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Stabilization of Monomial Maps

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Introduction

An important part of higher-dimensional complex dynamics concerns the construction of currents and measures that are invariant under a given meromorphic self-map $f\colon X \dashrightarrow X$ of a compact complex manifold X. In doing so, it is often desirable that the action of f on the cohomology of X be compatible with iteration; thus, following Sibony [Si] (see also [FoSi]) we call f (algebraically) *stable*.

If f is not stable, we can try to make a bimeromorphic change of coordinates $X' \to X$ such that the induced self-map of X' becomes stable. Understanding when this is possible seems to be a difficult problem. On the one hand, Favre [Fa] showed that stability is not always achievable. On the other hand, it can be achieved for bimeromorphic maps of surfaces [DiF] for a large class of monomial mappings in dimension 2 [F] (more on this below) and for polynomial maps of $\mathbf{C}^2$ [FJ2]. Beyond these classes, very little seems to be known.

In this paper we study the stabilization problem for monomial (or equivariant) maps of toric varieties, extending certain results of Favre to higher dimensions. A toric variety $X = X(\Delta)$ is defined by a lattice $N \cong \mathbf{Z}^m$ and a fan Δ in N. A monomial self-map $f\colon X \dashrightarrow X$ corresponds to a $\mathbf{Z}$-linear map $\phi\colon N \to N$. See Sections 1 and 2 for more details.

We work in codimension 1 and say that f is *1-stable* if $(f^n)^* = (f^*)^n$, where f^* denotes the action on the Picard group of X. Geometrically, this means that no iterate of f sends a hypersurface into the indeterminacy set of f [FoSi; Si].

THEOREM A. *Let Δ be a fan in a lattice $N \cong \mathbf{Z}^m$, and let $f\colon X(\Delta) \dashrightarrow X(\Delta)$ be a monomial map. Assume that the eigenvalues of the associated linear map $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$ are real and satisfy $\mu_1 > \mu_2 > \cdots > \mu_m > 0$. Then there exists a complete simplicial refinement Δ' of Δ such that $X(\Delta')$ is projective and the induced map $f\colon X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable.*

Here $N_{\mathbf{R}}$ denotes the vector space $N \otimes_{\mathbf{Z}} \mathbf{R}$. The variety $X' = X(\Delta')$ will not be smooth in general, but it will have at worst quotient singularities. We can pick X' smooth at the expense of replacing f with an iterate (but allowing more general ϕ) as follows.

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THEOREM A′. *Let Δ be a fan in a lattice N of rank m, and let $f\colon X(\Delta) \dashrightarrow X(\Delta)$ be a monomial map. Suppose that the eigenvalues of the associated linear map $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$ are real and satisfy $|\mu_1| > |\mu_2| > \cdots > |\mu_m| > 0$. Then there exist a complete (regular) refinement Δ' of Δ and $n_0 \in \mathbf{N}$ such that $X(\Delta')$ is smooth and projective and the induced map $f^n\colon X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable for all $n \geq n_0$.*

If the fan we start with is trivial—that is, if the initial toric variety is the torus $(\mathbf{C}^*)^m$—then we can relax the assumptions on the eigenvalues slightly and obtain the following statement.

THEOREM B. *Let $f\colon (\mathbf{C}^*)^m \to (\mathbf{C}^*)^m$ be a monomial map. Suppose that the associated linear map $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$ is diagonalizable with real eigenvalues $\mu_1 > \mu_2 \geq \mu_3 \geq \cdots \geq \mu_m > 0$. Then there exists a complete simplicial fan Δ such that $X(\Delta)$ is projective and $f\colon X(\Delta) \dashrightarrow X(\Delta)$ is 1-stable.*

It is unclear whether $X(\Delta)$ can be chosen smooth in Theorem B even if we replace f by an iterate; see Remark 5.1. Picking $X(\Delta')$ smooth in Theorem A (without passing to an iterate) also seems quite delicate. We address the latter problem only in dimension 2 as follows.

THEOREM C. *Let Δ be a fan in a lattice N of rank $m = 2$, let $f\colon X(\Delta) \dashrightarrow X(\Delta)$ be a monomial map, and let μ_1, μ_2 be the eigenvalues of the associated linear map $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$ and labeled so that $|\mu_1| \geq |\mu_2|$. Suppose that any of the following conditions hold*:

(a) $|\mu_2| < 1$;
(b) $\mu_1, \mu_2 \in \mathbf{Z}$ *and* $|\mu_1| > |\mu_2|$;
(c) $\mu_1, \mu_2 \in \mathbf{R}$ *and* $\mu_1\mu_2 > 0$.

Then there is a complete (regular) refinement Δ' of Δ such that $X(\Delta')$ is smooth (and projective) and $f\colon X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable.

Example 3.12 shows that Theorem C may fail when $|\mu_1| > |\mu_2| > 1$, while Example 3.14 and [F, Ex.2] show that it may fail when $|\mu_1| = |\mu_2|$ and μ_1/μ_2 is a root of unity different from 1.

Theorem C should be compared with the work of Favre [F], in which the following result is proved.

THEOREM C′. *Let Δ be a fan in a lattice N of rank $m = 2$, let $f\colon X(\Delta) \dashrightarrow X(\Delta)$ be a monomial map, and let μ_1, μ_2 be the eigenvalues of the associated linear map $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$. Then we are in precisely one of the following cases*:

(i) *μ_1, μ_2 are complex conjugate and μ_1/μ_2 is not a root of unity*;
(ii) *there exists a complete refinement Δ' of Δ such that the induced map $f\colon X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable.*

Here $X(\Delta')$ is not necessarily smooth. The main result in [F] also asserts that we can make f 1-stable on a smooth toric variety by allowing ramified covers, but there is a gap in this argument; see Remark 3.1.

Monomial maps are quite special, but they are interesting in their own right. We refer to the paper by Hasselblatt and Propp [HaP] for more information and to the paper by Bedford and Kim [BK2] for the problem—related to stability—of characterizing monomial maps whose degree growth sequence satisfies a linear recursion formula. For nonmonomial maps in higher dimensions, stability or degree growth is understood only in special cases [BK1; N].

We note that many of the results in this paper have been obtained independently by Jan-Li Lin. In particular, [L1, Thm. 5.7(a)] coincides with our Theorem B′ in Section 5.

There is a conjectured relationship between the eigenvalues μ_j and the *dynamical degrees* λ_j, $1 \le j \le m$, of f (see [DSi; G; RS] for a definition of dynamical degrees). Namely, the conjecture states that $\lambda_j = |\mu_1| \cdots |\mu_j|$. See [FW; L2] for work in this direction. Given this formula, the condition $|\mu_1| > \cdots > |\mu_m|$ is equivalent to $j \mapsto \log \lambda_j$ being strictly concave. Now the conjecture does hold in dimension 2. This means that part (a) of Theorem C is equivalent to (a′) $\lambda_2 < \lambda_1$ and that part (b) is equivalent to (b′) $\lambda_2 < \lambda_1^2$ and $\lambda_1 \in \mathbf{Z}$. Also observe that part (c) is satisfied for f^2 as soon as $\lambda_2 < \lambda_1^2$.

To prove the preceding theorems, we translate them into statements about the linear map $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$. What we ultimately prove is that we can refine the original fan Δ (by adding cones) so that the new fan Δ' contains a finite collection $\mathcal{T}$ of invariant cones (i.e., ϕ maps each cone into itself) that together attract all 1-dimensional cones in Δ'. More precisely, for every 1-dimensional cone $\rho \in \Delta'$ there exists an $n_0 \ge 0$ such that $\phi^n(\rho)$ lies in a cone in $\mathcal{T}$ for $n \ge n_0$ and $\phi^n(\rho)$ is a 1-dimensional cone in Δ' for $0 \le n < n_0$; see Corollary 2.3.

Constructing such a collection of cones is also the strategy used by Favre [F] to prove Theorem C′. In fact, the proof of Theorem B is a straightforward adaptation of arguments in [F]. Indeed, the dynamics of $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$ is easy to understand: under iteration, a typical vector v tends to move toward the 1-dimensional eigenspace associated to the largest eigenvalue μ_1 of ϕ. We can therefore find a simplicial cone σ of maximum dimension that is invariant under ϕ; it will contain an eigenvector e_1 corresponding to μ_1 in its interior. Using this cone, we easily construct a fan for which Theorem B holds.

On the other hand, Theorems A and A′ are much more delicate because we must take into account the original fan Δ. For example, the simple argument for Theorem B outlined previously will not work in general, since it is possible that the 1-dimensional cone $\mathbf{R}_+ e_1$ is rational and belongs to Δ. Moreover, there may be 1-dimensional rays in Δ that are not attracted to $\mathbf{R}_+ e_1$ under iteration. Thus we must proceed more systematically, and this is where the argument becomes significantly more involved in higher dimensions.

Our approach is to look at the set $T_{\mathrm{red}}(\phi)$ of all invariant rational subspaces $V \subseteq N_{\mathbf{R}}$. This means that $\phi(V) = V$ and that $V \cap N_{\mathbf{Q}}$ is dense in V. It turns out that $T_{\mathrm{red}}(\phi)$ is a finite set that admits a natural tree structure determined by the dynamics. Using this tree, we inductively construct a collection $\mathcal{T}$ of invariant rational cones that together attract any lattice point in N. The construction is flexible enough that the cones in $\mathcal{T}$ are "well positioned" with respect to the original

fan Δ. In particular, each cone in $\mathcal{T}$ is contained in a unique minimal cone in Δ. This allows us to refine Δ into a fan that contains all cones in $\mathcal{T}$. Significant care is called for, however, since the construction is done inductively over a tree and since incorporating a new cone into a given fan will require many cones of the original fan to be subdivided. The actual construction is therefore more technical than may be expected.

In dimension 2, these difficulties are largely invisible. They are the reason why, in Theorem A, we impose stronger conditions on the eigenvalues than did Favre in [F]. It would be interesting to try to weaken the conditions in Theorem A.

The proof of Theorem C is of a quite different nature; it uses the original ideas of Favre as well as some methods from classical number theory. Indeed, some of the arguments are parallel to the analysis of the continued fractions expansion of quadratic surds [HWr].

The paper is organized as follows. In Sections 1 and 2 we discuss toric varieties and monomial mappings. Section 3 is concerned with the 2-dimensional situation—namely, the proofs of Theorems C and C′—and examples showing that smooth stabilization is not always possible. Then in Sections 4 and 5 we return to the higher-dimensional case and prove Theorems A, A′, and B. Finally, in Section 6 we illustrate our proof of Theorem A in dimensions 2 and 3 and give a counterexample to the statement in Theorem A when the eigenvalues have mixed signs.

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1. Toric Varieties

A toric variety is a (partial) compactification of the torus $T \cong (\mathbf{C}^*)^m$ that contains T as a dense subset and admits an action of T that extends the natural action of T on itself. We briefly recall some of the basic definitions, referring to [Fu1] and [O] for details.

1.1. Fans and Toric Varieties

Let N be a lattice isomorphic to $\mathbf{Z}^m$ and let $M = \mathrm{Hom}(N, \mathbf{Z})$ denote the dual lattice. Set $N_{\mathbf{Q}} := N \otimes_{\mathbf{Z}} \mathbf{Q}$, $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$, and $N_{\mathbf{C}} := N \otimes_{\mathbf{Z}} \mathbf{C}$.

A *cone* σ in $N_{\mathbf{R}}$ is a set that is closed under positive scaling. If σ is convex and does not contain any line in $N_{\mathbf{R}}$ then it is said to be *strongly convex*. If σ is of the form $\sigma = \sum \mathbf{R}_+ v_i$ for some $v_i \in N$, we say that σ is a *convex rational cone* generated by the vectors v_i. A *face* of σ is the intersection of σ and a supporting hyperplane. The *dimension* of σ is the dimension of the linear space $\mathbf{R} \cdot \sigma$ spanned by σ. One-dimensional cones are called *rays.* Given a ray σ, the associated *primitive vector* is the first lattice point met along σ. A k-dimensional cone is *simplicial*

if it can be generated by k vectors. A cone is *regular* if it is generated by part of a basis for N. If σ is a rational cone, we denote by $\operatorname{Int}\sigma$ the *relative interior* of σ; that is, $\operatorname{Int}\sigma$ consists of the elements that are in σ but not in any proper face of σ. If σ is generated by v_i, then $\operatorname{Int}\sigma = \sum \mathbf{R}_+^* v_i$. Write $\partial\sigma := \sigma \setminus \operatorname{Int}\sigma$.

A *fan* Δ in N is a finite collection of rational strongly convex cones in $N_{\mathbf{R}}$ such that each face of a cone in Δ is also a cone in Δ and, moreover, the intersection of two cones in Δ is a face of both of them. The last condition could be replaced by requiring the relative interiors of the cones in Δ to be disjoint. Note that a fan is determined by its maximal cones with respect to inclusion. Let $\Delta(k)$ denote the set of k-dimensional faces of Δ. The *support* $|\Delta|$ of a fan Δ is the union of all cones of Δ. In fact, given any collection of cones Σ, we use $|\Sigma|$ to denote the union of the cones in Σ. If $|\Delta| = N_{\mathbf{R}}$, then the fan Δ is said to be *complete*. If all cones in Δ are simplicial then Δ is said to be *simplicial,* and if all cones in Δ are regular then Δ is said to be *regular.* A *subfan* of a fan Δ is a fan $\tilde{\Delta}$ with $\tilde{\Delta} \subseteq \Delta$. A fan Δ' is a *refinement* of Δ if each cone in Δ is a union of cones in Δ'. Every fan admits a regular refinement.

A strongly convex rational cone σ in N determines an affine toric variety U_σ, and a fan Δ determines a toric variety $X(\Delta)$ obtained by gluing together the U_σ for $\sigma \in \Delta$. The variety U_τ is dense in U_σ if τ is a face of σ. In particular, the torus $T_N := U_{\{0\}} = N \otimes_{\mathbf{Z}} \mathbf{C}^* \cong (\mathbf{C}^*)^m$ is dense in $X(\Delta)$. The torus acts on $X(\Delta)$, where the orbits are exactly the varieties U_σ, $\sigma \in \Delta$.

A toric variety $X(\Delta)$ is compact if and only if Δ is complete. Toric varieties are normal and Cohen–Macaulay. If Δ is simplicial then $X(\Delta)$ has at worst quotient singularities, and $X(\Delta)$ is nonsingular if and only if Δ is regular.

1.2. Incorporation of Cones

To prove Theorems A, A′, and C, we will refine fans by adding certain cones. The following lemma is probably well known; we learned it from A. Barvinok. The techniques in the proof will not be used elsewhere in the paper.

LEMMA 1.1. *Let Δ be a simplicial fan and let $\sigma_0 \in \Delta$. Assume that $\sigma_1 \subseteq \sigma_0$ is a simplicial cone such that $\partial\sigma_1 \cap \partial\sigma_0$ is a face of both σ_1 and σ_0. Then there exists a simplicial refinement Δ' of Δ such that $\sigma_1 \in \Delta'$, and if $\sigma \in \Delta$ satisfies $\sigma \not\supseteq \sigma_0$ then $\sigma \in \Delta'$. Moreover, all rays in $\Delta'(1) \setminus \Delta(1)$ are 1-dimensional faces of σ_1.*

For examples of cones $\sigma_1 \subseteq \sigma_0$, see Figure 1.

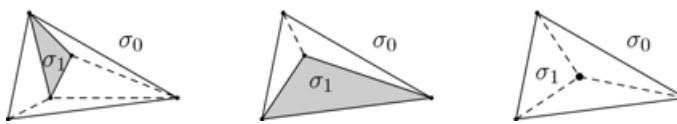


Figure 1 Examples of cones $\sigma_1 \subseteq \sigma_0$ in Lemma 1.1 (the cone σ_0 is 3-dimensional; the figure shows the intersection with an affine plane, and the dashed lines indicate the fan Δ_0 in the proof of Lemma 1.1)

Proof of Lemma 1.1. Following [Z, pp. 129ff], we construct a fan Δ_0 that contains σ_1 and whose support is σ_0. Embed $N_{\mathbf{R}}$ in $N_{\mathbf{R}} \oplus \mathbf{R}$ as the hyperplane $\{x_{m+1} = 0\}$, let τ_0 be the image of σ_0, and let $\pi : N_{\mathbf{R}} \oplus \mathbf{R} \to N_{\mathbf{R}}$ be the projection. For each ray $\mathbf{R}_+ v$ of σ_1 that is not in $\partial\sigma_0$, choose $t_v \in \mathbf{R}_+$ and let T be the convex hull in $N_{\mathbf{R}} \oplus \mathbf{R}$ of τ_0 and the rays $\mathbf{R}_+(v, t_v)$. Observe that, for a generic choice of t_v, the faces of T are simplicial cones. Let Δ_0 be the collection of images of faces of T excepting σ_0 itself. Note that π maps $\partial T \setminus \operatorname{Int}\tau_0$ one-to-one onto σ_0. It follows that Δ_0 is a simplicial fan with support equal to σ_0 and that σ_1 is one of the cones in Δ_0. If $\sigma \in \Delta_0$, then either σ is a face of σ_0 or $\operatorname{Int}\sigma \subseteq \operatorname{Int}\sigma_0$. Also, note that all rays in $\Delta_0 \setminus \Delta$ are 1-dimensional faces of σ_1.

