Michigan Math. J. 60 (2011), 545–559

The Rank of the Second Gaussian Map for General Curves

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Introduction

Let X be a smooth, projective curve of genus g and let $\mathcal{L}$ be a line bundle on X. Consider the product $X \times X$ with the projections p_1, p_2 to the factors and the natural morphism p to the symmetric product $X(2)$. One has $p_*(p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}) = \mathcal{L}^+ \oplus \mathcal{L}^-$, where $\mathcal{L}^\pm$ denotes the invariant and anti-invariant line bundles with respect to the involution $(x, y) \mapsto (y, x)$. One has $H^0(\mathcal{L}^+) \cong \operatorname{Sym}^2 H^0(\mathcal{L})$ and $H^0(\mathcal{L}^-) \cong \wedge^2 H^0(\mathcal{L})$. Restriction to the diagonal of $X(2)$ gives rise to the maps

$$\mu_{\mathcal{L},1}\colon \operatorname{Sym}^2 H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2}) \quad \text{and} \quad w_{\mathcal{L},1}\colon \wedge^2 H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X),$$

where K_X is the canonical bundle of X. Both maps have a well-known geometric meaning. The former is given by considering the map $\phi_{\mathcal{L}}\colon X \to \mathbb{P}^r := \mathbb{P}(H^0(\mathcal{L}))^*$ defined by the complete linear series determined by $\mathcal{L}$ and by pulling forms of degree 2 in $\mathbb{P}^r$ back to X. The latter is given by considering the composition γ of $\phi_{\mathcal{L}}$ with the *Gauss map* of X to the Grassmannian of lines $\mathbb{G}(1, r)$ and by pulling forms of degree 1 in $\mathbb{P}^{\binom{r+1}{2}-1}$ back to X via γ.

The maps $\mu_{\mathcal{L},1}$ and $w_{\mathcal{L},1}$ are the first instances of two hierarchies of maps $\mu_{\mathcal{L},k}$ and $w_{\mathcal{L},k}$, which are defined for all positive integers k and are called by some authors *higher Gaussian maps* of X. They are inductively defined by iterated restrictions to the diagonal of $X(2)$. Precisely, for all $k \geq 2$ one has

$$\mu_{\mathcal{L},k}\colon \ker(\mu_{\mathcal{L},k-1}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X^{\otimes 2(k-1)}),$$

$$w_{\mathcal{L},k}\colon \ker(w_{\mathcal{L},k-1}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X^{\otimes(2k-1)}).$$

These maps are particularly interesting when $\mathcal{L} \cong K_X$, in which case we will simply denote them as μ_k and w_k. They are both defined at a general point of the moduli space of curves $\mathcal{M}_g$, and it is natural to suppose that they have some modular meaning. Indeed, μ_1 is the codifferential, at the point corresponding to X, of the Torelli map $\tau\colon \mathcal{M}_g \to \mathcal{A}_g$, and Noether's theorem says that μ_1 is surjective if and only if X is nonhyperelliptic.

The map w_1 is called the *Wahl map,* and it is related to important deformation and extendability properties of the canonical image of the curve (cf. [BMé; W]). Because of this, it has been studied by various authors—too many to be quoted

Received January 7, 2010. Revision received May 6, 2010.

here. One the most interesting results concerning the Wahl map is perhaps a theorem first proved by Ciliberto, Harris, and Miranda [CiHM] to the effect that w_1 is surjective, as expected, for a general curve of genus $g = 10$ and $g \geq 12$. Moreover, this map is injective, as expected, for a general curve of genus $g \leq 8$ (cf. [CiM1]). Unexpectedly, however, the Wahl map is not of maximal rank for a general curve of genus $g = 9, 11$.

In general, all maps μ_k and w_k are supposed to be meaningful in the geometry of curves, especially of curves with general moduli. Here we will look in particular at the map $\mu_2 : \mathcal{I}_2(K_X) \to H^0(X, K_X^{\otimes 4})$, where $\mathcal{I}_2(K_X)$ is the vector space of forms of degree 2 vanishing on the canonical model of X. From now on we will simply denote this map by μ, and we will call it the *second Gaussian map* of X. This map was first considered by Green and Griffiths (see [Gr]), and its importance stems from its relation to the second fundamental form of the moduli space of curves $\mathcal{M}_g$ embedded in $\mathcal{A}_g$ via the Torelli map (cf. [CF1; CF2; CPT]).

