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Cubic Relations between Frequencies of Digits and Hausdorff Dimension

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1. Introduction

We consider the notion of frequency of digits, which is defined as follows. Given an integer $m > 1$, for each $x \in [0,1]$ we denote by $0.x_1x_2\cdots$ a base-m representation of x (the representation is unique except for countably many points, and thus the nonuniqueness does not affect the study of Hausdorff dimension, since countable sets have zero Hausdorff dimension). For each $k \in \{0,\dots,m-1\}$, $x \in [0,1]$, and $n \in \mathbb{N}$ we set

$$\tau_k(x,n) = \operatorname{card}\{i \in \{1,\dots,n\} : x_i = k\}$$

and

$$\tau_k(x) = \lim_{n\to\infty} \frac{\tau_k(x,n)}{n}$$

whenever the limit exists. The number $\tau_k(x)$ is called the *frequency* of the number k in the base-m representation of x. Now we consider the set

$$F_m(\alpha_0,\dots,\alpha_{m-1}) = \{x \in [0,1] : \tau_k(x) = \alpha_k \text{ for } k = 0,\dots,m-1\},$$

where $\alpha_k \in [0,1]$ for each k and $\sum_{k=0}^{m-1}\alpha_k = 1$. Eggleston showed in [11] that this set has Hausdorff dimension

$$\dim_H F_m(\alpha_0,\dots,\alpha_{m-1}) = -\frac{1}{\log m}\sum_{k=0}^{m-1}\alpha_k\log\alpha_k$$

(with the convention that $0\log 0 = 0$). A related result was obtained earlier by Besicovitch in [6] when $m = 2$. The work of Eggleston was further generalized by Billingsley with a more unified approach (see [7] for details and references). See also [3] for more recent developments and further references.

In this paper we consider sets defined in terms of *nonlinear* relations between the frequencies. Namely, for each ε and δ we consider the function

$$f_{\varepsilon,\delta}(x) = x + \varepsilon x^2 + \delta x^3$$

and the set

$$F_{\varepsilon,\delta} = \{x \in [0,1] : \tau_1(x) = f_{\varepsilon,\delta}(\tau_0(x))\}. \tag{1}$$

It follows from a general procedure described in [3] that

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$$\dim_H F_{\varepsilon,\delta} = \max \dim_H F_m(\alpha_0, f_{\varepsilon,\delta}(\alpha_0), \alpha_2, \ldots, \alpha_{m-1}), \tag{2}$$

where the maximum is taken over all numbers $\alpha_0, \alpha_2, \ldots, \alpha_{m-1} \in [0,1]$ such that

$$\alpha_0 + f_{\varepsilon,\delta}(\alpha_0) + \sum_{i=2}^{m-1} \alpha_i = 1.$$

We can always assume from the beginning that $\alpha_2 = \cdots = \alpha_{m-1}$ given the strict convexity of the function $-x \log x$. Using (2), we can show that the function $(\varepsilon, \delta) \mapsto \dim_H F_{\varepsilon,\delta}$ is analytic. In addition, we also obtain decay estimates for the coefficients $\beta_{k,n-k}$ in the power series

$$\dim_H F_{\varepsilon,\delta} = -\frac{1}{\log m} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta_{k,n-k} \varepsilon^k \delta^{n-k}$$

in terms of m (see Theorem 1). The class of perturbations in (1) can be seen as a model of nonlinear perturbations between frequencies of digits. We certainly would prefer to consider much more general, or even arbitrary, Taylor series instead of only those in $f_{\varepsilon,\delta}$. However, the computations are already so demanding in this particular case that it must be considered a quite reasonable step toward a general theory. The main aim of such a theory would be to understand how the Hausdorff dimension of sets defined in terms of relations between frequencies of digits varies with the coefficients in the Taylor series (of the functions defining the relations). A particular case of the present work was already considered in [5]—namely, setting $\delta = 0$, which makes the computations much simpler. In particular, in [5] we do not need a multivariate version of the Faá di Bruno formula.

A principal advantage of our approach is that we are able to compute explicitly the coefficients $\beta_{k,n-k}$ up to any order by solving recursively finitely many equations. Furthermore, the value of the Hausdorff dimension of a given set need not be guessed a priori. In some works this guess is crucial, since it is necessary to construct specific auxiliary measures sitting on the set. Our work is based on results in [3] belonging to the theory of multifractal analysis. These were obtained combining work in [4; 12; 14]. We refer to the books [2; 13] for details and further references on the theory of multifractal analysis. Incidentally, problems of a nature similar to that of those considered in [3] were addressed in [1; 9; 10].

2. Main Result

Fix an integer $m > 2$. For each ε and δ, we consider the set $F_{\varepsilon,\delta}$ in (1). It follows from a general procedure described in [3] that

$$\dim_H F_{\varepsilon,\delta} = -\frac{1}{\log m} \inf\{G_{\varepsilon,\delta}(\alpha) : \alpha \in [0,1] \text{ and } \alpha + f_{\varepsilon,\delta}(\alpha) \in [0,1]\},$$

where

$$\begin{aligned} G_{\varepsilon,\delta}(\alpha) &= \alpha \log \alpha + f_{\varepsilon,\delta}(\alpha) \log f_{\varepsilon,\delta}(\alpha) \\ &\quad + (1 - \alpha - f_{\varepsilon,\delta}(\alpha)) \log \frac{1 - \alpha - f_{\varepsilon,\delta}(\alpha)}{m-2}. \end{aligned}$$

The following is our main result.

THEOREM 1. *There is an analytic function*

$$\alpha(\varepsilon,\delta)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\alpha_{k,n-k}(m)\varepsilon^k\delta^{n-k}$$

in some neighborhood U *of* $(0,0)$ *such that*

$$\dim_H F_{\varepsilon,\delta}=-\frac{1}{\log m}G_{\varepsilon,\delta}(\alpha(\varepsilon,\delta)),\quad (\varepsilon,\delta)\in U.$$

Moreover, if we set

$$G_{\varepsilon,\delta}(\alpha(\varepsilon,\delta))=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\beta_{k,n-k}(m)\varepsilon^k\delta^{n-k},$$

then there exist constants $c_{n,k}>0$ *for* $n\geq 1$ *and* $k=0,\dots,n$ *such that*

$$\max\{|\alpha_{k,n-k}(m)|,|\beta_{k,n-k}(m)|\}\leq\frac{c_{n,k}}{m^{2n+1-k}}$$

for every $m>2$.

Proof. We consider the analytic function $J(\alpha,\varepsilon,\delta)=G'_{\varepsilon,\delta}(\alpha)$. We have

$$\begin{aligned}G'_{\varepsilon,\delta}(\alpha)&=\log\alpha+f'_{\varepsilon,\delta}(\alpha)\log f_\varepsilon(\alpha)-(1+f'_{\varepsilon,\delta}(\alpha))\log\frac{1-\alpha-f_{\varepsilon,\delta}(\alpha)}{m-2}\\&=\log\alpha+(1+2\varepsilon\alpha+3\delta\alpha^2)\log[\alpha(1+\varepsilon\alpha+\delta\alpha^2)]\\&\quad-(2+2\varepsilon\alpha+3\delta\alpha^2)\log\frac{1-2\alpha-\varepsilon\alpha^2-\delta\alpha^3}{m-2}.\end{aligned}$$

In particular, $J(1/m,0,0)=0$. Furthermore,

$$\begin{aligned}G''_{\varepsilon,\delta}(\alpha)=&\frac{1}{\alpha}+\frac{(1+2\varepsilon\alpha+3\delta\alpha^2)^2}{\alpha+\varepsilon\alpha^2+\delta\alpha^3}-\frac{(2+2\varepsilon\alpha+3\delta\alpha^2)^2}{-1+2\alpha+\varepsilon\alpha^2+\delta\alpha^3}\\&+2(\varepsilon+3\delta\alpha)\log(\alpha(1+\alpha(\varepsilon+\delta\alpha)))\\&-2(\varepsilon+3\delta\alpha)\log\frac{(1-\alpha(2+\alpha(\varepsilon+\delta\alpha)))}{m-2},\end{aligned}$$

and in particular,

$$\frac{\partial J}{\partial\alpha}(1/m,0,0)=G''_{0,0}(1/m)=\frac{2m^2}{m-2}\neq 0. \tag{3}$$

This shows that there is an analytic function $\alpha(\varepsilon,\delta)$ with $\alpha(0,0)=1/m$ such that

$$G'_{\varepsilon,\delta}(\alpha(\varepsilon,\delta))=J(\alpha(\varepsilon,\delta),\varepsilon,\delta)=0$$

for ε and δ in some neighborhood of zero.

Now we consider the constants $\alpha_{k,n-k}=\alpha_{k,n-k}(m)$. Set

$$\hat{J}(\varepsilon,\delta)=G'_{\varepsilon,\delta}(\alpha(\varepsilon,\delta)).$$

We note that each derivative $D^n\hat{J}(0,0)$ depends only on the constants $\alpha_{k,p-k}$ with $k=0,\dots,p$ and $p=0,\dots,n$ (that is, it can be determined from $G_{\varepsilon,\delta}$ and these numbers). Moreover, the constants can be computed by solving recursively the equations

$$D^n \hat{J}(0,0) = 0, \quad n \geq 0. \tag{4}$$

For simplicity, we write $G'_{\varepsilon,\delta}(\alpha) = \sum_{j=1}^{9} F_j(\alpha, \varepsilon, \delta)$, where

$$\begin{aligned}
F_1(\alpha,\varepsilon,\delta) &= 2\log[\alpha(m-2)], \\
F_2(\alpha,\varepsilon,\delta) &= \log(1+\varepsilon\alpha+\delta\alpha^2), \\
F_3(\alpha,\varepsilon,\delta) &= 2\varepsilon\alpha\log[\alpha(m-2)], \\
F_4(\alpha,\varepsilon,\delta) &= 3\delta\alpha^2\log[\alpha(m-2)], \\
F_5(\alpha,\varepsilon,\delta) &= 2\varepsilon\alpha\log(1+\alpha(\varepsilon+\delta\alpha)), \\
F_6(\alpha,\varepsilon,\delta) &= 3\delta\alpha^2\log(1+\alpha(\varepsilon+\delta\alpha)), \\
F_7(\alpha,\varepsilon,\delta) &= -2\log(1-2\alpha-\varepsilon\alpha^2-\delta\alpha^3), \\
F_8(\alpha,\varepsilon,\delta) &= -2\varepsilon\alpha\log(1-2\alpha-\varepsilon\alpha^2-\delta\alpha^3), \\
F_9(\alpha,\varepsilon,\delta) &= -3\delta\alpha^2\log(1-2\alpha-\varepsilon\alpha^2-\delta\alpha^3).
\end{aligned} \tag{5}$$

We first verify that $\alpha_{0,0} = 1/m$ is given by the equation

$$\hat{J}(0,0) = \sum_{j=1}^{9} F_j(\alpha_{0,0}, 0, 0) = 0. \tag{6}$$

Since

$$F_1(\alpha_{0,0},0,0) = 2\log[\alpha_{0,0}(m-2)], \qquad F_8(\alpha_{0,0},0,0) = -2\log(1-2\alpha_{0,0}),$$

and $F_j(\alpha_{0,0},0,0) = 0$ for $j = 2,3,4,5,6,7,9$, it follows from (6) that

$$2\log\frac{\alpha_{0,0}(m-2)}{1-2\alpha_{0,0}} = 0$$

and thus $\alpha_{0,0} = 1/m$. Now we consider the other coefficients.