We will now construct a fan Δ' that refines Δ and contains Δ_0 as a subfan. Let Δ'_1 be the collection of cones in Δ that do not contain σ_0. In addition, let Δ'_2 be the collection of cones of the form $\sigma + \tau$, where $\sigma \in \Delta_0$ and $\tau \in \Delta$ is a face of a cone $\tilde{\tau} \in \Delta$ such that $\tilde{\tau} \supseteq \sigma_0$ but $\tau \cap \sigma_0 = \{0\}$. Finally, let Δ' be the union of Δ'_1 and Δ'_2. Note that this union is not disjoint. Observe that all cones in Δ' are simplicial. We claim that Δ' is a simplicial fan.

To prove the claim, first note that if $\sigma \in \Delta$ does not contain σ_0 then clearly the faces of σ do not contain σ_0. In other words, a face of a cone in Δ'_1 is in Δ'_1. Moreover, a face of a simplicial cone $\sigma + \tau \in \Delta'_2$ is of the form $\sigma' + \tau'$, where σ' is a face of σ and τ' is a face of τ. Since Δ_0 is a fan, $\sigma' \in \Delta_0$, and since Δ is a fan, $\tau' \in \Delta$. Furthermore, $\{0\} \subseteq \tau' \cap \sigma_0 \subseteq \tau \cap \sigma_0 = \{0\}$ and so $\sigma' + \tau' \in \Delta'_2$. To conclude, a face of a cone in Δ' is in Δ'.

It remains to prove that if ρ and ρ' are two distinct cones in Δ' then $\operatorname{Int}\rho \cap \operatorname{Int}\rho' = \emptyset$. We have three cases to consider. In the first case, $\rho, \rho' \in \Delta'_1 \subseteq \Delta$. Then $\operatorname{Int}\rho \cap \operatorname{Int}\rho' = \emptyset$ because Δ is a fan and $\rho \neq \rho'$.

In the second case, $\rho \in \Delta'_2 \setminus \Delta'_1$ and $\rho' \in \Delta'_1$. Then we can write $\rho = \sigma + \tau$, where $\operatorname{Int}\sigma \subseteq \operatorname{Int}\sigma_0$. Indeed, if σ is a face of σ_0 then $\rho \in \Delta'_1$. It follows that $\operatorname{Int}\rho \cap \operatorname{Int}\rho' \subseteq \operatorname{Int}(\sigma_0 + \tau) \cap \operatorname{Int}\rho' = \emptyset$. Observe that $\sigma_0 + \tau$ is a cone in Δ because Δ is simplicial and, by construction, $\rho' \neq \sigma_0 + \tau$.

In the third case, $\rho = \sigma + \tau$ and $\rho' = \sigma' + \tau'$ are both in $\Delta'_2 \setminus \Delta'_1$. If $\tau \neq \tau'$, then by construction $\sigma_0 + \tau$ and $\sigma_0 + \tau'$ are two distinct cones in Δ. Hence $\operatorname{Int}\rho \cap \operatorname{Int}\rho' \subseteq \operatorname{Int}(\sigma_0 + \tau) \cap \operatorname{Int}(\sigma_0 + \tau') = \emptyset$. Next, assume that $\tau = \tau'$. Then $\sigma_0 \cap \tau = \{0\}$, which implies that each element $v \in \sigma_0 + \tau$ admits a unique representation $v = v_0 + w$, where $v_0 \in \sigma_0$ and $w \in \tau$. Assume that $v \in \operatorname{Int}\rho \cap \operatorname{Int}\rho'$. Then $v_0 \in \operatorname{Int}\sigma \cap \operatorname{Int}\sigma'$, since $\operatorname{Int}\rho = \operatorname{Int}\sigma + \operatorname{Int}\tau$. Hence $\sigma = \sigma'$, which implies that $\rho = \rho'$. To conclude, Δ' is a simplicial fan.

Now let us show that Δ' has the desired properties. First, observe that all cones in Δ_0 (and, in particular, σ_1) are in Δ'. Indeed, if $\sigma \in \Delta_0$ then $\sigma = \sigma + 0 \in \Delta'_2$. Next, by definition of Δ', each cone in Δ that does not contain σ_0 is in $\Delta'_1 \subseteq \Delta'$. Moreover, each ray in $\Delta' \setminus \Delta$ is in $\Delta_0 \setminus \Delta$ and hence is a 1-dimensional face of σ_1.

It remains to show that Δ' refines Δ. Consider $\rho \in \Delta$. If ρ does not contain σ_0 then ρ is itself in Δ', so assume that $\rho \supseteq \sigma_0$. Since Δ is simplicial, this means that $\rho = \sigma_0 + \tau$ for some face τ of ρ for which $\sigma_0 \cap \tau = \{0\}$. Thus $\rho = \sigma_0 + \tau = \left(\bigcup_{\sigma \in \Delta_0} \sigma\right) + \tau = \bigcup_{\sigma \in \Delta_0} (\sigma + \tau)$, and each $\sigma + \tau \in \Delta'_2 \subseteq \Delta'$ by construction. Hence Δ' refines Δ. □

1.3. Invariant Divisors and Support Functions

Let $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$ denote, respectively, the groups of Weil and Cartier divisors on X modulo linear equivalence. For $X = X(\Delta)$, $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$ are generated by divisors that are invariant under the action of the torus T_N. Since $X(\Delta)$ is normal, every Cartier divisor defines a Weil divisor.

Each ray ρ of Δ determines a prime Weil divisor $D(\rho)$ that is invariant under the action of T_N, and these divisors generate $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$. A T_N-invariant Weil divisor is of the form $\sum a_i D(\rho_i)$, where $a_i \in \mathbf{Z}$ and the sum runs over the rays in Δ.

A T_N-invariant Cartier divisor can be represented as a certain piecewise linear function. We say that $h\colon |\Delta| \to \mathbf{R}$ is a (Δ-*linear*) $\mathbf{Q}$-*support function* if it is linear on each cone of Δ and if $h(|\Delta| \cap N_{\mathbf{Q}}) \subseteq \mathbf{Q}$. If the restriction of h to a cone is given by some element of M (rather than $M_{\mathbf{Q}}$), then h is said to be a (Δ-*linear*) *support function*. There is a one-to-one correspondence between T_N-invariant $\mathbf{Q}$-Cartier divisors and $\mathbf{Q}$-support functions and between T_N-invariant Cartier divisors and support functions; see [M, Chap. 6, p. 6]. Moreover, a T_N-invariant $\mathbf{Q}$-Cartier divisor is a Weil divisor if and only if $h(|\Delta| \cap N) \subseteq \mathbf{Z}$. Given support functions h_1 and h_2, the corresponding divisors are linearly equivalent if and only if $h_1 - h_2$ is linear.

Note that a Δ-linear support function is determined by its values on primitive vectors of rays in Δ. In particular, if D is a $\mathbf{Q}$-Cartier divisor of the form $D = \sum a_i D(\rho_i)$ then the corresponding $\mathbf{Q}$-support function is determined by $h(v_i) = a_i$, where v_i is the primitive vector of ρ_i. Conversely, a support function h determines a Weil divisor $\sum h(v_i) D(\rho_i)$.

A Δ-linear support function h is said to be *strictly convex* if it is convex and defined by different elements $\xi_\sigma \in M$ for each $\sigma \in \Delta(m)$. A compact toric variety $X(\Delta)$ is projective if and only if there is a strictly convex Δ-linear support function (see [O, Cor. 2.16]). We will then say that Δ is *projective*.

LEMMA 1.2. *Any fan Δ admits a regular* (*and hence simplicial*) *projective refinement*.

Proof. This result is well known, so we only sketch the proof. First, by the toric Chow Lemma [O, Prop. 2.17], Δ admits a projective refinement. In general, this refinement is not regular or even simplicial. However, one can check that the standard procedure for desingularizing a toric variety by refining the fan (see [Fu1, Sec. 2.6]) preserves projectivity. □

LEMMA 1.3. *If the fan Δ in Lemma 1.1 is projective, then the refinement Δ' in that lemma can also be chosen projective*.

Proof. Assume that Δ is projective and let h be a strictly convex Δ-linear support function. We will show that we can modify h to a strictly convex Δ'-linear function h', where Δ' is the fan constructed in the proof of Lemma 1.1. We will use the notation from that proof.

The construction of the fan Δ_0 in the proof of Lemma 1.1 implies that there is a strictly convex Δ_0-linear support function h_0 that is zero on the boundary of $|\Delta_0| = \sigma_0$. Pick a norm on $M_{\mathbf{R}}$ and choose $0 < \varepsilon \ll \min_{\sigma,\tau\in\Delta(m),\sigma\neq\tau}\|\xi_\sigma - \xi_\tau\|$ if $h = \xi_\sigma \in M_{\mathbf{Q}}$ on $\sigma \in \Delta(m)$.

Consider $\tau \in \Delta(m)$. Either $\tau \cap \sigma_0 = \{0\}$ or $\tau \supseteq \sigma_0$. In the first case, $\tau \in \Delta'(m)$ and we let $h' = h$ on τ.

Next, assume that $\tau \supseteq \sigma_0$. Since Δ is simplicial, there exists a unique maximal face τ' of τ such that $\tau' \cap \sigma_0 = \{0\}$. It follows that an element in σ_0 admits a unique representation $s + t$ with $s \in \sigma_0$ and $t \in \tau'$. In Δ', τ will be subdivided into maximal cones of the form $\rho = \tau' + \sigma$, where σ is a cone of maximal dimension in Δ_0. On each ρ let h' be defined by $h'(s + t) = \varepsilon h_0(s)$. Clearly h' is piecewise linear and strictly convex on the subfan of Δ' that has support on τ. Moreover, since h_0 vanishes on the boundary of σ_0, the choice of ε ensures that h' is continuous and convex on Δ' and different on all $\sigma \in \Delta'(n)$. In other words, h' is Δ'-linear and strictly convex and hence Δ' is projective. □

If all maximal cones of Δ are of dimension m, then $H^2(X(\Delta), \mathbf{Z}) = \mathrm{Pic}(X(\Delta))$. If Δ is complete and regular, then $H^{1,1}(X(\Delta)) = H^2(X(\Delta), \mathbf{C})$. If Δ is simplicial, then $X(\Delta)$ is $\mathbf{Q}$-factorial; that is, a Weil-divisor is $\mathbf{Q}$-Cartier.

2. Monomial Maps

Let Δ and Δ' be fans in $N \cong \mathbf{Z}^m$ and $N' \cong \mathbf{Z}^{m'}$, respectively. Then any $\mathbf{Z}$-linear map $\phi\colon N \to N'$ gives rise to a rational map $f\colon X(\Delta) \dashrightarrow X(\Delta')$ that is equivariant with respect to the actions of T_N and $T_{N'}$. Let $e_1, \dots, e_m$ and $e'_1, \dots, e'_{m'}$ be bases of N and N', respectively, and let $x_1, \dots, x_m$ and $x'_1, \dots, x'_{m'}$ be the corresponding bases for the duals M and M'. Then $\phi = \sum_{1\le j\le m, 1\le k\le m'} a_{kj} e_j \otimes x'_k$, where $a_{kj} \in \mathbf{Z}$. Let $z_1, \dots, z_m$ and $z'_1, \dots, z'_{m'}$ be the induced coordinates on $T_N \cong (\mathbf{C}^*)^m$ and $T_{N'} \cong (\mathbf{C}^*)^{m'}$, respectively. Then $f\colon T_N \to T_{N'}$ is given by the monomial map $f\colon (z_1, \dots, z_m) \mapsto (z_1^{a_{11}} \cdots z_m^{a_{1m}}, \dots, z_1^{a_{m'1}} \cdots z_m^{a_{m'm}})$. Conversely, any rational and equivariant map $f\colon X(\Delta) \dashrightarrow X(\Delta')$ comes from a $\mathbf{Z}$-linear map $\phi\colon N \to N'$; see [O, p. 19].

The map $f\colon X(\Delta) \dashrightarrow X(\Delta')$ is holomorphic precisely if the extension $\phi\colon N_{\mathbf{R}} \to N'_{\mathbf{R}}$ satisfies the condition that, for each $\sigma \in \Delta$, there is a $\sigma' \in \Delta'$ such that $\phi(\sigma) \subseteq \sigma'$. Let $\phi_{\Delta\Delta'}\colon (N, \Delta) \to (N', \Delta')$ be the map that takes (v, σ) to $(\phi(v), \sigma')$, where σ' is the smallest cone in Δ' containing $\phi(\sigma)$. If f is holomorphic we say that $\phi_{\Delta\Delta'}$ is *regular*. If f is not holomorphic then $\phi_{\Delta\Delta'}$ is not defined everywhere; we write $\phi_{\Delta\Delta'}\colon (N, \Delta) \dashrightarrow (N', \Delta')$. Observe that there is a subfan $\tilde{\Delta}$ of Δ containing all rays of Δ such that $\phi_{\Delta\Delta'}$ is well-defined on $(N, \tilde{\Delta})$. Indeed, the image of a ray in Δ under ϕ is always contained in a cone in Δ'.

A $\mathbf{Z}$-linear map $\phi\colon N \to N'$ induces a $\mathbf{Z}$-linear map $\phi^*\colon M' \to M$ given by $\phi^*\xi' = \xi' \circ \phi$.

2.1. Desingularization

By regularizing fans, we can desingularize not only toric varieties but also equivariant maps between toric varieties. First, let $\tilde{\Delta}$ be a regular refinement of Δ and let

$\mathrm{id}_{\tilde{\Delta}\Delta}\colon (N,\tilde{\Delta}) \to (N,\Delta)$ be the map induced by $\mathrm{id}\colon N \to N$. Then the map $\pi\colon X(\tilde{\Delta}) \to X(\Delta)$ induced by $\mathrm{id}_{\tilde{\Delta}\Delta}$ is a resolution of singularities; see [Fu1, Chap. 2.5].

Second, let N and N' be lattices of the same rank, let Δ and Δ' be fans in N and N' (respectively), and let $\phi\colon N \to N'$ be a $\mathbf{Z}$-linear map of maximal rank. We claim that there is a regular refinement $\tilde{\Delta}$ of Δ such that the map $\phi_{\tilde{\Delta}\Delta'}\colon (N,\tilde{\Delta}) \dashrightarrow (N',\Delta')$ induced by $\phi\colon N \to N'$ is regular. We obtain the left-hand diagram of (2.1)

$$\begin{array}{ccc} (N,\tilde{\Delta}) & & X(\tilde{\Delta}) \\ {\scriptstyle \mathrm{id}_{\tilde{\Delta}\Delta}}\downarrow & \searrow^{\phi_{\tilde{\Delta}\Delta'}} & {\scriptstyle \pi}\downarrow \quad \searrow^{\tilde{f}} \\ (N,\Delta) & \xrightarrow{\phi_{\Delta\Delta'}} (N',\Delta') & X(\Delta) \xrightarrow{f} X(\Delta') \end{array} \tag{2.1}$$

inducing the right-hand diagram, where $\pi\colon X(\tilde{\Delta}) \to X(\Delta)$ and $\tilde{f}\colon X(\tilde{\Delta}) \to X(\Delta')$ are holomorphic.

To prove the claim, let $\phi^{-1}(\Delta')$ be the collection of cones $\phi^{-1}(\sigma')$ for $\sigma' \in \Delta'$. Since ϕ is of maximal rank, the cones in $\phi^{-1}(\Delta')$ are strongly convex and thus $\phi^{-1}(\Delta')$ is a fan. Now, any regular fan $\tilde{\Delta}$ that refines both Δ and $\phi^{-1}(\Delta')$ has the desired properties, and recall from Section 1.1 that such a fan always exists.

2.2. Pullback and Pushforward under Holomorphic Maps

Let N and N' be lattices of the same rank, let Δ and Δ' be fans in N and N' (respectively), and let $\phi\colon N \to N'$ be a $\mathbf{Z}$-linear map of maximal rank such that $\phi_{\Delta\Delta'}$ is regular. Let $f\colon X(\Delta) \to X(\Delta')$ be the corresponding holomorphic map on toric varieties.