Despite the unexpected behavior of the Wahl map for genus $g = 9, 11$, a reasonable working hypothesis is that the second Gaussian map μ should be of maximal rank for a general curve of any genus g. A dimension count shows that this is equivalent to saying that μ should be injective for a general curve of genus $g \leq 17$ and surjective if $g \geq 18$. So far, the best result in this direction has been proved by Colombo and Frediani in [CF3], where—by studying hyperplane sections of high genera of K3 surfaces—they show that μ is surjective for a general curve of genus $g > 152$. For other interesting results concerning μ, see also [CF2; CFPa].

In this paper, we prove the maximal rank property for every genus.

THEOREM 1. *The second Gaussian map* $\mu : \mathcal{I}_2(K_X) \to H^0(X, K_X^{\otimes 4})$ *for X a general curve of any genus g has maximal rank*; *namely, it is injective for g* ≤ 17 *and surjective for g* ≥ 18.

As shown in [CPT], the map μ has a lifting $\rho : \mathcal{I}_2(K_X) \to \mathrm{Sym}^2(H^0(K_X^{\otimes 2}))$, which is the datum of the second fundamental form of the Torelli embedding at the point corresponding to X in the nonhyperelliptic case. As proved in [CF2, Cor. 3.4], ρ is injective for *all* nonhyperelliptic curves X. Our result shows that if X is general then the image of ρ is transversal to the kernel of the multiplication map $\mathrm{Sym}^2(H^0(K_X^{\otimes 2})) \to H^0(K_X^{\otimes 4})$.

The proof of Theorem 1 is by degeneration to a reducible nodal curve for which the limit of μ, described in Section 1, has maximal rank. The theorem then follows by upper semicontinuity. We do not use graph curves here (i.e., the curves exploited in [CiHM]) because for them the limit of μ is more difficult to understand. We used instead a general *binary curve*—in other words, a stable curve of genus g consisting of two rational components meeting at $g + 1$ points that are general on both components. For such a curve C we explicitly write down the ideal $\mathcal{I}_2(K_C)$ in Section 2. In Section 3 we describe the second Gaussian map for C modulo torsion, and in Section 4 we deal with the torsion part. By direct computations performed with Maple (the script is presented and commented in the Appendix), we verify the injectivity for a general binary curve of genus $g \leq 17$

and the surjectivity for $g = 18$. Finally, in Section 5, we proceed by induction on g to complete the argument for $g \geq 19$.

The behavior of μ, and its connection with the curvature of $\mathcal{M}_g$ in $\mathcal{A}_g$, indicates possible relations of the surjectivity of μ with the Kodaira dimension of $\mathcal{M}_g$ being nonnegative. This, we think, would be a great subject for future research. Also interesting is the study of the Gaussian maps μ_k, w_k for higher values of k. The maps μ_k are related to higher fundamental forms of the Torelli immersion of $\mathcal{M}_g$ in $\mathcal{A}_g$ at a nonhyperelliptic point. Are these maps also of maximal rank for a general curve?

In this paper we work over the complex field and use standard notation in algebraic geometry. In particular, if X is a Gorenstein curve, then Ω^1_X will denote its sheaf of Kähler differentials and K_X will denote its dualizing sheaf or canonical bundle, or a canonical divisor. In general, we will indifferently use sheaf, bundle, or divisor notation. We will often write $H^i(\mathcal{L})$ instead of $H^i(X, \mathcal{L})$ for cohomology spaces.

ACKNOWLEDGMENTS. The second author wishes to thank G. P. Pirola for suggesting the problem solved in this paper as well as G. P. Pirola and P. Frediani for discussions on this subject.

1. The Second Gaussian Map for a Stable Curve

Let X be a stable curve of genus g. We will denote by $\mathcal{I}_2(K_X)$ the vector space of forms of degree 2 vanishing on the canonical model of X. If X is smooth, then the second Gaussian map $\mu\colon \mathcal{I}_2(K_X) \to H^0(X, K_X^{\otimes 4})$ is locally defined as follows.

