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Normal Forms, Hermitian Operators, and CR Maps of Spheres and Hyperquadrics

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1. Introduction

The purpose of this paper is to use normal forms of Hermitian operators in the study of CR maps. We strengthen a useful link between linear algebra and several complex variables and apply the techniques discussed to the theory of rational CR maps of spheres and hyperquadrics. After discussing the classification of such CR maps in terms of their Hermitian forms, we turn to the classification of Hermitian forms arising from degree-2 rational CR maps of spheres and hyperquadrics. For degree-2 maps, the classification is the same as that of a pair of Hermitian forms up to simultaneous $*$-congruence (matrices A and B are $*$-congruent if there exists a nonsingular matrix X such that $X^*AX = B$, where X^* is the conjugate transpose), which is a classical problem in linear algebra whose solution dates to the 1930s (see the survey [18]).

We will apply the theory developed in this paper to two problems. First, extending a result of Faran [12], in Theorem 1.2 we finish the classification of all CR maps of hyperquadrics in dimensions 2 and 3. Second, Ji and Zhang [17] classified degree-2 rational CR maps of spheres from source dimension 2. In Theorem 1.5 we extend this result to arbitrary source dimension and give an elegant version of the theorem by proving that all degree-2 rational CR maps of spheres in any dimension are spherically equivalent to a monomial map. We also study the real-algebraic version of the CR maps of hyperquadrics problem in dimensions 2 and 3, which arises in the case of diagonal Hermitian forms.

In CR geometry we often think of a real-valued polynomial $p(z, \bar{z})$ on complex space as the composition of the Veronese map $\mathcal{Z}$ with a Hermitian form B. That is, $p(z, \bar{z}) = \langle B\mathcal{Z}, \mathcal{Z}\rangle$. See Section 2 for more on this setup. Writing B as a sum of rank-1 matrices, we find that p can also be viewed as the composition of a holomorphic map composed with a diagonal Hermitian form. When we divide the form $\langle B\mathcal{Z}, \mathcal{Z}\rangle$ by the defining equation of the source hyperquadric, the result is a pair of Hermitian forms. When this pair is put into canonical form, we obtain a canonical form of the map up to a natural equivalence relation. A crucial point is that, for degree-2 maps, the two forms are linear in z.

We thus make a connection between real polynomials and holomorphic maps to hyperquadrics. A hyperquadric is the zero set of a diagonal Hermitian form and

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is a basic example of a real hypersurface in complex space. Usually given in non-homogeneous coordinates, the hyperquadric $Q(a,b)$ is defined as

$$Q(a,b) := \{z \in \mathbb{C}^{a+b} \mid |z_1|^2 + \cdots + |z_a|^2 - |z_{a+1}|^2 - \cdots - |z_{a+b}|^2 = 1\}. \quad (1)$$

Note that $Q(a,b)$ is a hypersurface only when $a \geq 1$. Also, $Q(n,0)$ is the sphere S^{2n-1}.

We introduce the natural notion of equivalence for CR maps of hyperquadrics, which we will call Q-equivalence. A map $f\colon Q(a,b) \to Q(c,d)$ is CR if it is continuously differentiable and satisfies the tangential Cauchy–Riemann equations. A real-analytic CR map is a restriction of a holomorphic map. See [2; 6] for more information. Let $U(a,b)$ be the set of automorphisms of the complex projective space $\mathbb{P}^{a+b}$ that preserve $Q(a,b)$. In homogeneous coordinates these automorphisms are invertible matrices, or linear fractional when working in $\mathbb{C}^{a+b}$. We will say that two CR maps f and g taking $Q(a,b)$ to $Q(c,d)$ are *Q-equivalent* if there exist $\tau \in U(c,d)$ and $\chi \in U(a,b)$ such that $f \circ \chi = \tau \circ g$. In the case of spheres, Q-equivalence is commonly called *spherical equivalence*.

The problem of classifying CR maps of hyperquadrics has a long history. For the sphere case, see for example [4; 5; 6; 8; 9; 10; 11; 16] and the references therein. For the general hyperquadric case, see [1; 3; 16] and the references therein.

Again, let $f\colon Q(a,b) \to Q(c,d)$ be a CR map. We will first study the case $a+b=2$ and $2 \leq c+d \leq 3$. We note that $Q(2,0)$ is equivalent to $Q(1,1)$ by the map $(z,w) \mapsto (1/z, w/z)$. Hence we need only consider $Q(2,0)$ as our source. Similarly, $Q(1,2)$ is equivalent to $Q(3,0)$. If $c+d=2$ then we need only consider maps from ball to ball, which are Q-equivalent to the identity by a theorem of Pinčuk [19]. Faran classified all planar maps in [12] and used this result to classify all ball maps in dimensions 2 and 3.

THEOREM 1.1 (Faran [12]). *Let $U \subset Q(2,0)$ be connected and open. Let $f\colon U \to Q(3,0)$ be a nonconstant C^3 CR map. Then f is Q-equivalent to exactly one of the following maps*:

(i) $(z,w) \mapsto (z,w,0)$;
(ii) $(z,w) \mapsto (z,zw,w^2)$;
(iii) $(z,w) \mapsto (z^2, \sqrt{2}zw, w^2)$; *or*
(iv) $(z,w) \mapsto (z^3, \sqrt{3}zw, w^3)$.

Although Faran stated this theorem while assuming that the map is defined on all of $Q(2,0)$, the slight generalization that we give follows from the same proof (or by appealing to the theorem of Forstnerič [14]). To completely classify all CR maps of hyperquadrics in dimensions 2 and 3, it remains to classify maps from $Q(2,0)$ to $Q(2,1)$.

THEOREM 1.2. *Let $U \subset Q(2,0)$ be connected and open. Let $f\colon U \to Q(2,1)$ be a nonconstant real-analytic CR map. Then f is Q-equivalent to exactly one of the following maps*:

(i) $(z,w)\mapsto(z,w,0)$;
(ii) $(z,w)\mapsto(z^2,\sqrt{2}w,w^2)$;
(iii) $(z,w)\mapsto\left(\frac{1}{z},\frac{w^2}{z^2},\frac{w}{z^2}\right)$;
(iv) $(z,w)\mapsto\left(\frac{z^2+\sqrt{3}zw+w^2-z}{w^2+z+\sqrt{3}w-1},\frac{w^2+z-\sqrt{3}w-1}{w^2+z+\sqrt{3}w-1},\frac{z^2-\sqrt{3}zw+w^2-z}{w^2+z+\sqrt{3}w-1}\right)$;
(v) $(z,w)\mapsto\left(\frac{\sqrt[4]{2}(zw-iz)}{w^2+\sqrt{2}iw+1},\frac{w^2-\sqrt{2}iw+1}{w^2+\sqrt{2}iw+1},\frac{\sqrt[4]{2}(zw+iz)}{w^2+\sqrt{2}iw+1}\right)$;
(vi) $(z,w)\mapsto\left(\frac{2w^3}{3z^2+1},\frac{z^3+3z}{3z^2+1},\sqrt{3}\frac{wz^2-w}{3z^2+1}\right)$; *or*
(vii) $(z,w)\mapsto(1,g(z,w),g(z,w))$ *for an arbitrary CR function* g.

Maps (i), (ii), and (iii) originate in monomial maps (where each component is a monomial) from $Q(2,0)$ or $Q(1,1)$. Classifying monomial maps leads to a problem in real-algebraic geometry, which we discuss in Section 5.

REMARK 1.3. It is interesting that maps (iii) and (iv) of Theorem 1.1 and (ii) of Theorem 1.2 are group invariant, but this fact is not used in the paper. Map (ii) of Theorem 1.2 is one of the family of CR maps of hyperquadrics invariant under a cyclic group obtained by D'Angelo [7].

REMARK 1.4. Faran proved his theorem with C^3 regularity. In order to apply Faran's result on classification of planar maps, we shall need real-analytic CR maps. For maps from the sphere to the $Q(2,1)$ hyperquadric we get the map $(1,g,g)$ for an arbitrary CR function g. Obviously, this map can have arbitrarily bad regularity.

We will also prove the following generalization of a theorem by Ji and Zhang [17]. The form of their maps was found by D'Angelo [4] and Huang, Ji, and Xu [16]. The statement and proof of the theorem by Ji and Zhang was more involved, and it covered only the case of source dimension 2. A *monomial map* is a map whose every component is a monomial. We do not allow any monomial to have negative exponents. The *degree* of a rational map is the maximum degree of the numerator and the denominator when the map is written in lowest terms.

The notion of spherical equivalence can be naturally extended to maps with different target dimensions. We will say two maps f and g with different target dimensions (e.g., the target dimension of f is smaller) are *spherically equivalent* if $f\oplus 0$ is spherically equivalent to g in the usual sense.

THEOREM 1.5. *Let* $f\colon S^{2n-1}\to S^{2N-1}$, $n\geq 2$, *be a rational CR map of degree 2. Then* f *is spherically equivalent to a monomial map.*

In particular, f *is equivalent to a map taking* $(z_1,\dots,z_n)$ *to*

$$\begin{aligned}\Big(&\sqrt{t_1}z_1,\sqrt{t_2}z_2,\dots,\sqrt{t_n}z_n,\sqrt{1-t_1}z_1^2,\sqrt{1-t_2}z_2^2,\dots,\sqrt{1-t_n}z_n^2,\\&\sqrt{2-t_1-t_2}z_1z_2,\sqrt{2-t_1-t_3}z_1z_3,\dots,\sqrt{2-t_{n-1}-t_n}z_{n-1}z_n\Big),\\&0\leq t_1\leq t_2\leq\cdots\leq t_n\leq 1,\quad (t_1,t_2,\dots,t_n)\neq(1,1,\dots,1).\end{aligned}\tag{2}$$

Furthermore, all maps of the form (2) *are mutually spherically inequivalent for different parameters* $(t_1,\dots,t_n)$.

It is worthwhile and generally more convenient to see what the maps (2) look like in more abstract language. D'Angelo [6] has shown that any degree-2 polynomial map taking the origin to the origin can be abstractly written as

$$Lz \oplus \left(\sqrt{I - L^*L}\, z\right) \otimes z, \tag{3}$$

where L is any linear map such that $I - L^*L$ is positive semidefinite and where z is the identity map in nonhomogeneous coordinates. A monomial map then corresponds to taking a diagonal L. We can make L have nonnegative entries, and by permuting the variables we can sort the diagonal entries of L. Hence our result could be stated as follows. *Every nonconstant degree-2 rational CR map of spheres is spherically equivalent to exactly one map of the form* (3), *where L is diagonal with nonnegative diagonal entries sorted by size, such that $I - L^*L$ also has nonnegative entries.* With the result of D'Angelo it is obvious that (3) and hence (2) is exhaustive; in other words, each gives all the monomial maps. We will give another proof.

