6]; see also [GIT, Apx. to Chap. 1, Sec. G]), the converse of Theorem 2.8 also holds: If X is Chow stable (resp. Chow semistable), then any lift F_W of $\bar{F}$ is also stable (resp. semistable) with respect to the induced action of $\mathrm{SL}(n+1, \bar{K})$.

3. Chow Stability of Nonsingular Hypersurfaces

In this section, X denotes a hypersurface of degree d in $\mathbb{P}^n_k$.

In Section 3.1 we prove the stability of nonsingular hypersurfaces of degree $\geq$ 3; this is an easy application of Theorem 2.8. In Section 3.2 we study the complement of the locus of nonsingular hypersurfaces via geometric arguments. It turns out that the complement is an irreducible divisor and that some multiple of its defining equation lifts to the usual discriminant in characteristic 0.

3.1. A Proof via Lifting to Characteristic 0

First we recall that the characteristic-0 case of Theorem 1.3 was settled in [GIT, Chap. 4, Sec. 2]. Thanks to a theorem by Matsumura and Monsky, the proof given there also works for characteristic-p cases if p does not divide d. We briefly recall the proof and see why it does not work for the cases when p does divide d.

Let $F(X_0, X_1, \dots, X_n)$ be a homogeneous polynomial of degree d. We have the following Euler lemma:

$$dF = \sum_{i=0}^{n} X_i \frac{\partial F}{\partial X_i}.$$

Therefore, we see that

$$V\left(F, \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_n}\right) = V\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_n}\right) \tag{7}$$

provided that p does not divide d. The emptiness of the latter is equivalent to the vanishing of the discriminant of F when $d \geq 2$. This shows the semistability of nonsingular hypersurfaces of degree ≥ 2. Furthermore, for $d \geq 3$ it is known (see [MaM, Thm. 1]) that only finitely many projective linear transformations preserve the given nonsingular hypersurface. This means that any nonsingular hypersurface is stable provided that $d \geq 3$ and p does not divide d.

The preceding argument does not work in general because the equality (7) may break down when p divides d. Actually, when p divides d and $d \geq 3$, the right-hand side of the equality (7) cannot be empty. This claim will be proved in Section 3.2 (see Proposition 3.4).

Even when p divides d, a closer look at the numerical criterion shows that nonsingular hypersurfaces are always (semi)stable if $d > n+1$ (resp. $d \geq n+1$); see [N, Lemma 4.2]. This fact may also be deduced from Theorem 4.1, since the pair $(\mathbb{P}^n_k, X)$ is log canonical when X is a nonsingular hypersurface.

Now we prove that stability always holds.

THEOREM 3.1. *If $d \geq 3$, then any nonsingular projective hypersurface of degree d is Chow stable.*

Proof. The theorem is already established for char $k = 0$, so we assume that char $k > 0$. We use Theorem 2.8. Let $X \subset \mathbb{P}^n_k$ be a nonsingular projective hypersurface of degree ≥ 3. Take any $g \in \mathrm{SL}(n+1, k)$ and let F_k be the equation of g^*X. Note that, in this case, F_k itself is the Chow form of g^*X. Take a lift F_W of F_k over the ring of Witt vectors W satisfying $\mathcal{I}(F_k) = \mathcal{I}(F_W)$. Then it is easy to see the following claim.

CLAIM. *$V(F_W)$ is an integral scheme.*

Proof. Since W is a DVR, $W[X_0, \dots, X_n]$ is a unique factorization domain. It is therefore enough to show that F_W is an irreducible element of $W[X_0, \dots, X_n]$. Suppose to the contrary that $F_W = G \cdot H$ holds for some $G, H \in W[X_0, \dots, X_n]$ such that neither G nor H is a unit. Note that both G and H are homogeneous and nonzero because F_W is. Hence G and H are homogeneous polynomials of degree ≥ 1, since neither is a unit. This means that either $\bar{G} = 0$ or $\deg \bar{G} = \deg G$ (here $\bar{G}$ denotes the reduction modulo m_W of G) must hold and similarly for H. On the other hand, $\bar{G} \cdot \bar{H} = \bar{F}_W = F_k \neq 0$ holds. Thus $\deg \bar{G} = \deg G \geq 1$ (resp. $\deg \bar{H} \geq 1$), contradicting the irreducibility of F_k. □

Since $V(F_W)$ dominates the generic point of Spec W, our claim means that $V(F_W)$ is flat over Spec W (see [Ha, Chap. III, Prop. 9.7]). It also is projective over Spec W.