Fix a basis $\{\omega_i\}$ of $H^0(K_X)$, and write it in a local coordinate z as $\omega_i = f_i(z)\,dz$. Let $Q \in \mathcal{I}_2(K_X)$ with $Q = \sum_{i,j} s_{ij}\omega_i \otimes \omega_j$, where the matrix (s_{ij}) is symmetric. Since $\sum_{i,j} s_{ij} f_i f_j \equiv 0$, one has $\sum_{i,j} s_{ij} f_i' f_j \equiv 0$. The local expression of $\mu(Q)$ is then (cf., e.g., [CF2])

$$\mu(Q) = \sum_{i,j} s_{ij} f_i'' f_j\,(dz)^4 = -\sum_{i,j} s_{ij} f_i' f_j'\,(dz)^4. \tag{1}$$

If X is nodal, one can similarly define the second Gaussian map

$$\mu\colon \mathcal{I}_2(K_X) \to H^0(X, \mathrm{Sym}^2(\Omega^1_X) \otimes K_X^{\otimes 2}),$$

which is locally defined in a similar way as in (1). Precisely, let $\{\omega_i\}$ be a basis of $H^0(K_X)$. In local coordinates we can write $\omega_i = f_i\xi$, where f_i is a regular function and ξ is a local generator of the canonical bundle K_X. Then μ is locally defined by

$$\mu(Q) = -\sum_{i,j} s_{ij}\,df_i\,df_j\,\xi^{\otimes 2}. \tag{2}$$

Given a flat degeneration over a disc of a general curve to a stable curve X, the second Gaussian map for X is the flat limit of the second Gaussian map for the general curve.

It is useful to describe in some detail the space $H^0(X, \mathrm{Sym}^2(\Omega^1_X) \otimes K_X^{\otimes 2})$. We first remark that $\mathrm{Sym}^2(\Omega^1_X)$ has torsion T supported at the nodes of X. Hence there is a short exact sequence

$$0 \to T \to \mathrm{Sym}^2(\Omega^1_X) \to \mathcal{F}_X \to 0,$$

where $\mathcal{F}_X$ is a nonlocally free, rank-1, torsion-free sheaf on X.

LEMMA 2. (a) *For every node p of X, T_p is a 3-dimensional vector space*; *if the local equation of X around p is $xy = 0$, then T_p is spanned by $dx\,dy$, $x\,dx\,dy$, and $y\,dx\,dy$.*

(b) *If X_i are the irreducible components of the normalization $\pi : \tilde{X} \to X$ of X, then*

$$\mathcal{F}_X \cong \bigoplus_i \pi_* K_{X_i}^{\otimes 2}.$$

Proof. Since $y\,dx = -x\,dy$, a local section of $\mathrm{Sym}^2(\Omega^1_X)$ around a node $xy = 0$ can be uniquely written as $f(x)\,(dx)^2 + g(x, y)\,dx\,dy + h(y)\,(dy)^2$, where $g(x, y)$ is linear. Then (a) is a local computation and (b) follows from (a). □

As a consequence, since $K_{X|X_i} = K_{X_i}(D_i)$ for D_i the divisor of nodes on X_i, it follows that

$$H^0(X, \mathrm{Sym}^2(\Omega^1_X) \otimes K_X^{\otimes 2}) \cong T \oplus \bigoplus_i H^0(X_i, K_{X_i}^{\otimes 4}(2D_i)); \tag{3}$$

here $T \cong \mathbb{C}^{3\delta}$, with δ the number of nodes of X.

2. Canonical Binary Curves

Let $[x_1, \dots, x_g]$ be homogenous coordinates in $\mathbb{P}^{g-1}$, $g \geq 3$. Let $p_h = [0, \dots, 0, 1, 0, \dots, 0]$, with 1 at the hth place, $1 \leq h \leq g$, be the coordinate points and let $u = [1, 1, \dots, 1]$ be the unit point. Take two distinct rational normal curves C_1, C_2 in $\mathbb{P}^{g-1}$ passing through p_h, $1 \leq h \leq g$, and u. Then C_1 and C_2 intersect transversally at these $g + 1$ points and have no further intersection.