The main point is that all degree-2 CR maps of spheres are spherically equivalent to monomial maps. Faran, Huang, Ji, and Zhang [13] have shown that, for $n = 2$, all degree-2 rational CR maps of spheres are equivalent to polynomial maps; hence we improve on their result by allowing the source dimension to be arbitrary and showing that the maps are actually monomial, not just polynomial. An explicit example is also given in [13] of a rational degree-3 CR map of S^3 to S^7 that is not spherically equivalent to any polynomial map. Therefore, our result is optimal in some sense. See also Proposition 4.7 for a slightly weaker statement.

The family of maps (2) in the simplest case $n = 2$ consists of the following maps that take S^3 to S^9:

$$(z, w) \mapsto \left(\sqrt{s}\, z, \sqrt{t}\, w, \sqrt{1-s}\, z^2, \sqrt{2-s-t}\, zw, \sqrt{1-t}\, w^2\right), \qquad 0 \leq s \leq t \leq 1, \quad (s,t) \neq (1,1). \tag{4}$$

This family has appeared in the work of Wono [20], who classified all monomial maps from S^3 to S^9. Note that $s = 0$ corresponds to maps of type (I) in [17]; $t = 1$ corresponds to type (IIA), and the other cases correspond to type (IIC). Since there is only one 2-dimensional family, it is not hard to see that this family corresponds to the maps (IIC) with the maps of type (I) and (IIA) located on the "boundary" of the family.

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2. Hermitian Forms

Let $\langle \cdot, \cdot \rangle$ denote the standard pairing $\langle z, w \rangle = z_0\bar{w}_0 + z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$, and let A be a Hermitian matrix. Then a Hermitian form is simply

$$\langle Az, z \rangle, \tag{5}$$

where $z \in \mathbb{C}^{n+1}$. We will often talk of the zero sets of Hermitian forms, so let us define

$$\mathbb{V}_A := \{z \in \mathbb{C}^{n+1} \mid \langle Az, z \rangle = 0\}. \tag{6}$$

In CR geometry we often think of a real polynomial as a Veronese map composed with a Hermitian form. Let p be a polynomial in $(z, w) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$:

$$p(z, w) = \sum_{\alpha\beta} a_{\alpha\beta} z^\alpha w^\beta. \tag{7}$$

Suppose that p is bihomogeneous of bidegree (d, d): let $p(tz, w) = p(z, tw) = t^d p(z, w)$. The polynomial is said to be *Hermitian symmetric* if $p(z, \bar{w}) = \overline{p(w, \bar{z})}$. In other words, $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$; that is, the matrix with the entries $a_{\alpha\beta}$ is Hermitian. It is not hard to see that p is Hermitian symmetric if and only if $p(z, \bar{z})$ is real valued.

Let $\mathcal{Z} = \mathcal{Z}_d$ be the degree-d Veronese map

$$(z_0, \ldots, z_n) \overset{\mathcal{Z}}{\mapsto} (z_0^d, z_0^{d-1} z_1, \ldots, z_n^d), \tag{8}$$

or the map whose components are all the degree-d monomials. We can think of p as

$$p(z, \bar{z}) = \langle A\mathcal{Z}, \mathcal{Z} \rangle, \tag{9}$$

where $A = [a_{\alpha\beta}]_{\alpha\beta}$ is the matrix of coefficients from (7). By the *signature* of p we mean the signature of A. Writing A as a sum of rank-1 matrices, we show that p is the composition of a diagonal Hermitian form with the same signature as A and a homogeneous holomorphic map of $\mathbb{C}^{n+1}$ to some $\mathbb{C}^{N+1}$. That is, we obtain a map taking the zero set of p to a hyperquadric.

Let f be a rational map taking $\mathbb{P}^n$ to $\mathbb{P}^N$. In homogeneous coordinates, f is given by $N+1$ homogeneous polynomials $f = f_0, f_1, \ldots, f_N$. We wish to formulate what it means for f to take $\mathbb{V}_J \subset \mathbb{P}^n$ to $\mathbb{V}_V \subset \mathbb{P}^N$ for two Hermitian matrices J and V. We can simply plug f into the equation for $\mathbb{V}_V$ and obtain

$$\langle Vf(z), f(z) \rangle = 0. \tag{10}$$

Then f takes $\mathbb{V}_J$ to $\mathbb{V}_V$ if and only if there exists a bihomogeneous real polynomial q such that

$$\langle Vf(z), f(z) \rangle = q(z, \bar{z}) \langle Jz, z \rangle. \tag{11}$$

To classify the f that take $\mathbb{V}_J$ to $\mathbb{V}_V$, we first need to identify those f that differ by an automorphism preserving $\mathbb{V}_V$.

It will be useful to represent $\langle Vf(z), f(z)\rangle$ slightly differently. Suppose that f is of degree d, and let $\mathcal{Z}$ be the degree-d Veronese map. Then we can write $f(z) = F\mathcal{Z}$ for some complex matrix F. We put

$$\langle Vf(z), f(z)\rangle = \langle VF\mathcal{Z}, F\mathcal{Z}\rangle = \langle F^*VF\mathcal{Z}, \mathcal{Z}\rangle. \tag{12}$$

LEMMA 2.1. *Let V be a nonsingular Hermitian matrix. Let f and g be homogeneous polynomial maps taking $\mathbb{C}^{n+1}$ to $\mathbb{C}^{N+1}$ such that*

$$\langle Vf(z), f(z)\rangle = \langle Vg(z), g(z)\rangle. \tag{13}$$

Suppose that f has linearly independent components. Then

$$g(z) = Cf(z) \tag{14}$$

for some invertible matrix C such that $C^*VC = V$.

Proof. As explained previously, we write $f(z) = F\mathcal{Z}$ for some complex matrix F with linearly independent rows. Similarly, $g(z) = G\mathcal{Z}$. Thus $\langle Vf(z), f(z)\rangle = \langle Vg(z), g(z)\rangle$ implies that

$$\langle (F^*VF - G^*VG)\mathcal{Z}, \mathcal{Z}\rangle = 0. \tag{15}$$

We have a real polynomial that is identically zero and so its coefficients are zero. Thus $F^*VF = G^*VG$. By permuting the monomials, we could suppose that $F = [F_1 \;\; F_2]$, where F_1 is an invertible matrix. We write $G = [G_1 \;\; G_2]$, where G_1 is square. Now $F^*VF = G^*VG$ implies $F_1^*VF_1 = G_1^*VG_1$. Let $C = G_1F_1^{-1}$. Clearly $C^*VC = V$ and $G_1 = CF_1$. So if $G_2 = CF_2$, then we are finished. Now $F^*VF = G^*VG$ also implies that $F_1^*VF_2 = G_1^*VG_2$. Replacing G_1 with CF_1 yields $F_1^*VF_2 = F_1^*VC^{-1}G_2$. Since F_1 and V are invertible, it follows that $G_2 = CF_2$ and thus $g(z) = Cf(z)$. □

Lemma 2.1 says that if $\langle Vf(z), f(z)\rangle = \langle Vg(z), g(z)\rangle$, then f and g are equal up to a linear map of the target space preserving the form defined by V. Note that these linear maps (up to a scalar multiple) correspond exactly to linear fractional transformations that take $\mathbb{V}_V$ to itself and preserve the sides of $\mathbb{V}_V$. Because we are working with homogeneous coordinates, we must also always consider the possibility $\langle Vf(z), f(z)\rangle = \lambda\langle Vg(z), g(z)\rangle$ for $\lambda > 0$. We can then rescale f or g and use the proposition. Furthermore, if there are equal numbers of positive and negative eigenvalues, then there exists a linear map that takes the form corresponding to V to the form corresponding to $-V$. In this case, it could be that $\langle Vf(z), f(z)\rangle = -\langle Vg(z), g(z)\rangle$ and still f and g differ by an automorphism of the target and take the same set to $\mathbb{V}_V$, with one swapping sides and the other not.

For the source and target we are mostly interested in hyperquadrics and spheres. When V is Hermitian, the set $\mathbb{V}_V$ is equivalent to a hyperquadric by an automorphism of $\mathbb{P}^N$. For the hyperquadric $Q(a,b)$ we let V be the matrix with a ones and $b+1$ negative ones on the diagonal. We have the following corollary.

COROLLARY 2.2. *Let f and g be rational CR maps of $Q(a,b)$ to $Q(c,d)$, and let V be a Hermitian form defining $Q(c,d)$. Let $\hat{f}$ and $\hat{g}$ be the corresponding*

homogeneous polynomial maps. Suppose that the components of $\hat{f}$ *are linearly independent. Then* f *and* g *are Q-equivalent if and only if there exist* χ *and* λ, *with* $\chi \in U(a,b)$ *and with either* $\lambda > 0$ *if* $c \neq d+1$ *or* $\lambda \in \mathbb{R} \setminus \{0\}$ *if* $c = d+1$, *such that*

$$\langle V\hat{f}(\chi z), \hat{f}(\chi z)\rangle = \lambda \langle V\hat{g}(z), \hat{g}(z)\rangle \quad \textit{for all } z. \tag{16}$$

Consequently, by working in projective space and with Hermitian forms rather than with the maps themselves, we reduce the Q-equivalence problem to an equivalence problem using only the group of automorphisms of the source.

Now suppose we start with any real polynomial $\langle B\mathcal{Z}_d, \mathcal{Z}_d\rangle$ for some Hermitian matrix B. We write B as a sum of rank-1 matrices. Since B is Hermitian, this can be done in the following way. We take the positive eigenvalues $\lambda_1, \ldots, \lambda_k$ and the negative eigenvalues $-\lambda_{k+1}, \ldots, -\lambda_r$; we ignore zero eigenvalues if present. We let $v_1, \ldots, v_r$ be an orthonormal set of corresponding eigenvectors:

$$B = \sum_{j=1}^{k} \lambda_j v_j v_j^* - \sum_{j=k+1}^{r} \lambda_j v_j v_j^*. \tag{17}$$

Now we define $f_j(z) = \sqrt{\lambda_j}\, v_j^* \mathcal{Z}$. We see that

$$\langle Vf(z), f(z)\rangle = \langle B\mathcal{Z}, \mathcal{Z}\rangle, \tag{18}$$

where V is the form with k ones and $r-k$ negative ones on the diagonal.