The closed fiber of $V(F_W) \to \operatorname{Spec} W$ is g^*X, which is nonsingular; hence the geometric generic fiber is also nonsingular (see [EGA, (12.2.4)(iii)]). Since the characteristic of the generic fiber is 0 and since $\deg(F_W) \geq 3$, we already know that it is stable. By Theorem 2.8, we see that X is stable, too. □

REMARK. Since the stabilizer subgroup of a stable hypersurface is finite, we can use Theorem 3.1 to bypass Case II of the proof of [MaM, Thm. 1]. Similarly, our Theorem 4.1 is stronger than [MaM, Thm. 1] when $d > n + 1$.

3.2. The Defining Equation

Let $\mathrm{Hyp}_d(n)$ be the projective space of degree-d hypersurfaces in $\mathbb{P}^n_k$, and let $\mathrm{U}_{\mathrm{ns}} \subset \mathrm{Hyp}_d(n)$ be the locus of nonsingular hypersurfaces. In this section we study the defining equation for the complement of the locus of nonsingular hypersurfaces, $\mathrm{Hyp}_d(n) \setminus \mathrm{U}_{\mathrm{ns}}$, via geometric arguments that are versions of those given in [Mu, Chap. 5, Sec. 2].

The defining equation is well known when p does not divide d, the discriminant. We are therefore interested in the cases when p divides d. Recall that the nonsingularity of $X = V(F)$ is equivalent to the emptiness of the left-hand side of (7). Using this equivalence yields the following result.

THEOREM 3.2. *Assume that p divides d. Then*

$$\mathrm{Hyp}_d(n) \setminus \mathrm{U}_{\mathrm{ns}}$$

is an irreducible divisor. Moreover, some multiple of its defining equation lifts to the discriminant in characteristic 0.

Example 3.3 (see [D, Chap. 10, Sec. 2] for details). Consider the case $(n, d) = (1, 4)$. Let

$$X = V(F) \quad \text{for } F = a_0 X_0^4 + a_1 X_0^3 X_1 + a_2 X_0^2 X_1^2 + a_3 X_0 X_1^3 + a_4 X_1^4$$

be a hypersurface in $\mathbb{P}_k^1$. If char $k \neq 2$ then the defining equation for $\mathrm{Hyp}_4(1) \setminus \mathrm{U}_{\mathrm{ns}}$ is given by $D = 4S^3 - T^2$, where

$$S = 2^2 \cdot 3a_0a_4 - 3a_1a_3 + a_2^2 \quad \text{and}$$
$$T = 2^3 \cdot 3^2 a_0a_2a_4 - 3^3 a_0a_3^2 + 3^2 a_1a_2a_3 - 3^3 a_1^2 a_4 - 2a_2^3.$$

If char $k = 2$, then $D \bmod 2 = (T \bmod 2)^2$ and the defining equation for $\mathrm{Hyp}_4(1) \setminus \mathrm{U}_{\mathrm{ns}}$ is given by $T \bmod 2 = a_0a_3^2 + a_1a_2a_3 + a_1^2a_4$.

Proof of Theorem 3.2. Let $W = W(k)$ be the ring of Witt vectors. Set

$$I = \left\{(x, X) \;\middle|\; x \in V\left(F, \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_n}\right)\right\} \subset \mathbb{P}_W^n \times_{\mathrm{Spec}\, W} \mathrm{Hyp}_d(n),$$

where $\mathrm{Hyp}_d(n) = |\mathcal{O}_{\mathbb{P}_W^n}(d)|$ is the projective space of families of degree-d projective hypersurfaces over Spec W. Let $p\colon I \to \mathbb{P}_W^n$ and $q\colon I \to \mathrm{Hyp}_d(n)$ be the natural projections.