We may and will assume that C_k, $k = 1, 2$, is the closure of the image of the map f_k given by

$$t \mapsto f_k(t) = \left[\frac{1}{t - \alpha_{k,1}}, \frac{1}{t - \alpha_{k,2}}, \dots, \frac{1}{t - \alpha_{k,g}} \right], \tag{4}$$

where $\alpha_{k,i} \in \mathbb{C}$ for $k = 1, 2$ and $i = 1, \dots, g$. In particular, $f_k(\alpha_{k,h}) = p_h$, $h = 1, \dots, g$, and $f_k(\infty) = u$. For our purposes, the $\alpha_{k,i}$ will be general in $\mathbb{C}$. Actually, we will often consider them as indeterminates on $\mathbb{C}$.

The curve $C = C_1 \cup C_2$ is the limit of a general canonical curve $X \subset \mathbb{P}^{g-1}$ of genus g, and C is canonical, too; that is, $\mathcal{O}_C(1) \cong K_C$. The curve C is usually called a *canonical binary curve*.

We sketch the proof of the following proposition, which is more than we need. Indeed, we will need only the quadratic normality of a general canonical binary curve C, which can be directly proved (see Remarks 4 and 8).

PROPOSITION 3. *A canonical binary curve $C = C_1 \cup C_2$ is projectively normal.*

Proof. The assertion is trivial for $g = 3$, which is the minimum allowed value of g. So we may assume $g \geq 4$. By Theorem 1.2 in [S], it suffices to show that there are $g - 2$ smooth points of C spanning a $\mathbb{P}^{g-3}$ that meets C scheme-theoretically at these $g - 2$ points only. Choose $g - 2$ general points on C_1 and let $\Lambda \cong \mathbb{P}^{g-3}$ be their span, which meets C_1 transversally at these points. We claim that Λ does not meet C_2. Otherwise, choose $g - 4$ general points on C_1 and project C down to $\mathbb{P}^3$ from their span. The image of C_1 is a rational normal cubic Γ_1, whereas C_2 projects birationally (cf. [CaCi]) to a nondegenerate rational curve Γ_2 of degree > 3; hence Γ_1 and Γ_2 are distinct. Moreover, the general secant line to Γ_1 would meet Γ_2, which is impossible by the trisecant lemma (see the *focal proof* in [ChCi]). □

REMARK 4. The only information that we will need from Proposition 3 is that C is quadratically normal, which is equivalent to

$$\dim(\mathcal{I}_2(K_C)) = \binom{g-2}{2}.$$

The simple argument in the proof of Proposition 3 relies on Schreyer's result, which requires a careful analysis following the classical approach of Petri. The same result would follow by proving that the general hyperplane section of C verifies the general position theorem (see [ACGH, p. 109]). This may be proved with the same argument as before, but we do not dwell on that here.

In case C is a general binary curve, it is quite simple to prove that C is quadratically normal. One way is to remark that the general trigonal binary curve is quadratically normal. For example, if $g = 2h$, embed $\mathbb{F}_0$ in $\mathbb{P}^{g-1}$ via the linear system of curves of type $(1, h-1)$. The general trigonal binary curve is the union of the images of a general curve of type $(1, h)$ and of a general curve of type $(2, 1)$. The case g odd is similar and is left to the reader.

We are now interested in explicitly describing the vector space $\mathcal{I}_2(K_C)$ of degree-2 forms vanishing on C (i.e., the domain of the map μ for C). The analysis we shall make provides another proof that the general binary curve C is quadratically normal.

For $k = 1, 2$, set

$$A_k(t) = \prod_{i=1}^{g} (t - \alpha_{k,i}). \tag{5}$$

For each $h = 0, \ldots, g$, the coefficients $c_{k,h}$ of t^{g-h} in $A_k(t)$ are, up to sign, the elementary symmetric functions

$$c_{k,0}=1, \qquad c_{k,h}=(-1)^h \sum_{1\le i_1<i_2<\cdots<i_h\le g} \alpha_{k,i_1}\alpha_{k,i_2}\cdots\alpha_{k,i_h}.$$

Note that the index h is the degree of $c_{k,h}$ as a polynomial in the $\alpha_{k,i}$.