LEMMA 1. *The equations in* (4) *determine recursively the constants* $\alpha_{k,n-k}$ *for* $n \geq 1$ *and* $k = 0, \dots, n$. *Moreover, for each* $n \geq 1$ *and* $k = 0, \dots, n$ *there exists a* $C_{k,n-k} > 0$ (*independent of* m) *such that*

$$|\alpha_{k,n-k}| \leq C_{k,n-k}/m^{2n+1-k} \quad \text{for every } m > 2. \tag{7}$$

Proof. We use induction on n. Set $\hat{F}_j(\varepsilon,\delta) = F_j(\alpha(\varepsilon,\delta),\varepsilon,\delta)$ for each j. To obtain $\alpha_{0,1}$ and $\alpha_{1,0}$ we solve respectively $\partial_\varepsilon \hat{J}(0,0) = 0$ and $\partial_\delta \hat{J}(0,0) = 0$. We have

$$\partial_\varepsilon \hat{J}(0,0) = \sum_{j=1}^{9} \frac{\partial \hat{F}_j}{\partial \varepsilon}(0,0) = \sum_{j=1}^{9}\left(\frac{\partial F_j}{\partial \alpha}\alpha_{1,0} + \frac{\partial F_j}{\partial \varepsilon}\right)$$

and

$$\partial_\delta \hat{J}(0,0) = \sum_{j=1}^{9} \frac{\partial \hat{F}_j}{\partial \delta}(0,0) = \sum_{j=1}^{9}\left(\frac{\partial F_j}{\partial \alpha}\alpha_{0,1} + \frac{\partial F_j}{\partial \delta}\right),$$

where the partial derivatives of F_j are computed at $(\alpha_{0,0}, 0, 0)$. Since $\alpha_{0,0} = 1/m$, we obtain

$$\frac{\partial \hat{F}_1}{\partial \varepsilon} = 2m\alpha_{1,0}, \quad \frac{\partial \hat{F}_2}{\partial \varepsilon} = \frac{1}{m}, \quad \frac{\partial \hat{F}_3}{\partial \varepsilon} = \frac{2}{m}\log\left(\frac{m-2}{m}\right),$$
$$\frac{\partial \hat{F}_4}{\partial \varepsilon} = \frac{\partial \hat{F}_5}{\partial \varepsilon} = \frac{\partial \hat{F}_6}{\partial \varepsilon} = 0, \qquad \frac{\partial \hat{F}_7}{\partial \varepsilon} = \frac{2+4m^2\alpha_{1,0}}{m(m-2)},$$
$$\frac{\partial \hat{F}_8}{\partial \varepsilon} = -\frac{2}{m}\log\left(\frac{m-2}{m}\right), \qquad \frac{\partial \hat{F}_9}{\partial \varepsilon} = 0$$

at the point $(0,0)$. Thus,

$$0 = \sum_{j=1}^{9} \frac{\partial \hat{F}_j}{\partial \varepsilon}(0,0) = \frac{1+2m^2\alpha_{1,0}}{m-2} \quad \text{and} \quad \alpha_{1,0} = -\frac{1}{2m^2}.$$

Similarly,

$$\frac{\partial \hat{F}_1}{\partial \delta} = 2m\alpha_{0,1}, \quad \frac{\partial \hat{F}_2}{\partial \delta} = \frac{1}{m^2}, \quad \frac{\partial \hat{F}_3}{\partial \delta} = 0,$$
$$\frac{\partial \hat{F}_3}{\partial \delta} = \frac{\partial \hat{F}_4}{\partial \delta} = \frac{3}{m^2}\log\left(\frac{m-2}{m}\right), \qquad \frac{\partial \hat{F}_5}{\partial \delta} = \frac{\partial \hat{F}_6}{\partial \delta} = 0,$$
$$\frac{\partial \hat{F}_7}{\partial \delta} = \frac{2+4m^3\alpha_{0,1}}{m^2(m-2)}, \quad \frac{\partial \hat{F}_8}{\partial \delta} = 0, \quad \frac{\partial \hat{F}_9}{\partial \delta} = -\frac{3}{m^2}\log\left(\frac{m-2}{m^2}\right),$$

which yields

$$0 = \sum_{j=1}^{9} \frac{\partial \hat{F}_j}{\partial \delta}(0,0) = \frac{1+2m^3\alpha_{0,1}}{m(m-2)} \quad \text{and} \quad \alpha_{0,1} = -\frac{1}{2m^3}.$$

This shows that (7) holds for $n = 1$ and $k = 0, 1$ with $C_{0,1} = C_{1,0} = 1/2$.

Now we assume that the statement in the lemma holds for $n-1$ (including that we have solved the equations in (4) to obtain the constants $\alpha_{k,n-k}$), and we prove it for n. We first recall a multivariate version of the Faá di Bruno formula (see [8, Cor. 2.10]). Given a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$, the nth derivatives of the composition $g = f \circ \alpha$ are given by

$$\frac{\partial^n g}{\partial_\varepsilon^j \partial_\delta^{n-j}} = j!\,(n-j)! \sum_{\lambda=1}^{n} f^{(\lambda)} \sum_{s=1}^{n} \sum_{p_s(j,\lambda)} \prod_{i=1}^{s} \left(\frac{1}{l_{i,1}!\,l_{i,2}!} \cdot \frac{\partial^{l_{i,1}+l_{i,2}}\alpha}{\partial\varepsilon^{l_{i,1}}\partial\delta^{l_{i,2}}}\right)^{k_i} \frac{1}{k_i!}, \quad (8)$$

where

$$p_s(j,\lambda) = \Bigg\{(k_1,\dots,k_s; l_1,\dots,l_s) \in \mathbb{N}^s \times \mathbb{N}_0^{2s} : 0 < l_1 < \cdots < l_s, \\ \sum_{i=1}^{s} k_i = \lambda,\ \sum_{i=1}^{s} k_i l_{i,1} = j,\ \sum_{i=1}^{s} k_i l_{i,2} = n-j\Bigg\},$$

writing $l_i = (l_{i,1}, l_{i,2})$, and with the order $(a_1,a_2) < (b_1,b_2)$ provided that $a_1 + a_2 < b_1 + b_2$ or $a_1 + a_2 = b_1 + b_2$ with $a_1 < b_1$.

In what follows we consider separately each term F_j in the function $G'_{\varepsilon,\delta}$. For simplicity we shall write

$$(\hat{F}_j)^{(k,l)} = \frac{\partial^{k+l}\hat{F}_j}{\partial\varepsilon^k\partial\delta^l},$$

and analogously for any other functions of (ε, δ).

Step 1. By (5) we have

$$\frac{\partial^\lambda F_1}{\partial\alpha^\lambda} = \frac{2(-1)^{\lambda+1}(\lambda-1)!}{\alpha^\lambda}.$$

Since $\alpha_{0,0} = 1/m$, using (8) we find that

$$\begin{aligned}(\hat{F}_1)^{(j,n-j)}(0,0) &= \frac{2j!\,(n-j)!}{\alpha_{0,0}}\alpha_{j,n-j} + j!\,(n-j)!\,A_{j,n-j} \\ &= 2mj!\,(n-j)!\,\alpha_{j,n-j} + j!\,(n-j)!\,A_{j,n-j}, \end{aligned} \tag{9}$$

where

$$A_{j,n-j} = 2\sum_{\lambda=2}^{n}\frac{(-1)^{\lambda+1}(\lambda-1)!}{\alpha_{0,0}^\lambda}\sum_{s=1}^{n}\sum_{p_s(j,\lambda)}\prod_{i=1}^{s}\left(\frac{\alpha^{(l_{i,1},l_{i,2})}(0,0)}{l_{i,1}!\,l_{i,2}!}\right)^{k_i}\frac{1}{k_i!}.$$

Using the induction hypothesis and setting $\Gamma_\lambda = 2(\lambda-1)!$, we obtain

$$\begin{aligned}|A_{j,n-j}| &\le \sum_{\lambda=2}^{n}\Gamma_\lambda m^\lambda\sum_{s=1}^{n}\sum_{p_s(j,\lambda)}\prod_{i=1}^{s}\left(\frac{C_{l_{i,1},l_{i,2}}}{m^{2(l_{i,1}+l_{i,2})+1-l_{i,1}}}\right)^{k_i}\frac{1}{k_i!} \\ &\le \sum_{\lambda=2}^{n}\Gamma_\lambda m^\lambda\sum_{s=1}^{n}\sum_{p_s(j,\lambda)}\frac{1}{m^{\sum_{i=1}^{s}[2k_i(l_{i,1}+l_{i,2})+k_i-l_{i,1}k_i]}}\prod_{i=1}^{s}\frac{C_{l_{i,1},l_{i,2}}^{k_i}}{k_i!} \\ &\le \sum_{\lambda=2}^{n}\sum_{s=1}^{n}\sum_{p_s(j,\lambda)}\frac{\Gamma_\lambda m^\lambda}{m^{2n+\lambda-j}}\prod_{i=1}^{s}\frac{C_{l_{i,1},l_{i,2}}^{k_i}}{k_i!} \\ &= \frac{1}{m^{2n-j}}\sum_{\lambda=2}^{n}\sum_{s=1}^{n}\sum_{p_s(j,\lambda)}\Gamma_\lambda\prod_{i=1}^{s}\frac{C_{l_{i,1},l_{i,2}}^{k_i}}{k_i!} = \frac{D_{1,j,n}}{m^{2n-j}} \end{aligned} \tag{10}$$

for some constant $D_{1,j,n} > 0$.