First we prove the following claim.

Claim. *p is a smooth morphism with connected fibers.*

Proof. Let $x\colon \mathrm{Spec}\,\Omega \to \mathbb{P}_W^n$ be a geometric point, where Ω is an algebraically closed field. By the preceding definition of I, it is easy to see that $I_x \subset \mathrm{Hyp}_d(n)_\Omega := \mathrm{Hyp}_d(n) \times_{\mathrm{Spec}\, W} \mathrm{Spec}\,\Omega$ is a linear subspace.

Next we calculate the fiber $I_{(1:0:\cdots:0)}$, where $(1 : 0 : \cdots : 0) \in \mathbb{P}_\Omega^n$. Note that writing

$$F(X) = \sum_{|\alpha|=d} C_\alpha X^\alpha$$

with multi-indices $(C_\alpha \mid |\alpha| = d)$ gives a system of coordinates for the projective space $\mathrm{Hyp}_d(n)_\Omega$. Then we can show the following equality, which is independent of the characteristic of Ω:

$$I_{(1:0:\cdots:0)} = V(C_{(d0\cdots0)}, C_{((d-1)10\cdots0)}, C_{((d-1)010\cdots0)}, \dots, C_{((d-1)0\cdots01)}) \subset \mathrm{Hyp}_d(n)_\Omega.$$

In order to show that the dimension of the linear subspace I_x is independent of x, we show that it is isomorphic to $I_{(1:0:\cdots:0)}$. Consider the action of $\mathrm{SL}_\Omega(n+1)$ on $\mathbb{P}_W^n \times_{\mathrm{Spec}\, W} \mathrm{Hyp}_d(n)_\Omega$; this action is defined by $g \cdot (x, X) = (gx, g_* X)$ for $g \in \mathrm{SL}(n+1, \Omega)$. It can be easily checked that this action preserves $I \times_{\mathrm{Spec}\, W} \mathrm{Spec}\,\Omega$ and that, via this action, we obtain an isomorphism between $I_{(1:0:\cdots:0)}$ and I_x. □

By the claim we see that both I and I_k, the restriction of I over the closed point $\mathrm{Spec}\, k \subset \mathrm{Spec}\, W$, are integral schemes.

Now consider the integral closed subscheme $q(I) \subset \mathrm{Hyp}_d(n)$. Observe that the defining equation for $q(I)$ is the usual discriminant and that $q(I)_k$, the restriction of $q(I)$ over $\operatorname{Spec} k \subset \operatorname{Spec} W$, coincides with $q(I_k)$ as sets. □

Similar arguments yield the following result.

PROPOSITION 3.4. *Assume that d is divisible by p. Let $X = V(F) \subset \mathbb{P}^n_k$ be an arbitrary hypersurface of degree d. Then*

$$V\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_n}\right) \neq \emptyset$$

holds except when $d = p = 2$ and n is odd.

Proof. Set

$$Z = \left\{(x, X) \;\middle|\; x \in V\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_n}\right)\right\} \subset \mathbb{P}^n_k \times \mathrm{Hyp}_d(n)$$

and let $p\colon Z \to \mathbb{P}^n_k$ and $q\colon Z \to \mathrm{Hyp}_d(n)$ be the natural projections. As in the proof of Theorem 3.2, we can show the following.