Let D be a $T_{N'}$-invariant $\mathbf{Q}$-Cartier divisor on $X(\Delta')$, and let h_D be the corresponding $\mathbf{Q}$-support function. Then the pullback f^*D is a well-defined $\mathbf{Q}$-Cartier divisor (see [Fu2, Chap. 2.2]) and $h_{f^*D} = \phi^* h_D$ (see e.g. [M, Chap. 6, Exer. 8]). If D is Cartier, then f^*D is Cartier. To see this, assume that h is a support function on Δ'. Pick $\sigma \in \Delta$ and assume that $\phi(\sigma) \subseteq \sigma'$. On $\sigma' \in \Delta'$, h is defined by $h = \xi'$ for some $\xi' \in M'$. It follows that, on σ, $\phi^* h = \phi^* \xi' \in M$.

Next, let $D = \sum a_i D(\rho_i)$ be a T_N-invariant Weil divisor on $X(\Delta)$. Then $f_* D$ is a well-defined $T_{N'}$-invariant Weil divisor on $X(\Delta')$ (see e.g. [Fu2, Chap. 1.4]) and $f_* D = \sum a_i n_i D(\phi(\rho_i))$, where the sum is over all i such that $\phi(\rho_i) \in \Delta'$ and where $n_i \in \mathbf{N}^*$. Note that the pushforward of a Cartier divisor is in general only $\mathbf{Q}$-Cartier. Both pullback and pushforward respect linear equivalence.

2.3. Pullback under Rational Maps

Let N and N' be lattices of the same rank, let Δ and Δ' be fans in N and N' (respectively), and let $\phi\colon N \to N'$ be a $\mathbf{Z}$-linear map of maximal rank. Let D be a $T_{N'}$-invariant Cartier divisor on $X(\Delta')$. In terms of the right-hand diagram of (2.1), we can define the pullback of D under f as $f^*D := \pi_* \tilde{f}^* D$. In fact, this definition does not depend on the particular choice of $\tilde{\Delta}$. Observe that f^*D is the T_N-invariant Weil divisor $\sum -h_D(\phi(v_i)) D(\rho_i)$, where the sum is over the rays $\rho_i \in \Delta(1)$.

Assume that Δ is simplicial, and let h be a Δ'-linear $\mathbf{Q}$-support function. Let $\phi^*_{\Delta\Delta'}h$ be the Δ-linear support function defined by $(\phi^*_{\Delta\Delta'}h)(v) = h(\phi(v))$ if v is a primitive vector of a ray in Δ. In other words, $\phi^*_{\Delta\Delta'}h$ is obtained from ϕ^*h by using Δ-linear interpolation. If D is a $T_{N'}$-invariant $\mathbf{Q}$-Cartier divisor on $X(\Delta')$ and if h_D is the corresponding $\mathbf{Q}$-support function, then f^*D is determined by the $\mathbf{Q}$-support function $\phi^*_{\Delta\Delta'}h_D$.

Let $\sigma \in \Delta$ and let h be a Δ'-linear $\mathbf{Q}$-support function. Assume that $\phi(\sigma)$ is contained in a cone $\sigma' \in \Delta'$. Then h is linear on $\phi(\sigma)$, which implies that ϕ^*h is linear on σ and

$$\phi^*_{\Delta\Delta'}h|_\sigma = \phi^*h|_\sigma. \tag{2.2}$$

In particular, if $\phi_{\Delta\Delta'}$ is regular then $\phi^*_{\Delta\Delta'}h = \phi^*h$ for all Δ'-linear $\mathbf{Q}$-support functions. This is not the case in general if $\phi_{\Delta\Delta'}$ is not regular. Assume that there are cones $\sigma \in \Delta$ and $\sigma'_1, \sigma'_2 \in \Delta'$ such that $(\operatorname{Int}\phi(\sigma)) \cap \sigma'_j \neq \emptyset$ for $j = 1, 2$, and assume also that $h|_{\sigma'_1}$ and $h|_{\sigma'_2}$ are not defined by the same element in M'. Then $\phi^*_{\Delta\Delta'}h \neq \phi^*h$; indeed, $\phi^*_{\Delta\Delta'}h$ is linear on σ whereas h is not linear on $\phi(\sigma)$.

2.4. Criteria for Stability

We can express the stability of $f\colon X(\Delta) \dashrightarrow X(\Delta)$ in terms of $\phi\colon N \to N$.

LEMMA 2.1. *Assume that N, N', and N'' are lattices of the same rank, that Δ, Δ', and Δ'' are complete simplicial fans in N, N', and N'' (respectively), and that*

$$N \xrightarrow{\phi} N' \xrightarrow{\phi'} N''$$

are $\mathbf{Z}$-linear maps of maximal rank and with corresponding rational equivariant maps

$$X(\Delta) \overset{f}{\dashrightarrow} X(\Delta') \overset{f'}{\dashrightarrow} X(\Delta'').$$

Moreover, let f^* and f'^* be the corresponding pullback maps

$$\operatorname{Pic}(X(\Delta'')) \xrightarrow{f'^*} \operatorname{Pic}(X(\Delta')) \xrightarrow{f^*} \operatorname{Pic}(X(\Delta)).$$

For $\rho \in \Delta(1)$, let σ'_ρ be the (unique) cone in Δ' such that $\operatorname{Int}\phi(\rho) \subseteq \operatorname{Int}\sigma'_\rho$. Then $(f' \circ f)^* = f^*f'^*$ if $\phi'(\sigma'_\rho)$ is contained in a cone in Δ'' for all $\rho \in \Delta(1)$. The converse holds when Δ is a projective fan.

Proof. Note that $f^*f'^* = (f' \circ f)^*$ on $\operatorname{Pic}(X(\Delta''))$ if and only if, for every Δ''-linear support function h, the function $\phi^*_{\Delta\Delta'}\phi^*_{\Delta'\Delta''}h - (\phi' \circ \phi)^*_{\Delta\Delta''}h$ on $N_{\mathbf{Q}}$ is linear—that is, belongs to $M_{\mathbf{Q}}$. However, when Δ is projective, this is equivalent (as pointed out to us by Jan-Li Lin) to the stronger condition $\phi^*_{\Delta\Delta'}\phi^*_{\Delta'\Delta''}h = (\phi' \circ \phi)^*_{\Delta\Delta''}h$ for every Δ''-linear support function h; see [Li1, Prop. 5.5]. Furthermore, the latter condition can be written as $\phi^*_{\Delta\Delta'}\phi^*_{\Delta'\Delta''}h(v) = \phi^*\phi'^*h(v)$ for all primitive vectors of rays of Δ.

Let ρ be a ray of Δ with corresponding primitive vector v, and let $\sigma'_\rho \in \Delta'$ be the (unique) cone for which $\phi(v) \in \operatorname{Int}\sigma'_\rho$. First assume there is a cone $\sigma'' \in \Delta''$ such that $\phi'(\sigma'_\rho) \subseteq \sigma''$, and let h be a Δ''-linear support function. Then

$$\phi^*_{\Delta\Delta'}\phi'^*_{\Delta'\Delta''}h(v) = \phi^*(\phi'^*_{\Delta'\Delta''}h)|_{\sigma'_\rho}(v) = \phi^*(\phi'^*h)|_{\sigma'_\rho}(v) = \phi^*\phi'^*h(v);$$

here we have used that v is a primitive vector of ρ for the first equality and (2.2) for the second equality. Hence the "if" direction of the lemma follows.

Now assume $\phi'(\sigma'_\rho)$ is not contained in any cone in Δ''. It follows that there exist cones $\sigma''_1, \sigma''_2 \in \Delta''$ such that $\dim(\phi'(\sigma'_\rho) \cap \sigma''_j) = \dim \sigma'_\rho$ for $j = 1, 2$. Note that then $\sigma''_1 \not\subseteq \sigma''_2$. Pick $\rho''_1 \in \Delta''(1)$ such that ρ''_1 is a face of σ''_1 but not of σ''_2, and let v''_1 be the corresponding primitive vector. Let h be the Δ''-linear $\mathbf{Q}$-support function that is determined by $h(v''_1) = 1$, but $h(v''_i) = 0$ for all other primitive vectors of rays in Δ''. Then $h \not\equiv 0$ on σ''_2 but $h \equiv 0$ on σ''_1, which implies that $\phi'^*_{\Delta'\Delta''}h$ is linear on σ'_ρ whereas ϕ'^*h is not. In particular, $\phi'^*_{\Delta\Delta'}h(\phi(v)) > \phi'^*h(\phi(v))$ because $\phi(v) \in \operatorname{Int}\sigma'_\rho$. Hence $\phi^*_{\Delta\Delta'}\phi'^*_{\Delta'\Delta''}h(v) \neq \phi^*\phi'^*h(v)$, proving the "only if" direction of the lemma (when Δ is projective). □

The following results are immediate consequences of Lemma 2.1.

COROLLARY 2.2. *Assume that Δ is a complete simplicial fan in N and that $\phi\colon N \to N$ is a $\mathbf{Z}$-linear map with corresponding rational equivariant map $f\colon X(\Delta) \dashrightarrow X(\Delta)$. Then f is 1-stable if all cones σ in Δ for which there is a ray $\rho \in \Delta(1)$ such that $\operatorname{Int}\phi^n(\rho) \subseteq \operatorname{Int}\sigma$ for some n satisfy the condition that $\phi^{n'}(\sigma)$ is contained in a cone in Δ for all $n' \in \mathbf{N}$. The converse holds when Δ is projective.*

When $\phi\colon N \to N$ satisfies the assumption in Corollary 2.2, we say that it is *torically stable* on Δ.

COROLLARY 2.3. *Assume that Δ is a complete simplicial fan and that $\phi\colon N \to N$ is a $\mathbf{Z}$-linear map. Assume there is a collection $\mathcal{S} \subseteq \Delta$ such that $\phi(\sigma) \subseteq \sigma$ for $\sigma \in \mathcal{S}$ and such that each ray in Δ is either mapped onto another ray in Δ or mapped into one of the cones in $\mathcal{S}$. Then $f\colon X(\Delta) \dashrightarrow X(\Delta)$ is 1-stable.*

REMARK 2.4. In Corollary 2.3, it suffices to require that ϕ maps every cone in $\mathcal{S}$ into some other cone in $\mathcal{S}$.

In Theorems A and B we require that the eigenvalues of ϕ be positive. The reason is that, when some eigenvalues are negative, it may be impossible to find invariant cones.

PROPOSITION 2.5. *Let $\phi\colon N \to N$ be a $\mathbf{Z}$-linear map, where $N \cong \mathbf{Z}^m$. If σ is a simplicial cone of dimension m such that $\phi(\sigma) \subseteq \sigma$, then the trace of ϕ (i.e., the sum of the eigenvalues) must be nonnegative.*

This result is presumably known, but we include a proof for completeness.

Proof. Let $v_1, \ldots, v_m$ be a basis for $N_{\mathbf{R}}$ such that $\sigma = \sum_{j=1}^m \mathbf{R}_+ v_j$. We may assume $\det(v_1, \ldots, v_m) = 1$. Since $\phi(\sigma) \subseteq \sigma$, we must have

$$\det(v_1, \ldots, v_{j-1}, \phi(v_j), v_{j+1}, \ldots, v_m) \geq 0 \quad \text{for } 1 \leq j \leq m. \tag{2.3}$$

The left-hand side of (2.3) equals the jth diagonal element in the matrix of ϕ in the chosen basis, so the lemma follows by summing (2.3) over j. □

3. Smooth Stabilization in Dimension 2

We now look at 2-dimensional monomial mappings so that $N \simeq \mathbf{Z}^2$. In this case, any fan in N is simplicial and projective. Recall that a $\mathbf{Z}$-linear map $\phi\colon N \to N$ is said to be torically stable on a fan Δ if it satisfies the condition in Corollary 2.2.

Note that both eigenvalues μ_1, μ_2 are either real or complex conjugates of each other. When they are real, they are either both integers or both irrational.

In [Fa], Favre gave a complete characterization of when we can make ϕ torically stable on a possibly irregular fan; see Theorem C′ in the introduction. For the rest of this section we analyze whether we can make ϕ torically stable on a *regular* fan. We will prove Theorem C and give several examples showing that this result is essentially sharp. We also recover Theorem C′. The main new ideas are contained in Section 3.2.2.

REMARK 3.1. The statement in [Fa, Théorème Principal] does deal with smooth toric varieties, but there is a gap in the proof. Suppose μ_2/μ_1 is not of the form $e^{i\pi\theta}$ with $\theta \in \mathbf{R} \setminus \mathbf{Q}$. What Favre proves is that we can find a (not necessarily regular) refinement Δ' of Δ on which ϕ is torically stable. He then asserts that Δ' becomes a regular fan by passing to a sublattice $N' \subseteq N$. However, this last step does not work in general, see [Fu1, Sec. 2.2, p. 36].

3.1. Basic Facts on Fans in Dimension 2

We need a few basic results about refinements of fans in dimension 2. Let us call a fan Δ *symmetric* if $-\sigma \in \Delta$ for every cone $\sigma \in \Delta$.

First, as described in [Fu1, Sec. 2.6], there exists a canonical procedure for making an irregular fan regular. In fact, we have the following statement.

LEMMA 3.2. *Every fan Δ admits a regular refinement Δ' such that*:

(i) *any regular cone in Δ is also a cone in Δ'*;
(ii) *if Δ is symmetric, then so is Δ'.*

Both the lemma and its proof are valid in any dimension.

Sketch of Proof. The construction of Δ' proceeds inductively as follows. Pick an irregular 2-dimensional cone τ in Δ, let σ_1, σ_2 be its 1-dimensional faces, and let $v_j \in \sigma_j \cap N$ be the associated primitive vectors. Since τ is irregular, there exist $t_i \in (0,1) \cap \mathbf{Q}$, $i = 1, 2$, such that $v := t_1 v_1 + t_2 v_2 \in N$. Let $\sigma = \mathbf{R}_+ v$, and let τ_i $(i = 1, 2)$ be the 2-dimensional cones whose 1-dimensional faces are σ_i and σ. Applying this procedure finitely many times yields a regular fan Δ'. If Δ is symmetric, then we may simultaneously subdivide τ and $-\tau$ into τ_1, τ_2 and $-\tau_1, -\tau_2$, respectively. In this way, Δ' will remain symmetric. □

Second, we need to refine fans that may already be regular. Let τ be a regular 2-dimensional cone, and let σ_1, σ_2 be its 1-dimensional faces with corresponding primitive vectors v_1, v_2. Then v_1, v_2 generate N. Let $\sigma = \mathbf{R}_+(v_1 + v_2)$ and let τ_i, $i = 1, 2$, be the 2-dimensional cones whose 1-dimensional faces are σ_i and σ. Then the *barycentric subdivision* of τ consists of replacing τ with τ_1 and τ_2.

REMARK 3.3. Both the barycentric subdivision and the closely related procedure in the proof of Lemma 3.2 are special cases of Lemma 1.1.

LEMMA 3.4. *Let $(\tau_n)_{n\geq 0}$ be a sequence of regular 2-dimensional cones such that τ_{n+1} is one of the two cones obtained by barycentric subdivision of τ_n, $n \geq 0$. Then $\bigcap_{n\geq 0} \tau_n = \mathbf{R}_+ w$ for some $w \in N_{\mathbf{R}}$.*

Proof. Since τ_0 is regular it follows that $\tau_0 = \mathbf{R}_+ v + \mathbf{R}_+ v'$, where v, v' generate N. Write $\tau_n = \mathbf{R}_+ v_n + \mathbf{R}_+ v_n'$, where v_n, v_n' generate N. We can assume $v_{n+1} = v_n + v_n'$ and $v_{n+1}' \in \{v_n, v_n'\}$. Inductively, we see that $v_n = p_n v + q_n v'$ and $v_n' = p_n' v + q_n' v'$, where $p_n, q_n, p_n', q_n' \geq 0$ and $|p_n q_n' - p_n' q_n| = 1$. The lemma follows because $\max\{p_n, q_n\} \to \infty$ or $\max\{p_n', q_n'\} \to \infty$ as $n \to \infty$. □

LEMMA 3.5. *Consider regular 2-dimensional cones $\tau, \tau_0 \subseteq N_{\mathbf{R}}$ such that $\tau \subsetneq \tau_0$. Then τ is obtained from τ_0 by performing finitely many barycentric subdivisions.*

Proof. By Lemma 3.4 and induction it suffices to show that τ is contained in one of the 2-dimensional cones obtained by barycentric subdivision of τ_0.