Hence any real polynomial corresponds to a holomorphic map $f\colon \mathbb{P}^n \to \mathbb{P}^N$ taking the zero set of the polynomial to a hyperquadric with the same signature as the coefficient matrix B. Thus, in order to classify all maps taking one hyperquadric to another hyperquadric up to Q-equivalence, we need only classify real polynomials that vanish on the source hyperquadric and have the correct signature.

Note that we will generally scale all the v_j by the same number to make the expressions easier to work with. Other methods can be employed to construct Q-equivalent f by writing B differently as a sum of rank-1 matrices, but the procedure just outlined is the one we will use. Any other method will produce the same number of positive and of negative terms—assuming the components are linearly independent.

If we have a Hermitian matrix B such that $\langle B\mathcal{Z}_d, \mathcal{Z}_d\rangle$ is zero on $\mathbb{V}_J$, then

$$\langle B\mathcal{Z}_d, \mathcal{Z}_d\rangle = \langle A\mathcal{Z}_{d-1}, \mathcal{Z}_{d-1}\rangle \langle Jz, z\rangle \tag{19}$$

for some Hermitian matrix A. We take an automorphism preserving the set defined by J—in other words, a matrix X such that $X^*JX = J$. To find all maps equivalent by an automorphism of the source, we compute the canonical form of

$$\langle A\mathcal{Z}_{d-1}(Xz), \mathcal{Z}_{d-1}(Xz)\rangle \langle Jz, z\rangle = \langle \hat{X}^* A \hat{X} \mathcal{Z}_{d-1}, \mathcal{Z}_{d-1}\rangle \langle Jz, z\rangle, \tag{20}$$

where $\hat{X}$ is the matrix defined by $\mathcal{Z}_{d-1}(Xz) = \hat{X}\mathcal{Z}_{d-1}(z)$. Hence, we will look for a canonical form of the pair (J, A) under $*$-conjugation by $\hat{X}$ and X. If J has the same number of positive and negative eigenvalues, we would also have to consider linear maps such that $X^*JX = -J$. If $d = 2$, then $\hat{X} = X$ and so matters become simpler; we shall next describe the method in more detail.

In general, we could start with any algebraic source manifold—for example, one defined by the zero set of the real polynomial $\langle C\mathcal{Z}_k, \mathcal{Z}_k\rangle$. Then the existence of a map f taking the manifold to a hyperquadric implies the existence of a Hermitian matrix A such that

$$\langle B\mathcal{Z}_d, \mathcal{Z}_d\rangle = \langle A\mathcal{Z}_{d-k}, \mathcal{Z}_{d-k}\rangle\langle C\mathcal{Z}_k, \mathcal{Z}_k\rangle, \tag{21}$$

where B is the form resulting from composing the map f with the defining equation of the target hyperquadric. Classifying such maps is then equivalent to finding normal forms for the pair (C, A) under the automorphism group fixing C. In this paper we study the case of a hyperquadric source manifold.

3. Degree-2 Maps

In this section we consider degree-2 maps. Let f be a degree-2 map taking a hyperquadric to a hyperquadric. Then, for the proper V and J, we have $\langle Vf(z), f(z)\rangle = q(z, \bar{z})\langle Jz, z\rangle$. The polynomial q is real valued of bidegree $(1, 1)$, so it can also be written as a Hermitian form. We write $f(z) = F\mathcal{Z}$ as before. We obtain

$$\langle F^*VF\mathcal{Z}, \mathcal{Z}\rangle = \langle Az, z\rangle\langle Jz, z\rangle \tag{22}$$

for some Hermitian matrix A.

On the other hand, if we start with $\langle Az, z\rangle\langle Jz, z\rangle$ and then multiply, the result will be a Hermitian matrix B such that $\langle B\mathcal{Z}, \mathcal{Z}\rangle = \langle Az, z\rangle\langle Jz, z\rangle$. If B and V have the same signature, then we can use linear algebra (as explained in Section 2) to find a matrix F such that $B = F^*VF$.

Therefore, examining all the possible Hermitian matrices A yields all the degree-2 maps of $\mathbb{V}_J$ to all hyperquadrics. We restrict our attention to those matrices A for which the resulting matrix B has the correct signature.

We assume that the components of f are linearly independent. This limitation turns out not to be a problem for CR maps of spheres, but it is a problem for CR maps of hyperquadrics. Namely, we miss maps such as $(1, g, g)$—but that is essentially all we miss.

If $Q(a, b)$ is the source hyperquadric (corresponding to J) and if we take an automorphism χ of $Q(a, b)$, then χ is represented by a matrix X such that $X^*JX = J$. Therefore,

$$\langle AXz, Xz\rangle\langle JXz, Xz\rangle = \langle X^*AXz, z\rangle\langle Jz, z\rangle. \tag{23}$$

Thus we first find all the canonical forms for a pair of Hermitian matrices under the equivalence $(A, B) \sim (C, D)$ whenever there exists a nonsingular matrix X such that $X^*AX = C$ and $X^*BX = D$ (simultaneous $*$-congruence). Next we collect those canonical pairs for which one of the matrices is $*$-congruent to J; such pairs therefore give canonical forms of degree-2 CR maps of hyperquadrics from the J hyperquadric up to Q-equivalence. In the special case when the number of positive and negative eigenvalues of J is the same (i.e., $a = b+1$), we must also consider those X for which $X^*JX = -J$. However, we will usually assume that J defines the sphere and that $a + b \geq 2$.

The problem of classifying a pair of matrices has a long history that goes back to Kronecker. The first results for Hermitian matrices and $*$-congruence were proved in the 1930s by several authors independently (see the survey by Lancaster and Rodman [18]). We could use these older results, but for convenience we use a more recent paper by Horn and Sergeichuk [15], whose canonical form is easy to work with. Thse authors also demonstrate an algorithm for computing the canonical form, so we likewise have an algorithm for producing a normal form for maps of hyperquadrics. That is, when deciding whether two degree-2 CR maps of hyperquadrics are Q-equivalent, we first find their corresponding Hermitian forms. Then we follow the procedure of Horn and Sergeichuk to generate a normal form of the matrices, after which we simply check and see whether the normal forms are the same (up to a multiple, of course). This method works only if the components of the maps are linearly independent. However, reducing to this case is not hard in general.

Before giving the result of Horn and Sergeichuk, we must first define the "building block" matrices used for their canonical form. Let us define M_n as the $n \times n$ matrix with ones on the superdiagonal and the subdiagonal and similarly define N_n as the $n \times n$ matrix with ones on the superdiagonal but negative ones on the subdiagonal:

$$M_n := \begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix}; \qquad N_n := \begin{bmatrix} 0 & 1 & & 0 \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & 0 \end{bmatrix}. \tag{24}$$

Let $J_n(\lambda)$ be the $n \times n$ Jordan block with eigenvalue λ (i.e., with λ on the diagonal and ones on the superdiagonal). Finally, define

$$\Delta_n(\alpha, \beta) := \begin{bmatrix} 0 & & & & \alpha \\ & & & \alpha & \beta \\ & & \iddots & \iddots & \\ & \alpha & \beta & & \\ \alpha & \beta & & & 0 \end{bmatrix}. \tag{25}$$

We can now give the classification theorem, or at least that part of the theorem that is useful for us. Let I_n be the $n \times n$ identity matrix.

THEOREM 3.1 (Horn and Sergeichuk [15]). *Let A, B be a pair of Hermitian matrices. These matrices are simultaneously $*$-congruent to a direct sum of blocks of the following four types, determined uniquely up to permutation*:

(i) (M_n, iN_n);
(ii) $\pm(\Delta_n(1,0), \Delta_n(\alpha, 1))$, $\alpha \in \mathbb{R}$;
(iii) $\pm(\Delta_n(0,1), \Delta_n(1,0))$; *and*
(iv) $\left(\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(\alpha+i\beta)^* \\ J_n(\alpha+i\beta) & 0 \end{bmatrix}\right)$, $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta i \neq i$, $\beta > 0$.

We use $\pm(\Delta_n(1,0), \Delta_n(\alpha,1))$ to denote the pair $(\Delta_n(1,0), \Delta_n(\alpha,1))$ or the pair $(\Delta_n(-1,0), \Delta_n(-\alpha,-1))$, and similarly for $\pm(\Delta_n(0,1), \Delta_n(1,0))$.

Let us use this classification theorem to study the case when the source hyperquadric is a sphere. Use a degree-2 rational CR map $f\colon S^{2n-1} \to Q(c,d)$ to obtain the pair of Hermitian matrices (J,A). We will now put the pair (J,A) into canonical form. If we can diagonalize J and A simultaneously by $*$-congruence, then the map is equivalent to a monomial map. That is, the map in homogeneous coordinates is monomial. It remains to be seen whether the map is monomial (not allowing for negative exponents) in nonhomogeneous coordinates. Suppose that, for $k = 0, \ldots, N$, the kth component of the homogenized map $\hat{f}$ is $c_k z^{\alpha_k}$ for some degree-2 multi-index α_k. Write $\langle V\hat{f}(z), \hat{f}(z)\rangle = \langle Az, z\rangle\langle Jz, z\rangle$. Since A is not identically zero, there exists a $j = 0, \ldots, n$ such that $\langle Az, z\rangle \neq 0$ when $z_j = 1$ and $z_m = 0$ for all $m \neq j$. Since J is invertible, it follows that $\langle Jz, z\rangle \neq 0$ and thus $\langle V\hat{f}(z), \hat{f}(z)\rangle \neq 0$. Because

$$\langle V\hat{f}(z), \hat{f}(z)\rangle = \sum_{k=0}^{N} \pm|c_k|^2|z^{\alpha_k}|^2, \tag{26}$$

we see that $z^{\alpha_k} = z_j^2$ for at least one k; that is, $\hat{f}$ contains a pure monomial term. Therefore, in the nonhomogeneous coordinates obtained by $z_j = 1$, we obtain a monomial (no negative exponents) CR map from $Q(a,b)$ to $Q(c,d)$, where $Q(a,b)$ is equivalent to S^{2n-1}. We will deal with the classification of monomial maps for each problem separately in later sections.

So let us see what happens in the case when we cannot simultaneously diagonalize J and A. From now on, assume that at least one block in the canonical form is larger than 1×1.

We remark that, since J defines the sphere, all blocks except one in the canonical form for J must be 1×1 blocks. Indeed, all the possible blocks of larger size have at least one negative eigenvalue and hence there can be only one of these. Furthermore, J has no zero eigenvalues. The only blocks that have no zero eigenvalues and at most one negative eigenvalue are M_2, $\pm\Delta_2(1,0)$, $\Delta_3(1,0)$, and $\left[\begin{smallmatrix} 0 & I_1 \\ I_1 & 0 \end{smallmatrix}\right]$. All these blocks are 2×2 or 3×3. It will be sufficient to consider only source dimension $n = 2$ in all that follows. So let us suppose that J and A are both 3×3 as we compute all the normal forms for the Hermitian forms that arise from degree-2 maps.