CLAIM. *p is a smooth morphism with connected fibers.*

Next we calculate the dimension of $Z_{(1:0:\cdots:0)}$. With notation as in the proof of Theorem 3.2, we can write

$$Z_{(1:0:\cdots:0)} = V(C_{((d-1)10\cdots 0)}, C_{((d-1)010\cdots 0)}, \dots, C_{((d-1)0\cdots 01)}) \subset \mathrm{Hyp}_d(n).$$

Therefore,

$$\begin{aligned}\dim Z &= \dim \mathbb{P}^n_k + \dim Z_{(1:0:\cdots:0)} \\ &= n + (\dim \mathrm{Hyp}_d(n) - n) \\ &= \dim \mathrm{Hyp}_d(n).\end{aligned}$$

Now we need only show that $q\colon Z \to q(Z)$ is generically finite, because then $\dim q(Z) = \dim Z = \dim \mathrm{Hyp}_d(n)$ and so $q(Z) = \mathrm{Hyp}_d(n)$. In order to show it, we check the finiteness of the fiber of q at

$$F(X) = X_0^{d-1}X_1 + X_1^{d-1}X_2 + \cdots + X_{n-1}^{d-1}X_n + X_n^{d-1}X_0.$$

First of all, note that

$$\frac{\partial F}{\partial X_i} = 0 \iff X_{i-1}^{d-1} = X_i^{d-2}X_{i+1} \tag{8}$$

for all $i = 0, 1, \dots, n$, where $X_{-1} = X_n$ and $X_{n+1} = X_0$. Suppose $a = (a_0 : \cdots : a_n) \in V\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_n}\right)$.

From (8), one see that $a_0, \dots, a_n \neq 0$. Hence we assume $a_0 = 1$. Using (8) recursively, we obtain the following equation:

$$a_1^{1-(1-d)^{n+1}} = 1. \tag{9}$$

Here the exponent of a_1 is nonzero under our assumptions on (d, p, n). Hence (9) imposes a nontrivial condition on a_1. Given (8), $a_2, a_3, \dots, a_n$ are uniquely determined from a_1. Thus the finiteness is proved. □

REMARK 3.5. When $d = p = 2$ and n is odd, we have the following counterexample:

$$F = X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n.$$

It is easy to see that $V\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_n}\right) = \emptyset$.

4. Lee's Criterion in Characteristic *p*

In this section we prove the following result.

THEOREM 4.1. *Let X be an effective cycle of dimension r and degree d in $\mathbb{P}^n_k$. Let $(\mathbb{G}, Z(X))$ be the log pair defined by the Chow divisor $Z(X)$ of X. If* $\operatorname{lct}(\mathbb{G}, Z(X))$ *is greater than* (*respectively, is greater than or equal to*) $\frac{n+1}{d}$, *then X is Chow stable* (*respectively, Chow semistable*).

See Section 2.3 for notation. Our proof follows the same line as the original one by Lee [Le], but we need to modify several points.

Before proving Theorem 4.1 in Section 4.2, we point out that it could be proved via Theorem 2.8 as in Section 3.2—provided the following conjecture were true (here W is the ring of Witt vectors, and K and k are, respectively, the field of fractions and the residue field of W).

CONJECTURE 4.2. *Let $X_W \to$* Spec W *be a smooth proper morphism, where X_W is an integral scheme. Let D_W be an effective $\mathbb{R}$-divisor on X_W such that no irreducible component is contained in a fiber of the projection to* Spec W. *By X_K and D_K we denote the restrictions of X_W and D_W over the generic point of* Spec W; *similarly, X_k and D_k denote the restrictions of X_W and D_W over the closed point of* Spec W. *Now, if (X_k, D_k) is log canonical then so is (X_K, D_K). In particular,* $\operatorname{lct}(X_k, D_k) \le \operatorname{lct}(X_K, D_K)$.

We remark that this conjecture can be proved, following [Mus], when (X_k, D_k) has a good log resolution ("good" means that it is isomorphic outside of the support of D_k). In [Mus], the lower semicontinuity of log canonical thresholds in a family of projective log pairs with nonsingular ambient varieties is demonstrated when the base scheme of the family is defined in characteristic 0. We need the last assumption because the existence of good log resolution is not yet established in positive characteristics in full generality. Basic results on motivic integrations have been established over arbitrary perfect fields (see [Y]), so the arguments in [Mus] can be applied to our case without change under the existence of good log resolutions.