Fix $k\in\{1,2\}$. Since C_k passes through the coordinate points, the equation of a quadric $Q\subset\mathbb{P}^{g-1}$ containing C_k has the form

$$\sum_{1\le i<j\le g} s_{ij}x_ix_j=0 \tag{6}$$

with the conditions

$$P_k(t):=\sum_{1\le i<j\le g}\frac{A_k(t)}{(t-\alpha_{k,i})(t-\alpha_{k,j})}s_{ij}=\sum_{n=0}^{g-2}P_{k,n}t^n\equiv 0,$$

where $P_k(t)$ is a polynomial in t of degree $g-2$ whose coefficients are linear polynomials $P_{k,n}(s_{ij})$ in the s_{ij}, $n=0,\dots,g-2$. By expanding the product $A_k(t)$, one sees that the coefficients $p_{k,h;i,j}$ of s_{ij} in $P_{k,g-2-h}$, $h=0,\dots,g-2$, are

$$\begin{gathered} p_{k,0;i,j}=1, \qquad p_{k,1;i,j}=-\sum_{i_1\ne i,j}\alpha_{k,i_1}, \\ p_{k,h;i,j}=(-1)^h\sum_{\substack{i_1<i_2<\cdots<i_h\\ \text{all}\ne i,j}}\alpha_{k,i_1}\alpha_{k,i_2}\cdots\alpha_{k,i_h}, \quad 2\le h\le g-2, \end{gathered} \tag{7}$$

namely, the elementary symmetric functions (removing the i and j terms) up to sign. Again the index h coincides with the degree of $p_{k,h;i,j}$ as a homogeneous polynomial in the $\alpha_{k,i}$.

Consider also the polynomials

$$Q_{k,n}(s_{ij}):=\sum_{1\le i<j\le g}\left(\sum_{m=0}^{g-2-n}\alpha_{k,i}^m\alpha_{k,j}^{g-2-n-m}\right)s_{ij}, \quad n=0,\dots,g-2,$$

and let $q_{k,h;i,j}=\sum_{m=0}^h\alpha_{k,i}^m\alpha_{k,j}^{h-m}$ be the coefficient of s_{ij} in $Q_{k,g-2-h}$, $h=0,\dots,g-2$. In this case, too, the index h coincides with the degree of $q_{k,h;i,j}$ as a homogeneous polynomial in the $\alpha_{k,i}$.

REMARK 5. The coefficient $q_{k,h;i,j}$ of s_{ij} in $Q_{k,g-2-h}$ can be recursively computed by

$$\begin{gathered} q_{k,0;i,j}=1, \qquad q_{k,1;i,j}=\alpha_{k,i}+\alpha_{k,j}, \\ q_{k,h;i,j}=q_{k,1;i,j}q_{k,h-1;i,j}-\alpha_{k,i}\alpha_{k,j}q_{k,h-2;i,j}, \quad 2\le h\le g-2. \end{gathered}$$

Note that all the monomials $\alpha_{k,j}^m\alpha_{k,i}^{h-m}$, $m=0,\dots,h$—in particular, $\alpha_{k,i}^h$ and $\alpha_{k,j}\alpha_{k,i}^{h-1}$—appear in $q_{k,h;i,j}$ with coefficient 1. Note also the recursive formula

$$q_{k,h;i,j}=\alpha_iq_{k,h-1;i,j}+\alpha_j^h, \quad 1\le h\le g-2. \tag{8}$$

We will need the following lemma.

LEMMA 6. *Fix $k \in \{1,2\}$. For each $n = 0, \dots, g-2$, one has*

$$P_{k,n} = \sum_{m=0}^{g-2-n} c_{k,m} Q_{k,n+m}. \tag{9}$$

In particular, the linear system

$$P_{k,n}(s_{ij}) = 0, \quad n = 0, \dots, g-2, \tag{10}$$

in the s_{ij} is equivalent to the linear system

$$Q_{k,n}(s_{ij}) = 0, \quad n = 0, \dots, g-2. \tag{11}$$

Proof. One has $P_{k,g-2} = Q_{k,g-2}$ and $P_{k,g-3} = Q_{k,g-3} + c_{k,1} Q_{k,g-2}$. Now we proceed by induction. Equation (9) is equivalent to

$$p_{k,h;i,j} = \sum_{l=0}^{h} c_{k,l} q_{k,h-l;i,j} \quad \text{for } h = 0, \dots, g-2. \tag{12}$$