Note that $\left[\begin{smallmatrix} 0 & I_1 \\ I_1 & 0 \end{smallmatrix}\right] = M_2$ and that iN_2 can be obtained by swapping variables and letting $\alpha + i\beta = i$ in the block $\left[\begin{smallmatrix} 0 & \alpha - i\beta \\ \alpha + i\beta & 0 \end{smallmatrix}\right]$. Thus we need not consider the block M_2 and can simply allow $\alpha + i\beta = i$.

We are now ready to compute. First let us take the situation corresponding to the block Δ_2 in J. That is, we have the following canonical form for (J,A):

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta & 1 \end{bmatrix}\right) \tag{27}$$

for $\alpha, \beta \in \mathbb{R}$. Take $\mathcal{Z} = (z_0^2, z_0z_1, z_0z_2, z_1^2, z_1z_2, z_2^2)$. The form corresponding to the product $\langle Az, z\rangle\langle Jz, z\rangle = \langle B\mathcal{Z}, \mathcal{Z}\rangle$ is

$$B = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha+\beta & 0 & 0 & 0 \\ 0 & \alpha+\beta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 & 2\beta & 1 \\ 0 & 0 & 0 & \beta & 1 & 0 \end{bmatrix}. \tag{28}$$

The forms for the related canonical pair $\left(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta & -1 \end{smallmatrix}\right]\right)$ are similar.

The next pair to consider is

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 1 & 0 \end{bmatrix}\right) \tag{29}$$

for $\alpha \in \mathbb{R}$. Multiplying the forms yields

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 2\alpha & 1 \\ 0 & 0 & 2\alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 2\alpha & 1 & 1 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

We finally arrive at the canonical pair

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta - i\gamma \\ 0 & \beta + i\gamma & 0 \end{bmatrix}\right) \tag{31}$$

for $\alpha, \beta \in \mathbb{R}$ and $\gamma > 0$. Observe that we can simply rescale if we need $\gamma = 1$. After multiplying the forms, we have

$$B = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha+\beta-i\gamma & 0 & 0 & 0 \\ 0 & \alpha+\beta+i\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta-i\gamma \\ 0 & 0 & 0 & 0 & 2\beta & 0 \\ 0 & 0 & 0 & \beta+i\gamma & 0 & 0 \end{bmatrix}. \tag{32}$$

Given this matrix B, we can write it as a sum of rank-1 matrices to obtain a representative of the class of maps given by B. In addition, we must change variables so that J is in the form $\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$; thus we will always have the same ball as our source. (We could also change variables first and then write B as a sum of rank-1 matrices.)

4. Degree-2 Maps of S^{2n-1} to S^{2N-1}

In this section we prove that all degree-2 maps of spheres are Q-equivalent (i.e., spherically equivalent) to a monomial map and then give the classification of those

monomial maps. First let us prove that all degree-2 maps are equivalent to monomial maps.

LEMMA 4.1. *Let $f\colon S^{2n-1} \to S^{2N-1}$, $n \geq 2$, be a rational CR map of degree 2. Then f is spherically equivalent to a monomial map.*

Proof. Suppose we have a degree-2 map of spheres. In other words, we have a Hermitian matrix A such that

$$\langle Az, z\rangle\langle Jz, z\rangle \tag{33}$$

has only one negative eigenvalue. Here $J = \operatorname{diag}(1, 1, \ldots, 1, -1)$ is the matrix that gives us a sphere; that is, J has ones along the diagonal except for the last diagonal element, which is negative one. Unlike before, in this case we start in arbitrary source dimension; however, we will see that considering $n = 2$ is sufficient.

If A and J can be simultaneously diagonalized by $*$-congruence, then we are essentially done. The map will be equivalent to a monomial map if we momentarily allow some exponents to be negative in nonhomogeneous coordinates. We have already remarked that the map has no negative exponents in some set of nonhomogeneous coordinates, but we wish it to be so in the nonhomogeneous coordinates where $\mathbb{V}_J$ is S^{2n-1}.

Negative exponents in nonhomogeneous coordinates mean that the monomial corresponding to the negative eigenvalue in the form for f is divisible by one of the variables corresponding to a positive eigenvalue for J. That is, take the homogeneous coordinates $z = (z_0, z', z_n) \in \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C} = \mathbb{C}^{n+1}$ and write

$$\|f'(z)\|^2 - |z^\alpha|^2 = q(z, \bar{z})(|z_0|^2 + \|z'\|^2 - |z_n|^2) \tag{34}$$

for some bihomogeneous real polynomial q. We denote by f' that part of the monomial map corresponding to positive eigenvalues. We can assume that the components of f' have no factor in common with the monomial z^α. If $z^\alpha = z_n^d$ then we are done.

Therefore, assume (after, perhaps, renaming of variables) that z^α is divisible by z_0. We set $z_0 = 0$ to obtain $\|f'(0, z', z_n)\|^2 = q(0, z', z_n, 0, \bar{z}', z_n)(\|z'\|^2 - |z_n|^2)$. Note that $f'(0, z', z_n)$ vanishes on a real hypersurface and hence is identically zero. This means that f' is divisible by z_0; that is, the map is not given in lowest terms—a contradiction. So if we can diagonalize, the map is monomial (with the usual understanding that in nonhomogeneous coordinates, where the source is the sphere, no exponent is negative).

We focus on the case when one cannot diagonalize. Note that we are seeking a contradiction. As before, we find the canonical form for the pair (J, A) and assume that there is some block larger than 1×1 in the canonical form for J. Again we note that all such blocks for J that have at most one negative eigenvalue are either 2×2 or 3×3. Therefore, if we prove the result for source dimension 2 then we are done: if there were a map not equivalent to a monomial map for higher source dimension, then we could set to zero all but the three variables corresponding to 1×1 blocks. We obtain a CR map of spheres with source dimension 2 that is not equivalent to a monomial map, since the surviving 2×2 or 3×3 block is

canonical and so the corresponding block in A is not simultaneously diagonalizable. Absent the existence of such a map, all blocks in the canonical form of the pair (J, A) must have been 1×1 and so the matrices must have been simultaneously diagonalizable. Hence we assume that J and A are 3×3 just as before.

We have already computed the matrices B resulting from $\langle Az, z\rangle\langle Jz, z\rangle = \langle B\mathcal{Z}, \mathcal{Z}\rangle$. Let us consider these matrices one by one.

First we take the situation corresponding to the block Δ_2 in J. That is, for (J, A) consider the canonical form $\left(\left[\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}\alpha&0&0\\0&0&\beta\\0&\beta&1\end{smallmatrix}\right]\right)$. The corresponding B is computed in (28). If $\alpha + \beta \neq 0$, then this form clearly has at least two negative eigenvalues (notice the block structure). If $\beta = -\alpha \neq 0$, then it is also not hard to see that there must still be at least two negative eigenvalues. If $\beta = \alpha = 0$, then A has rank 1 and so we have a first-degree map. Similar calculations yield the same result for the related canonical form $\left(\left[\begin{smallmatrix}1&0&0\\0&0&-1\\0&-1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}\alpha&0&0\\0&0&\beta\\0&\beta&-1\end{smallmatrix}\right]\right)$.

Next let us consider the pair $\left(\left[\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0&\alpha\\0&\alpha&1\\\alpha&1&0\end{smallmatrix}\right]\right)$. The corresponding B is computed in (30). This form, too, always has at least two negative eigenvalues. To see this, note that the form has a zero eigenvalue only when $\alpha = 0$. It is therefore enough to check the signature of the matrix for $\alpha = 0$.

Finally we get to the canonical pair $\left(\left[\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}\alpha&0&0\\0&0&\beta-i\gamma\\0&\beta+i\gamma&0\end{smallmatrix}\right]\right)$ for $\gamma > 0$. The corresponding B is computed in (32). Once again, given the block structure and $\gamma \neq 0$, we easily see that this form must have at least two negative eigenvalues.

We are done. We have dealt with all the canonical forms and have shown that the canonical form for (J, A) must contain only 1×1 blocks. Hence it must be monomial, so the lemma is proved. □

REMARK 4.2. Lemma 4.1 does not hold when $n = 1$, and it is easy to create counterexamples using the techniques just described. Take the canonical pair $\left(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&i\\-i&0\end{smallmatrix}\right]\right)$. With $\mathcal{Z} = (z_0^2, z_0z_1, z_1^2)$, the form B becomes

$$\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}. \tag{35}$$

We find the eigenvalues 1 and -1 and the corresponding orthonormal eigenvectors $[1\ \ 0\ \ -i]^T/\sqrt{2}$ and $[1\ \ 0\ \ i]^T/\sqrt{2}$. When constructing the map, we scale both eigenvectors by the same number to get rid of the $\sqrt{2}$. In homogeneous coordinates, the map is $(z_0^2 - iz_1^2, z_0^2 + iz_1^2)$. Yet we are not finished until we change to the standard coordinates for the sphere. Hence we must precompose with a linear map that takes J to $\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$. We also multiply the first component by i for simplicity. In nonhomogeneous coordinates, the map is

$$z \mapsto \frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}. \tag{36}$$

We know this map cannot be equivalent to a monomial map because we used the canonical blocks that were not diagonalizable. It is not hard to see that any polynomial degree-2 map of the disc to the disc must be $e^{i\theta}z^2$ for some $\theta \in \mathbb{R}$, so the map is not equivalent to any polynomial map, either. Other examples can be constructed in a similar way.

Next let us classify the degree-2 monomial maps. The classification in general follows from the work of D'Angelo [4; 5]; see also [9; 10; 11]. First, we have the following lemma.

LEMMA 4.3. *Let f and g be monomial CR maps of spheres that take 0 to 0,*

$$f(z) = \bigoplus_{\alpha} a_\alpha z^\alpha, \qquad g(z) = \bigoplus_{\alpha} b_\alpha z^\alpha, \tag{37}$$

where each monomial is distinct. Then f and g are spherically equivalent if and only if there exists a permutation σ of the variables taking α to $\sigma(\alpha)$ and $|a_\alpha| = |b_{\sigma(\alpha)}|$.