Step 2. We set $\tilde{F}_2(z) = \log(1+z)$ and $z(\varepsilon,\delta) = \varepsilon\alpha + \delta\alpha^2$. We have

$$\tilde{F}_2^{(k)}(0) = (-1)^{k+1}(k-1)!.$$

Now we recall that, given differentiable functions $f, g\colon \mathbb{R} \to \mathbb{R}$, we have

$$(fg)^{(k)} = \sum_{j=0}^{k}\binom{k}{j}f^{(j)}g^{(k-j)}. \tag{11}$$

Therefore,

$$\begin{aligned} z^{(l_1,l_2)}(0,0) &= (\varepsilon\alpha)^{(l_1,l_2)}(0,0) + (\delta\alpha^2)^{(l_1,l_2)}(0,0) \\ &= l_1\alpha^{(l_1-1,l_2)}(0,0) + l_2(\alpha^2)^{(l_1,l_2-1)}(0,0) \\ &= l_1!\,l_2!\,\alpha_{l_1-1,l_2} + l_2(\alpha^2)^{(l_1,l_2-1)}(0,0). \end{aligned}$$

We have

$$(\alpha^2)^{(l_1,l_2)} = \frac{\partial^{l_2}}{\partial\delta^{l_2}}(\alpha^2)^{(l_1,0)} = \frac{\partial^{l_2}}{\partial\delta^{l_2}}\sum_{l=0}^{l_1}\binom{l_1}{l}\alpha^{(l,0)}\alpha^{(l_1-l,0)}$$
$$= \sum_{l=0}^{l_1}\binom{l_1}{l}\frac{\partial^{l_2}}{\partial\delta^{l_2}}(\alpha^{(l,0)}\alpha^{(l_1-l,0)})$$
$$= \sum_{l=0}^{l_1}\sum_{r=0}^{l_2}\binom{l_1}{l}\binom{l_2}{r}\alpha^{(l,r)}\alpha^{(l_1-l,l_2-r)} \tag{12}$$

and thus

$$(\alpha^2)^{(l_1,l_2)}(0,0) = l_1!\,l_2!\sum_{l=0}^{l_1}\sum_{r=0}^{l_2}\alpha_{l,r}\alpha_{l_1-l,l_2-r}. \tag{13}$$

Therefore, using (13) with l_2 replaced by l_2-1, we obtain

$$z^{(l_1,l_2)}(0,0) = l_1!\,l_2!\,\alpha_{l_1-1,l_2} + l_1!\,l_2!\sum_{l=0}^{l_1}\sum_{r=0}^{l_2-1}\alpha_{l,r}\alpha_{l_1-l,l_2-1-r}.$$

By the induction hypothesis, we have

$$|z^{(l_1,l_2)}(0,0)| \le l_1!\,l_2!\left(\frac{C_{l_1-1,l_2}}{m^{2(l_1+l_2)-l_1}} + \sum_{l=0}^{l_1}\sum_{r=0}^{l_2-1}\frac{C_{l,r}C_{l_1-l,l_2-1-r}}{m^{2(l_1+l_2)-l_1}}\right)$$
$$\le \frac{l_1!\,l_2!}{m^{2(l_1+l_2)-l_1}}\tilde{D}_{2,l_1,l_2}$$

for some constant $\tilde{D}_{2,l_1,l_2} > 0$. Using again the Faá di Bruno formula in (8) with $f = \tilde{F}_2$ and α replaced by z, setting $\Gamma_\lambda = (\lambda-1)!$ yields

$$\frac{|\hat{F}_2^{(j,n-j)}(0,0)|}{j!\,(n-j)!} \le \sum_{\lambda=1}^{n}\Gamma_\lambda\sum_{s=1}^{n}\sum_{p_s(j,\lambda)}\prod_{i=1}^{s}\left(\frac{\tilde{D}_{2,l_{i,1},l_{i,2}}}{m^{2(l_{i,1}+l_{i,2})-l_{i,1}}}\right)^{k_i}\frac{1}{k_i!}$$
$$\le \sum_{\lambda=1}^{n}\Gamma_\lambda\sum_{s=1}^{n}\sum_{p_s(j,\lambda)}\frac{1}{m^{\sum_{i=1}^{s}[2k_i(l_{i,1}+l_{i,2})-k_i l_{i,1}]}}\prod_{i=1}^{s}\frac{\tilde{D}_{2,l_{i,1},l_{i,2}}^{k_i}}{k_i!}$$
$$\le \frac{1}{m^{2n-j}}\sum_{\lambda=1}^{n}\Gamma_\lambda\sum_{s=1}^{n}\sum_{p_s(j,\lambda)}\prod_{i=1}^{s}\frac{\tilde{D}_{2,l_{i,1},l_{i,2}}^{k_i}}{k_i!} = \frac{D_{2,j,n}}{m^{2n-j}} \tag{14}$$

for some constant $D_{2,j,n} > 0$.

Step 3. We write $\hat{F}_3(\varepsilon,\delta) = \varepsilon h(\varepsilon,\delta)$ for some function h, and we note that

$$\hat{F}_3^{(l_1,l_2)} = l_1 h^{(l_1-1,l_2)} + \varepsilon h^{(l_1,l_2)} \tag{15}$$

for $l_2 \ge 0$ and $l_1 \ge 1$. Clearly, (15) holds for $l_1 = 1$. Furthermore,

$$\frac{\partial}{\partial\varepsilon}[(l_1-1)h^{(l_1-2,l_2)} + \varepsilon h^{(l_1-1,l_2)}] = l_1 h^{(l_1-1,l_2)} + \varepsilon h^{(l_1,l_2)},$$

and it follows by induction on l_1 that (15) holds. By (11) we thus have

$$\begin{aligned}
\hat{F}_3^{(j,n-j)}(0,0) &= jh^{(j-1,n-j)}(0,0) = j(\alpha\hat{F}_1)^{(j-1,n-j)}(0,0) \\
&= j\frac{\partial^{n-j}}{\partial\delta^{n-j}}\sum_{l=0}^{j-1}\binom{j-1}{l}l!\,\alpha_{l,0}\hat{F}_1^{(j-1-l,0)}(0,0) \\
&= j!\sum_{l=0}^{j-1}\frac{\partial^{n-j}}{\partial\delta^{n-j}}\left(\alpha_{l,0}\frac{\hat{F}_1^{(j-1-l,0)}(0,0)}{(j-1-l)!}\right) \\
&= j!\sum_{l=0}^{j-1}\sum_{r=0}^{n-j}\binom{n-j}{r}r!\,\alpha_{l,r}\frac{\hat{F}_1^{(j-1-l,n-j-r)}(0,0)}{(j-1-l)!} \\
&= j!\,(n-j)!\sum_{l=0}^{j-1}\sum_{r=0}^{n-j}\alpha_{l,r}\frac{\hat{F}_1^{(j-1-l,n-j-r)}(0,0)}{(j-1-l)!\,(n-j-r)!}. \qquad (16)
\end{aligned}$$

It follows from (9), and (10), and the induction hypothesis that

$$\left|\frac{\hat{F}_1^{(j-1-l,n-j-r)}(0,0)}{(j-1-l)!\,(n-j-r)!}\right| \le \frac{\tilde{D}_{3,j-1-l,n-j-r}}{m^{2(n-l-r-1)-j+1+l}}, \qquad |\alpha_{l,r}| \le \frac{C_{l,r}}{m^{2(l+r)+1-l}},$$

where

$$\tilde{D}_{3,j-1-l,n-j-r} = D_{1,j-1-l,n-j-r} + 2C_{j-1-l,n-j-r} > 0.$$

By (16) we finally get

$$\frac{|\hat{F}_3^{(j,n-j)}(0,0)|}{j!\,(n-j)!} \le \sum_{l=0}^{j-1}\sum_{r=0}^{n-j}\frac{\tilde{D}_{3,j-1-l,n-j-r}C_{l,r}}{m^{2n-j}} = \frac{j!\,(n-j)!\,D_{3,j,n}}{m^{2n-j}} \qquad (17)$$

for some constant $D_{3,j,n} > 0$.

Step 4. We write $\hat{F}_4(\varepsilon,\delta) = \delta h(\varepsilon,\delta)$. In a similar manner to that in (15) we can show that

$$\hat{F}_4^{(l_1,l_2)} = l_2 h^{(l_1,l_2-1)} + \delta h^{(l_1,l_2)} \qquad (18)$$

for $l_1 \ge 0$ and $l_2 \ge 1$. By (11) we thus obtain

$$\begin{aligned}
\hat{F}_4^{(j,n-j)}(0,0) &= (n-j)h^{(j,n-j-1)}(0,0) = (n-j)(3\alpha^2\hat{F}_1)^{(j,n-j-1)}(0,0) \\
&= 3(n-j)\frac{\partial^j}{\partial\varepsilon^j}\sum_{l=0}^{n-j-1}\binom{n-j-1}{l}l!\,(\alpha^2)^{(0,l)}\hat{F}_1^{(0,n-j-1-l)}(0,0).
\end{aligned}$$

In a similar manner to that in (12) we get

$$
\begin{aligned}
&\hat{F}_4^{(j,n-j)}(0,0) \\
&= 3(n-j)! \sum_{l=0}^{n-j-1} \sum_{r=0}^{l} \frac{\partial^j}{\partial \varepsilon^j} \left(\alpha^{(0,r)} \alpha^{(0,l-r)} \frac{\hat{F}_1^{(0,n-j-1-l,0)}}{(n-j-1-l)!} \right)(0,0) \\
&= 3(n-j)! \sum_{l=0}^{n-j-1} \sum_{r=0}^{l} \sum_{s=0}^{j} \binom{j}{s} (\alpha^{(0,r)} \alpha^{(0,l-r)})^{(s,0)} \frac{\hat{F}_1^{(j-s,n-j-1-l)}(0,0)}{(n-j-1-l)!} \\
&= 3j!\,(n-j)! \sum_{l=0}^{n-j-1} \sum_{r=0}^{l} \sum_{s=0}^{j} \sum_{t=0}^{s} \alpha_{t,r} \alpha_{s-t,l-r} \frac{\hat{F}_1^{(j-s,n-j-1-l)}(0,0)}{(j-s)!\,(n-j-1-l)!}. \qquad (19)
\end{aligned}
$$

It follows from (9), (10), and the induction hypothesis that

$$
\left| \frac{\hat{F}_1^{(j-s,n-j-1-l)}(0,0)}{(j-s)!\,(n-j-1-l)!} \right| \le \frac{\tilde{D}_{4,j-s,n-j-1-l}}{m^{2(n-1-l-s)-j+s}}
$$

and

$$
|\alpha_{t,r}| \cdot |\alpha_{s-t,l-r}| \le \frac{C_{t,r} C_{s-t,l-r}}{m^{2(s+l)+2-s}},
$$

where

$$
\tilde{D}_{4,j-s,n-j-1-l} = D_{1,j-s,n-j-1-l} + 2C_{j-s,n-j-1-l} > 0.
$$

By equation (19) we finally get

$$
\begin{aligned}
\frac{|\hat{F}_4^{(j,n-j)}(0,0)|}{j!\,(n-j)!} &\le 3 \sum_{l=0}^{n-j-1} \sum_{r=0}^{l} \sum_{s=0}^{j} \sum_{t=0}^{s} \frac{\tilde{D}_{4,j-s,n-j-1-l} C_{t,r} C_{s-t,l-r}}{m^{2n-j}} \\
&= \frac{D_{4,j,n}}{m^{2n-j}} \qquad (20)
\end{aligned}
$$

for some constant $D_{4,j,n} > 0$.