Write $\tau_0 = \mathbf{R}_+ v_1 + \mathbf{R}_+ v_2$ and $\tau = \mathbf{R}_+ w_1 + \mathbf{R}_+ w_2$, where v_1, v_2 and w_1, w_2 are generators of N. Since τ is regular and $\tau \subseteq \tau_0$, we may assume that $w_i = p_i v_1 + q_i v_2$, where $p_i, q_i \geq 0$ $(i = 1, 2)$, $p_1 q_2 - p_2 q_1 = 1$, $p_1 > p_2$, and thus $q_2 > q_1$.

It suffices to prove that $v_1 + v_2 \notin \operatorname{Int} \tau$ unless $\tau = \tau_0$. Assume that $v_1 + v_2 \in \operatorname{Int} \tau$. Then $p_1 > q_1$ and $p_2 < q_2$, which implies that $p_1 q_2 - p_2 q_1 \geq q_1 + p_2 + 1$. It follows that $q_1 = p_2 = 0$ and hence $\tau = \tau_0$. □

COROLLARY 3.6. *Let $\tau_n, n \geq 0$, be regular 2-dimensional cones such that $\tau_{n+1} \subsetneq \tau_n$ for all n. Then $\bigcap_{n\geq 0} \tau_n = \mathbf{R}_+ w$ for some $w \in N_{\mathbf{R}}$.*

3.2. The Case $|\mu_1| > |\mu_2|$

We consider first the case when $|\mu_1| > |\mu_2|$. Then the μ_i are real and the corresponding eigenspaces $E_i \subseteq N_{\mathbf{R}}$ are 1-dimensional. Either $\mu_1, \mu_2 \in \mathbf{Z}$ or $\mu_1, \mu_2 \notin \mathbf{Q}$. Since $n \to \infty$, we have $\phi^n(v) \to E_1$ for any $v \in N_{\mathbf{R}} \setminus E_2$ and $\phi^{-n}(v) \to E_2$ for any $v \in N_{\mathbf{R}} \setminus E_1$.

We will use the following criterion for making ϕ torically stable.

LEMMA 3.7. *Suppose that Δ is a regular fan and define $U_i \subseteq N_{\mathbf{R}}$ as the union of all cones in Δ that intersect $E_i \setminus \{0\}$ for $i = 1, 2$. Assume that $\phi(U_1) \subseteq U_1$ and $\phi^{-1}(U_2) \subseteq U_2$. Then there exists a regular refinement Δ' of Δ on which ϕ is torically stable. If Δ is symmetric, then we can choose Δ' symmetric.*

Conversely, suppose that ϕ is torically stable with respect to a regular fan Δ and define U_i as before. Then $\phi(U_1) \subseteq U_1$ and $\phi^{-1}(U_2) \subseteq U_2$.

Proof. We may assume $U_1 \cup U_2 \neq N_{\mathbf{R}}$ since otherwise $f\colon X(\Delta) \dashrightarrow X(\Delta)$ is 1-stable by Corollary 2.3. We define an integer $J \geq 1$ and a sequence of (not necessarily convex) cones

$$\Omega_0 = U_2, \Omega_1, \dots, \Omega_J, \Omega_{J+1} = U_1$$

as follows. The set $\Omega_1 := \phi(\Omega_0) \setminus (U_1 \cup U_2)$ is nonempty, and there exists a $J \geq 1$ minimal such that $\phi^J(\Omega_1) \subseteq U_1$. Set $\Omega_j = \phi^{j-1}(\Omega_1) \setminus U_1$ for $1 < j \leq J$. Then $\{\Omega_j \setminus \{0\}\}_{j=0}^{J+1}$ defines a partition of $N_{\mathbf{R}} \setminus \{0\}$. Note that $\phi(\Omega_j) \subseteq \Omega_{j+1} \cup U_1$ for $1 \leq j \leq J$.

Let Δ_1 be the fan obtained from Δ by adding all rays of the form $\phi(\sigma) \in \Omega_1$, where σ is a ray in Δ contained in U_2. Let Δ_1' be a regular refinement of Δ_1 in which the regular cones of Δ_1 are kept, as described in Lemma 3.2. Note that this refinement procedure does not subdivide any cone contained in $U_1 \cup U_2$.

Inductively, for $1 < j \leq J$, let Δ_j be the fan obtained from Δ_{j-1}' by adding all rays of the form $\phi(\sigma) \in \Omega_j$, where σ is a ray in Δ_{j-1}' contained in Ω_{j-1}, and let Δ_j' be the regular refinement of Δ_j given by Lemma 3.2. This refinement procedure does not modify any cone contained in $U_1 \cup U_2 \cup \Omega_1 \cup \cdots \cup \Omega_{j-1}$. Then we can use $\Delta' = \Delta_J'$. If Δ is symmetric, then so is Δ'.

For the second part of the lemma, note that if σ is a ray in Δ that is not contained in $E_1 \cup E_2$, then $\phi^n(\operatorname{Int}\sigma) \subseteq \operatorname{Int} U_1$ for $n \gg 1$. Thus, for ϕ to be torically stable on Δ, ϕ must map any cone contained in U_1 into another cone contained in U_1. This implies $\phi(U_1) \subseteq U_1$. A similar argument shows that $\phi^{-1}(U_2) \subseteq U_2$. □

3.2.1. *Integer Eigenvalues*

The first subcase is when $|\mu_1| > |\mu_2|$ and $\mu_1, \mu_2 \in \mathbf{Z}$; thus the corresponding eigenspaces E_1, E_2 are rational. We claim that ϕ can always be made torically stable on a regular fan in this case. To see this, we need only satisfy the hypotheses of Lemma 3.7. After refining, we may assume that Δ is symmetric and regular and that the eigenspaces $E_i \subseteq N_{\mathbf{R}}$ are unions of cones in Δ.

For $i = 1, 2$, let U_i be the union of all cones in Δ intersecting E_i. If $\mu_1, \mu_2 > 0$, then $\phi(U_1) \subseteq U_1$ and $\phi^{-1}(U_2) \subseteq U_2$. Hence Lemma 3.7 applies. The same is true also when $\mu_1, \mu_2 < 0$, since Δ is symmetric.

When μ_1 and μ_2 have opposite signs, we have to be more careful. For definiteness, let us assume $\mu_1 > 0 > \mu_2$. (The case $\mu_1 < 0 < \mu_2$ is handled the same way as long as all fans we construct are symmetric.) Let σ_1 and σ_2 be 2-dimensional cones in Δ sharing a common face τ contained in E_1. Provided Δ is symmetric, $\phi(U_1) \subseteq U_1$ is equivalent to $\phi(\sigma_1) \subseteq \sigma_2$ and $\phi(\sigma_2) \subseteq \sigma_1$, which will happen only if σ_1 and σ_2 are of roughly the same size. We claim that this can be arranged by subdividing the cones σ_i. Pick generators v_1, v_2 for N such that $v_1 \in \tau = \sigma_1 \cap \sigma_2$. Then ϕ is given by the matrix $\left(\begin{smallmatrix} a & b \\ 0 & -d \end{smallmatrix}\right)$, where $a = \mu_1 > 0$ and $0 < d = |\mu_2| < a$. The 1-dimensional faces of σ_1 (resp. σ_2) are τ and a ray whose primitive vector is of the form $r_1 v_1 + v_2$ (resp. $r_2 v_1 - v_2$), where $r_1, r_2 \in \mathbf{Z}$. By making a barycentric

subdivision of σ_i and replacing σ_i with the subcone containing τ, we replace r_i with $r_i + 1$. Repeating this procedure finitely many times, we can achieve $r_1 = r_2 = r \gg 0$. After picking $r > |b|/(a-d)$, it is straightforward to verify that $\phi(\sigma_1) \subseteq \sigma_2$ and $\phi(\sigma_2) \subseteq \sigma_1$. Making the construction symmetric, we obtain $\phi(U_1) \subseteq U_1$. A similar construction gives $\phi^{-1}(U_2) \subseteq U_2$. Thus Lemma 3.7 applies.

3.2.2. *Irrational Eigenvalues*

The second subcase is when $|\mu_1| > |\mu_2|$ and μ_1, μ_2 are both real irrational. Then the corresponding eigenspaces $E_i \subseteq N_{\mathbf{R}}$, $i = 1, 2$, contain no nonzero lattice points.

PROPOSITION 3.8. *If μ_1, μ_2 are of the same sign, then any fan Δ admits a regular refinement Δ' on which ϕ is torically stable.*

Proof. We may assume Δ is symmetric. The assumption that μ_1 and μ_2 have the same sign implies that *any* symmetric cone U_i $(i = 1, 2)$ for which $E_i \setminus \{0\} \subseteq \operatorname{Int} U_i$ must satisfy $\phi(U_1) \subseteq U_1$ and $\phi^{-1}(U_2) \subseteq U_2$. Hence the proposition follows from Lemma 3.7. □

Now assume μ_1 and μ_2 have different signs. This case is quite delicate. Let us assume for now that $\mu_1 > 0 > \mu_2$.

Our starting point is a regular 2-dimensional cone σ_1 containing an eigenvector associated to μ_1 but not containing any eigenvector associated to μ_2. Such a cone exists and can be constructed using repeated barycentric subdivisions and invoking Lemma 3.4. Write $\sigma_1 = \mathbf{R}_+ v_1 + \mathbf{R}_+ v_2$, where v_1, v_2 are generators for N. Then ϕ admits eigenvectors of the form $v_1 + z_i v_2$ associated to μ_i, $i = 1, 2$, where $z_2 < 0 < z_1$. After exchanging v_1 and v_2 if necessary, we may and will assume that $\max\{|z_1|, |z_2|\} > 1$.

We now inductively define a sequence $(v_{1,n}, v_{2,n})_{n \in \mathbf{Z}}$ of generators for N. They will have the property that ϕ admits an eigenvector of the form $v_{1,n} + z_{i,n} v_{2,n}$ associated to μ_i, $i = 1, 2$, where $z_{2,n} < 0 < z_{1,n}$ and $\max\{|z_{1,n}|, |z_{2,n}|\} > 1$. Set $v_{i,0} := v_i$ and $z_{i,0} = z_i$, $i = 1, 2$.

First suppose $n > 0$. If $z_{1,n-1} > 1$ set $(v_{1,n}, v_{2,n}) := (v_{1,n-1} + v_{2,n-1}, v_{2,n-1})$, and if $0 < z_{1,n-1} < 1$ set $(v_{1,n}, v_{2,n}) := (v_{2,n-1}, v_{1,n-1})$. This leads to

$$(z_{1,n}, z_{2,n}) = \begin{cases} (z_{1,n-1} - 1, z_{2,n-1} - 1) & \text{if } z_{1,n-1} > 1, \\ (z_{1,n-1}^{-1}, z_{2,n-1}^{-1}) & \text{if } 0 < z_{1,n-1} < 1. \end{cases} \tag{3.1}$$

Now suppose $n < 0$. If $z_{2,n+1} < -1$ set $(v_{1,n}, v_{2,n}) := (v_{1,n+1} - v_{2,n+1}, v_{2,n+1})$, and if $-1 < z_{2,n+1} < 0$ set $(v_{1,n}, v_{2,n}) := (-v_{2,n+1}, -v_{1,n+1})$. We obtain

$$(z_{1,n}, z_{2,n}) = \begin{cases} (z_{1,n+1} + 1, z_{2,n+1} + 1) & \text{if } z_{2,n+1} < -1, \\ (z_{1,n+1}^{-1}, z_{2,n+1}^{-1}) & \text{if } -1 < z_{2,n+1} < 0. \end{cases} \tag{3.2}$$

Notice that (3.1) and (3.2) actually hold for *all* $n \in \mathbf{Z}$, which follows because $\max\{|z_{1,n}|, |z_{2,n}|\} > 1$. For example, suppose that $n \leq 0$ and $z_{1,n-1} > 1$. To verify (3.1) we must show that $(z_{1,n}, z_{2,n}) = (z_{1,n-1} - 1, z_{2,n-1} - 1)$. This follows

from (3.2) applied to $n-1$ if we know that $z_{2,n} < -1$. But if $-1 < z_{2,n} < 0$, then $z_{1,n} > 1$ and so (3.2) would give $z_{1,n-1} = z_{1,n}^{-1} < 1$—a contradiction.

For any $n \in \mathbf{Z}$, $\sigma_{1,n} := \mathbf{R}_+ v_{1,n} + \mathbf{R}_+ v_{2,n}$ and $\sigma_{2,n} := \mathbf{R}_+ v_{1,n} + \mathbf{R}_+(-v_{2,n})$ are regular cones containing eigenvectors associated to μ_1 and μ_2 (respectively) in their interiors. For $n > 0$, $\sigma_{1,n}$ is obtained by barycentric subdivision of $\sigma_{1,n-1}$. For $n < 0$, $\sigma_{2,n}$ is obtained by barycentric subdivision of $\sigma_{2,n+1}$. This implies that the sequences $(\sigma_{i,n})_{n\in\mathbf{Z}}$, $i = 1, 2$ are largely independent of the initial choice of cone σ_1. Indeed, suppose we start with another cone σ_1' and obtain corresponding sequences $(\sigma'_{i,n})_{n\in\mathbf{Z}}$. By Lemma 3.5 there exist $l_i \in \mathbf{Z}$, $i = 1, 2$, such that $\sigma'_{1,n} = \sigma_{1,n+l_1}$ and $\sigma'_{2,n} = \sigma_{2,n+l_2}$ for $n \gg 0$ and $n \ll 0$, respectively.

Let $A_n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$ be the matrix of ϕ in the basis $(v_{1,n}, v_{2,n})$. We are interested in whether A_n has nonnegative entries. A direct computation shows $b_n = (\mu_1 - \mu_2)/(z_{1,n} - z_{2,n}) > 0$ and $c_n = (\mu_1 - \mu_2)/(z_{1,n}^{-1} - z_{2,n}^{-1}) > 0$ for all n. As for the diagonal entries, note that $a_n + d_n = \mu_1 + \mu_2 =: \gamma > 0$ is independent of n. Set $\delta_n = a_n - d_n$. We see that A_n has nonnegative entries if and only if $|\delta_n| \le \gamma$.

LEMMA 3.9. *The sequence $(A_n)_{n\in\mathbf{Z}}$ is periodic. Furthermore, the following conditions are equivalent*:

(i) *there exists an n such that $a_n, b_n, c_n, d_n \ge 0$;*
(ii) *there exist infinitely many $n \ge 0$ such that $a_n, b_n, c_n, d_n \ge 0$;*
(iii) *any fan Δ admits a regular refinement Δ' on which ϕ is torically stable.*

Proof. Note that $z_{i,n}$, $i = 1, 2$, are the roots of $b_n z^2 + \delta_n z - c_n = 0$. It follows from (3.1) that

$$(b_{n+1}, \delta_{n+1}, c_{n+1}) = \begin{cases} (b_n, \delta_n + 2b_n, c_n - b_n - \delta_n) & \text{if } c_n > b_n + \delta_n, \\ (c_n, -\delta_n, b_n) & \text{if } c_n < b_n + \delta_n. \end{cases} \tag{3.3}$$

We see that the quantity $D := D_n = \delta_n^2 + 4b_n c_n$ is independent of n; in fact, D is the discriminant of $b_n z^2 + \delta_n z - b_n$. Since b_n, c_n, δ_n are integers and since $b_n, c_n > 0$, it follows that the sequence $(b_n, \delta_n, c_n)_{n\in\mathbf{Z}}$, and therefore also the sequence $(A_n)_{n\in\mathbf{Z}}$, must be periodic. This immediately shows that (i) and (ii) are equivalent.

Before showing that (i) and (ii) are equivalent to (iii), we recall that the data constructed so far are essentially independent of the initial choice of regular cone σ_1. In particular, the sequence $(A_n)_{n\in\mathbf{Z}}$ is independent of this choice, up to an index shift, and so the validity of (ii) is independent of σ_1.

To show that (ii) implies (iii), suppose A_n has nonnegative entries. Then the regular cone $\sigma_{1,n} := \mathbf{R}_+ v_{1,n} + \mathbf{R}_+ v_{2,n}$ is invariant: $\phi(\sigma_{1,n}) \subseteq \sigma_{1,n}$. Similarly, the regular cone $\sigma_{2,n} := \mathbf{R}_+ v_{1,n} + \mathbf{R}_+(-v_{2,n})$ satisfies $\phi^{-1}(\sigma_{2,n}) \subseteq \sigma_{2,n}$.

Pick $n_1 \gg 0$ and $n_2 \ll 0$ such that A_{n_i} has nonnegative entries for $i = 1, 2$. We may assume that σ_{1,n_1} and σ_{2,n_2} are arbitrarily small regular cones containing eigenvectors associated to μ_1 and μ_2, respectively. By Lemma 3.5 we may, after replacing Δ by a suitable symmetric regular refinement, assume that $\pm\sigma_{1,n_1}$ and

$\pm\sigma_{2,n_2}$ are cones in Δ. We may then apply Lemma 3.7 to $U_i = \sigma_{i,n_i} \cup (-\sigma_{i,n_i})$ and conclude that (iii) holds.