The proof follows by results of [4]; polynomial proper maps of balls taking origin to origin are spherically equivalent if and only if they are unitarily equivalent. In other words, f and g are spherically equivalent if and only if $f = g \circ U$ for a unitary matrix U. By setting all but one of the variables to zero—that is, $z = (0, \ldots, 0, 1, 0, \ldots, 0)$ with $z_k = 1$—we find that z is in the sphere and therefore $f(z)$ is in the sphere. Hence at least one component of $f(z)$ must be nonzero; that is, at least one of the monomials must depend only on z_k. This was true for all k and so there must be at least one pure monomial for each variable. The result follows by application of the multinomial theorem. We give the following, slightly different proof addressing degree-2 maps for convenience and to illustrate the methods of this paper.

Proof for degree-2 maps. One direction is simple: if there is a permutation of the variables such that $|a_\alpha| = |b_{\sigma(\alpha)}|$, then obviously f and g are spherically equivalent.

For the other direction, suppose that f and g are monomial, degree-2, and spherically equivalent. We write down the forms corresponding to $\|f(z)\|^2 - 1$ and $\|g(z)\|^2 - 1$ in homogeneous coordinates and find the matrix A_f for f and A_g for g. Note that these matrices must be diagonal. They are in canonical form, and the canonical form is canonical up to permutation of the blocks (and hence up to permutation of the variables). A permutation of variables then shows that the diagonal entries of A_f and A_g are equal and hence $|a_\alpha|^2 = |b_{\sigma(\alpha)}|^2$. □

Therefore, the maps in (2) are all spherically inequivalent. We will give a simple proof that the list is exhaustive, but we also note that this follows easily from the general theory of polynomial proper maps developed by D'Angelo. In [6] it is shown that all degree-2 polynomial maps preserving the origin are given in the form $Lz \oplus \left(\sqrt{I - L^*L}\, z\right) \otimes z$. By Lemma 4.1, we need only consider diagonal L.

See also [8; 11] for more details on the classification of monomial maps. In general, all polynomial (and hence monomial) maps are obtained by a finite series of partial "tensorings" and "untensorings". It is not hard to see that, in this terminology also, the list (2) is exhaustive. For simplicity, we will not use those terms here.

We will give an elementary argument in the real-algebraic language. In the form $\|f(z)\|^2$, replace $|z_1|^2$ with x_1, $|z_2|^2$ with $x_2, \ldots$. Thus, for a monomial map f, the form $\|f(z)\|^2$ will become a real polynomial in $x_1, \ldots, x_n$ with nonnegative coefficients. This polynomial gives the equivalence class of all monomial maps up to postcomposing (with a diagonal unitary matrix, of course). So we must classify all real polynomials $p(x)$ of degree 2 with nonnegative coefficients such that $p(x) = 1$ when $x_1 + \cdots + x_n = 1$.

Without loss of generality, we allow adding zero components to the map. We have defined spherical equivalence of maps with different target dimensions. The following lemmas (and the theorem) could be given without this convenience, but they would then be more complicated to state.

LEMMA 4.4. *Any monomial map of spheres is spherically equivalent to a monomial map taking the origin to the origin by postcomposing with a diagonal matrix.*

Proof. Suppose that $p(x)$ is a polynomial with nonnegative coefficients such that $p(x) = 1$ whenever $x_1 + \cdots + x_n = 1$ and $p(0) \neq 0$. (Note that $p(0) < 1$ unless p is trivial.) Then $\frac{p(x)-p(0)}{1-p(0)}$ is a polynomial with nonnegative coefficients with no constant term that is one on $x_1 + \cdots + x_n = 1$. The equivalence of the induced maps is by a diagonal matrix. □

LEMMA 4.5. *All degree-2 monomial maps of spheres taking the origin to the origin are spherically equivalent to a map of the form* (2) *by composing with permutation matrices and a unitary matrix.*

Proof. Let us show how to construct all the degree-2 monomial examples in (2). Suppose that we have a polynomial $p(x)$ in $(x_1, \ldots, x_n)$ of degree 2 such that $p-1$ is divisible by $(x_1 + \cdots + x_n - 1)$, $p(0) = 0$, and all coefficients of p are nonnegative. Write $p = p_1 + p_2$, where p_1 is of degree 1 and p_2 is of degree 2. We claim that p_2 must be divisible by $(x_1 + \cdots + x_n)$. This fact is easy to see by homogenizing to $p_1(x)t + p_2(x) - t^2$, noting that this polynomial is divisible by $(x_1 + \cdots + x_n - t)$, and then setting $t = 0$. We find $q = p_1 + p_2/(x_1 + \cdots + x_n)$. The polynomial $q - 1$ is also divisible by $(x_1 + \cdots + x_n - 1)$. Since q is of degree 1, it follows that $q - 1$ is a constant multiple of $(x_1 + \cdots + x_n - 1)$. Since $q(0) = 0$, we must have $q = x_1 + \cdots + x_n$. So p was constructed from q by a partial tensoring operation, which in this language is simply multiplication of certain terms of q by $(x_1 + \cdots + x_n)$.

Therefore, the list in (2) is exhaustive because all the listed monomial maps are obtained in the fashion just described. □

REMARK 4.6. We mentioned that there exists a third-degree rational CR map of spheres that is not spherically equivalent to a polynomial one [13]. Let us give an

alternative simple argument that uses our previous reasoning to generate a slightly weaker statement.

PROPOSITION 4.7. *There exists a third-degree CR rational map* $f\colon S^{2n-1} \to S^{2N-1}$, $n \geq 2$, *that is not spherically equivalent to a polynomial map taking origin to origin. In fact, when* $n = 2$, *there exists a real 2-dimensional family of such maps.*

Proof. First we describe the Hermitian form for a polynomial map taking origin to origin—that is, the matrix B in $\langle B\mathcal{Z}_3, \mathcal{Z}_3\rangle$ for $\mathcal{Z}_3$ the degree-3 Veronese map. It is not hard to see that $B = B_1 \oplus \mathbf{1}$ for some matrix B_1 (with one less row and one less column than B) and the 1×1 matrix $\mathbf{1}$. We therefore let $\mathcal{Z}_2$ be the degree-2 Veronese map and A the matrix such that $\langle B\mathcal{Z}_3, \mathcal{Z}_3\rangle = \langle A\mathcal{Z}_2, \mathcal{Z}_2\rangle\langle Jz, z\rangle$. Then $A = A_1 \oplus \mathbf{1}$ (as for B). Suppose, for simplicity, that $n = 2$; then A is 6×6.

By results of D'Angelo (see e.g. [9, Prop. 3] for an explicit statement), a polynomial map of spheres of degree d is determined by its $d-1$ jet (the coefficients of all monomials of degree $d-1$ or less). That is, no two distinct polynomial CR maps of spheres have the same $d-1$ jet. Furthermore, there exists an open neighborhood of the origin in the $d-1$ jet space such that each such jet gives a CR map of spheres. Hence there exists an open set of A in the space of 6×6 Hermitian matrices that correspond to degree-3 rational maps of spheres. For the map to be equivalent to a polynomial map taking the origin to the origin, we must set five complex parameters in A to zero. That is, in order for A to correspond to a polynomial map preserving the origin, it must be of the form $A_1 \oplus \mathbf{1}$ (where A_1 is a 5×5 Hermitian matrix). The set of linear maps of the source preserving J has complex dimension 3. Therefore, since an open set of the $d-1$ jets is possible, we will not always be able to set all five parameters to zero. In fact, there will be a whole family of such examples. In nonhomogeneous coordinates, we see that there must be a real 2-dimensional family of examples of maps not spherically equivalent to a polynomial map taking origin to origin. □

5. Monomial Maps of Hyperquadrics

We shall study the monomial version of the problem of classifying monomial maps between hyperquadrics in dimensions 2 and 3. The proofs in this simplified case illustrate the combinatorics of the more general problem. Furthermore, we also use the monomial classification for the general case. It appears that the combinatorics governing the monomial situation govern the general situation of CR maps of hyperquadrics in some sense. For example, even allowing for higher source and target dimension, there is no CR map of spheres known to the author that is not homotopic to a monomial example. We have also shown in this paper that all degree-2 maps of spheres are equivalent to monomial maps.

The setup translates into a simpler problem in real-algebraic geometry, much as it did for monomial CR map of spheres. That being said, the different equivalence relation results in a somewhat more complicated statement. In this case $Q(1,1)$ is no longer equivalent to $Q(2,0)$ and, moreover, we have a perfectly valid interpretation of $Q(0,2)$.

See [8; 11] for more on the following setup. Let $f\colon Q(a,b) \to Q(c,d)$ be a monomial CR map. We have $|f_1|^2+\cdots+|f_c|^2-|f_{c+1}|^2-\cdots-|f_{c+d}|^2=1$ when $|z|^2+\varepsilon|w|^2=1$, where $\varepsilon=1$ if the source is $Q(2,0)$ and $\varepsilon=-1$ if the source is $Q(1,1)$. Since every f_j is a monomial, we can replace $x=|z|^2$ and $y=|w|^2$.

We can now formulate the following real-algebraic problem. Let $p(x,y)$ be a real polynomial that has $N=N(p)=N_+(p)+N_-(p)$ distinct monomials, N_+ monomials with a positive coefficient, and N_- monomials with a negative coefficient. We call the tuple $(N_+(p),N_-(p))$ the *signature* of p, and we consider two polynomials p and q to be *equivalent* if either $p(x,y)=q(x,y)$ or $p(x,y)=q(y,x)$. Then we have the following classification. Note that the signature $(0,k)$ makes sense in the real-algebraic problem.

We will state the general version in homogeneous coordinates. First, however, we give the specific result for spheres in nonhomogeneous coordinates.

PROPOSITION 5.1. *Let $p(x,y)$ be a real polynomial such that $p(x,y)=1$ whenever $x+y=1$. Then the possible polynomials (up to swapping of variables) with signature $(3,0)$ are*

(i) $x^3+3xy+y^3$,
(ii) $x^2+2xy+y^2$,
(iii) $x+xy+y^2$, *and*
(iv) $\alpha x+\alpha y+(1-\alpha)$ *for some* $\alpha\in(0,1)$.

For signature $(2,1)$, we have

(i) x^2+2y-y^2 *and*
(ii) $\alpha x+\alpha y-(\alpha-1)$ *for some* $\alpha>1$.

For signature $(1,2)$ we have

(i) $(1+\alpha)-\alpha x-\alpha y$ *for some* $\alpha>0$.

There exists no such polynomial with signature $(0,3)$.

Instead of directly proving this particular case, we give a more general statement. Suppose we have a real homogeneous polynomial $p(x,y,t)$ such that $p=0$ on $x+y+t=0$. If we can classify all such polynomials with four (or fewer) terms, then we will also have proved the proposition. We take $p(x,y,-t)$, set $t=1$, and consider only those polynomials with a constant term equal to one.