Step 5. We write $\hat{F}_5(\varepsilon,\delta) = \varepsilon h(\varepsilon,\delta)$. Proceeding as in (15), we obtain

$$
\hat{F}_5^{(j,n-j)}(0,0) = jh^{(j-1,n-j)}(0,0) = j(2\alpha \hat{F}_2)^{(j-1,n-j)}(0,0).
$$

Also, in a similar manner to that in (16) we have

$$
\frac{\hat{F}_5^{(j,n-j)}(0,0)}{j!\,(n-j)!} = \sum_{l=0}^{j-1} \sum_{r=0}^{n-j} \alpha_{l,r} \frac{2\hat{F}_2^{(j-1-l,n-j-r)}(0,0)}{(j-1-l)!\,(n-j-r)!}. \qquad (21)
$$

It follows from (14) and the induction hypothesis that

$$
\left| \frac{2\hat{F}_2^{(j-1-l,n-j-r)}(0,0)}{(j-1-l)!\,(n-j-r)!} \right| \le \frac{2D_{2,j-1-l,n-j-r}}{m^{2(n-l-r-1)-j+1+l}}, \qquad |\alpha_{l,r}| \le \frac{C_{l,r}}{m^{2(l+r)+1-l}}.
$$

Finally, by (21) we get

$$\frac{|\hat F_5^{(j,n-j)}(0,0)|}{j!\,(n-j)!} \le \sum_{l=0}^{j-1}\sum_{r=0}^{n-j} \frac{2D_{2,\,j-1-l,n-j-r}C_{l,r}}{m^{2n-j}} = \frac{j!\,(n-j)!\,D_{5,\,j,n}}{m^{2n-j}} \tag{22}$$

for some constant $D_{5,\,j,n} > 0$.

Step 6. We write $\hat F_6(\varepsilon,\delta) = \delta h(\varepsilon,\delta)$. Proceeding as in (18), we have

$$\begin{aligned}\hat F_6^{(j,n-j)}(0,0) &= (n-j)h^{(j,n-j-1)}(0,0)\\ &= 3(n-j)(\alpha^2\hat F_2)^{(j,n-j-1)}(0,0).\end{aligned}$$

Also, in the same manner as in (19) we obtain

$$\begin{aligned}&\frac{\hat F_6^{(j,n-j)}(0,0)}{j!\,(n-j)!}\\ &\quad = 3\sum_{l=0}^{n-j-1}\sum_{r=0}^{l}\sum_{s=0}^{j}\sum_{t=0}^{s}\alpha_{t,r}\alpha_{s-t,l-r}\frac{\hat F_2^{(j-s,n-j-1-l)}(0,0)}{(j-s)!\,(n-j-1-l)!}.\end{aligned} \tag{23}$$

It follows from (14) and the induction hypothesis that

$$\left|\frac{\hat F_2^{(j-s,n-j-1-l)}(0,0)}{(j-s)!\,(n-j-1-l)!}\right| \le \frac{D_{2,\,j-s,n-j-1-l}}{m^{2(n-1-l-s)-j+s}}$$

and

$$|\alpha_{t,r}|\cdot|\alpha_{s-t,l-r}| \le \frac{C_{t,r}C_{s-t,l-r}}{m^{2(s+l)+2-s}}.$$

By equation (23) we thus get

$$\begin{aligned}\frac{|\hat F_6^{(j,n-j)}(0,0)|}{j!\,(n-j)!} &\le 3\sum_{l=0}^{n-j-1}\sum_{r=0}^{l}\sum_{s=0}^{j}\sum_{t=0}^{s}\frac{D_{2,\,j-s,n-j-1-l}C_{t,r}C_{s-t,l-r}}{m^{2n-j}}\\ &= \frac{D_{6,\,j,n}}{m^{2n-j}}\end{aligned} \tag{24}$$

for some constant $D_{6,\,j,n} > 0$.

Step 7. We set $\tilde F_7(z) = -\log z$ and

$$z(\varepsilon,\delta) = 1 - 2\alpha - \varepsilon\alpha^2 - \delta\alpha^3,$$

where $\alpha = \alpha(\varepsilon,\delta)$. We have

$$\tilde F_7^{(1)}(1-2\alpha_{0,0}) = -\frac{1}{1-2\alpha_{0,0}} = -\frac{m}{m-2},$$

and for $\lambda \ge 2$,

$$\tilde F_7^{(\lambda)}(1-2\alpha_{0,0}) = \frac{(-1)^\lambda(\lambda-1)!}{(1-2\alpha_{0,0})^\lambda} = \frac{(-1)^\lambda(\lambda-1)!\,m^\lambda}{(m-2)^\lambda}.$$

Moreover, we have

$$\begin{aligned}z^{(l_1,l_2)}(0,0) &= -2\alpha^{(l_1,l_2)}(0,0) - (\varepsilon\alpha^2)^{(l_1,l_2)}(0,0) - (\delta\alpha^3)^{(l_1,l_2)}(0,0)\\ &= -2l_1!\,l_2!\,\alpha_{l_1,l_2} - l_1(\alpha^2)^{(l_1-1,l_2)}(0,0) - l_2(\alpha^3)^{(l_1,l_2-1)}.\end{aligned}$$

Proceeding as in (12), we obtain

$$
\begin{aligned}
&(\alpha^3)^{(l_1,l_2)}\\
&= \frac{\partial^{l_2}}{\partial\delta^{l_2}}(\alpha^3)^{(j,0)} = \frac{\partial^{l_2}}{\partial\delta^{l_2}}\sum_{l=0}^{l_1}\binom{l_1}{l}\alpha^{(l,0)}(\alpha^2)^{(l_1-l,0)}\\
&= \sum_{l=0}^{l_1}\sum_{r=0}^{l_2}\binom{l_1}{l}\binom{l_2}{s}\alpha^{(l,s)}(\alpha^2)^{(l_1-l,l_2-r)}\\
&= \sum_{l=0}^{l_1}\sum_{r=0}^{l_2}\sum_{s=0}^{l_1-l}\sum_{t=0}^{l_2-r}\binom{l_1}{l}\binom{l_2}{s}\binom{l_1-l}{s}\binom{l_2-r}{t}\alpha^{(l_1,r)}\alpha^{(s,t)}\alpha^{(l_1-l-s,l_2-r-t)}
\end{aligned}
$$

and hence

$$
(\alpha^3)^{(l_1,l_2)}(0,0) = l_1!\,l_2!\sum_{l=0}^{l_1}\sum_{r=0}^{l_2}\sum_{s=0}^{l_1-l}\sum_{t=0}^{l_2-r}\alpha_{l,r}\alpha_{s,t}\alpha_{l_1-l-s,l_2-r-t}.
$$

This implies that

$$
\begin{aligned}
\frac{z^{(l_1,l_2)}(0,0)}{l_1!\,l_2!} &= -2\alpha_{l_1,l_2} - \sum_{l=0}^{l_1-1}\sum_{r=0}^{l_2}\alpha_{l,r}\alpha_{l_1-1-l,l_2-r}\\
&\quad - \sum_{l=0}^{l_1}\sum_{r=0}^{l_2-1}\sum_{s=0}^{l_1-l}\sum_{t=0}^{l_2-1-r}\alpha_{l,r}\alpha_{s,t}\alpha_{l_1-l-s,l_2-1-r-t}. \qquad (25)
\end{aligned}
$$

Using the Faà di Bruno formula (8) and the induction hypothesis, we obtain

$$
\begin{aligned}
&\hat{F}_7^{(j,n-j)}(0,0)\\
&\quad = -\frac{m}{m-2}\Bigg(-2j!\,(n-j)!\,\alpha_{j,n-j} - j!\,(n-j)!\\
&\qquad\qquad \sum_{l=0}^{j-1}\sum_{r=0}^{n-j}\alpha_{l,r}\alpha_{j-1-l,n-j-r} - j!\,(n-j)!\\
&\qquad\qquad \sum_{l=0}^{j}\sum_{r=0}^{n-1-j}\sum_{s=0}^{j-l}\sum_{t=0}^{n-1-j-r}\alpha_{l,r}\alpha_{s,t}\alpha_{j-l-s,n-1-j-r-t}\Bigg) + C_{j,n-j}\\
&\quad = \frac{2mj!\,(n-j)!}{m-2}\alpha_{j,n-j} + B_{j,n-j} + C_{j,n-j}, \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
B_{j,n-j} &= \frac{2mj!\,(n-j)!}{m-2}\sum_{l=0}^{j-1}\sum_{r=0}^{n-j}\alpha_{l,r}\alpha_{j-1-l,n-j-r}\\
&\quad + \frac{2mj!\,(n-j)!}{m-2}\sum_{l=0}^{j}\sum_{r=0}^{n-1-j}\sum_{s=0}^{j-l}\sum_{t=0}^{n-1-j-r}\alpha_{l,r}\alpha_{s,t}\alpha_{j-l-s,n-1-j-r-t}
\end{aligned}
$$

and

$$C_{j,n-j} = j!\,(n-j)! \sum_{\lambda=2}^{n} \frac{(-1)^\lambda (\lambda-1)!\, m^\lambda}{(m-2)^\lambda} \sum_{s=1}^{n} \sum_{p_s(j,\lambda)} \prod_{i=1}^{s} \left(\frac{z^{(l_{i,1}, l_{i,2})}}{l_{i,1}!\, l_{i,2}!} \right)^{k_i} \frac{1}{k_i!}.$$

By the induction hypothesis we have

$$\begin{aligned} |B_{j,n-j}| \le {} & \frac{2mj!\,(n-j)!}{m-2} \frac{1}{m^{2n+1-j}} \\ & \times \left(\sum_{l=0}^{j-1} \sum_{r=0}^{n-j} C_{l,r} C_{j-1-l,n-j-r} \right. \\ & \left. + \sum_{l=0}^{j} \sum_{r=0}^{n-1-j} \sum_{s=0}^{j-l} \sum_{t=0}^{n-1-j-r} C_{l,r} C_{s,t} C_{j-l-s,n-1-j-r-t} \right). \end{aligned}$$