Finally, to show that (iii) implies (i), assume that ϕ is torically stable on a regular refinement Δ' of Δ. We can then use as our initial cone σ_1 a cone in Δ' containing an eigenvector associated to μ_1. Indeed, it follows from Lemma 3.7 that $\phi(\sigma_1) \subseteq \sigma_1$ and that σ_1 cannot contain any eigenvector associated to μ_2. Because $\phi(\sigma_1) \subseteq \sigma_1$, A_0 must have nonnegative entries. □

REMARK 3.10. The sequence $(z_{1,n})_{n\geq 0}$ encodes the continued fractions expansion of z_1, and the proof that $(A_n)_{n\geq 0}$ is periodic corresponds to the classical proof of the (pre)periodicity of the continued fractions expansion of a quadratic surd (a result due to Lagrange; see [HWr, Thm. 177, p. 185]).

PROPOSITION 3.11. *When $|\mu_2| < 1$, any fan Δ admits a regular refinement Δ' on which ϕ is torically stable.*

Proof. Note that μ_1 and μ_2 must be real irrational and that $|\mu_1| > 1 > |\mu_2|$. By Proposition 3.8 we may assume they have different signs. First suppose $\mu_1 > 0 > \mu_2$. The condition $-1 < \mu_2 < 0$ easily translates into $\sqrt{D} - 2 < \gamma < \sqrt{D}$, where $\gamma = \mu_1 + \mu_2$ and D is as in the proof of Lemma 3.9; indeed, $\mu_j = \left(\gamma \pm \sqrt{D}\right)/2$. As noted previously, $b_n, c_n > 0$ for all n and by Lemma 3.9 we need only find an $n \in \mathbf{N}$ such that $|\delta_n| \leq \gamma$, where $\delta_n = a_n - d_n$.

First suppose there exists an n such that $c_n = 1$ and $|\delta_n| = b_n$. Then $D = \delta_n^2 + 4b_nc_n = (b_n + 2)^2 - 4$, so $\mathbf{Z} \ni \gamma > \sqrt{D} - 2$ implies $\gamma > b_n + 2 - 2 = b_n = |\delta_n|$ and we are done.

In general, it suffices to find n with $|\delta_n| \leq \sqrt{D} - 2$, a condition equivalent to $|\delta_n| < b_nc_n$. There exists an $n_0 \geq 0$ such that $\delta_{n_0} < 0$; otherwise, we would be able to apply the first transformation in (3.3) infinitely many times in a row, which is clearly not possible. Indeed, the second transformation changes the sign of δ_n. Successively applying (3.3) yields $n \geq n_0$ with $-b_n \leq \delta_n < b_n$. Then $|\delta_n| < b_nc_n$ unless $\delta_n = -b_n$ and $c_n = 1$, a case we have already addressed.

Finally, consider the case $\mu_1 < 0 < \mu_2$. By the foregoing, we can find a *symmetric* regular refinement Δ' of Δ on which the map $-\phi\colon N \to N$ is torically stable. Then ϕ is also torically stable on Δ'. □

The following example shows that Theorem C fails in general when $|\mu_1| > |\mu_2| > 1$.

EXAMPLE 3.12. It follows from Lemma 3.9 that the linear map $\phi\colon N \to N$ given by the matrix $A = A_\phi = \left(\begin{smallmatrix} -1 & 3 \\ 3 & 2 \end{smallmatrix}\right)$ cannot be made torically stable for any complete regular fan. Indeed, we have $A_0 = A$, $A_1 = \left(\begin{smallmatrix} 2 & 3 \\ 3 & -1 \end{smallmatrix}\right)$, and $A_n = A_{n-2}$. Here $\mu_j = \left(1 \pm 3\sqrt{5}\right)/2$.

We record the following consequence of our analysis in Section 3.2.2.

COROLLARY 3.13. *Assume that the eigenvalues of $\phi\colon N \to N$ satisfy $\mu_1 > -\mu_2 > 0$ and $\mu_i \notin \mathbf{Z}$ for $i = 1, 2$. Moreover, assume N has generators v_1, v_2*

such that ϕ is given by a matrix with nonnegative coefficients in the associated basis for $N_{\mathbf{R}}$. Then ϕ admits an eigenvector $e_1 = v_1 + z_1 v_2$ in the first quadrant $\sigma_0 = \mathbf{R}_+ v_1 + \mathbf{R}_+ v_2$ and there exists a sequence $(\sigma_j)_{j\geq 0}$ of regular cones such that $\mathbf{R}_+ e_1 \subseteq \sigma_{j+1} \subseteq \sigma_j$, $\bigcap_{j=0}^{\infty} \sigma_j = \mathbf{R}_+ e_1$, and $\phi(\sigma_j) \subseteq \sigma_j$ for $j \geq 0$.

We thus obtain an independent proof of [FJ1, Lemma 7]; see also [FJ2].

3.3. The Case $|\mu_1| = |\mu_2|$

Write $\lambda = |\mu_1| = |\mu_2|$. There are two subcases.

3.3.1. *The Diagonalizable Case*
First consider the case when $\phi\colon N_{\mathbf{C}} \to N_{\mathbf{C}}$ is diagonalizable. When μ_1/μ_2 is not a root of unity, Favre [F] observed that f cannot be made torically stable even on an irregular fan. Indeed, the orbit $\bigcup_{n\geq 0} \phi^n(\rho)$ of any ray ρ is dense in $N_{\mathbf{R}}$, so stability is impossible in view of Lemma 2.2.

Now suppose μ_1/μ_2 is a root of unity. Then $\phi^n = \lambda^n \operatorname{Id}$ for some $n > 0$, where $\lambda = |\mu_1| = |\mu_2|$. This implies that when f is stable, ϕ must map any ray to another ray in the fan. We can achieve this only in special cases, such as when $\mu_1 = \mu_2$. Indeed, then $\phi = \pm\lambda \operatorname{Id}$ and any symmetric fan is invariant.

The following example illustrates the problems that may arise when μ_1/μ_2 is a root of unity different from 1. See also [F, Ex. 2].

EXAMPLE 3.14. Let $\phi\colon N \to N$ be given by the matrix $A = A_\phi = \left(\begin{smallmatrix} -1 & -1 \\ 3 & -1 \end{smallmatrix}\right)$. Then $\mu_j = 2e^{2\pi i j/3}$, $j = 1, 2$. In particular, $\phi^3 = 8 \operatorname{Id}$. We claim that no complete regular fan Δ can be invariant by ϕ. To see this, consider any ray in Δ and let $v \in N$ be the corresponding primitive vector. Then $\phi(v) = lv'$, where v' is another primitive vector and $l = l(v) \in \mathbf{N}$. If v_1 and v_2 are the primitive vectors of two adjacent rays in Δ, then $l(v_1)l(v_2) = |\det \phi| = 4$, since Δ is regular. Thus there are two cases: either $l(v) = 2$ for all v; or $\{l(v_1), l(v_2)\} = \{1, 4\}$ for any two adjacent primitive vectors v_1, v_2. The first case is not possible because all entries in ϕ would have to be even. The second case cannot occur because $\phi^3 = 8 \operatorname{Id}$.

3.3.2. *The Nondiagonalizable Case*
Finally, assume that $\phi\colon N_{\mathbf{C}} \to N_{\mathbf{C}}$ is not diagonalizable. Then $\mu_1 = \mu_2 = \pm\lambda$, where $\lambda \in \mathbf{N}$. There exists a primitive lattice point $v \in N$ such that $\mathbf{R}v$ is the eigenspace for $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$. After subdividing, we may assume that Δ is regular and symmetric and that $\sigma := \mathbf{R}_+ v$ and $-\sigma$ are cones in Δ. Pick $w \in N$ such that (v, w) are generators for N and such that $\mathbf{R}_+ w$ and $-\mathbf{R}_+ w$ are cones in Δ. The matrix of ϕ is given by $A = A_\phi = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$, where $a = \pm\lambda$ and $b \in \mathbf{Z}$, $b \neq 0$. Replacing w with $-w$ if necessary, we have $b > 0$.

First assume $a = \lambda$. Let $\tau \in \Delta(2)$ be the unique cone contained in $\mathbf{R}_+ v + \mathbf{R}_+ w$ and having σ as one of its faces. Then $\phi(\tau) \subsetneq \tau$ and $\phi(-\tau) \subsetneq -\tau$. We can now proceed as in the proof of Lemma 3.7 and refine Δ into a symmetric regular fan Δ' such that $\pm\tau \in \Delta'$ and such that, for all rays $\sigma' \in \Delta(1)$ and all $n \geq 0$, we have either $\phi^n(\sigma') \in \Delta(1)$ or $\phi^n(\sigma') \subseteq \pm\tau$. Then ϕ is torically stable on Δ'. The case when $a = -\lambda$ is handled in the same way, keeping all fans symmetric.

3.4. Proof of Theorems C and C′

We now have all ingredients necessary to complete the proof of Theorem C. In case (a), where $|\mu_2| < 1$, we are done by Proposition 3.11. In case (b), where $|\mu_1| > |\mu_2|$ and $\mu_1, \mu_2 \in \mathbf{Z}$, the result follows as explained in Section 3.2.1.

Finally, consider case (c); that is, $\mu_1, \mu_2 \in \mathbf{R}$ and $\mu_1\mu_2 > 0$. If μ_1 and μ_2 are irrational then we are done by Proposition 3.8 and so, having treated cases (a) and (b), we may assume $\mu_1 = \mu_2 \in \mathbf{Z}$. Then either $\phi = \mu_1 \mathrm{Id}$, in which case the theorem is trivial, or ϕ is not diagonalizable over $\mathbf{C}$, in which case the theorem follows from the discussion in Section 3.3.2.

In fact, we have also proved Theorem C′ except for the case where μ_1, μ_2 are real, irrational, and of different sign. We can then refine the original fan Δ so that it contains (possibly irregular) cones σ_1, σ_2 for which $\phi(\sigma_1) \subseteq \pm\sigma_1$ and $\phi^{-1}(\sigma_2) \subseteq \pm\sigma_2$. The proof of Lemma 3.7 now goes through and produces a refinement Δ' of Δ on which ϕ is torically stable. In fact, the only irregular cones in Δ' are $\pm\sigma_1$ and $\pm\sigma_2$.

4. Stabilization: Proof of Theorems A and A′

Throughout this section we assume that $\phi\colon N \to N$ has distinct and positive eigenvalues. To prove Theorems A and A′, we will use the criterion in Corollary 2.3.

The mapping $\phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}}$ induces a mapping $\phi^*\colon M_{\mathbf{R}} \to M_{\mathbf{R}}$ defined by $\langle \phi^*\xi, v\rangle := \langle \xi, \phi(v)\rangle$ and with the same eigenvalues as ϕ. Given a 1-dimensional eigenspace $E \subseteq N_{\mathbf{R}}$ of ϕ, let $\tilde{E} \subseteq M_{\mathbf{R}}$ denote the corresponding eigenspace of ϕ^* and let $E^\perp := \{v \in N_{\mathbf{R}} \mid \langle \xi, v\rangle = 0\ \forall \xi \in \tilde{E}\} \subseteq N_{\mathbf{R}}$. Note that, since the eigenvalues of ϕ are distinct, $E^\perp$ is spanned by the eigenvectors that are not in E.

4.1. Real Dynamics

We say that a set $Z \subseteq N_{\mathbf{R}}$ is *invariant* (under ϕ) if $\phi(Z) \subseteq Z$.

The following result is well known; see, for example, [La, Exer. 13, p. 552].

LEMMA 4.1. *Any invariant subspace $V \subseteq N_{\mathbf{R}}$ is spanned by eigenvectors of ϕ.*

Given an invariant subspace $V \subseteq N_{\mathbf{R}}$, let $E_1, \ldots, E_{\dim V}$ be the invariant eigenspaces of ϕ corresponding to the eigenvectors with eigenvalues $\nu_1 > \cdots > \nu_{\dim V} > 0$ that span V. For $1 \le j \le \dim V$, let $V_j := E_j \oplus \cdots \oplus E_{\dim V}$. Then we have a filtration $V = V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{\dim V+1} := \{0\}$, and if $v \in V_j \setminus V_{j+1}$ then $\phi^n(v) \to E_j$ when $n \to \infty$.

4.2. Invariant Rational Subspaces

We say that a subspace $V \subseteq N_{\mathbf{R}}$ is *rational* if $\overline{V \cap N_{\mathbf{Q}}} = V$. This is equivalent to the lattice $N \cap V$ having rank equal to $\dim V$. A subspace $V \subseteq N_{\mathbf{R}}$ is rational if and only if its annihilator $V^o := \{\xi \in M_{\mathbf{R}} \mid \xi|_V \equiv 0\} \subseteq M_{\mathbf{R}}$ is rational. Note that $(V^o)^o = V$.

Assume that V and W are rational subspaces. Then $V+W$ is rational and hence so is $V \cap W = (V^o + W^o)^o$. Given $V \subseteq N_{\mathbf{R}}$, there is a minimal rational subspace of $N_{\mathbf{R}}$ that contains V and a maximal rational subspace contained in V.

LEMMA 4.2. *Assume that $V \subseteq N_{\mathbf{R}}$ is invariant under ϕ. Then the minimal rational subspace that contains V and the maximal rational subspace contained in V are both invariant.*

Proof. Let W be the minimal rational subspace that contains V. Then $V \subseteq W \cap \phi(W) = W$, since W is minimal. To conclude, $\phi(W) = W$. The second statement follows from the first by using annihilators. □

The mapping ϕ induces a binary tree $T(\phi)$ of rational invariant subspaces of $N_{\mathbf{R}}$, which should be compared to the real filtration in Section 4.1. The nodes of $T(\phi)$ are of the form (V, W), where V and W are rational invariant subspaces of $N_{\mathbf{R}}$ such that $V \subseteq W$. The root of $T(\phi)$ is $(\{0\}, N_{\mathbf{R}})$, and (V, W) is a leaf if $V = W$. Assume that $V \neq W$. Among all 1-dimensional eigenspaces E of ϕ such that $E \subseteq W$ but $E \not\subseteq V$, let $E(V, W)$ be the one with the largest eigenvalue. Let V' be the smallest rational subspace that contains $V + E(V, W)$, and let W' be the largest rational subspace contained in $W \cap E(V, W)^{\perp}$. Then the two children of (V, W) are (V', W) and (V, W'). Note that $V' \subseteq W$ since V and $E(V, W)$ are contained in W and since W is rational, and note that $V \subseteq W'$ since W and $E(V, W)^{\perp}$ contain V and since V is rational. Observe, in light of Lemma 4.2, that V' and W' are invariant.

LEMMA 4.3. *Let (V, W) be a node in $T(\phi)$ and let U be a rational invariant subspace such that $V \subseteq U \subseteq W$. Then either $E(V, W) \subseteq U$ or $U \subseteq E(V, W)^{\perp}$.*

Proof. Pick $x \in M_{\mathbf{R}}$ such that $E^{\perp} = \{x = 0\}$, where $E := E(V, W)$. Assume $U \not\subseteq E^{\perp}$, and pick $v \in U$ such that $x(v) \neq 0$. Then $v \notin E^{\perp} \supseteq V$. Let $\widetilde{W} = W/V$, let $\tilde{\phi}\colon \widetilde{W} \to \widetilde{W}$ be the map induced by ϕ, and let $\tilde{E}$, $\tilde{U}$, and $\tilde{v}$ be (respectively) the images of E, U, and v under the quotient map $W \to \widetilde{W}$. Then $\tilde{E}$ is an eigenspace for $\tilde{\phi}$ with eigenvalue ν dominating all other eigenvalues of $\tilde{\phi}$. Thus $\nu^{-n}\tilde{\phi}^n(\tilde{v})$ converges to a nonzero element of $\tilde{E}$. This implies that $\tilde{E} \subseteq \tilde{U}$, since $\tilde{v} \in \tilde{U}$ and $\tilde{U}$ is invariant under $\tilde{\phi}$. It follows that $E \subseteq U$. □

Let us create a new tree from $T(\phi)$. Replace each node (V, W) in $T(\phi)$ by V and thereafter remove all loops joining a node V with itself. We will refer to the tree so obtained as the *reduced tree* induced by ϕ and denote it by $T_{\mathrm{red}}(\phi)$. Observe that the nodes in $T_{\mathrm{red}}(\phi)$ are in one-to-one correspondence with the leaves in $T(\phi)$. Given a node V in $T_{\mathrm{red}}(\phi)$ with parent V', among all 1-dimensional eigenspaces of ϕ in $V \setminus V'$ let $E(V)$ be the one corresponding to the largest eigenvalue. Then, by construction, V is the smallest (invariant) rational subspace of $N_{\mathbf{R}}$ that contains $V' + E(V)$.