In this way we also easily obtain all monomial maps from $Q(1,1)$ to other hyperquadrics. Therefore, we have only to prove the following lemma.

LEMMA 5.2. *Let $p(x,y,t)$ be a nonzero homogeneous polynomial with four or fewer distinct monomials such that $p(x,y,t)=0$ on $x+y+t=0$. Then, up to permutation of variables, we have $p=qm$, where m is an arbitrary monomial and q is one of the following polynomials*:

(i) $x^3+y^3+t^3-3xyt$;
(ii) $x^2+y^2+2xy-t^2$;
(iii) $x^2+xy-yt-t^2$;
(iv) $x+y+t$.

Before proving this lemma we first establish the following proposition, which is a modification of the classification of homogeneous monomial CR map of spheres. See [8; 11] for the sphere version and a slightly different proof.

PROPOSITION 5.3. *For some real number α, let $p(x,y,t) = \varphi(x,y) - \alpha t^d$ be a homogeneous polynomial such that $p = 0$ when $x + y + t = 0$. Then $\varphi(x,y) = \alpha(-1)^d(x+y)^d$.*

Proof. Note that $-(x+y) = t$ on $x + y + t = 0$. Hence we can write

$$\varphi(x,y) - \alpha(-1)^d(x+y)^d. \tag{38}$$

This is a polynomial that is zero on $x + y + t = 0$, but it does not depend on t and so must be identically zero. □

Proof of Lemma 5.2. After dividing through by m we can assume that the monomials in p have no common factor. Next we observe that $p(x,y,t)$ is divisible by $x + y + t$. First suppose that $p(x,y,t)$ has three or fewer monomials. For each variable, there must (by our assumption) be a monomial containing it. Thus, for example, $p(x,y,0)$ has at most two monomials but also cannot be identically zero by assumption. Since it is divisible by $x + y$, we see that $p(x,y,0) = m(x,y)(x^k \pm y^k)$ for some monomial $m(x,y)$ depending only on x and y. This fact is easy to see, and it can be proved in similar fashion as Proposition 5.3. The same reasoning applies to the other variables. Since there can be at most three monomials, it follows that p must be a constant multiple of $x^k + y^k + t^k$. Now, by application of Proposition 5.3, $k = 1$.

Hence we can assume that there are exactly four distinct monomials. It is elementary to see that at least two variables divide two or more terms in p. Without loss of generality, suppose these are the variables x and y. By the same logic as before, we have $p(x,0,t) = m_1(x,t)(x^k \pm t^k)$ and $p(0,y,t) = m_2(y,t)(y^\ell \pm t^\ell)$ for some monomials m_1 and m_2. Since t cannot divide all terms and since we have exactly four terms, we conclude (without loss of generality) that $m_1 = \mu x^{d-k}$. We can scale p so that $\mu = 1$. We have two possibilities: the polynomial p is either

$$x^d \pm x^{d-k}t^k + \alpha y^{a+\ell}t^b \pm \alpha y^a t^{b+\ell} \quad \text{or} \quad x^d \pm t^d \pm y^\ell t^{d-\ell} + \alpha x^a y^b t^c. \tag{39}$$

To discount the first possibility, note that $k \geq 1$ and $\ell \geq 1$. If $t = 0$ then we must have at least two terms as before, so $b = 0$. Note also that this must mean $\alpha = \pm 1$ (again after setting $t = 0$). We are left with essentially a special case of the second possibility, so let us focus on that.

First suppose that $\ell = d$; then $x^d \pm t^d \pm y^d + \alpha x^a y^b t^c$. If $c = 0$, then we apply Proposition 5.3 to conclude that $d = 2$ and the polynomial is $x^2 - t^2 + y^2 + 2xy$. We proceed similarly for a and b.

Next we replace t in the last term with $(-x - y)$, which yields $x^d \pm t^d \pm y^d + \alpha x^a y^b(-x-y)^c$. This polynomial fulfills the hypothesis of Proposition 5.3, so $x^d \pm y^d + \alpha x^a y^b(-x-y)^c$ must have the same number of terms as $(x+y)^d$. We conclude that $c = d - 2$. Similarly, we repeat the argument for x and y and conclude that $a = d - 2$ and $b = d - 2$. Hence $d = 3$. Dividing by $x + y + t$, we see that the polynomial must be $x^3 + t^3 + y^3 - 3xyt$.

So suppose that $\ell < d$. Then, after setting $t = 0$, we must have two terms and so $c = 0$. By setting $t = 0$ we also see that $\alpha = \pm 1$. Therefore, $x^d \pm t^d \pm y^\ell t^{d-\ell} \pm x^a y^b$. We replace x with $(-y-t)$ and apply Proposition 5.3. After counting monomials we find that $a = d-2$ or $a = d-1$. Letting $t = (-x-y)$ and proceeding as before, we find that $d - \ell = d-2$ or $d-\ell = d-1$. Hence y appears to be of at most second power. If we substitute $y = -x - t$ then we must obtain an identically zero polynomial. Counting monomials yields a finite list of possibilities, and the only one divisible by $x + y + t$ is $x^2 - t^2 + yt - xy$. □

We can now obtain monomial CR maps of hyperquadrics by (possibly permuting variables and) substituting $\pm|z|^2$ for x, $\pm|w|^2$ for y, and ± 1 for t. We can also obtain CR maps of hyperquadrics by substituting $\pm z\bar{w}$ for x, $\pm w\bar{z}$ for y, and ± 1 for t before changing variables so that the source hyperquadric is in the standard form. We can follow this procedure to find the degree-3 CR map of $Q(2,0)$ to $Q(2,1)$. Note, however, that we will never get a CR map of spheres by using this nonstandard substitution.

6. Degree-2 Maps from S^3 to $Q(2,1)$

Let $f\colon S^3 = Q(2,0) \to Q(2,1)$ be a degree-2 rational map. That is, let $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \operatorname{diag}(1,1,-1)$ and let A be a Hermitian 3×3 matrix such that, in homogeneous coordinates (using our previous notation), we have

$$\langle Vf(z), f(z)\rangle = \langle F^* VF\mathcal{Z}, \mathcal{Z}\rangle = \langle Az, z\rangle\langle Jz, z\rangle \tag{40}$$

for $V = \operatorname{diag}(1,1,-1,-1)$. In other words, the 6×6 matrix $B = F^*VF$ is rank 4 or less. Because the sphere contains no complex varieties, it is easy to see that the rank cannot be less than 3. If B is rank 3, then the map would be a CR map of spheres in the same dimension and hence Q-equivalent to the identity map, which is not degree 2. So assume that B is of rank 4 with two positive and two negative eigenvalues.

As explained previously, we use the automorphism group of the ball to put A into a canonical form. We have already computed a list of all possible canonical forms and the resulting matrices B. All we have to do is find those canonical forms for which B is rank 4 and has two positive and two negative eigenvalues.

If we can diagonalize the pair (J, A) simultaneously by $*$-congruence, then the map is equivalent to a monomial map—that is, monomial in homogeneous coordinates. As mentioned before, we know there is a monomial map from some hyperquadric equivalent to the ball. So by the classification of monomial CR maps of hyperquadrics, we are essentially done. Applying Lemma 5.2, we find the following list of monomial CR maps of hyperquadrics in homogeneous coordinates (z, w, t):

$$(z, w, t) \mapsto (z^2, \sqrt{2}wt, w^2, t^2); \tag{41}$$

$$(z, w, t) \mapsto (tz, w^2, wt, z^2). \tag{42}$$

The matrix A for these two maps is different even after permutation or negation (we need to handle negation, since V has two positive and two negative eigenvalues). Consequently, the two maps (41) and (42) are not Q-equivalent.

We focus on the case where we cannot diagonalize. Let us again assume that there is some block in the canonical form greater than 1×1. We will consider all the computed canonical forms for B for all the matrix pairs (J, A) from Section 3.

Let us take the situation corresponding to the block Δ_2 in J. The canonical pair for (J, A) is then $\left(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta & 1 \end{smallmatrix}\right]\right)$. The corresponding B is computed in (28). This matrix can have rank 3, 5, or 6, but it can never have rank 4. Therefore, we need not consider this case. Similar calculations give the same result for the related canonical form $\left(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta & -1 \end{smallmatrix}\right]\right)$.

Next consider the pair $\left(\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 1 & 0 \end{smallmatrix}\right]\right)$, for which the corresponding B is computed in (30). This matrix is rank 4 if and only if $\alpha = 0$. The matrix has two positive and two negative eigenvalues, so we do get a map to the $Q(2,1)$ hyperquadric.

We use the following procedure to obtain a map. First change variables to put J into the form $\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$. Next, find a set of orthonormal eigenvectors and then follow the procedure outlined before to get the map in homogeneous coordinates:

$$(z, w, t) \mapsto (z^2 + \sqrt{3}zw + w^2 - zt, w^2 + zt - \sqrt{3}wt - t^2, \\ z^2 - \sqrt{3}zw + w^2 - zt, w^2 + zt + \sqrt{3}wt - t^2). \quad (43)$$

Note that we have scaled the eigenvectors to avoid ugly expressions.

Finally, we get to the canonical pair $\left(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta - i\gamma \\ 0 & \beta + i\gamma & 0 \end{smallmatrix}\right]\right)$ for $\gamma > 0$. The corresponding B is computed in (32). The matrix B can have rank 4 only if $\alpha = \beta = 0$. We can rescale so that $\gamma = 1$. Following the same procedure as before, we change variables and find a set of orthonormal eigenvectors to obtain the map

$$(z, w, t) \mapsto \big(\sqrt[4]{2}(zw - izt), w^2 - \sqrt{2}iwt + t^2, \\ \sqrt[4]{2}(zw + izt), w^2 + \sqrt{2}iwt + t^2\big). \quad (44)$$

We are done. We have proved the following lemma.

LEMMA 6.1. *Let $f\colon Q(2,0) \mapsto Q(2,1)$ be a rational degree-2 CR map. Then f is Q-equivalent to exactly one of the following maps*:

(i) $(z, w) \mapsto (z^2, \sqrt{2}w, w^2)$;

(ii) $(z, w) \mapsto \left(\frac{1}{z}, \frac{w^2}{z^2}, \frac{w}{z^2}\right)$;

(iii) $(z, w) \mapsto \left(\frac{z^2+\sqrt{3}zw+w^2-z}{w^2+z+\sqrt{3}w-1}, \frac{w^2+z-\sqrt{3}w-1}{w^2+z+\sqrt{3}w-1}, \frac{z^2-\sqrt{3}zw+w^2-z}{w^2+z+\sqrt{3}w-1}\right)$; *or*

(iv) $(z, w) \mapsto \left(\frac{\sqrt[4]{2}(zw-iz)}{w^2+\sqrt{2}iw+1}, \frac{w^2-\sqrt{2}iw+1}{w^2+\sqrt{2}iw+1}, \frac{\sqrt[4]{2}(zw+iz)}{w^2+\sqrt{2}iw+1}\right)$.