Furthermore, since $m \ge 3$,

$$\frac{m}{m-2} = \frac{1}{1-2/m} \le 3 \tag{27}$$

and hence

$$\begin{aligned} |B_{j,n-j}| & \le \frac{6j!\,(n-j)!}{m^{2n+1-j}} \left(\sum_{l=0}^{j-1} \sum_{r=0}^{n-j} C_{l,r} C_{j-1-l,n-j-r} \right. \\ & \qquad \left. + \sum_{l=0}^{j} \sum_{r=0}^{n-1-j} \sum_{s=0}^{j-l} \sum_{t=0}^{n-1-j-r} C_{l,r} C_{s,t} C_{j-l-s,n-1-j-r-t} \right) \\ & = \frac{j!\,(n-j)!\, \hat{D}_{7,j,n-j}}{m^{2n+1-j}} \end{aligned}$$

for some constant $\hat{D}_{7,l_{i,1},l_{i,2}} > 0$. In a similar manner, using the induction hypothesis and (25) we obtain

$$\begin{aligned} \left| \frac{z^{(l_{i,1},l_{i,2})}(0,0)}{l_{i,1}!\, l_{i,2}!} \right| & \le \frac{1}{m^{2(l_{i,1}+l_{i,2})+1-l_{i,1}}} \\ & \qquad \times \left(2 + \sum_{l=0}^{l_{i,1}-1} \sum_{r=0}^{l_{i,2}} C_{l,r} C_{l_{i,1}-1-l,l_{i,2}-r} \right. \\ & \qquad \left. + \sum_{l=0}^{l_{i,1}} \sum_{r=0}^{l_{i,2}-1} \sum_{s=0}^{l_{i,1}-l} \sum_{t=0}^{l_{i,2}-1-r} C_{l,r} C_{s,t} C_{l_{i,1}-l-s,l_{i,2}-1-r-t} \right) \\ & = \frac{\tilde{D}_{7,l_{i,1},l_{i,2}}}{m^{2(l_{i,1}+l_{i,2})+1-l_{i,1}}} \end{aligned}$$

for some constant $\tilde{D}_{7,l_{i,1},l_{i,2}} > 0$. Therefore, using (27) we have

$$\begin{aligned}\frac{|C_{j,n-j}|}{j!\,(n-j)!} &\le \sum_{\lambda=2}^{n} 3^{\lambda}(\lambda-1)! \sum_{s=1}^{n} \sum_{p_s(j,\lambda)} \prod_{i=1}^{s} \left(\frac{\tilde{D}_{7,l_{i,1},l_{i,2}}}{m^{2(l_{i,1}+l_{i,2})+1-l_{i,1}}} \right)^{k_i} \frac{1}{k_i!} \\ &\le \frac{1}{m^{2n+2-j}} \sum_{\lambda=2}^{n} 3^{\lambda}(\lambda-1)! \sum_{s=1}^{n} \sum_{p_s(j,\lambda)} \prod_{i=1}^{s} \frac{\tilde{D}^{k_i}_{7,l_{i,1},l_{i,2}}}{k_i!} \\ &= \frac{D^*_{7,j,n-j}}{m^{2n+2-j}}\end{aligned}$$

for some constant $D^*_{7,j,n-j} > 0$. Hence

$$|B_{j,n-j} + C_{j,n-j}| \le \frac{j!\,(n-j)!\,D_{7,j,n-j}}{m^{2n+1-j}}, \tag{28}$$

where $D_{7,j,n-j} = \hat{D}_{7,j,n-j} + D^*_{7,j,n-j} > 0$.

Step 8. We write $\hat{F}_8(\varepsilon,\delta) = \varepsilon h(\varepsilon,\delta)$. Proceeding as in (15), we get

$$\hat{F}_8^{(j,n-j)}(0,0) = jh^{(j-1,n-j)}(0,0) = j(\alpha\hat{F}_7)^{(j-1,n-j)}(0,0).$$

Then, in a similar manner to that in (16), we have

$$\frac{\hat{F}_8^{(j,n-j)}(0,0)}{j!\,(n-j)!} = \sum_{l=0}^{j-1}\sum_{r=0}^{n-j} \alpha_{l,r} \frac{\hat{F}_7^{(j-1-l,n-j-r)}(0,0)}{(j-1-l)!\,(n-j-r)!}. \tag{29}$$

It follows from (26), (28), and the induction hypothesis that

$$\left| \frac{\hat{F}_7^{(j-1-l,n-j-r)}(0,0)}{(j-1-l)!\,(n-j-r)!} \right| \le \frac{\tilde{D}_{8,j-1-l,n-j-r}}{m^{2(n-l-r)-j+l}}, \qquad |\alpha_{l,r}| \le \frac{C_{l,r}}{m^{2(l+r)+1-l}},$$

where

$$\tilde{D}_{8,j-1-l,n-j-r} = D_{7,j-1-l,n-j-r} + 6C_{j-1-l,n-j-r}$$

(here we have used that $m \ge 3$). Then by (29) we finally get

$$\frac{|\hat{F}_8^{(j,n-j)}(0,0)|}{j!\,(n-j)!} \le \sum_{l=0}^{j-1}\sum_{r=0}^{n-j} \frac{\tilde{D}_{8,j-1-l,n-j-r}C_{l,r}}{m^{2n+1-j}} = \frac{D_{8,j,n}}{m^{2n+1-j}} \tag{30}$$

for some constant $D_{8,j,n} > 0$.

Step 9. Finally we write $\hat{F}_9(\varepsilon,\delta) = \delta h(\varepsilon,\delta)$. Proceeding as in (18), we get

$$\begin{aligned}\hat{F}_9^{(j,n-j)}(0,0) &= (n-j)h^{(j,n-j-1)}(0,0) \\ &= \frac{3}{2}(n-j)(\alpha^2\hat{F}_7)^{(j,n-j-1)}(0,0).\end{aligned}$$

Then, in a similar manner as in (19), we have

$$\frac{\hat{F}_9^{(j,n-j)}(0,0)}{j!\,(n-j)!} = \frac{3}{2}\sum_{l=0}^{n-j-1}\sum_{r=0}^{l}\sum_{s=0}^{j}\sum_{t=0}^{s} \alpha_{t,r}\alpha_{s-t,l-r} \frac{\hat{F}_7^{(j-s,n-j-1-l)}(0,0)}{(j-s)!\,(n-j-1-l)!}. \tag{31}$$

It follows from (26), (28), and the induction hypothesis that

$$\left|\frac{\hat{F}_7^{(j-s,n-j-1-l)}(0,0)}{(j-s)!\,(n-j-1-l)!}\right| \le \frac{\tilde{D}_{8,j-s,n-j-1-l}}{m^{2(n-1-l-s)+1-j+s}}$$

and

$$|\alpha_{t,r}|\cdot|\alpha_{s-t,l-r}| \le \frac{C_{t,r}C_{s-t,l-r}}{m^{2(s+l)+2-s}}.$$

Then by (31) we get

$$\begin{aligned}\frac{|\hat{F}_9^{(j,n-j)}(0,0)|}{j!\,(n-j)!} &\le \frac{3}{2}\sum_{l=0}^{n-j-1}\sum_{r=0}^{l}\sum_{s=0}^{j}\sum_{t=0}^{s}\frac{\tilde{D}_{8,j-s,n-j-1-l}C_{t,r}C_{s-t,l-r}}{m^{2n+1-j}}\\ &= \frac{D_{9,j,n}}{m^{2n+1-j}} \qquad (32)\end{aligned}$$

for some constant $D_{9,j,n} > 0$.

Now we observe that, given (9) and (26), the identity $\hat{J}^{(j,n-j)}(0,0) = 0$ is equivalent to

$$\begin{aligned}-\frac{2j!\,(n-j)!\,m(m-1)}{m-2}\alpha_{j,n-j} = A_{j,n-j} &+ B_{j,n-j} + C_{j,n-j} - \hat{F}_2^{(j,n-j)}(0,0)\\ &- \hat{F}_3^{(j,n-j)}(0,0) - \hat{F}_4^{(j,n-j)}(0,0)\\ &- \hat{F}_5^{(j,n-j)}(0,0) - \hat{F}_6^{(j,n-j)}(0,0)\\ &- \hat{F}_8^{(j,n-j)}(0,0) - \hat{F}_9^{(j,n-j)}(0,0). \qquad (33)\end{aligned}$$

In particular, the constant $\alpha_{j,n-j}$ is determined by (33) (after determining $\alpha_{l,k-l}$ for $k = 1,\dots,n-1$ and $l = 0,\dots,k$). By (10), (14), (17), (20), (22), (24), (28), (30), and (32) we obtain

$$\begin{aligned}|\alpha_{j,n-j}| &\le \frac{m-2}{(m-1)m^{2n+2-j}}\sum_{i=1}^{9}D_{i,j,n}\\ &\le \frac{1}{m^{2n+2-j}}\sum_{i=1}^{9}D_{i,j,n} \le \frac{C_{j,n-j}}{m^{2n+1-j}}\end{aligned}$$

with $C_{j,n-j} = \sum_{i=1}^{9} D_{i,j,n}$. This completes the proof of Lemma 1. □

LEMMA 2. *For every sufficiently small ε and δ, the function $G_{\varepsilon,\delta}$ attains its minimum at $\alpha = \alpha(\varepsilon,\delta)$.*

Proof. By construction we have $G'_{\varepsilon,\delta}(\alpha(\varepsilon,\delta)) = 0$. To complete the proof it is enough to show that $G''_{\varepsilon,\delta}(\alpha(\varepsilon,\delta)) > 0$ for every sufficiently small ε and δ. For this it is enough to note that, by (3),

$$G''_{0,0}(\alpha(0,0)) = \frac{2m^2}{m-2} > 0,$$

since by continuity this implies the desired statement. □

LEMMA 3. *For each $n \geq 1$ and $k = 0, \ldots, n$, there exists a $D_{k,n} > 0$ (independent of m) such that $|\beta_{k,n-k}(m)| \leq D_{k,n}/m^{2n+1-k}$ for every $m > 2$.*

Proof. By (8) and (11), for the functions $f(z) = z \log z$ and $z = z(\varepsilon, \delta)$ and for each $n \geq 1$ and $j = 0, \ldots, n$, we obtain

$$\begin{aligned} f^{(j,n-j)}(0,0) &= \frac{\partial^{n-j}}{\partial \delta^{n-j}} \sum_{l=0}^{j} \binom{j}{l} z^{(l,0)} (\log z)^{(j-l,0)}(0,0) \\ &= \sum_{l=0}^{j} \binom{j}{l} \sum_{r=0}^{n-j} \binom{n-j}{r} z^{(l,r)} (\log z)^{(j-l,n-j-r)}(0,0) \\ &= \sum_{l=0}^{j} \binom{j}{l} \sum_{r=0}^{n-j} \binom{n-j}{r} z^{(l,r)}(0,0) \\ &\quad \times \left(\frac{(j-l)!\,(n-j-r)!}{z(0,0)} z^{(j-l,n-j-r)}(0,0) + h(m)_{j-l,n-j-r} \right), \end{aligned}$$

where

$$h(m)_{j-l,n-j-r} = (j-l)!\,(n-j-r)!\,\hat{h}(m)_{j-l,n-j-r},$$

with

$$\hat{h}(m)_{j-l,n-j-r} = \sum_{\lambda=2}^{n-l-r} \Gamma_\lambda m^\lambda \sum_{s=1}^{n-l-r} \sum_{p_s(j-l,\lambda)} \prod_{i=1}^{s} \left(\frac{1}{l_{i,1}!\,l_{i,2}!} \cdot \frac{\partial^{l_{i,1}+l_{i,2}} z}{\partial \varepsilon^{l_{i,1}} \partial \delta^{l_{i,2}}} \right)^{k_i} \frac{1}{k_i!}$$

and $\Gamma_\lambda = (-1)^{\lambda-1}(\lambda-1)!$.