We claim that all rational invariant subspaces of $N_{\mathbf{R}}$ are in $T_{\mathrm{red}}(\phi)$. To see this, given a rational invariant subspace U, let $S(U) = \{(V, W) \in T(\phi) \mid V \subseteq U \subseteq W\}$. Note that $S(U)$ is nonempty because $(\{0\}, N_{\mathbf{R}}) \in S(U)$. Pick $(V, W) \in S(U)$. By

Lemma 4.3, either $E := E(V,W) \subseteq U$ or $U \subseteq E^{\perp}$. In the first case, $V' \subseteq U$ for V' the smallest rational invariant subspace of $N_{\mathbf{R}}$ that contains $V + E$. In the second case, $U \subseteq W'$ for W' the largest rational invariant subspace contained in $W \cap E^{\perp}$. Thus, exactly one of the children of (V,W) is in $S(U)$. It follows that $S(U)$ is a maximal chain in $T(\phi)$. In particular, $S(U)$ contains a leaf of $T(\phi)$ that must be of the form (U,U). Hence U is a node in $T_{\mathrm{red}}(\phi)$. However, it is not true in general that (V,W) is a node in $T(\phi)$ as soon as $V \subseteq W$ and V, W are rational and invariant.

4.3. Invariant Chambers

To each node (V,W) in $T(\phi)$ we associate a *chamber* $C(V,W)$. The chamber $C(V,W)$ is an invariant open dense subset of W and is defined recursively as follows. First let $C(\{0\}, N_{\mathbf{R}}) = N_{\mathbf{R}}$. Then, if $C(V,W)$ is defined and if (V',W) and (V,W') are the children of (V,W), let $C(V',W) := C(V,W) \setminus E(V,W)^{\perp}$ and $C(V,W') := C(V,W) \cap W'$. Note that $C(V,W) \cap N_{\mathbf{Q}}$ is the disjoint union of $C(V',W) \cap N_{\mathbf{Q}}$ and $C(V,W') \cap N_{\mathbf{Q}}$. In particular, the chambers associated with the leaves of $T(\phi)$ induce invariant partitions of $N_{\mathbf{Q}}$ and N (but not of $N_{\mathbf{R}}$, in general).

To the node V in $T_{\mathrm{red}}(\phi)$ we associate the chamber $C(V) := C(V,V)$. Then the chambers $C(V)$ provide partitions of $N_{\mathbf{Q}}$ and N. More precisely, given a node V' in $T_{\mathrm{red}}(\phi)$, the chambers $C(V)$—where V ranges over ancestors of V' in $T_{\mathrm{red}}(\phi)$—give partitions of $V' \cap N_{\mathbf{Q}}$ and $V' \cap N$. Assume that the genealogy of V is the chain of nodes in $T_{\mathrm{red}}(\phi)$:

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V. \tag{4.1}$$

For $1 \le k \le s$, pick $x_k \in M_{\mathbf{R}}$ such that $E(V_k)^{\perp} = \{x_k = 0\}$. Then

$$C(V) = V \setminus \bigcup_{k=1}^{s} E(V_k)^{\perp} = V \cap \bigcap_{k=1}^{s} \{x_k \neq 0\}.$$

Observe that V is the smallest rational subspace of $N_{\mathbf{R}}$ that contains the subspaces $\{E(V_k)\}_{1 \le k \le s}$.

LEMMA 4.4. *Let V be a node in $T_{\mathrm{red}}(\phi)$ and let $v \in C(V)$. Then there is no rational invariant proper subspace $U \subsetneq V$ containing v.*

Proof. Let (4.1) be the genealogy of V in $T_{\mathrm{red}}(\phi)$, with corresponding $x_k \in M_{\mathbf{R}}$, and let U be the smallest rational invariant subspace of V containing v. Pick r maximal such that $V_r \subseteq U$. Assume $r < s$. Since $v \in C(V)$, we have $x_{r+1}(v) \neq 0$. By arguments as in the proof of Lemma 4.3, one can show that $E(V_{r+1}) \subseteq U$. Since V_{r+1} is the smallest rational invariant subspace of V that contains $V_r + E(V_{r+1})$, it follows that $V_{r+1} \subseteq U$, which contradicts the maximality of r. Hence $U = V$, proving the lemma. □

The chamber $C(V)$ admits a further decomposition into 2^s connected components. Given $x_1, \ldots, x_s \in M_{\mathbf{R}}$ and $\eta = (\eta_1, \ldots, \eta_s) \in \{\pm 1\}^s$, let $C(V,\eta) := V \cap \bigcap_{j=1}^{s} \{\eta_j x_j > 0\}$; we will refer to η as a *sign vector.* Then the $C(V,\eta)$ are

clearly disjoint and $C(V) = \bigcup_{\eta\in\{\pm 1\}^s} C(V,\eta)$. Hence the chamber components $C(V,\eta)$, where V ranges over the nodes in $T_{\mathrm{red}}(\phi)$ and η ranges over possible sign vectors, establish partitions of N and $N_{\mathbf{Q}}$. Moreover, if the eigenvalues of ϕ are positive, then each $C(V,\eta)$ is invariant under ϕ. If V' is an ancestor of V—say, $V' = V_{s'}$—then we will refer to $\eta' := (\eta_1,\dots,\eta_{s'})$ as the *truncation* of $\eta = (\eta_1,\dots,\eta_{s'},\eta_{s'+1},\dots,\eta_s)$.

For each (V,η), let $E(V,\eta) := E(V) \cap C(V,\eta)$ and let $e(V,\eta) \in N_{\mathbf{R}}$ be a generator for the ray $E(V,\eta)$.

4.4. Adapted Systems of Cones

We define an *adapted system of cones* to be a collection of simplicial cones $\sigma(V,\eta)$, where V runs over the vertices in $T_{\mathrm{red}}(\phi)$ and η ranges over possible sign vectors, that satisfies the following conditions.

(A1) $\operatorname{Int}\sigma(V,\eta) \subseteq C(V,\eta)$ and $\sigma(V,\eta)$ spans V.

(A2) If $V' \subseteq V$ is the parent of V in $T_{\mathrm{red}}(\phi)$ and if $\pi : V \to V/V'$ is the natural projection, then $\sigma(V,\eta) \cap V' = \sigma(V',\eta')$, where η' is the truncation of η and $\pi(e(V,\eta)) \in \operatorname{Int}\pi(\sigma(V,\eta))$.

We say that the system is *rational* if all cones $\sigma(V,\eta)$ are rational and *invariant* (under ϕ) if each cone is invariant (under ϕ).

LEMMA 4.5. *Let $\mathcal{S} = \{\sigma(V,\eta)\}$ be an adapted system of cones, and let $v \in N$. Then there exists an $n_0 = n_0(v) \in \mathbf{N}$ such that, for $n \geq n_0$, $\phi^n(v) \in \sigma(V,\eta)$ for some $\sigma(V,\eta) \in \mathcal{S}$. More precisely, if $v \in C(V,\eta)$ then $\phi^n(v) \in \sigma(V,\eta)$ for $n \geq n_0$.*

Proof. From Section 4.3 we know that v is contained in a unique chamber $C(V,\eta)$. Let $\sigma = \sigma(V,\eta)$ be the corresponding cone in $\mathcal{S}$. Assume that (4.1) is the genealogy of V in $T_{\mathrm{red}}(\phi)$. Write $e_k := e(V_k,\eta^k)$ and $\sigma_k := \sigma(V_k,\eta^k)$, where η^k is the truncation of η. Then for $1 \leq k \leq s$ the cone $\sigma_k \in \mathcal{S}$ is of the form $\sigma_k = \sigma_{k-1} + \sum_{j=1}^{m_k} \mathbf{R}_+ v_{k,j}$, where $\sigma_0 = \{0\}$, $m_k := \dim V_k - \dim V_{k-1}$, and $v_{k,j} \in C(V_k,\eta^k)$, so that $\sigma = \sigma_s = \sum_{k=1}^{s}\sum_{j=1}^{m_k} \mathbf{R}_+ v_{k,j}$. Moreover, $\pi_k(e_k) \in \operatorname{Int}\pi_k\big(\sum_{j=1}^{m_k} \mathbf{R}_+ v_{k,j}\big)$, where $\pi_k : V_k \to V_k/V_{k-1}$ is the natural projection.

For $1 \leq k \leq s$, choose $x_k \in V^*$ such that $\langle x_k, e_j\rangle = \delta_{kj}$. We identify $(V/V_{k-1})^*$ with $\{\xi \in V^* \mid \xi|_{V_{k-1}} = 0\}$. Then ϕ^* induces a self-mapping on $(V/V_{k-1})^*$, and if $\langle \xi, e_k\rangle > 0$ then $\phi^{*n}\xi \to \mathbf{R}_+ x_k$ when $n \to \infty$. Indeed, the subspace $\mathbf{R}_+ x_k \subseteq (V/V_{k-1})^*$ is the one with the largest eigenvalue.

The dual cone σ^* of σ is of the form $\sigma^* = \sum_{k=1}^{s}\sum_{j=1}^{m_k} \mathbf{R}_+ \xi_{k,j}$, where $\xi_{k,j} \in \ker(V^* \to V_{k-1}^*) \cong (V/V_{k-1})^*$ and $\langle \xi_{k,j}, e_k\rangle > 0$; in particular, $\phi^{*n}\xi_{k,j} \to \mathbf{R}_+ x_k$.

Since $v \in C(V,\eta)$, we have $\langle x_k, v\rangle > 0$ for $1 \leq k \leq s$. By continuity, there is an $n_0 = n_0(v) \in \mathbf{N}$ such that $\langle \xi_{k,j}, \phi^n(v)\rangle = \langle \phi^{*n}\xi_{k,j}, v\rangle > 0$ for $1 \leq k \leq s$, $1 \leq j \leq m_k$, and $n \geq n_0$. Thus $\phi^n(v) \in \operatorname{Int}\sigma$ for $n \geq n_0$. □

REMARK 4.6. As can be seen from the proof, the first part of Lemma 4.5 remains valid if some of the eigenvalues $(\mu_i)_{i=1}^m$ of ϕ are negative provided that $|\mu_1| > \cdots > |\mu_m| > 0$. In general, if $v \in C(V)$ then, for $n \geq n_0$, $\phi^n(v) \in \sigma(V,\eta)$ for some sign vector η.

LEMMA 4.7. *Let $\mathcal{S} = \{\sigma(V,\eta)\}$ be an adapted system of cones. Then there exists an $n_0 \in \mathbf{N}$ such that $\mathcal{S}$ is invariant under ϕ^n for $n \geq n_0$.*

Proof. To each node V in $T_{\mathrm{red}}(\phi)$ we will associate an $n_0(V) \in \mathbf{N}$ such that $\phi^n(\sigma(V,\eta)) \subseteq \sigma(V,\eta)$ for all sign vectors η and $n \geq n_0(V)$; this is done by induction over $T_{\mathrm{red}}(\phi)$.

Set $n_0(\{0\}) = 0$. Let V be a node in $T_{\mathrm{red}}(\phi)$ such that $n_0(V')$ is defined, where V' is the parent of V. Pick a sign vector η and let η' be the truncation. Then $\sigma(V,\eta)$ is of the form $\sigma(V,\eta) = \sigma(V',\eta') + \sum_{j=1}^{m'} \mathbf{R}_+ v_j$ for some $v_j \in C(V,\eta)$ and $m' = \dim V - \dim V'$. From Lemma 4.5 we know that, for $1 \leq j \leq m'$, there is an $n_0(v_j) \in \mathbf{N}$ such that $\phi^n(v_j) \in \operatorname{Int}\sigma(V,\eta)$ for $n \geq n_0(v_j)$. Let $n_0(V,\eta) := \max_{1\leq j\leq m'} n_0(v_j)$, and let $n_0(V) := \max(n_0(V'), \max_\eta n_0(V,\eta))$.

Finally, set $n_0 := \max_{V\in T_{\mathrm{red}}(\phi)} n_0(V)$. Then n_0 has the desired properties. □

REMARK 4.8. Following the proof of Lemma 4.7, one can show that if the eigenvalues of ϕ satisfy $|\mu_1| > \cdots > |\mu_m| > 0$ then there exists an $n_0 \in \mathbf{N}$ such that, for $n \geq n_0$, ϕ^n maps each $\sigma(V,\eta)$ into $\sigma(V,\eta')$ for some sign vector η'.

The idea of the proofs of Theorems A and A′ is to refine Δ so that it contains an invariant (under ϕ and ϕ^n, respectively) adapted system of rational cones. Then the results follow by applying Corollary 2.3. First we need a few preliminary results on adapted systems of cones.

LEMMA 4.9. *Let Δ be a fan in N and write $V = N_{\mathbf{R}}$. Given $\sigma' \in \Delta$, let $V' = \operatorname{span}\sigma'$ and let $\pi : V \to V/V'$ be the natural projection. Then, for each $v \in V/V'$, there exists at most one σ such that $\sigma \supseteq \sigma'$ and $v \in \operatorname{Int}\pi(\sigma)$. If Δ is complete, there is a unique such σ.*

Proof. Let $\operatorname{Star}(\sigma') := \{\sigma \in \Delta \mid \sigma \supseteq \sigma'\}$. Then $\Delta_{\sigma'} := \{\pi(\sigma) \mid \sigma \in \operatorname{Star}(\sigma')\}$ is a fan in N/N', where N' is the sublattice of N generated by $\sigma' \cap N$; see [Fu1, Sec. 3.1]. If Δ is complete, then so is $\Delta_{\sigma'}$. Moreover, there is a one-to-one correspondence between the cones in $\operatorname{Star}(\sigma')$ and the cones in $\Delta_{\sigma'}$. In particular, there is at most one cone in $\operatorname{Star}(\sigma')$ such that $\pi(\sigma)$ contains v, and if Δ is complete then there exists such a σ. □

LEMMA 4.10. *Any fan Δ admits at most one rational adapted system of cones.*

Proof. Let V be a node in $T_{\mathrm{red}}(\phi)$ and let η be a sign vector. Note that the collection of cones in Δ that are contained in V form a fan. Suppose that $\sigma' \in \Delta$ spans $V' \subseteq V$. Then, by Lemma 4.9, there is at most one cone $\sigma \in \Delta$ such that $\sigma' \subseteq \sigma \subseteq V$ and $\operatorname{Int}\pi(\sigma) \ni \pi(e(V,\eta))$, where π is the projection $\pi : V \to V/V'$. Hence there is at most one cone $\sigma(V,\eta)$ satisfying conditions (A1) and (A2). □

LEMMA 4.11. *Let Δ be a fan in N that contains an adapted system of cones, and let Δ' be a refinement of Δ. Assume that, for every invariant rational subspace V of $N_{\mathbf{R}}$, there is a subfan of Δ' whose support equals V. Then Δ' contains a unique adapted system of cones.*

Proof. Uniqueness follows from Lemma 4.10. Assume that Δ and Δ' satisfy the assumptions of the lemma. Let $\mathcal{S} = \{\sigma(V,\eta)\}$ denote the adapted system of cones in Δ.

We will inductively find cones $\tau(V,\eta) \in \Delta'$ that satisfy (A1) and (A2). Let $\tau(\{0\},\eta) := \{0\}$. Let V be a node in $T_{\mathrm{red}}(\phi)$ with parent V', and let η be a sign vector. Assume that we have found $\tau' = \tau(V',\eta')$, where η' is the truncation of η. Note that $\sigma(V',\eta')$ is then the smallest cone in Δ that contains τ'. Let $\pi : V \to V/V'$ be the natural projection. Since there is a subfan of Δ' with support V, Lemma 4.9 asserts the existence of a unique cone $\tau \subseteq V$ that contains τ' and satisfies the condition $\pi(e(V,\eta)) \in \operatorname{Int}\pi(\tau)$. In particular, there is a $v \in \tau$ such that $\pi(v) = \pi(e(V,\eta))$; that is, $v = e(V,\eta) + v'$ for some $v' \in V'$. Thus $\operatorname{span}\tau$ contains $E(V)$ and, since it also contains V' and is rational, it follows that $\operatorname{span}\tau$ contains V. Hence τ spans V.