7. Proof of Theorem 1.2

Before proving the classification of CR maps of hyperquadrics, we revisit some of Faran's [12] setup.

Suppose that a holomorphic function $f: U \subset \mathbb{C}^2 \to \mathbb{C}^3$ is such that an image of any complex line is contained in some plane. We then call f a *planar* map. If f is a planar map defined on a domain $U \subset \mathbb{P}^2$, we can define the *dual* map. For a domain U define the dual domain U^*, where U^* is composed of lines $L \subset \mathbb{P}^2$ such that $f(L \cap U)$ is contained in a unique plane in $\mathbb{P}^3$. Let $\mathbb{P}^{2*}$ and $\mathbb{P}^{3*}$ be the dual spaces—that is, the spaces of hyperplanes in $\mathbb{P}^2$ and $\mathbb{P}^3$, respectively. We can then define $f^*: U^* \subset \mathbb{P}^{2*} \to \mathbb{P}^{3*}$ such that, if f takes a line L into the unique plane P, then $f^*(L) = P$. See [12] for basic properties of duals. For example, the dual of a dual is the map itself. Let f and g be two maps of $\mathbb{P}^2$ to $\mathbb{P}^3$. If there exist conjugate-linear isomorphisms ψ and φ such that $\psi f = g\varphi$, then f and g are said to be conjugate isomorphic. If f and f^* are conjugate isomorphic, then f is said to be self-dual.

The following lemma gives a useful property of CR maps of hyperquadrics. Faran proved this lemma for CR maps of sphere. The proof for hyperquadrics is almost exactly the same, but we restate it here for convenience.

LEMMA 7.1. *If $f: U \subset Q(2,0) \to Q(2,1)$ is a nonconstant real-analytic CR map, then f is planar and self-dual.*

Proof. Let $f: U \subset Q(2,0) \to Q(2,1)$ be a nonconstant real-analytic CR map. Since f is real-analytic, it extends to a holomorphic map of a neighborhood of U in $\mathbb{C}^2$ to $\mathbb{C}^3$. Let us call this extension f for simplicity. Since f maps $Q(2,0)$ to $Q(2,1)$, it follows that (in homogeneous coordinates) we have $\langle Vf(z), f(z)\rangle = 0$ whenever $\langle Jz, z\rangle = 0$, where V defines $Q(2,1)$ and J defines $Q(2,0)$. Then, by polarization, the line defined by $\langle Jz, w\rangle = 0$ for a fixed point w is mapped to the plane defined by $\langle V\zeta, f(w)\rangle = 0$. Hence f is a planar map.

Let ξ_0, ξ_1, ξ_2 be the homogeneous coordinates for $\mathbb{P}^2$ and let the image by f be the point $\xi_0', \xi_1', \xi_2', \xi_3'$ in homogeneous coordinates for $\mathbb{P}^3$. Suppose that our non-homogeneous coordinates correspond to $z_1 = \frac{\xi_1}{\xi_0}$ and $z_2 = \frac{\xi_2}{\xi_0}$ and that for $\mathbb{P}^3$ we have $z_1' = \frac{\xi_1'}{\xi_0'}$, $z_2' = \frac{\xi_2'}{\xi_0'}$, and $z_3' = \frac{\xi_3'}{\xi_0'}$.

We let $w_1 = \frac{\eta_1}{\eta_0}, \ldots$. By polarization, the statement that f takes $Q(2,0)$ to $Q(2,1)$ becomes:

$$\xi_0\bar{\eta}_0 - \xi_1\bar{\eta}_1 - \xi_2\bar{\eta}_2 = 0 \tag{45}$$

implies

$$\xi_0'\bar{\eta}_0' + \xi_1'\bar{\eta}_1' - \xi_2'\bar{\eta}_2' - \xi_3'\bar{\eta}_3' = 0. \tag{46}$$

Equation (45) defines a line that in dual coordinates is $(\bar{\eta}_0, -\bar{\eta}_1, -\bar{\eta}_2)$. Similarly, (46) defines a plane that in dual coordinates is $(\bar{\eta}_0', \bar{\eta}_1', -\bar{\eta}_2', -\bar{\eta}_3')$. Because f^* takes $(\bar{\eta}_0, -\bar{\eta}_1, -\bar{\eta}_2)$ to $(\bar{\eta}_0', \bar{\eta}_1', -\bar{\eta}_2', -\bar{\eta}_3')$, we have defined the conjugate isomorphisms ψ and φ such that $\psi f^* = f\varphi$. That is, f is self-dual. □

Faran proved the following complete classification of planar maps. The designations of nondegenerate, partially degenerate, developable, degenerate, and flat maps are not relevant to us but are retained for consistency. We say that two maps $f, g: U \subset \mathbb{P}^2 \to \mathbb{P}^3$ are *equivalent* if there exist $\tau \in \mathrm{Aut}(\mathbb{P}^3)$ and $\chi \in \mathrm{Aut}(\mathbb{P}^2)$ such that $f \circ \chi = \tau \circ g$.

THEOREM 7.2 (Faran [12]). *Let $f\colon U \subset \mathbb{P}^2 \to \mathbb{P}^3$ be a planar immersion. Then f is equivalent to one of the following maps.*

A. *Nondegenerate maps*
   1. $(z_0, z_1, z_2) \mapsto (z_0^3, z_1^3, z_2^3, z_0 z_1 z_2)$;
   2. (a) $(z_0, z_1, z_2) \mapsto \left(z_0^2 + \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2, z_0 z_1 + \frac{1}{2}z_2^2, z_0 z_2 + \frac{1}{2}z_1^2, z_1 z_2\right)$,
      (b) $(z_0, z_1, z_2) \mapsto \left(z_0^2 - \frac{15}{16}z_1^2 - \frac{3}{4}z_2^2, z_0 z_1 + \frac{1}{2}z_2^2, z_0 z_2 + \frac{1}{2}z_1^2, z_1 z_2\right)$,
      (c) $(z_0, z_1, z_2) \mapsto \left(z_0^2, z_0 z_1 + \frac{1}{2}z_2^2, z_0 z_2 + \frac{1}{2}z_1^2, z_1 z_2\right)$.

B. *Partially degenerate maps*
   1. (a) *dual to* A.2(a),
      (b) *dual to* A.2(b),
      (c) *dual to* A.2(c);
   2. $(z_0, z_1, z_2) \mapsto (z_0^2, z_1^2, z_0 z_2, z_1 z_2)$;
   3. $(z_0, z_1, z_2) \mapsto \left(z_0^2, z_0 z_1, z_0 z_2 + \frac{1}{2}z_1^2, z_1 z_2\right)$.

C. *Developable map* $(z_0, z_1, z_2) \mapsto (z_0^2, z_1^2, z_1 z_2, z_2^2)$.

D. *Degenerate maps* $(z_0, z_1, z_2) \mapsto (z_0, z_1, z_2, z_0 g(z_1/z_0, z_2/z_0))$ *for any function g.*

E. *Flat maps* $(z_0, z_1, z_2) \mapsto (1, g_1(z_1/z_0, z_2/z_0), g_2(z_1/z_0, z_2/z_0), 0)$ *for any functions g_1 and g_2.*

Except for degenerate and flat maps, no listed map is equivalent to any other.

Two CR maps of hyperquadrics could be equivalent yet not Q-equivalent. So before using the classification of planar maps for CR hyperquadric maps, we must check (i) whether the class of planar maps contains a CR hyperquadric map and (ii) whether the class contains other non–Q-equivalent CR hyperquadric maps.

We can apply our result for degree-2 maps to handle all the degree-2 cases. Hence we need only study the cases A.1, D, and E. Case A.1 is handled by the following lemma.

LEMMA 7.3. *Suppose that a homogeneous polynomial map of $\mathbb{P}^2 \to \mathbb{P}^3$ induced by $f\colon Q(2,0) \mapsto Q(2,1)$ is equivalent to $(z_0, z_1, z_2) \mapsto (z_0^3, z_1^3, z_2^3, z_0 z_1 z_2)$. Then f is Q-equivalent to $(z, w) \mapsto \left(\frac{2w^3}{3z^2+1}, \frac{z^3+3z}{3z^2+1}, \sqrt{3}\frac{wz^2-w}{3z^2+1}\right)$.*

Proof. Faran demonstrates that f is self-dual. After changing coordinates, the map

$$(z_0, z_1, z_2) \mapsto (z_0^3, z_1^3, z_2^3, \sqrt{3}\, z_0 z_1 z_2) \tag{47}$$

takes the sphere to the sphere. Now f is self-dual by Lemma 7.1, and it is not hard to compute that the dual of f is $(\eta_0, \eta_1, \eta_2) \mapsto (\eta_0^3, \eta_1^3, \eta_2^3, -\sqrt{3}\eta_0\eta_1\eta_2)$.

If we have conjugate isomorphisms $\psi_n\colon \mathbb{P}^n \to \mathbb{P}^{n*}$ such that $f^*\psi_2 = \psi_3 f$, then f takes the set $z \cdot \psi_2(z) = 0$ into the set $w \cdot \psi_3(w) = 0$. These zero sets are hyperquadrics if the corresponding forms are Hermitian. We then check the signature of the forms. Thus, to find all the ways that f maps a hyperquadric to a hyperquadric, we need only find all the ways that f is conjugate isomorphic to f^*.

We have already found that

$$\psi_2(z) = (-\bar{z}_0, \bar{z}_1, \bar{z}_2), \qquad \psi_3(w) = (-\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3), \tag{48}$$

which give the CR map of spheres.

Suppose that (ψ_2, ψ_3) and (ψ_2', ψ_3') are conjugate isomorphisms of f and f^* that give Hermitian forms (i.e., give maps of hyperquadrics). Hence we have $f^* = \psi_3 f \psi_2^{-1} = \psi_3' f \psi_2'^{-1}$ and so $f\psi_2^{-1}\psi_2' = \psi_3^{-1}\psi_3' f$. Therefore, the pair

$$(\alpha_2, \alpha_3) = (\psi_2^{-1}\psi_2', \psi_3^{-1}\psi_3') \tag{49}$$

is a pair of automorphisms of the source and target that fix f; that is, $f\alpha_2 = \alpha_3 f$. We denote by $\mathrm{Aut}(f)$ the set of such pairs (α_2, α_3). We have already found (ψ_2, ψ_3); hence, by computing $\mathrm{Aut}(f)$, we find all hyperquadrics that f takes to other hyperquadrics. We need only check that they are of the right signature and that we do not get Q-equivalent maps.