Now we write $G_{\varepsilon,\delta}(\alpha) = \sum_{i=1}^{3} G_i(\alpha, \varepsilon, \delta)$, where

$$\begin{aligned} G_1(\alpha, \varepsilon, \delta) &= \alpha \log \alpha, \\ G_2(\alpha, \varepsilon, \delta) &= (\alpha + \varepsilon\alpha^2 + \delta\alpha^3) \log(\alpha + \varepsilon\alpha^2 + \delta\alpha^3), \\ G_3(\alpha, \varepsilon, \delta) &= (1 - 2\alpha - \varepsilon\alpha^2 - \delta\alpha^3) \log\left(\frac{1 - 2\alpha - \varepsilon\alpha^2 - \delta\alpha^3}{m-2} \right). \end{aligned}$$

We also set $\hat{G}_j(\varepsilon, \delta) = G_j(\alpha(\varepsilon, \delta), \varepsilon, \delta)$ for each j.

Step 1. For G_1 we set $z(\varepsilon, \delta) = \alpha(\varepsilon, \delta)$. We have $z(0,0) = \alpha_{0,0} = 1/m$ and

$$|z^{(l_1,l_2)}(0,0)| = |l_1!\,l_2!\,\alpha_{l_1,l_2}| \leq l_1!\,l_2!\,C_{l_1,l_2}/m^{2(l_1+l_2)+1-l_1}.$$

Therefore,

$$\begin{aligned} |\hat{G}_1^{(j,n-j)}(0,0)| &\leq \sum_{l=0}^{j} \binom{j}{l} \sum_{r=0}^{n-j} \binom{n-j}{r} l!\,r!\,\frac{C_{l,r}}{m^{2(l+r)+1-l}} (j-l)!\,(n-j-r)! \\ &\quad \times \left(m \frac{C_{j-l,n-j-r}}{m^{2(n-l-r)+1-j+l}} + \hat{h}_1(m)_{j-l,n-j-r} \right) \\ &\leq j!\,(n-j)! \sum_{l=0}^{j} \sum_{r=0}^{n-j} \frac{C_{l,r} C_{j-l,n-j-r}}{m^{2n+1-j}} \\ &\quad + j!\,(n-j)! \sum_{l=0}^{j} \sum_{r=0}^{n-j} \frac{C_{l,r}}{m^{2(l+r)+1-l}} \hat{h}_1(m)_{j-l,n-j-r}, \end{aligned}$$

with

$$\begin{aligned}
\hat{h}_1(m)_{j-l,n-j-r} &\le \sum_{\lambda=2}^{n-l-r} |\Gamma_\lambda| m^\lambda \sum_{s=1}^{n-l-r} \sum_{p_s(j-l,\lambda)} \prod_{i=1}^{s} \frac{C_{l_{i,1},l_{i,2}}^{k_i}}{m^{(2k_i(l_{i,1}+l_{i,2})+k_i-l_{i,1}k_i)} k_i!} \\
&\le \sum_{\lambda=2}^{n-l-r} \frac{|\Gamma_\lambda| m^\lambda}{m^{2(n-l-r)+\lambda-j+l}} \sum_{s=1}^{n-l-r} \sum_{p_s(j-l,\lambda)} \prod_{i=1}^{s} \frac{C_{l_{i,1},l_{i,2}}^{k_i}}{k_i!} \\
&\le \frac{1}{m^{2(n-l-r)-j+l}} \sum_{\lambda=2}^{n-l-r} |\Gamma_\lambda| \sum_{s=1}^{n-l-r} \sum_{p_s(j-l,\lambda)} \prod_{i=1}^{s} \frac{C_{l_{i,1},l_{i,2}}^{k_i}}{k_i!} \\
&= \frac{D_{1,j-l,n-j-r}}{m^{2(n-l-r)-j+l}}
\end{aligned}$$

(see (10)). This implies that

$$\begin{aligned}
|\hat{G}_1^{(j,n-j)}(0,0)| &\le \frac{j!\,(n-j)!}{m^{2n+1-j}} \sum_{l=0}^{j} \sum_{r=0}^{n-j} C_{l,r}(C_{j-l,n-j-r} + D_{1,j-l,n-j-r}) \\
&= \frac{j!\,(n-j)!\,E_{1,j,n-j}}{m^{2n+1-j}} \qquad (34)
\end{aligned}$$

for some constant $E_{1,j,n-j} > 0$.

Step 2. For G_2 we set

$$z(\varepsilon,\delta) = \alpha(\varepsilon,\delta) + \varepsilon\alpha(\varepsilon,\delta)^2 + \delta\alpha(\varepsilon,\delta)^3.$$

We again have $z(0,0) = \alpha_{0,0} = 1/m$. Proceeding as in Step 7 of the proof of Lemma 1 (see (25)), we obtain

$$\begin{aligned}
\left|\frac{z^{(l_1,l_2)}(0,0)}{l_1!\,l_2!}\right| &\le |\alpha_{l_1,l_2}| + 2\sum_{l=0}^{l_1-1}\sum_{r=0}^{l_2} |\alpha_{l,r}\alpha_{l_1-1-l,l_2-r}| \\
&\quad + \sum_{l=0}^{l_1}\sum_{r=0}^{l_2-1}\sum_{s=0}^{l_1-l}\sum_{t=0}^{l_2-1-r} |\alpha_{l,r}\alpha_{s,t}\alpha_{l_1-l-s,l_2-1-r-t}| \\
&\le \frac{C_{l_1,l_2}}{m^{2(l_1+l_2)+1-l_1}} + \sum_{l=0}^{l_1-1}\sum_{r=0}^{l_2} \frac{C_{l,r}C_{l_1-1-l,l_2-r}}{m^{2(l_1+l_2)+1-l_1}} \\
&\quad + \sum_{l=0}^{l_1}\sum_{r=0}^{l_2-1}\sum_{s=0}^{l_1-l}\sum_{t=0}^{l_2-1-r} \frac{C_{l,r}C_{s,t}C_{l_1-l-s,l_2-1-r-t}}{m^{2(l_1+l_2)+1-l_1}} \\
&= \frac{\hat{F}_{2,l_1,l_2}}{m^{2(l_1+l_2)+1-l_1}}
\end{aligned}$$

for some constant $\hat{F}_{2,l_1,l_2} > 0$. Therefore,

$$
\begin{aligned}
|\hat{G}_2^{(j,n-j)}(0,0)| &\le \sum_{l=0}^{j}\binom{j}{l}\sum_{r=0}^{n-j}\binom{n-j}{r} l!\,r!\,\frac{\hat{F}_{2,l,r}}{m^{2(l+r)+1-l}}(j-l)!\,(n-j-r)! \\
&\quad \times \left(m\frac{\hat{F}_{2,j-l,n-j-r}}{m^{2(n-l-r)+1-j+l}}+\hat{h}_2(m)_{j-l,n-j-r}\right) \\
&\le j!\,(n-j)!\sum_{l=0}^{j}\sum_{r=0}^{n-j}\frac{\hat{F}_{2,l,r}\hat{F}_{2,j-l,n-j-r}}{m^{2n+1-j}} \\
&\quad + j!\,(n-j)!\sum_{l=0}^{j}\sum_{r=0}^{n-j}\frac{\hat{F}_{2,l,r}}{m^{2(l+r)+1-l}}\hat{h}_2(m)_{j-l,n-j-r},
\end{aligned}
$$

with

$$
\begin{aligned}
\hat{h}_2(m)_{j-l,n-j-r} &\le \sum_{\lambda=2}^{n-l-r}|\Gamma_\lambda| m^\lambda \sum_{s=1}^{n-l-r}\sum_{p_s(j-l,\lambda)}\prod_{i=1}^{s}\frac{\hat{F}_{2,l_{i,1},l_{i,2}}^{k_i}}{m^{(2k_i(l_{i,1}+l_{i,2})+k_i-l_{i,1}k_i)}k_i!} \\
&\le \sum_{\lambda=2}^{n-l-r}\frac{|\Gamma_\lambda| m^\lambda}{m^{2(n-l-r)+\lambda-j+l}}\sum_{s=1}^{n-l-r}\sum_{p_s(j-l,\lambda)}\prod_{i=1}^{s}\frac{\hat{F}_{2,l_{i,1},l_{i,2}}^{k_i}}{k_i!} \\
&\le \frac{1}{m^{2(n-l-r)-j+l}}\sum_{\lambda=2}^{n-l-r}|\Gamma_\lambda|\sum_{s=1}^{n-l-r}\sum_{p_s(j-l,\lambda)}\prod_{i=1}^{s}\frac{\hat{F}_{2,l_{i,1},l_{i,2}}^{k_i}}{k_i!} \\
&= \frac{F_{2,j-l,n-j-r}}{m^{2(n-l-r)-j+l}}
\end{aligned}
$$

for some constant $F_{2,j-l,n-j-r} > 0$. This implies that

$$
\begin{aligned}
|\hat{G}_2^{(j,n-j)}(0,0)| &\le \frac{j!\,(n-j)!}{m^{2n+1-j}}\sum_{l=0}^{j}\sum_{r=0}^{n-j}\hat{F}_{2,l,r}(\hat{F}_{2,j-l,n-j-r}+F_{2,j-l,n-j-r}) \\
&= \frac{j!\,(n-j)!\,E_{2,j,n-j}}{m^{2n+1-j}} \qquad (35)
\end{aligned}
$$

for some constant $E_{2,j,n-j} > 0$.

Step 3. Finally, for G_3, since

$$
\begin{aligned}
&\frac{\partial^n}{\partial\varepsilon^j\partial\delta^{n-j}}\log\frac{1-2\alpha(\varepsilon,\delta)-\varepsilon\alpha(\varepsilon,\delta)^2-\delta\alpha(\varepsilon,\delta)^3}{m-2} \\
&\qquad = \frac{\partial^n}{\partial\varepsilon^j\partial\delta^{n-j}}\log[1-2\alpha(\varepsilon,\delta)-\varepsilon\alpha(\varepsilon,\delta)^2-\delta\alpha(\varepsilon,\delta)^3],
\end{aligned}
$$

we can set

$$
z(\varepsilon,\delta) = 1-2\alpha(\varepsilon,\delta)-\varepsilon\alpha(\varepsilon,\delta)^2-\delta\alpha(\varepsilon,\delta)^3.
$$