4.1. Log Canonicity in Positive Characteristics

In this section we discuss how the log canonicity of log pairs is preserved under finite morphisms. Some properties of log canonicity that hold in characteristic 0 fail to hold in characteristic $p > 0$; however, we can circumvent those difficulties and obtain Proposition 4.9, which is the key for the proof of Theorem 4.1.

It is well known that, when the characteristic of the base field is 0, log canonicity is preserved under finite dominant morphisms (see [KoMo, Prop. 5.20(4)]). We state this formally as our next theorem.

THEOREM 4.3. *Let $g\colon X' \to X$ be a finite dominant morphism of normal varieties over a field of characteristic 0. Let Δ (resp. Δ') be a $\mathbb{Q}$-divisor on X (resp. X') such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $g^*(K_X + \Delta) = K_{X'} + \Delta'$. Then (X, Δ) is log canonical if and only if (X', Δ') is.*

We should note that the canonical divisors K_X and $K_{X'}$ in Theorem 4.3 are chosen such that $K_{X'} = g^*K_X + R$, where R is the ramification divisor of g.

When the characteristic of the base field is positive, we need to modify Theorem 4.3. First we consider when g is separable. In this case we may have wild ramifications, so we have a weaker version of the ramification formula as follows.

LEMMA 4.4. *Let $g\colon X \to Y$ be a finite separable morphism between normal varieties over k. Let $E \subset X$ be a prime divisor on X and let r be the ramification index of g along E. Then there exists a nonnegative integer $b \geq r - 1$ such that $K_X = g^*K_Y + bE$ around the generic point of E.*

Proof. Set $V = Y \setminus \operatorname{Sing} Y$ and $U = g^{-1}(V) - \operatorname{Sing} X$. Note that the closed subsets we have discarded all have codimension > 1. Over U, we have the following exact sequence:

$$0 \to g^*\Omega_V \xrightarrow{f} \Omega_U \to \Omega_{U/V} \to 0. \tag{10}$$

Since g is separable, $\Omega_{U/V}$ generically vanishes (see [Ma, Thm. 59]). Hence f is generically isomorphic.

Let $F\colon g^*\mathcal{O}_V(K_V) \to \mathcal{O}_U(K_U)$ be the highest exterior product of the morphism f in (10). This product is also generically isomorphic. Therefore, $\ker F$ is a torsion subsheaf of the torsion-free sheaf $g^*\mathcal{O}_V(K_V)$ and so is trivial. Hence we see that F is injective.

Take a generic closed point e of $E \cap U$ that is contained in no other irreducible component of $\operatorname{Supp} \Omega_{U/V}$ except for E. Set $e' = g(e)$ and $E' = g(E)$. Choose systems of local coordinates $x_1, \ldots, x_n$ at e and $y_1, \ldots, y_n$ at e' that satisfy the following conditions:

(a) $E = \operatorname{div}(x_1)$ near e (resp. $E' = \operatorname{div}(y_1)$);
(b) $g^*y_i = x_i$ holds for all $i = 2, \ldots, n$; and
(c) there exists an invertible function u at e such that $g^*y_1 = u \cdot x_1^r$.

In (c), r denotes the ramification index of g along E. Now

$$\begin{aligned} F(g^*(dy_1 \wedge \cdots \wedge dy_n)) &= d(u \cdot x_1^r) \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n \\ &= \left(\frac{\partial u}{\partial x_1} x_1 + ru\right) x_1^{r-1} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n; \end{aligned} \tag{11}$$

hence there exists some nonnegative integer b such that

$$K_X = g^* K_Y + bE$$

in a neighborhood of e.

If $r \not\equiv 0 \pmod p$ then $b = r - 1$. Now assume that E is wildly ramifying. Then

$$\begin{aligned}(11) &= \frac{\partial u}{\partial x_1} x_1^r dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\ &\neq 0,\end{aligned}$$

since otherwise F is not generically isomorphic. In this case we see that

$$b = \mathrm{val}_E\left(\frac{\partial u}{\partial x_1}\right) + r \geq r,$$

where val_E denotes the valuation corresponding to E. □

REMARK 4.5. Ramification formulas for inseparable morphisms are discussed in [RŠa]. In this case, the ramification divisor is defined only up to linear equivalence. If we adopt this version of the ramification formula, then the "only if" part of Theorem 4.3 does not hold in general. For our purpose we need not deal with inseparable cases.