Now we note that the Jacobian of f drops rank at precisely the coordinates $(1,0,0), (0,1,0), (0,0,1)$ in $\mathbb{P}^2$. Hence any automorphism preserves these points. Any permutation of the source variables (z_0, z_1, z_2) is part of an automorphism of f. Therefore, any automorphism of f is a composition of a permutation of the variables and the map

$$(z_0, z_1, z_2; w_0, w_1, w_2, w_3) \mapsto (az_0, bz_1, cz_2; a^3 w_0, b^3 w_1, c^3 w_2, abcw_3). \tag{50}$$

Here the first three components (in the z_j variables) of the map represent the self-map of $\mathbb{P}^2$ and the last four components (in the w_k variables) represent the self-map of $\mathbb{P}^3$.

Now we need to check all the permutations of variables to see what maps we get by considering different elements of $\mathrm{Aut}(f)$, which has six components (corresponding to the permutations). However, symmetry in the mappings means that we need only check the identity, the permutation of two variables, and the cyclic permutation.

Faran establishes that the identity component gives only maps that are Q-equivalent to the CR map of spheres. It is also possible to use the following computation to show the same result. The cyclic permutation does not give Hermitian forms and hence does not give CR maps to hyperquadrics.

Therefore, we need only check the permutation of two variables. For example,

$$(z_0, z_1, z_2; w_0, w_1, w_2, w_3) \mapsto (az_2, bz_1, cz_0; a^3 w_2, b^3 w_1, c^3 w_0, abcw_3). \tag{51}$$

Then ψ_2' and ψ_3' give the forms

$$z \cdot \psi_2'(z) = -\bar{a} z_0 \bar{z}_2 + \bar{b} z_1 \bar{z}_1 + \bar{c} z_2 \bar{z}_0, \tag{52}$$

$$w \cdot \psi_3'(w) = -\bar{a}^3 w_0 \bar{w}_2 + \bar{b}^3 w_1 \bar{w}_1 + \bar{c}^3 w_2 \bar{w}_0 + \bar{a}\bar{b}\bar{c} w_3 \bar{w}_3. \tag{53}$$

These forms are Hermitian if $b \in \mathbb{R}$ and $c = -\bar{a}$. The first form has two positive eigenvalues only if $b > 0$. We can now change variables $z \mapsto (c^{-1} z_0, b^{-1/2} z_1, z_2)$ to eliminate the variables a, b, c from the problem. That is, we obtain

$$z \cdot \psi_2'(z) = z_0\bar{z}_2 + z_1\bar{z}_1 + z_2\bar{z}_0, \tag{54}$$

$$w \cdot \psi_3'(w) = w_0\bar{w}_2 + z_1\bar{w}_1 + w_2\bar{w}_0 - w_3\bar{w}_3. \tag{55}$$

The forms are of correct signature and hence we have a map taking $Q(2,0)$ to $Q(2,1)$.

As a side note, we get the same map that we obtain by looking at the real-algebraic version of the monomial problem. Take the negative of the homogenized version of the degree-3 monomial map. We have that

$$p(x,y,t) = x^3 + y^2 + t^3 - 3xyt \quad \text{is zero on} \quad x + y + t = 0. \tag{56}$$

Now we compute $p(z_0\bar{z}_2, z_1\bar{z}_1, z_2\bar{z}_0)$, which will give the form $w \cdot \psi_3'(w)$ after removing the Veronese map. Hence, even this nonmonomial map comes about from a monomial map.

After changing variables we get the map (in nonhomogeneous coordinates)

$$(z,w) \mapsto \left(\frac{2w^3}{3z^2+1}, \frac{z^3+3z}{3z^2+1}, \sqrt{3}\frac{wz^2 - w}{3z^2+1} \right). \tag{57}$$

Because we found no other maps, all degree-3 maps taking $Q(2,0)$ to $Q(2,1)$ are Q-equivalent to the map (57). □

Proof of Theorem 1.2. Let $f\colon U \subset Q(2,0) \to Q(2,1)$ be a nonconstant real-analytic CR map. Since f is real-analytic, it extends to a holomorphic map of a neighborhood of U in $\mathbb{C}^2$ to $\mathbb{C}^3$. Let us call this extension f for simplicity. The map f is planar by Lemma 7.1.

First suppose that the derivative of the map is at most 1. Then the derivative must be of rank 1 at generic points of a neighborhood of U. That is, at a generic point, f takes a germ of $\mathbb{C}^2$ into a germ of a 1-dimensional subvariety of $\mathbb{C}^3$. This behavior occurs on the complement of a complex analytic set in $\mathbb{C}^2$. Hence there exists a point $p \in U \subset Q(2,0)$ such that f takes a neighborhood of p in $\mathbb{C}^2$ into a 1-dimensional complex subvariety in $\mathbb{C}^3$. It is easy to see that f takes a neighborhood W of p in $\mathbb{C}^2$ into $Q(2,1)$. If $f(W) \cap Q(2,1)$ were of less than two real dimensions then we could pull back a point and get a complex subvariety of $Q(2,0)$, which is impossible.

Therefore, f takes a neighborhood of p into an irreducible complex subvariety contained in $Q(2,1)$. After a change of coordinates on the target side, we can assume that $f(p) = (1,0,0)$ and that the complex variety inside $Q(2,1)$ containing the image of f is given by $\{z \in \mathbb{C}^3 \mid z_2 = z_3, z_1 = 1\}$. We immediately get that f is Q-equivalent to the map $(z,w) \mapsto (1,g,g)$ for some CR function g.

So suppose that, at generic points, the rank of the derivative is 2. Recall that the rank can drop only on a complex variety, which means that the rank of the derivative must be 2 on an open and dense subset of $U \subset Q(2,0)$. Taking a perhaps smaller neighborhood U, we can simply assume that the rank of the derivative is identically 2 and that f is an immersion. We know that f is equivalent to one of the maps in Theorem 7.2. However, as we said before, the type of equivalence is not quite correct (it is not Q-equivalence). We need to check each class to see

whether it contains a CR map of hyperquadrics. If it does, we need to see whether it contains several non–Q-equivalent maps.

If f is of type E, then f must map into a plane P. We know that $Q(2,1) \cap P$ is equivalent to $Q(2,0)$ or $S^1 \times \mathbb{C}$. If $Q(2,1) \cap P \cong Q(2,0)$ then f must be linear. If $Q(2,1) \cap P \cong S^1 \times \mathbb{C}$ then f cannot be an immersion (essentially because the inverse image of $\{p\} \times \mathbb{C}$ cannot be contained in a sphere). We have already handled this case.

If f is of type D, then f is not self-dual by an easy computation. Being self-dual is a requirement for maps to take a hyperquadric to a hyperquadric, so f is not equivalent to any map of hyperquadrics. See Faran [12] for more details.

Therefore, f must be rational of degree ≤ 3. If f is linear then it is obviously Q-equivalent to $(z,w) \mapsto (z,w,0)$. If f is of degree 2, we apply Lemma 6.1. If f is of degree 3, we apply Lemma 7.3. □

References

[1] M. S. Baouendi, P. Ebenfelt, and X. Huang, *Super-rigidity for CR embeddings of real hypersurfaces into hyperquadrics,* Adv. Math. 219 (2008), 1427–1445.

[2] M. S. Baouendi, P. Ebenfelt, and L. Preiss Rothschild, *Real submanifolds in complex space and their mappings,* Princeton Math. Ser., 47, Princeton Univ. Press, Princeton, NJ, 1999.

[3] M. S. Baouendi and X. Huang, *Super-rigidity for holomorphic mappings between hyperquadrics with positive signature,* J. Differential Geom. 69 (2005), 379–398.

[4] J. P. D'Angelo, *Proper holomorphic maps between balls of different dimensions,* Michigan Math. J. 35 (1988), 83–90.

[5] ———, *Polynomial proper holomorphic mappings between balls. II,* Michigan Math. J. 38 (1991), 53–65.

[6] ———, *Several complex variables and the geometry of real hypersurfaces,* Stud. Adv. Math., CRC Press, Boca Raton, FL, 1993.

[7] ———, *Invariant holomorphic mappings,* J. Geom. Anal. 6 (1996), 163–179.

[8] J. P. D'Angelo, Š. Kos, and E. Riehl, *A sharp bound for the degree of proper monomial mappings between balls,* J. Geom. Anal. 13 (2003), 581–593.

[9] J. P. D'Angelo and J. Lebl, *On the complexity of proper holomorphic mappings between balls,* Complex Var. Elliptic Equ. 54 (2009), 187–204.

[10] ———, *Complexity results for CR mappings between spheres,* Internat. J. Math. 20 (2009), 149–166.

[11] J. P. D'Angelo, J. Lebl, and H. Peters, *Degree estimates for polynomials constant on a hyperplane,* Michigan Math. J. 55 (2007), 693–713.

[12] J. J. Faran, *Maps from the two-ball to the three-ball,* Invent. Math. 68 (1982), 441–475.

[13] J. J. Faran, X. Huang, S. Ji, and Y. Zhang, *Polynomial and rational maps between balls,* Q. J. Pure Appl. Math. 6 (2010), 829–842.

[14] F. Forstnerič, *Extending proper holomorphic mappings of positive codimension,* Invent. Math. 95 (1989), 31–61.

[15] R. A. Horn and V. V. Sergeichuk, *Canonical forms for complex matrix congruence and *congruence,* Linear Algebra Appl. 416 (2006), 1010–1032.

[16] X. Huang, S. Ji, and D. Xu, *A new gap phenomenon for proper holomorphic mappings from B^n into B^N*, Math. Res. Lett. 13 (2006), 515–529.

[17] S. Ji and Y. Zhang, *Classification of rational holomorphic maps from B^2 into B^N with degree 2,* Sci. China Ser. A 52 (2009), 2647–2667.
[18] P. Lancaster and L. Rodman, *Canonical forms for Hermitian matrix pairs under strict equivalence and congruence,* SIAM Rev. 47 (2005), 407–443.
[19] S. I. Pinčuk, *Proper holomorphic maps of strictly pseudoconvex domains,* Sibirsk. Mat. Zh. 15 (1974), 909–917, 959 (Russian).
[20] M. W. Setya-Budhi, *Proper holomorphic mappings in several complex variables,* Ph.D. thesis, University of Illinois, Urbana-Champaign, 1993.

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