Then $z(0,0) = 1-2\alpha_{0,0} = (m-2)/m$ and, proceeding as for G_2, we obtain

$$\left|\frac{z^{(l_1,l_2)}(0,0)}{l_1!\,l_2!}\right| \le 2|\alpha_{l_1,l_2}| + \sum_{l=0}^{l_1-1}\sum_{r=0}^{l_2}|\alpha_{l,r}\alpha_{l_1-1-l,l_2-r}| + \sum_{l=0}^{l_1}\sum_{r=0}^{l_2-1}\sum_{s=0}^{l_1-l}\sum_{t=0}^{l_2-1-r}|\alpha_{l,r}\alpha_{s,t}\alpha_{l_1-l-s,l_2-1-r-t}| \le \frac{\hat{F}_{3,l_1,l_2}}{m^{2(l_1+l_2)+1-l_1}}$$

for some constant $\hat{F}_{3,l_1,l_2} > 0$. Therefore, since $1/(m-2)^\lambda \le 1$ for $m \ge 3$, for any $\lambda \ge 1$ we have

$$|\hat{G}_3^{(j,n-j)}(0,0)| \le \sum_{l=0}^{j}\binom{j}{l}\sum_{r=0}^{n-j}\binom{n-j}{r} l!\,r!\,\frac{\hat{F}_{3,l,r}}{m^{2(l+r)+1-l}}(j-l)!\,(n-j-r)! \times\left(\frac{m}{m-2}\,\frac{\hat{F}_{3,j-l,n-j-r}}{m^{2(n-l-r)+1-j+l}} + \hat{h}_3(m)_{j-l,n-j-r}\right) \le j!\,(n-j)!\sum_{l=0}^{j}\sum_{r=0}^{n-j}\frac{\hat{F}_{3,l,r}\hat{F}_{2,j-l,n-j-r}}{m^{2n+1-j}} + j!\,(n-j)!\sum_{l=0}^{j}\sum_{r=0}^{n-j}\frac{\hat{F}_{3,l,r}}{m^{2(l+r)+1-l}}\hat{h}_3(m)_{j-l,n-j-r}$$

with

$$\hat{h}_2(m)_{j-l,n-j-r} \le \sum_{\lambda=2}^{n-l-r}|\Gamma_\lambda|\frac{m^\lambda}{(m-2)^\lambda}\sum_{s=1}^{n-l-r}\sum_{p_s(j-l,\lambda)}\prod_{i=1}^{s}\frac{\hat{F}^{k_i}_{2,l_{i,1},l_{i,2}}}{m^{(2k_i(l_{i,1}+l_{i,2})+k_i-l_{i,1}k_i)}k_i!} \le \sum_{\lambda=2}^{n-l-r}\frac{|\Gamma_\lambda|m^\lambda}{m^{2(n-l-r)+\lambda-j+l}}\sum_{s=1}^{n-l-r}\sum_{p_s(j-l,\lambda)}\prod_{i=1}^{s}\frac{\hat{F}^{k_i}_{3,l_{i,1},l_{i,2}}}{k_i!} = \frac{1}{m^{2(n-l-r)-j+l}}\sum_{\lambda=2}^{n-l-r}|\Gamma_\lambda|\sum_{s=1}^{n-l-r}\sum_{p_s(j-l,\lambda)}\prod_{i=1}^{s}\frac{\hat{F}^{k_i}_{3,l_{i,1},l_{i,2}}}{k_i!} = \frac{F_{3,j-l,n-j-r}}{m^{2(n-l-r)-j+l}}$$

for some constant $F_{3,j-l,n-j-r} > 0$. This implies that

$$|\hat{G}_3^{(j,n-j)}(0,0)| \le \frac{j!\,(n-j)!}{m^{2n+1-j}}\sum_{l=0}^{j}\sum_{r=0}^{n-j}\hat{F}_{3,l,r}(\hat{F}_{3,j-l,n-j-r} + F_{3,j-l,n-j-r}) = \frac{j!\,(n-j)!\,E_{3,j,n-j}}{m^{2n+1-j}} \tag{36}$$

for some constant $E_{3,j,n-j} > 0$.

Now we observe that, for $n \geq 1$ and $j = 0, \ldots, n$, we have

$$\beta_{j,n-j}(m) = \frac{1}{j!\,(n-j)!} \frac{\partial^n}{\partial \varepsilon^j \partial \delta^{n-j}} G_\varepsilon(\alpha(\varepsilon))|_{(\varepsilon,\delta)=(0,0)}$$
$$= \frac{1}{j!\,(n-j)!} \sum_{l=1}^{3} \hat{G}_l^{(j,n-j)}(0,0),$$

and by (34), (35), and (36) we obtain

$$|\beta_{j,n-j}(m)| \leq \frac{1}{m^{2n+1-l}} (E_{1,j,n-l} + E_{2,j,n-l} + E_{3,j,n-l}).$$

This establishes the desired estimate, proving Lemma 3. □

The statement of Theorem 1 follows readily from Lemmas 1–3. □

3. Further Results

Our approach also allows to compute the Taylor coefficients explicitly.

PROPOSITION 1. *We have*

$$\alpha(\varepsilon,\delta) = \frac{1}{m} - \frac{1}{2m^2}\varepsilon - \frac{1}{2m^3}\delta + \frac{6+m}{4m^4}\varepsilon^2 + \frac{8+m}{4m^5}\varepsilon\delta + \frac{10+m}{4m^6}\delta^2 + O(\|(\varepsilon,\delta)\|^3) \tag{37}$$

and

$$\dim_H F_{\varepsilon,\delta} = -\frac{G_{\varepsilon,\delta}(\alpha(\varepsilon,\delta))}{\log m}$$
$$= 1 - \frac{1}{2m^3 \log m}\left(\varepsilon^2 + \frac{\varepsilon\delta}{m} + \frac{\delta^2}{m^2}\right) + O(\|(\varepsilon,\delta)\|^3).$$

Proof. We know from the proof of Theorem 1 that

$$\alpha_{0,0} = 1/m, \quad \alpha_{1,0} = -1/(2m^2), \quad \text{and} \quad \alpha_{0,1} = -1/(2m^3). \tag{38}$$

To determine additional coefficients we use (33). For example: to compute $\alpha_{2,0}$ we solve $\partial^2 \hat{J}/\partial\varepsilon^2(0,0) = 0$; to compute $\alpha_{1,1}$ we solve $\partial^2 \hat{J}/\partial\varepsilon\partial\delta(0,0) = 0$; and, finally, to compute $\alpha_{0,2}$ we solve $\partial^2 \hat{J}/\partial\delta^2(0,0) = 0$.

It follows from (38) that

$$\frac{\partial^2 \hat{F}_1}{\partial\varepsilon^2}(0,0) = -\frac{1}{2m^2} + 2m\alpha_{2,0}, \qquad \frac{\partial^2 \hat{F}_2}{\partial\varepsilon^2}(0,0) = -\frac{2}{m^2},$$
$$\frac{\partial^2 \hat{F}_3}{\partial\varepsilon^2}(0,0) = -\frac{2}{m^2}\left(1 + \log\left(\frac{m-2}{m}\right)\right), \qquad \frac{\partial^2 \hat{F}_4}{\partial\varepsilon^2}(0,0) = 0,$$
$$\frac{\partial^2 \hat{F}_5}{\partial\varepsilon^2}(0,0) = \frac{4}{m^2}, \quad \frac{\partial^2 \hat{F}_6}{\partial\varepsilon^2}(0,0) = 0, \quad \frac{\partial^2 \hat{F}_7}{\partial\varepsilon^2}(0,0) = \frac{4(-1+m^3\alpha_{2,0})}{m^2(m-2)},$$
$$\frac{\partial^2 \hat{F}_8}{\partial\varepsilon^2}(0,0) = \frac{2}{m^2}\log\left(\frac{m-2}{m}\right), \qquad \frac{\partial^2 \hat{F}_9}{\partial\varepsilon^2}(0,0) = 0.$$

Thus,

$$0 = \sum_{j=1}^{9} \frac{\partial^2 \hat{F}_j}{\partial \varepsilon^2}(0,0) = \frac{6+m-4m^4\alpha_{2,0}}{4m^2-2m^3} \quad \text{and} \quad \alpha_{2,0} = \frac{6+m}{4m^4}.$$

Furthermore,

$$\frac{\partial^2 \hat{F}_1}{\partial \varepsilon \partial \delta}(0,0) = -\frac{1}{2m^3} + 2m\alpha_{1,1}, \qquad \frac{\partial^2 \hat{F}_2}{\partial \varepsilon \partial \delta}(0,0) = -\frac{5}{2m^3},$$

$$\frac{\partial^2 \hat{F}_3}{\partial \varepsilon \partial \delta}(0,0) = -\frac{1}{m^3}\left(1 + \log\left(\frac{m-2}{m}\right)\right),$$

$$\frac{\partial^2 \hat{F}_4}{\partial \varepsilon \partial \delta}(0,0) = -\frac{3}{2m^3}\left(3 + 6\log\left(\frac{m-2}{m}\right)\right), \qquad \frac{\partial^2 \hat{F}_5}{\partial \varepsilon \partial \delta}(0,0) = \frac{2}{m^3},$$

$$\frac{\partial^2 \hat{F}_6}{\partial \varepsilon \partial \delta}(0,0) = \frac{3}{m^3}, \qquad \frac{\partial^2 \hat{F}_7}{\partial \varepsilon \partial \delta}(0,0) = \frac{-5+4m^4\alpha_{1,1}}{m^3(m-2)},$$

$$\frac{\partial^2 \hat{F}_8}{\partial \varepsilon \partial \delta}(0,0) = \frac{1}{m^3}\log\left(\frac{m-2}{m}\right), \qquad \frac{\partial \hat{F}_9}{\partial \varepsilon \partial \delta}(0,0) = \frac{3}{m^3}\log\left(\frac{m-2}{m^2}\right).$$

Thus,

$$0 = \sum_{j=1}^{9} \frac{\partial^2 \hat{F}_j}{\partial \varepsilon \partial \delta}(0,0) = \frac{8+m-4m^5\alpha_{1,1}}{4m^3-2m^4} \quad \text{and} \quad \alpha_{1,1} = \frac{8+m}{4m^5}.$$

Finally,

$$\frac{\partial^2 \hat{F}_1}{\partial \delta^2}(0,0) = -\frac{1}{2m^4} + 2m\alpha_{0,2}, \qquad \frac{\partial^2 \hat{F}_2}{\partial \delta^2}(0,0) = -\frac{3}{m^4},$$

$$\frac{\partial^2 \hat{F}_3}{\partial \delta^2}(0,0) = 0, \qquad \frac{\partial^2 \hat{F}_4}{\partial \delta^2}(0,0) = -\frac{1}{m^4}\left(3 + 6\log\left(\frac{m-2}{m}\right)\right),$$

$$\frac{\partial \hat{F}_5}{\partial \delta^2}(0,0) = 0, \quad \frac{\partial^2 \hat{F}_6}{\partial \delta^2}(0,0) = \frac{6}{m^4}, \quad \frac{\partial^2 \hat{F}_7}{\partial \delta^2}(0,0) = \frac{-6+4m^5\alpha_{0,2}}{m^4(m-2)},$$

$$\frac{\partial^2 \hat{F}_8}{\partial \delta^2}(0,0) = 0, \qquad \frac{\partial \hat{F}_9}{\partial \delta^2}(0,0) = \frac{6}{m^4}\log\left(\frac{m-2}{m^2}\right).$$

Thus,

$$0 = \sum_{j=1}^{9} \frac{\partial^2 \hat{F}_j}{\partial \varepsilon \partial \delta}(0,0) = \frac{10+m-4m^6\alpha_{0,2}}{4m^4-2m^5} \quad \text{and} \quad \alpha_{0,1} = \frac{10+m}{4m^6}.$$

This establishes (37).