It remains to show $\operatorname{Int}\tau \subseteq C(V,\eta)$. Let Σ be the collection of cones in Δ' that are contained in $\sigma(V,\eta)$ and that contain τ'. Using notation as in the proof of Lemma 4.9, note that $\{\pi(\sigma) \mid \sigma \in \Sigma\}$ is a subfan of $\Delta_{\tau'}$ whose support equals $\pi(\sigma(V,\eta))$. Arguing as in that proof, since $\pi(e(V,\eta)) \in \pi(\sigma(V,\eta))$ there is a unique cone $\tilde{\sigma}$ in Σ such that $\pi(e(V,\eta)) \in \operatorname{Int}\pi(\tilde{\sigma})$. On the other hand, by Lemma 4.9, τ is the unique cone in Δ' that satisfies $\pi(e(V,\eta)) \in \operatorname{Int}\pi(\tau)$. Hence $\tau = \tilde{\sigma} \subseteq \sigma(V,\eta)$ and, since $\operatorname{span}\tau = \operatorname{span}\sigma(V,\eta)$, we have $\operatorname{Int}\tau \subseteq \operatorname{Int}\sigma(V,\eta) \subseteq C(V,\eta)$. To conclude, $\tau(\sigma,V) := \tau$ satisfies conditions (A1) and (A2). □

LEMMA 4.12. *There exists a rational adapted system of cones.*

Proof. We will construct rational cones $\sigma(V,\eta)$ inductively. First let $\sigma(\{0\},\eta) = \{0\}$. Now let $V \neq \{0\}$ be a node in $T_{\mathrm{red}}(\phi)$ and let η be a sign vector. Assume that $\sigma(V',\eta')$ is constructed, where V' is the parent of V in $T_{\mathrm{red}}(\phi)$ and η' is the truncation of η. Moreover, assume that the genealogy of $V' = V_s$ is given by (4.1). Write $m' := \dim V - \dim V'$ and $V = V' + \tilde{V}$, and pick $x \in M_{\mathbf{R}}$ such that $E(V)^{\perp} = \{x = 0\}$ and $x(e(V,\eta)) > 0$. For $1 \leq i \leq m'$, pick $\tilde{s}_i \in \tilde{V}$ such that $e(V,\eta) \in \operatorname{Int}\sum_{i=1}^{m'} \mathbf{R}_+\tilde{s}_i$ and $x(\tilde{s}_i) > 0$. Next, let v_i be a rational perturbation of

$$e(V_1,\eta^1) + \cdots + e(V_{s-1},\eta^{s-1}) + e(V_s,\eta^s) + \tilde{s}_i, \tag{4.2}$$

where the η^k are truncations of $\eta = \eta^s$. Since V is rational, we can find arbitrarily small such perturbations. Note that, provided the perturbation is small enough, $v_i \in C(V,\eta)$. Finally, let $\sigma(V,\eta) := \sigma(V',\eta') + \sum_{i=1}^{m'} \mathbf{R}_+ v_i$. If the perturbations v_i of (4.2) are small enough, then $\sigma(V,\eta)$ satisfies conditions (A1) and (A2). □

4.5. Invariant Adapted Systems of Real Cones

In this section we will construct a real (not necessarily rational) invariant adapted system of cones $\mathcal{G} = \{\Gamma(V,\eta)\}$. Later, in Section 4.8, we will perturb the cones in $\mathcal{G}$ into rational cones.

Let (4.1) be the genealogy of V in $T_{\mathrm{red}}(\phi)$ and pick a sign vector $\eta \in \{\pm 1\}^s$. For $1 \leq k \leq s$, let η^k be the truncation of η, let $e_k := e(V_k,\eta^k)$ with corresponding eigenvalue ν_k, and let $m_k := \dim V_k - \dim V_{k-1}$. Moreover, choose nonzero

eigenvectors $e_{k,i}$ labeled so that $v_{k,1} > \cdots > v_{k,m_k}$ and $e_k = e_{k,1}$. Then $V_k = V_{k-1} \oplus \tilde{V}_k$, where $\tilde{V}_k = \bigoplus_{i=1}^{m_k} \mathbf{R} e_{k,i}$.

Given parameters $\delta_1, \delta_2, \ldots, \delta_s > 0$ and $\varepsilon_2, \ldots, \varepsilon_s > 0$, let $\gamma_1 := 1$ and $\gamma_k := \varepsilon_2 \cdots \varepsilon_k$ for $k \geq 2$. For $1 \leq k \leq s$ and $1 \leq i \leq m_k$, set

$$v_{1,i} = e_1 + \delta_1 \tilde{v}_{1,i} \quad \text{and} \tag{4.3}$$

$$v_{k,i} = e_1 + 2^{-1}\gamma_2 e_2 + \cdots + 2^{2-k}\gamma_{k-1} e_{k-1} + 2^{2-k}\gamma_k (e_k + \delta_k \tilde{v}_{k,i}) \quad \text{if } k > 1, \tag{4.4}$$

where

$$\tilde{v}_{k,i} = \begin{cases} e_{k,2} + \cdots + e_{k,i} - e_{k,i+1} & \text{if } 1 \leq i < m_k, \\ e_{k,2} + \cdots + e_{k,m_k} & \text{if } i = m_k. \end{cases}$$

Here $\tilde{v}_{k,1}$ should be interpreted as being equal to $-e_{k,2}$ if $m_k \geq 2$ and equal to 0 otherwise. Note that $v_{k,i} \in V_k$ and $\tilde{v}_{k,i} \in \tilde{V}_k$. Also note that $v_{k,i}$ and $\tilde{v}_{k,i}$ depend on the sign vector η because the e_k do. Finally, note that $\sum_{i=1}^{m_k} \mathbf{R}_+ (e_k + \delta_k \tilde{v}_{k,i})$ is a simplicial real cone in $\tilde{V}_k$, of dimension $m_k = \dim \tilde{V}_k$, that contains e_k in its interior.

Now let

$$\Gamma(V_k, \eta^k) := \sum_{j=1}^{k} \sum_{i=1}^{m_j} \mathbf{R}_+ v_{j,i} = \Gamma(V_{k-1}, \eta^{k-1}) + \sum_{i=1}^{m_k} \mathbf{R}_+ v_{k,i} \subseteq V_k.$$

Observe that $\Gamma(V_k, \eta^k) \cap V_{k-1} = \Gamma(V_{k-1}, \eta^{k-1})$, since the coefficients of e_k in $v_{k,i}$ are positive for $1 \leq i \leq m_k$.

To show that $\Gamma(V, \eta) = \Gamma(V_s, \eta^s)$ satisfies conditions (A1) and (A2), we give a dual description of $\Gamma(V, \eta)$ in $V = V_s$. Let $\{x_{\ell,j}\}_{1 \leq \ell \leq s, 1 \leq j \leq m_\ell}$ be the basis of V^* dual to $\{e_{k,i}\}_{1 \leq k \leq s, 1 \leq i \leq m_k}$, so that $\langle x_{\ell,j}, e_{k,i} \rangle = 1$ if $\ell = k$ and $j = i$ and $\langle x_{\ell,j}, e_{k,i} \rangle = 0$ otherwise. Write $x_\ell := x_{\ell,1}$.

For $1 \leq \ell < j \leq s$, let $a_{\ell,j} = \varepsilon_{\ell+1}^{-1} \cdots \varepsilon_j^{-1}$ and let

$$\xi_{\ell,j} := \delta_\ell^{-1} \tilde{\xi}_{\ell,j} + x_\ell - (a_{\ell,\ell+1} x_{\ell+1} + \cdots + a_{\ell,s} x_s),$$

where

$$\tilde{\xi}_{\ell,j} := \begin{cases} x_{\ell,2} + 2x_{\ell,3} + \cdots + 2^{j-2} x_{\ell,j} - 2^{j-1} x_{\ell,j+1} & \text{if } 1 \leq j < m_\ell, \\ x_{\ell,2} + 2x_{\ell,3} + \cdots + 2^{m_\ell - 2} x_{\ell,m_\ell} & \text{if } j = m_\ell. \end{cases}$$

Here $\tilde{\xi}_{\ell,1}$ should be interpreted as $-x_{\ell,2}$ if $m_\ell \geq 2$ and as 0 otherwise.

A computation yields that $\langle \xi_{\ell,j}, v_{k,i} \rangle > 0$ if $\ell = k$ and $i = j$ and that $\langle \xi_{\ell,j}, v_{k,i} \rangle = 0$ otherwise, so the dual cone $\Gamma(V, \eta)^* = \sum_{\ell=1}^{s} \sum_{j=1}^{m_\ell} \mathbf{R}_+ \xi_{\ell,j}(\eta)$.

We claim that ϕ maps the open rays $\mathbf{R}_+^* v_{s,1}, \ldots, \mathbf{R}_+^* v_{s,m_s}$ into $\operatorname{Int} \Gamma(V, \eta)$. To prove the claim, observe first that

$$\langle \xi_{s,j}, \phi(v_{s,i}) \rangle = 2^{2-s} \gamma_s (v_s + \langle \tilde{\xi}_{s,j}, \phi(\tilde{v}_{s,i}) \rangle); \tag{4.5}$$

here

$$\langle \tilde{\xi}_{s,j}, \phi(\tilde{v}_{s,i}) \rangle = \begin{cases} v_{s,2} + \cdots + 2^{m_s - 2} v_{s,m_s} & \text{if } i = j = m_s, \\ v_{s,2} + \cdots + 2^{i-1} v_{s,i+1} & \text{if } i = j < m_s, \\ v_{s,2} + \cdots + 2^{I-2} v_{s,I} - 2^{I-1} v_{s,I+1} & \text{if } i \neq j, \end{cases}$$

where $I = \min(i, j)$. The second of these displayed cases should be interpreted as 0 if $i = j = 1$. Now the right-hand side of (4.5) is strictly positive because $\nu_s > \nu_{s,2} > \cdots > \nu_{s,m_s}$. Moreover, for $\ell < s$ we have

$$\langle \xi_{\ell,j}, \phi(v_{s,i}) \rangle = 2^{1-\ell}\gamma_\ell(\nu_\ell - 2^{-1}\nu_{\ell+1} - \cdots - 2^{\ell+1-s}\nu_{s-1} - 2^{\ell+1-s}\nu_s),$$

which is strictly positive since $\nu_1 > \cdots > \nu_s$.

To conclude, $\langle \xi_{\ell,j}, \phi(v_{s,i}) \rangle > 0$ for $1 \le \ell \le s$ and $1 \le j \le m_\ell$; hence we have proved that $\phi(\mathbf{R}_+^* v_{s,i})$ lies in the interior of $\Gamma(V, \eta)$. In particular, by induction, $\Gamma(V, \eta)$ is invariant under ϕ.

4.6. Preparation of the Fan

In order to prove Theorems A and A′, we first refine Δ so that, for each rational invariant subspace V (i.e., each node in $T_{\mathrm{red}}(\phi)$), there exists a subfan of Δ whose support is V. In particular, Δ is complete. This is possible because the rational rays are dense in V.

Next, we refine Δ so that it contains an adapted system of cones. This can be done as follows. Let $\mathcal{S}$ be a rational adapted system of cones, whose existence is guaranteed by Lemma 4.12. Let $\Delta_{\mathcal{S}}$ be the fan generated by the cones in $\mathcal{S}$, and let Δ' be a fan that refines both Δ and $\Delta_{\mathcal{S}}$. Then, by Lemma 4.11, Δ' contains an adapted system of cones.

Finally, by Lemma 1.2 we can refine Δ' so that it becomes regular and projective. The resulting fan will contain a unique adapted system of cones by Lemma 4.11.

4.7. Proof of Theorem A′

Let Δ' be the refined fan in Section 4.6 and let $\mathcal{S} = \{\sigma(V, \eta)\}$ denote the unique adapted system of cones. Consider $\rho \in \Delta'(1)$. According to Lemma 4.5 and Remark 4.6, there is an $n_0(\rho) \in \mathbf{N}$ such that, for $n \ge n_0(\rho)$, we have $\operatorname{Int} \phi^n(\rho) \subseteq \operatorname{Int} \sigma(V, \eta)$ for some $\sigma(V, \eta) \in \mathcal{S}$. Let $n_0 := \max_{\rho \in \Delta'(1)} n_0(\rho)$.

According to Lemma 4.5 and Remark 4.8, for $n \ge n_0$ (with n_0 possibly replaced by a larger number), V a node in $T_{\mathrm{red}}(\phi)$, and η a sign vector, we have $\phi^n(\sigma(V, \eta)) \subseteq \sigma(V, \eta')$ for some sign vector η'. Now Corollary 2.3 and Remark 2.4 assert that $f^n \colon X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable for $n \ge n_0$, which concludes the proof of Theorem A′.

4.8. Incorporation of Cones

We will now prove Theorem A. Given a fan Δ in N, replace Δ by the refined fan in Section 4.6 and let $\mathcal{S} = \{\sigma(V, \eta)\}$ be the (unique) adapted system of cones in Δ. We will construct and incorporate into Δ a rational invariant adapted system of cones $\mathcal{T} = \{\tau(V, \eta)\}$. This will be done inductively over the reduced tree $T_{\mathrm{red}}(\phi)$. In fact, the cones in $\mathcal{T}$ will be perturbations of the cones in the real invariant adapted system $\mathcal{G}$ constructed in Section 4.5.

Let V be a node in $T_{\mathrm{red}}(\phi)$ with genealogy (4.1). We will construct and incorporate cones $\tau(V, \eta)$ for all possible sign vectors η by inductively constructing and incorporating cones $\tau_k = \tau(V_k, \eta^k)$ for $1 \le k \le s$ and all possible choices of sign

vectors η^k. Let us use the notation from Section 4.5 and write $\Gamma_k := \Gamma(V_k, \eta_k)$. Moreover, let $w_k := e_1 + 2^{-1}\gamma_2 e_2 + \cdots + 2^{2-k}\gamma_{k-1}e_{k-1} + 2^{2-k}\gamma_k e_k$ and $u_k := e_1 + \cdots + 2^{2-k}\gamma_{k-1}e_{k-1} + 2^{1-k}\gamma_k e_k$, so that $w_k = u_{k-1} + 2^{2-k}\gamma_k e_k$.

Write $\sigma_1 := \sigma(V_1, \eta^1)$. Note that $w_1 = e_1 \in \operatorname{Int}\sigma_1$. By continuity we can choose δ_1 such that $v_{1,i} = w_1 + \delta_1 \tilde{v}_{1,i} \in \operatorname{Int}\sigma_1$ for $1 \le i \le m_1$. Moreover, since the rational rays are dense in V_1, we can find rational perturbations $t_{1,i}$ of $v_{1,i}$ such that $t_{1,i} \in \operatorname{Int}\sigma_1$ for $1 \le i \le m_1$. Write $\tilde{t}_{1,i} = t_{1,i} - v_{1,i}$, and let $\tau_1 = \sum_{i=1}^{m_1} \mathbf{R}_+ t_{1,i}$. Then τ_1 is a perturbation of Γ_1 and, if the $\tilde{t}_{1,i}$ are small enough, then $\phi(\operatorname{Int}\tau_1) \subseteq \operatorname{Int}\tau_1$ since $\phi(\operatorname{Int}\Gamma_1) \subseteq \operatorname{Int}\Gamma_1$. Also, τ_1 satisfies conditions (A1) and (A2) and $u_1 \in \operatorname{Int}\tau_1$. By Lemmas 1.1 and 1.3, we can find a simplicial and projective refinement Δ' of Δ such that $\tau_1 \in \Delta'$ and all cones in Δ that do not contain σ_1 are in Δ'. Replace Δ by Δ' and replace $\mathcal{S}$ by the unique adapted system of cones in Δ'. Such a system exists by Lemma 4.11.

Write $\sigma_2 := \sigma(V_2, \eta^2)$, let $\pi_1 \colon N_{\mathbf{R}} \to N_{\mathbf{R}}/V_1$ be the natural projection, and let $\operatorname{Star}(\tau_1) := \{\sigma \in \Delta \mid \sigma \supseteq \tau_1\}$. Since $u_1 \in \operatorname{Int}\tau_1$, it follows that $|\operatorname{Star}(\tau_1)|$ contains a neighborhood of u_1 in $N_{\mathbf{R}}$. In particular, $w_2 = u_1 + \gamma_2 e_2$ is in the interior of some cone in $\operatorname{Star}(\tau_1)$ if γ_2 is small enough; since $\pi_1(e_2) \in \pi_1(\sigma_2)$, this cone must be σ_2.