Given the weaker version of the ramification formula (Lemma 4.4), we can prove that the "only if" part of the Theorem 4.3 holds for separable morphisms.

PROPOSITION 4.6. *Let k be an algebraically closed field of characteristic $p > 0$, and let $g\colon X' \to X$ be a finite separable morphism of normal varieties over k. Let Δ (resp. Δ') be a $\mathbb{Q}$-divisor on X (resp. X') such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $g^*(K_X + \Delta) = K_{X'} + \Delta'$. If (X, Δ) is log canonical, then so is (X', Δ').*

Proof. The proof proceeds as in the proof of [KoMo, Prop. 5.20(4)] once we replace the ramification formula by the weaker version given as Lemma 4.4. □

REMARK 4.7. In general, the "if" part of Theorem 4.3 holds only when there exists no wildly ramifying divisor. In this case, the proof goes as in characteristic 0. If some of the ramification divisors are wildly ramifying then the "if" part may not hold, as the following example shows.

EXAMPLE 4.8. Let $X = X' = \mathbb{A}^1_k$. Set $g\colon X' \to X$, $g(x) = x^p(x+1)$, $\Delta = \frac{p+1}{p}\,\mathrm{div}(x)$, and $\Delta' = \mathrm{div}(x)$. Since $g^* dx = x^p dx$, we obtain

$$g^*(K_X + \Delta) = K_{X'} + \Delta'.$$

Note that (X, Δ) is not lc whereas (X', Δ') is lc.

Using Proposition 4.6, we can extend [Ko2, Prop. 8.13] over arbitrary fields as follows.

PROPOSITION 4.9. *Take any $f \in k[x_1, \ldots, x_n]$. Assign a weight*

$$w = (w(x_i))_{i=1,\ldots,n} \in (\mathbb{Z}_{\geq 0})^n \setminus \{0\}$$

to the variables $x_1, \dots, x_n$ and let $w(f)$ be the weighted multiplicity of f (i.e., the lowest weight of the monomials occurring in f). Then

$$\frac{1}{\operatorname{lct}_0(\mathbb{A}^n, \operatorname{div}(f))} \geq \frac{w(f)}{\sum_{i=1}^n w(x_i)}.$$

Proof. The proof proceeds in two steps.

Step 1. First we establish the inequality for those w such that $w(x_i) > 0$ holds for all $i = 1, \dots, n$ and p divides none of the $w(x_i)$. In this case, the inequality can be established along the same lines as in the original proof, since now we already have Proposition 4.6. For the sake of completeness, we reestablish the argument.

Consider $g\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ given by $g(x_i) = x_i^{w(x_i)}$. By our assumptions on the $w(x_i)$, g is dominant and separable. Take a real number $c \in \mathbb{R}_{\geq 0}$ and assume that $(\mathbb{A}^n_k, c \cdot \operatorname{div}(f))$ is lc at 0. Now calculate the pull-back of $K_{\mathbb{A}^n_k} + c \cdot \operatorname{div}(f)$ by g:

$$\begin{aligned} &g^*(K_{\mathbb{A}^n_k} + c \cdot \operatorname{div}(f)) \\ &\quad = K_{\mathbb{A}^n_k} + \sum_{i=1}^n (1 - w(x_i)) \operatorname{div}(x_i) + c \cdot \operatorname{div}(f(x_1^{w(x_1)}, \dots, x_n^{w(x_n)})) \\ &\quad =: K_{\mathbb{A}^n_k} + \Delta'. \end{aligned}$$

By Proposition 4.6, we have that $(\mathbb{A}^n_k, \Delta')$ is lc at 0. Let E be the exceptional divisor of the blow-up of $\mathbb{A}^n_k$ at the origin. We know that $a(E; \mathbb{A}^n_k, \Delta') \geq -1$ holds. We can calculate that $a(E; \mathbb{A}^n_k, \Delta')$ is equal to $-1 + \sum_i w(x_i) - cw(f)$ and thereby obtain the inequality.