To find the Taylor series of $(\varepsilon, \delta) \mapsto \dim_H F_{\varepsilon,\delta}$, we first note that

$$\hat{G}_1(0,0) = -\frac{\log m}{m}, \quad \hat{G}_2(0,0) = -\frac{\log m}{m}, \quad \hat{G}_3(0,0) = -\frac{(m-2)\log m}{m}$$

and thus $\beta_{0,0}(m) = -\log m$. By (38), we have

$$\frac{\partial \hat{G}_1}{\partial \varepsilon} = -\frac{1-\log m}{2m^2}, \quad \frac{\partial \hat{G}_2}{\partial \varepsilon} = \frac{1-\log m}{2m^2}, \quad \frac{\partial \hat{G}_3}{\partial \varepsilon} = 0$$

and thus

$$\beta_{1,0}(m) = \sum_{j=1}^{3} \frac{\partial \hat{G}_j}{\partial \varepsilon}(0,0) = 0.$$

In a similar manner, we have

$$\frac{\partial \hat{G}_1}{\partial \delta} = -\frac{1-\log m}{2m^3}, \quad \frac{\partial \hat{G}_2}{\partial \delta} = \frac{1-\log m}{2m^3}, \quad \frac{\partial \hat{G}_3}{\partial \delta} = 0$$

and

$$\beta_{0,1}(m) = \sum_{j=1}^{3} \frac{\partial \hat{G}_j}{\partial \delta}(0,0) = 0.$$

Now we consider the terms of second order. Since

$$\begin{aligned}
\frac{\partial^2 \hat{G}_1}{\partial \varepsilon^2} &= \frac{1}{4m^4}(2(m+3) - (6+m)\log m), \\
\frac{\partial^2 \hat{G}_2}{\partial \varepsilon^2} &= \frac{1}{4m^4}(6 - 6m + (6-7m)\log m), \\
\frac{\partial^2 \hat{G}_3}{\partial \varepsilon^2} &= \frac{3(m-2)}{2m^4}(1 - \log m),
\end{aligned}$$

we obtain

$$\beta_{2,0}(m) = \frac{1}{2}\sum_{j=1}^{3} \frac{\partial^2 \hat{G}_j}{\partial \varepsilon^2}(0,0) = \frac{1}{2m^3}.$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 \hat{G}_1}{\partial \varepsilon \partial \delta} &= \frac{1}{4m^5}(2(m+4) - (8+m)\log m), \\
\frac{\partial^2 \hat{G}_2}{\partial \varepsilon \partial \delta} &= \frac{1}{4m^5}(8 - 8m + (8-9m)\log m), \\
\frac{\partial^2 \hat{G}_3}{\partial \varepsilon \partial \delta} &= \frac{2(m-2)}{2m^5}(1 - \log m)
\end{aligned}$$

and thus

$$\beta_{1,1}(m) = \sum_{j=1}^{3} \frac{\partial^2 \hat{G}_j}{\partial \varepsilon \partial \delta}(0,0) = \frac{1}{2m^4}.$$

Finally, since

$$\begin{aligned}
\frac{\partial^2 \hat{G}_1}{\partial \delta^2} &= \frac{1}{4m^6}(2(m+5) - (10+m)\log m), \\
\frac{\partial^2 \hat{G}_2}{\partial \delta^2} &= -\frac{1}{4m^6}(10(m-1) + (10-11m)\log m), \\
\frac{\partial^2 \hat{G}_3}{\partial \delta^2} &= \frac{5(m-2)}{2m^6}(1 - \log m),
\end{aligned}$$

we obtain

$$\beta_{0,2}(m) = \frac{1}{2} \sum_{j=1}^{3} \frac{\partial^2 \hat{G}_j}{\partial \delta^2}(0,0) = \frac{1}{2m^5}.$$

This completes the proof of the proposition. □

Now, as an illustration, we consider the function

$$f_{\varepsilon,\delta,\mu}(x) = x + \varepsilon x^2 + \delta x^3 + \mu x^4$$

and the set

$$F_{\varepsilon,\delta,\mu} = \{x \in [0,1] : \tau_1(x) = f_{\varepsilon,\delta,\mu}(\tau_0(x))\}.$$

In a similar manner to that as before, we have

$$\dim_H F_{\varepsilon,\delta,\mu} = -\frac{1}{\log m} \inf\{G_{\varepsilon,\delta,\mu}(\alpha) : \alpha \in [0,1] \text{ and } \alpha + f_{\varepsilon,\delta,\mu}(\alpha) \in [0,1]\},$$

where

$$\begin{aligned} G_{\varepsilon,\delta,\mu}(\alpha) &= \alpha \log \alpha + f_{\varepsilon,\delta,\mu}(\alpha) \log f_{\varepsilon,\delta,\mu}(\alpha) \\ &\quad + (1 - \alpha - f_{\varepsilon,\delta,\mu}(\alpha)) \log \frac{1 - \alpha - f_{\varepsilon,\delta,\mu}(\alpha)}{m-2}. \end{aligned}$$

One can also look for an analytic function

$$\alpha(\varepsilon,\delta,\mu) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \alpha_{k,l,n-k-l}(m) \varepsilon^k \delta^l \mu^{n-k-l}$$

in a neighborhood of $(0,0)$ such that

$$\dim_H F_{\varepsilon,\delta,\mu} = -\frac{1}{\log m} G_{\varepsilon,\delta,\mu}(\alpha(\varepsilon,\delta,\mu)).$$

We compute the Taylor coefficients assuming that the existence of such an analytic function has already been established.

PROPOSITION 2. *We have*

$$\begin{aligned} \alpha(\varepsilon,\delta,\mu) &= \frac{1}{m} - \frac{1}{2m^2}\varepsilon - \frac{1}{2m^3}\delta - \frac{1}{2m^4}\mu + \frac{6+m}{4m^4}\varepsilon^2 + \frac{8+m}{4m^5}\varepsilon\delta \\ &\quad + \frac{10+m}{4m^6}\varepsilon\mu + \frac{10+m}{4m^6}\delta^2 + \frac{12+m}{4m^7}\delta\mu + \frac{14+m}{4m^8}\mu^2 \\ &\quad + O(\|(\varepsilon,\delta,\mu)\|^3) \end{aligned} \tag{39}$$

and

$$\begin{aligned} \dim_H F_{\varepsilon,\delta} = 1 - \frac{1}{2m^3 \log m} \left(\varepsilon^2 + \frac{1}{m}\varepsilon\delta + \frac{1}{m^2}\varepsilon\mu + \frac{1}{m^2}\delta^2 + \frac{1}{m^3}\delta\mu + \frac{1}{m^4}\mu^2 \right) \\ + O(\|(\varepsilon,\delta,\mu)\|^3). \end{aligned}$$

Proof. We consider the analytic function

$$J(\varepsilon,\delta,\mu) = G'_{\varepsilon,\delta,\mu}(\alpha(\varepsilon,\delta,\mu)),$$

which must be identically zero. We have

$$\begin{aligned}G'_{\varepsilon,\delta,\mu}(\alpha) = \log(\alpha(m-2))&\\ &+ \log(\alpha+\varepsilon\alpha^2+\delta\alpha^3+\mu\alpha^4)(1+2\varepsilon\alpha+3\delta\alpha^2+4\mu\alpha^3)\\ &- \log(1-2\alpha-\varepsilon\alpha^2-\delta\alpha^3-\mu\alpha^4)(2+2\varepsilon\alpha+3\delta\alpha^2+4\delta\alpha^3);\end{aligned}$$

in particular, $J(1/m,0,0)=0$. This shows that $\alpha_{0,0,0}(m)=1/m$. For the terms of first order, we note that

$$\frac{\partial J}{\partial \varepsilon}(0,0,0) = \frac{1+2m^2\alpha_{1,0,0}(m)}{m-2}$$

and hence $\alpha_{1,0,0}(m) = -1/(2m^2)$. In a similar manner, since

$$\frac{\partial J}{\partial \delta}(0,0,0) = \frac{1+2m^3\alpha_{0,1,0}(m)}{m(m-2)},$$

$$\frac{\partial J}{\partial \mu}(0,0,0) = \frac{1+2m^4\alpha_{0,0,1}(m)}{m^2(m-2)}$$

we obtain $\alpha_{0,1,0}(m) = -1/(2m^3)$ and $\alpha_{0,0,1}(m) = -1/(2m^4)$.

For the terms of second order, since

$$\frac{\partial^2 J}{\partial \varepsilon^2}(0,0,0) = \frac{-6-m+4m^4\alpha_{2,0,0}(m)}{2m^2(m-2)},$$

$$\frac{\partial^2 J}{\partial \varepsilon\partial\delta}(0,0,0) = \frac{-8-m+4m^5\alpha_{1,1,0}(m)}{2m^4(m-2)},$$

$$\frac{\partial^2 J}{\partial \varepsilon\partial\mu}(0,0,0) = \frac{-10-m+4m^6\alpha_{1,0,1}(m)}{2m^5(m-2)},$$

$$\frac{\partial^2 J}{\partial \delta^2}(0,0,0) = \frac{-10-m+4m^6\alpha_{0,2,0}(m)}{2m^5(m-2)},$$

$$\frac{\partial^2 J}{\partial \delta\partial\mu}(0,0,0) = \frac{-12-m+4m^7\alpha_{0,1,1}(m)}{2m^6(m-2)},$$

$$\frac{\partial^2 J}{\partial \mu^2}(0,0,0) = \frac{-14-m+4m^8\alpha_{0,0,2}(m)}{2m^6(m-2)}$$

we obtain

$$\alpha_{2,0,0}(m) = \frac{6+m}{4m^4}, \quad \alpha_{1,1,0}(m) = \frac{8+m}{4m^5}, \quad \alpha_{1,0,1}(m) = \frac{10+m}{4m^6},$$

$$\alpha_{0,2,0}(m) = \frac{10+m}{4m^6}, \quad \alpha_{0,1,1}(m) = \frac{12+m}{4m^7}, \quad \alpha_{0,0,2}(m) = \frac{14+m}{4m^8}.$$

This establishes (39). Finally, to obtain the last formula, we note that for the function

$$G(\varepsilon,\delta,\mu) = G_{\varepsilon,\delta,\mu}(\alpha(\varepsilon,\delta,\mu))$$

we have $G(0,0,0) = -\log m$,

$$\frac{\partial G}{\partial \varepsilon}(0,0,0) = \frac{\partial G}{\partial \delta}(0,0,0) = \frac{\partial G}{\partial \mu}(0,0,0) = 0,$$

and

$$\frac{\partial^2 G}{\partial \varepsilon^2}(0,0,0) = \frac{1}{m^3}, \qquad \frac{\partial^2 G}{\partial \varepsilon \partial \delta}(0,0,0) = \frac{1}{2m^4},$$
$$\frac{\partial^2 G}{\partial \varepsilon \partial \mu}(0,0,0) = \frac{1}{2m^5}, \qquad \frac{\partial^2 G}{\partial \delta^2}(0,0,0) = \frac{1}{m^5},$$
$$\frac{\partial^2 G}{\partial \delta \partial \mu}(0,0,0) = \frac{1}{2m^6}, \qquad \frac{\partial^2 G}{\partial \mu^2}(0,0,0) = \frac{1}{m^7}.$$

This completes the proof of Proposition 2. □

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