For $h = 0, 1$, (12) clearly holds. Since the index k is fixed, we omit it. For $2 \le h \le g-2$, one has

$$\begin{aligned} p_{h;i,j} - c_h q_{0;i,j} &= (\alpha_i + \alpha_j) p_{h-1;i,j} - \alpha_i \alpha_j p_{h-2;i,j} \\ &= c_{h-1} q_{1;i,j} + \sum_{l=0}^{h-2} c_l (q_{h-l-1;i,j} q_{1;i,j} - \alpha_i \alpha_j q_{h-l-2;i,j}) \\ &= \sum_{l=0}^{h-1} c_l q_{h-l;i,j}, \end{aligned}$$

where the second equality follows by induction. This expression proves (12) and therefore (9). Since $c_{k,0} = 1$, the base change matrix between the $Q_{k,n}$ and the $P_{k,n}$ is unipotent triangular; hence it is invertible. The equivalence between (10) and (11) follows. □

Next we can give the announced description of $\mathcal{I}_2(K_C)$.

PROPOSITION 7. *Let $g \ge 3$. For a general choice of $\alpha_{k,i}$, $1 \le k \le 2$, $1 \le i \le g$, one has that*:

(a) *the linear system* (11) *has maximal rank $g-1$; and*
(b) *the linear system*

$$Q_{1,0}(s_{ij}) = \cdots = Q_{1,g-2}(s_{ij}) = Q_{2,0}(s_{ij}) = \cdots = Q_{2,g-3}(s_{ij}) = 0 \tag{13}$$

has maximal rank $2g-3$.

Proof. (a) Since the index k is fixed, we drop it here. Let us consider the matrix

$$U := U(\alpha_1, \dots, \alpha_g) = (q_{h;i,j})_{0 \le h \le g-2, 1 \le i < j \le g}$$

of size $(g-1) \times \binom{g}{2}$, where the pairs (i,j) are lexicographically ordered. We have to prove that there is a minor of U of order $g-1$ that is not identically zero. We show this for the minor $D := D(\alpha_1, \dots, \alpha_g)$ determined by the first $g-1$ columns, indexed by $(1,i)$ with $2 \le i \le g$. This is true if $g = 3$, so we proceed by induction on g. Look at D as a polynomial in α_g: it has degree $g-2$ and the coefficient of α_g^{g-2} is $D(\alpha_1, \dots, \alpha_{g-1})$ (cf. Remark 5), which is nonzero by induction. This proves the assertion.

Equivalently, by subtracting from each row the previous one multiplied by α_1 and using (8) (cf. Remark 5), one sees that D is the Vandermonde determinant $V(\alpha_2, \dots, \alpha_g) = \prod_{2 \le i < j \le g}(\alpha_j - \alpha_i)$ of $\alpha_2, \dots, \alpha_g$.

(b) We use the same idea as in the proof of (a). Form a matrix

$$Z := Z(\alpha_{k,i})_{1 \le k \le 2, 1 \le i < j \le g}$$

of size $(2g-3) \times \binom{g}{2}$ by concatenating vertically U (for $k = 1$) and the matrix

$$W := W(\alpha_{2,1}, \dots, \alpha_{2,g}) = (q_{2,h;i,j})_{1 \le h \le g-2, 1 \le i < j \le g}.$$

It suffices to prove that the minor $M := M(\alpha_{k,i})_{1 \le k \le 2, 1 \le i < j \le g}$ of Z determined by the first $2g-3$ columns, indexed by $(1,i), (2,j)$ with $2 \le i \le g$ and $3 \le j \le g$, is not identically zero as a polynomial in the $\alpha_{k,i}$. This is clearly true for $g = 3$, so we proceed by induction on g. Look at M as a polynomial in $\alpha_{1,g}$ and $\alpha_{2,g}$: one sees that the monomial $\alpha_{1,g}^{g-2}\alpha_{2,g}^{g-3}$ appears in M with the coefficient $(\alpha_{2,2} - \alpha_{2,1})M(\alpha_{k,i})_{1 \le k \le 2, 1 \le i < j \le g-1}$, which is nonzero by induction; this proves the assertion.

Equivalently, looking at M as a polynomial in $\alpha_{1,1}$, one sees that the coefficient of the monomial $\alpha_{1,1}^{g-2}$ is the product of two Vandermonde determinants: $V(\alpha_{2,2}, \dots, \alpha_{2,g})V(\alpha_{1,3}, \dots, \alpha_{1,g})$. □

REMARK 8. The solutions of the linear system (11), as well as those of (10), give us the quadrics containing the rational normal curve C_k, whereas the solutions of (13) give us the quadrics in $\mathcal{I}_2(K_C)$ for the binary curve $C = C_1 \cup C_2$.