Step 2. Now consider the continuous function $\varphi\colon (\mathbb{Q}_{\geq 0})^n \setminus \{0\} \to \mathbb{Q}$ defined by

$$\varphi(w) = \frac{w(f)}{\sum_i w(x_i)},$$

as in the case where the $w(x_i)$ are integers. If we replace w by some positive multiple of w, then the value of φ never changes. Therefore, φ factors through the quotient space

$$S := (\mathbb{Q}_{\geq 0})^n \setminus \{0\} / \mathbb{Q}_{>0}$$

and so induces the continuous function $\bar{\varphi}\colon S \to \mathbb{Q}$.

The set of points represented by those w satisfying the assumptions in Step 1 is dense in S. Hence, by the continuity of $\bar{\varphi}$, we see that

$$\bar{\varphi}(s) \leq \frac{1}{\operatorname{lct}_0(\mathbb{A}^n, \operatorname{div}(f))}$$

for arbitrary $s \in S$. This concludes the proof of Proposition 4.9. □

REMARK 4.10. Step 2 in the preceding proof is necessary because $(\mathbb{A}^n_k, \Delta')$ need not be lc if g is inseparable. For example, consider the case $n = 2$, $w(x_1) = w(x_2) = p$, and $f(x_1, x_2) = x_1 - x_2$. In this case $\operatorname{lct}_0(\mathbb{A}^2_k, \operatorname{div}(f)) = 1$. On the other hand,

$$\Delta' = (1 - p) \operatorname{div}(x_1 x_2) + p \cdot \operatorname{div}(x_1 - x_2)$$

and so $(\mathbb{A}^2_k, \Delta')$ is not lc at the origin. Note in addition that, even in this case, $a(E; \mathbb{A}^2_k, \Delta') \geq -1$ by Proposition 4.9.

4.2. Proof of Theorem 4.1 and Corollary

Proof of Theorem 4.1. We discuss only the stable case; the semistable case can be proven in exactly the same way. We have only to confirm the inequality (5) of Lemma 2.7. Yet by the assumptions and Proposition 4.9, the inequality clearly holds. □

REMARK 4.11. Theorem 4.1 has the following direct corollary, which is slightly weaker.

COROLLARY 4.12. *If* $\mathrm{Fpt}(\mathbb{G}, Z(X))$ *is greater than (resp., is greater than or equal to)* $\frac{n+1}{d}$, *then X is Chow stable (resp., Chow semistable).*

Here Fpt denotes the *F-pure threshold* of the pair $(\mathbb{G}, Z(X))$.

Corollary 4.12 is deduced from Theorem 4.1 via [HW, Thm. 3.3], which states that

$$F\text{-pure} \implies \text{log canonical.}$$

We can also show Corollary 4.12 by directly proving the Fpt version of Proposition 4.9 via the Fedder-type criterion for F-purity due to [HW].

Appendix: Chow Stability of the Sum

In this appendix we show that the sum of two Chow semistable cycles of the same dimension is itself Chow semistable. Moreover, if one of the cycles is stable then it follows that the sum also becomes stable.

PROPOSITION A.1. *Let Y and Z be Chow semistable cycles of the same dimension in a projective space* $\mathbb{P}^n_k$. *Then* $Y + Z$ *is Chow semistable. Furthermore, if Y is Chow stable then so is* $Y + Z$.

In the proof we freely use such notation as $\lim_{t\to 0} \lambda(t) \cdot F$ (as in [GIT]), since doing so clarifies the ideas involved. Of course, to be logically complete we must modify the argument suitably. It is a routine work, so we omit the details.

Proof of Proposition A.1. Let d and e be the degrees of Y and Z, respectively. Let $F \in \mathcal{B}_d$ and $G \in \mathcal{B}_e$ be the Chow forms of Y and Z, respectively. Then the Chow form of $Y + Z$ is given by $F \cdot G \in \mathcal{B}_{d+e}$.