By continuity, we can choose δ_2 small enough that $v_{2,i} = w_2 + \delta_2\gamma_2\tilde{v}_{2,i} \in \operatorname{Int}\sigma_2$ for $1 \le i \le m_2$. Furthermore, we can replace $v_{2,i}$ by rational perturbations $t_{2,i} \in \operatorname{Int}\sigma_2$; write $\tilde{t}_{2,i} = v_{2,i} - t_{2,i}$. Now let $\tau_2 := \tau_1 + \sum_{i=1}^{m_2} \mathbf{R}_+ t_{2,i}$. Since the rays $v_{2,i}$ are mapped into the interior of Γ_2, we have $\phi(t_{2,i}) \in \operatorname{Int}\tau_2$ if the $\tilde{t}_{2,i}$ are small enough. Hence τ_2 is invariant. If the $\tilde{t}_{2,i}$ are small enough, then τ_2 satisfies conditions (A1) and (A2) in Section 4.4 and $u_2 = e_1 + 2^{-1}\gamma_2 e_2 \in \operatorname{Int}\tau_2$. Since $t_{2,i} \in \operatorname{Int}\sigma_2$ it follows that $\partial\sigma_2 \cap \partial\tau_2 = \tau_1$, which is a face of both σ_2 and τ_2. Thus, according to Lemmas 1.1 and 1.3, we can find a simplicial and projective refinement Δ' of Δ such that $\tau_2 \in \Delta'$ and such that the cones in Δ that do not contain σ_2 are in Δ'. Replace Δ by such a refinement and replace $\mathcal{S}$ by the new adapted system.

Inductively assume that we have constructed and incorporated τ_{k-1} so that $u_{k-1} = e_1 + \cdots + 2^{3-k}\gamma_{k-2}e_{k-2} + 2^{2-k}\gamma_{k-1}e_{k-1} \in \operatorname{Int}\tau_{k-1}$; here $\varepsilon_2, \ldots, \varepsilon_{k-1}$, and hence $\gamma_2, \ldots, \gamma_{k-1}$, are chosen along the way. By arguments as before we can choose ε_k (and hence γ_k) such that $w_k = u_{k-1} + 2^{2-k}\gamma_k e_k \in \operatorname{Int}\sigma_k$, where $\sigma_k := \sigma(V_k, \eta^k)$. Moreover, we can choose δ_k and $\tilde{t}_{k,i}$ such that $t_{k,i} := v_{k,i} + \tilde{t}_{k,i}$ are rational and contained in σ_k. Now let $\tau_k := \tau_{k-1} + \sum_{i=1}^{m_k} \mathbf{R}_+ t_{k,i}$. If $\tilde{t}_{k,i}$ are small enough, then τ_k is invariant and satisfies conditions (A1) and (A2) as well as $u_k \in \operatorname{Int}\tau_k$. Because $\partial\sigma_k \cap \partial\tau_k = \tau_{k-1}$ is a face of both σ_k and τ_k, we can incorporate τ_k into Δ according to Lemma 1.1; the resulting fan will have a unique adapted system of cones. By Lemma 1.3 we can choose the resulting fan projective.

We need to show that, when incorporating a cone $\tau(V, \eta) \in \mathcal{T}$ into Δ, we do not affect the cones in $\mathcal{T}$ already created and incorporated. So assume that $\hat{\tau} := \tau(\hat{V}, \hat{\eta})$ is in Δ. We claim that $\hat{\tau}$ is not affected when we incorporate τ. By Lemma 1.1 it suffices to show that $\hat{\tau}$ does not contain σ. If $\hat{V} = V$ but $\hat{\eta} = (\hat{\eta}_1, \ldots, \hat{\eta}_s) \ne \eta$, then $\hat{\tau} \not\supseteq \sigma$ because $\operatorname{Int}\hat{\tau} \subseteq C(V, \hat{\eta})$ and $\operatorname{Int}\sigma \subseteq C(V, \eta)$ are contained in different components of $C(V)$.

Therefore, assume that $\hat{V} \neq V$. Let $\{0\} = \hat{V}_0 \subsetneq \hat{V}_1 \subsetneq \cdots \subsetneq \hat{V}_r = \hat{V}$ be the genealogy of $\hat{V}$. By assumption, we have constructed and incorporated cones $\tau(\hat{V}_k, \hat{\eta}^k)$ for $1 \leq k \leq r$ and all possible sign vectors $\hat{\eta}^k$. Thus V is not among the $\hat{V}_k$. By construction, $\hat{\tau}$ is of the form $\hat{\tau} = \sum_{k=1}^{r} \sum_{i=1}^{m_k} \mathbf{R}_+ \hat{t}_{k,i}$, where $\hat{t}_{k,i} \in C(\hat{V}_k)$. From Lemma 4.4 we know that the smallest rational invariant subspace containing $\hat{t}_{j,i}$ is $\hat{V}_j$. Hence the smallest rational invariant subspace of $N_{\mathbf{R}}$ containing a given face of $\hat{\tau}$ is among the $\hat{V}_k$. Since the smallest rational invariant subspace containing σ is V, we conclude that σ is not a face of $\hat{\tau}$. Thus the claim is proved.

4.9. Incorporation of Rays

Let Δ be the fan in Section 4.8 and let $\mathcal{T}$ be the rational adapted systems of cones. We claim that we can find a further refinement Δ' of Δ such that, if ρ is a ray in Δ' and if $n \geq 1$, then either $\phi^n(\rho) \in \Delta'(1)$ or $\phi^n(\rho)$ is contained in a cone in $\mathcal{T}$. By Corollary 2.3, $f\colon X(\Delta') \dashrightarrow X(\Delta')$ is then 1-stable, which proves Theorem A.

It remains to prove the claim. Since (by Lemma 4.5) every ray in Δ is eventually mapped into one of the cones in $\mathcal{T}$, it is enough to add to Δ the finitely many rays of the form $\phi^n(\rho)$, where $\rho \in \Delta(1)$ and $\phi^n(\rho)$ is not contained in any of the cones in $\mathcal{T}$.

Let $\rho' = \phi^n(\rho)$ be such a ray, and let σ' be the unique cone in Δ such that $\operatorname{Int}\rho' \subseteq \operatorname{Int}\sigma'$. By Lemma 1.1, we can find a refinement Δ' of Δ such that $\rho' \in \Delta'(1)$ and $(\Delta'(1) \setminus \rho') \subseteq \Delta(1)$ and such that, if $\sigma \in \Delta$ does not contain σ', then $\sigma \in \Delta'$. Moreover, by Lemma 1.3, Δ' can be chosen projective. By assumption, ρ' is not contained in any cone in $\mathcal{T}$; hence σ' cannot be a face of a cone in $\mathcal{T}$. Thus, all cones in $\mathcal{T}$ are in Δ'.

This proves the claim and thereby concludes the proof of Theorem A.

5. Proof of Theorem B

Let $E \subseteq N_{\mathbf{R}}$ be the 1-dimensional eigenspace of ϕ associated with μ_1, choose $x \in M_{\mathbf{R}}$ such that $E^{\perp} = \{x = 0\}$, and let e be a generator of E such that $x(e) > 0$.

By techniques similar to those in Sections 4.5 and 4.8, we can choose $v_1, \ldots, v_m \in N$ such that $x(v_j) > 0$ for $1 \leq j \leq m$, e lies in the interior of the cone $\sigma := \sum_{j=1}^{m} \mathbf{R}_+ v_j$, and σ is invariant. Let $\Delta := \left\{\sum_{j=1}^{m} \mathbf{R}_+ \varepsilon_j v_j\right\}_{\varepsilon_j \in \{0,-1,+1\}^m}$. Then Δ is a complete simplicial fan. The cones σ and $\sum_{j=1}^{m} \mathbf{R}_+(-v_j)$ are invariant, and all rays in Δ are mapped into one of these cones. Thus Corollary 2.3 asserts that $f\colon X(\Delta) \dashrightarrow X(\Delta)$ is 1-stable. Also, Δ admits a strictly convex Δ-linear support function of the form $\max_j |v_j^*|$, so $X(\Delta)$ is projective; see Section 1.3. This completes the proof of Theorem B.

We have the following partial analogue of Theorem A′.

THEOREM B′. *Let $f\colon (\mathbf{C}^*)^m \to (\mathbf{C}^*)^m$ be a monomial map. Suppose that the associated eigenvalues satisfy $|\mu_1| > |\mu_2| \geq |\mu_3| \geq \cdots \geq |\mu_m| > 0$. Then there exist a complete simplicial fan Δ' and an $n_0 \in \mathbf{N}$ such that $X(\Delta')$ is projective and $f^n\colon X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable for $n \geq n_0$.*

Proof. Let E, e, and x be as in the proof of Theorem B. Choose $v_1, \dots, v_m \in N$ such that $x(v_j) > 0$ and $e \in \operatorname{Int} \sum_{j=1}^m \mathbf{R}_+ v_j$, and construct a fan Δ as in the proof of Theorem B. Then there is an $n_0 \in \mathbf{N}$ such that, for $n \geq n_0$, the union of $\sum_{j=1}^m \mathbf{R}_+ v_j$ and $\sum_{j=1}^m \mathbf{R}_+(-v_j)$ is invariant under ϕ^n; in particular, ϕ^n maps all rays in Δ into one of these cones. Now Theorem B′ follows from Corollary 2.3. □

REMARK 5.1. If we could find a regular refinement of Δ' not containing any rays in $E^\perp$, then we would get a smooth toric variety on which f^n is 1-stable as in Theorem B′. However, when regularizing Δ' it seems difficult to control where the new rays appear; compare Section 3.

By slightly modifying the proof of Theorem A, we could solve the problem of making $f\colon X(\Delta) \dashrightarrow X(\Delta)$ 1-stable in more general situations than the one in Theorem A. Let us mention a result in the same vein as Theorem B.

PROPOSITION 5.2. *Let Δ be a (complete) simplicial fan in a lattice N, and let $f\colon X(\Delta) \dashrightarrow X(\Delta)$ be a monomial map. Assume that the associated eigenvalues satisfy $\mu_1 > \mu_2 \geq \cdots \geq \mu_m > 0$.*

Let E be the 1-dimensional eigenspace of ϕ associated with μ_1, and let e be a generator of E. Assume that there are cones $\sigma^+, \sigma^- \in \Delta(m)$ such that $E^\perp \cap \sigma^\pm = \{0\}$, $\pm e \in \operatorname{Int} \sigma^\pm$, and $E^\perp$ contains no rays of Δ. Then there exists a simplicial refinement Δ' of Δ such that $f\colon X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable. If Δ is projective, then Δ' can be chosen projective.

Proof. Following Sections 4.5 and 4.8, we can find rational invariant simplicial cones τ^+ and τ^- of dimension m such that $\tau^\pm \subseteq \sigma^\pm$ and $\pm e \in \operatorname{Int} \tau^\pm$. By Lemma 1.1 we can incorporate $\tau^\pm$ into Δ without adding extra rays.

Since, by assumption, $E^\perp$ contains no rays of Δ, it follows that all rays of Δ are eventually mapped into τ^+ or τ^-. Following Section 4.9, we can incorporate the rays $\phi^n(\rho)$—where $\rho \in \Delta$ and $\phi^n(\rho)$ is not contained in $\tau^\pm$—into Δ. More precisely, we can find a simplicial refinement Δ' of Δ such that $\tau^\pm \in \Delta'$ and such that each ray in Δ' is either mapped onto another ray in Δ' or into τ^+ or τ^-. Now $f\colon X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable by Corollary 2.3.

By Lemma 1.3, $X(\Delta')$ in Proposition 5.2 can be chosen projective provided that Δ is projective (cf. Section 4). □

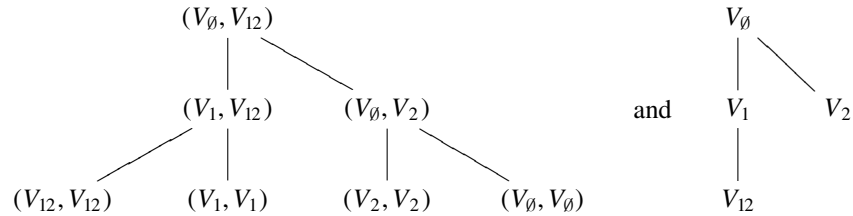
Observe that, in light of this proof, the way of constructing the fan Δ' in the proof of Theorem A is far from optimal in the sense that, in general, we refine Δ more than necessary. Indeed, if we followed the strategy in Section 4, we would typically start out by adding rays inside the hyperplane $E^\perp$; see Section 4.6.

6. Examples

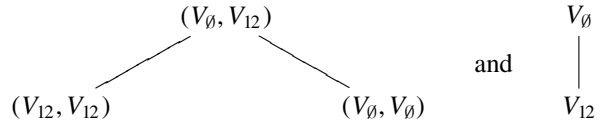
We now illustrate our method for proving Theorem A in dimensions 2 and 3. We also give examples illustrating the difficulties when the eigenvalues have different signs.

Let μ be an eigenvalue of $\phi\colon N \to N$. Recall that either $\mu \in \mathbf{Z}$ or $\mu \notin \mathbf{Q}$. Suppose that μ is a simple eigenvalue with corresponding 1-dimensional eigenspace E. If $\mu \in \mathbf{Z}$ then E and $E^\perp$ are rational. On the other hand, if $\mu \notin \mathbf{Q}$ then E is not rational.

EXAMPLE 6.1. Let $N \cong \mathbf{Z}^2$ and let $\phi\colon N \to N$ be a $\mathbf{Z}$-linear map with eigenvectors $\mu_1 > \mu_2 > 0$ and corresponding eigenspaces E_1, E_2. Then either $\mu_1, \mu_2 \in \mathbf{Z}$ or $\mu_1, \mu_2 \notin \mathbf{Q}$. In the first case, E_1 and E_2 are both rational and thus $T(\phi)$ and $T_{\mathrm{red}}(\phi)$ are given by

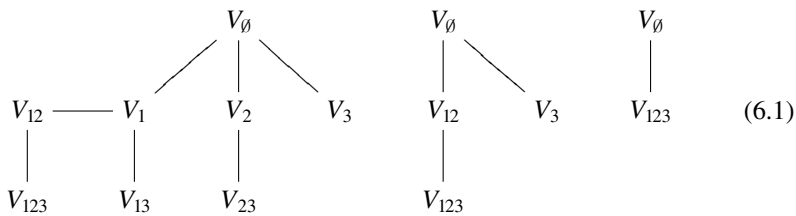


respectively. Here $V_{\mathcal{I}} = \sum_{i\in\mathcal{I}} E_i$ for $\mathcal{I} \subseteq \{1,2\}$; in particular, $V_\emptyset = \{0\}$ and $V_{12} = N_{\mathbf{R}}$. In the second case, neither E_1 nor E_2 is rational and so the trees are given by



In the first case, the associated chambers are given by $C(V_\emptyset) = \{0\}$, $C(V_j) = V_j \setminus \{0\}$, and $C(V_{12}) = N_{\mathbf{R}} \setminus (V_1 \cup V_2)$. In the second case, $C(V_\emptyset) = \{0\}$ and $C(V_{12}) = N_{\mathbf{R}} \setminus V_2$. Note that $N \cap C(V_{12}) = N \setminus \{0\}$.

EXAMPLE 6.2. Let $N \cong \mathbf{Z}^3$ and let $\phi\colon N \to N$ be a $\mathbf{Z}$-linear map with eigenvalues $\mu_1 > \mu_2 > \mu_3 > 0$. Depending on whether or not the eigenvalues are rational, there are five possibilities for $T_{\mathrm{red}}(\phi)$, of which three are the following:



(6.1)

Here we have used the notation from Example 6.1. The first tree in (6.1) is obtained when all eigenvalues are integers, the second tree when μ_3 is the unique integer eigenvalue, and the third tree when all eigenvalues are irrational. If μ_1 or μ_2 is the unique integer eigenvalue, we get a tree of the same structure as the second tree but with V_{12} replaced by V_1 or V_{13} (respectively) and V_3 replaced by V_{23} or V_2 (respectively).

If some of the eigenvalues of ϕ are negative then stabilization may not be possible, as the following example shows.

EXAMPLE 6.3. Let $N \cong \mathbf{Z}^3$ and assume that $\phi\colon N \to N$ is a $\mathbf{Z}$-linear map with real, irrational eigenvalues satisfying $\mu_1 > -\mu_2 > -\mu_3 > 0$ and $\mu_1+\mu_2+\mu_3 < 0$. Then there is no simplicial fan Δ on which ϕ is torically stable.

Indeed, let Δ be any complete simplicial fan and let $\sigma_1 \in \Delta$ be a cone containing a nonzero eigenvector associated to the eigenvalue μ_1 in its interior. Assume that ϕ is torically stable on Δ. It then follows from Corollary 2.2 that $\phi(\sigma_1) \subseteq \sigma_1$. Since the only invariant rational subspaces of $N_{\mathbf{R}}$ are $\{0\}$ and $N_{\mathbf{R}}$, it follows that σ_1 must have dimension 3. By Proposition 2.5, this contradicts the assumption $\mu_1 + \mu_2 + \mu_3 < 0$.

A concrete example is given by ϕ associated to the matrix $A = A_\phi = \left[\begin{smallmatrix} 3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{smallmatrix}\right]$. Then $\mu_1 \approx 3.1997$, $\mu_2 \approx -3.0855$, and $\mu_3 \approx -1.1142$.

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