Choose a nontrivial 1-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(n+1,k)$. Via λ we pull back the canonical actions of $\mathrm{SL}(n+1,k)$ onto $\mathcal{B}_d, \mathcal{B}_e$, and $\mathcal{B}_{d+e}$ to $\mathbb{G}_m$. Now consider the natural multiplication map $\mu\colon \mathcal{B}_d \times \mathcal{B}_e \to \mathcal{B}_{d+e}$ given by $(F,G) \mapsto F \cdot G$. If we impose the diagonal action of $\mathbb{G}_m$ on the source, then μ becomes equivariant.

Assume that Y and Z are both Chow semistable. Then $\lim_{t\to 0} \lambda(t) \cdot F \neq 0$ and also $\lim_{t\to 0} \lambda(t) \cdot G \neq 0$. Now, since we know that μ is continuous,

$$\begin{aligned}\lim_{t\to 0} \lambda(t) \cdot (F \cdot G) &= \Big(\lim_{t\to 0} \lambda(t) \cdot F\Big) \cdot \Big(\lim_{t\to 0} \lambda(t) \cdot G\Big)\\ &\neq 0\end{aligned}$$

because $\mathcal{B} = \bigoplus_{d\geq 0} \mathcal{B}_d$ is an integral domain. Therefore, $Y + Z$ is again Chow semistable.

Second, assume further that Y is Chow stable. Then $\lim_{t\to 0} \lambda(t)\cdot F = \infty$ and so

$$\begin{aligned}\lim_{t\to 0} \lambda(t)\cdot (F\cdot G) &= \Big(\lim_{t\to 0}\lambda(t)\cdot F\Big)\cdot\Big(\lim_{t\to 0}\lambda(t)\cdot G\Big)\\ &= \infty\cdot\Big(\lim_{t\to 0}\lambda(t)\cdot G\Big)\\ &= \infty,\end{aligned}$$

since $(\lim_{t\to 0}\lambda(t)\cdot G)$ is not 0. Therefore, $Y + Z$ is Chow stable. □

REMARK A.2. We have no reason to expect that the converse of Proposition A.1 will hold. The reason is that there exists a semistable cycle such that all of its subcycles are unstable, as in the following example.

EXAMPLE A.3. Take the union of three lines on a plane that are in general position. The union itself is Chow semistable (see [GIT, Chap. 4, Sec. 2]), but lines and reducible conics on a plane are Chow unstable.

However, the following statement holds.

PROPOSITION A.4. *Let Z be a cycle of $\mathbb{P}^n_k$. Then the following are equivalent*:

(i) *Z is Chow (semi)stable*;

(ii) *mZ is Chow (semi)stable for any positive integer $m \in \mathbb{Z}_{>0}$*;

(iii) *mZ is Chow (semi)stable for some positive integer $m \in \mathbb{Z}_{>0}$*.

Proof. We have only to prove that (3) $\Rightarrow$ (1). Let G be the Chow form of Z, in which case G^m gives the Chow form of mZ. Now assume that mZ is Chow semistable, and take any 1-PS λ as in the proof of the Proposition A.1. Then $0 \neq \lim_{t\to 0}\lambda(t)\cdot G^m = (\lim_{t\to 0}\lambda(t)\cdot G)^m$ and so $\lim_{t\to 0}\lambda(t)\cdot G \neq 0$. Therefore, Z is semistable. The stable case can be shown via a similar argument. □

EXAMPLE A.5. Let $Y \subset \mathbb{P}^n_k$ be a nonsingular hypersurface of degree 3 that is Chow stable by Theorem 3.1. By Proposition A.1, mY is also Chow stable for all the positive integers m. On the other hand, $\mathrm{lct}(\mathbb{P}^n_k, mY) = \frac{1}{m}$ and hence $\frac{1}{m} < \frac{n+1}{3m}$ if $n \geq 3$. Thus we obtain a sequence of examples of Chow stable hypersurfaces whose stability cannot be detected by Theorem 4.1.

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