3. Binary Curves: The Second Gaussian Map Modulo Torsion

Let $C = C_1 \cup C_2$ be a general binary curve. In this section we will consider the composition ν of the second Gaussian map for C with the projection to the nontorsion part of $H^0(C, \mathrm{Sym}^2(\Omega^1_C) \otimes K_C^{\otimes 2})$ (cf. (3) in Section 1). Specifically, for $k = 1, 2$, we will look at the map

$$\nu_k : \mathcal{I}_2(K_C) \to H^0(C_k, K_{C_k}^{\otimes 4}(2D_k)),$$

where D_k is a divisor of degree $g+1$ on C_k; therefore, $\nu = \nu_1 \oplus \nu_2$ and

$$H^0(C_k, K_{C_k}^{\otimes 4}(2D_k)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-6)).$$

The map ν_k can be explicitly written down by taking into account (2) and the description of the ideal $\mathcal{I}_2(K_C)$ (see Section 2). Precisely, let $Q \in \mathcal{I}_2(K_C)$ be of the form (6), where the s_{ij} are solutions of (13). Then

$$\nu_k(Q) = \sum_{1 \le i < j \le g} \frac{1}{(t-\alpha_{k,i})^2(t-\alpha_{k,j})^2} s_{ij}\,(dt)^4 \in H^0(C_k, K_{C_k}^{\otimes 4}(2D_k)).$$

To look at this as a section of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-6))$, we multiply by $A_k^2(t)$. Then

$$\nu_k(Q) = \sum_{1 \le i < j \le g} \frac{A_k^2(t)}{(t-\alpha_{k,i})^2(t-\alpha_{k,j})^2} s_{ij} =: R_k(t) \tag{14}$$

is a polynomial in t whose apparent degree is $2g-4$. However, its coefficient of degree $2g-4$ is $P_{k,g-2}$ and the one of degree $2g-5$ is proportional to $P_{k,g-3}$, so they vanish and $R_k(t)$ has actual degree $2g-6$.

Using this explicit description (14) of ν, we asked Maple to compute its rank for low values of g (see the Appendix for the Maple script). The result is as follows.

PROPOSITION 9. *The map ν has maximal rank for $g \le 18$; in other words, ν is injective for $g \le 10$ and is surjective for $11 \le g \le 18$.*

COROLLARY 10. *The second Gaussian map μ is injective for the general curve of genus $g \le 10$.*

4. Binary Curves: The Torsion

Let $C = C_1 \cup C_2$ be a general binary curve as in Section 2. In (4) we may replace f_k, $1 \le k \le 2$, with

$$A_k(t) f_k(t) = [\phi_{k,1}(t), \dots, \phi_{k,g}(t)], \quad \phi_{k,i}(t) = \frac{A_k(t)}{(t-\alpha_{k,i})}. \tag{15}$$

Now we consider the restriction τ of the second Gaussian map for C to $\ker(\nu)$, which lands in the torsion part T of $H^0(C, \mathrm{Sym}^2(\Omega_C^1) \otimes K_C^{\otimes 2})$ (cf. (3)). Once we take Lemma 2(a) into account, a direct computation shows that the composition of τ with the projection on the torsion part T_{p_h} at the coordinate point p_h is as follows: if $Q \in \ker(\nu)$ is of the form (6), then Q is mapped to

$$dx\,dy \sum_{i \ne j} s_{ij} \phi'_{1,i}(\alpha_{1,h}) \phi'_{2,j}(\alpha_{2,h}) + 2x\,dx\,dy \sum_{i \ne j} s_{ij} \phi''_{1,i}(\alpha_{1,h}) \phi'_{2,j}(\alpha_{2,h}) + 2y\,dx\,dy \sum_{i \ne j} s_{ij} \phi'_{1,i}(\alpha_{1,h}) \phi''_{2,j}(\alpha_{2,h}), \tag{16}$$

where $s_{ji} = s_{ij}$ and where x and y are local coordinates around p_h such that $C_1 : y = 0$ and $C_2 : x = 0$. The description of the torsion at the unitary point u is similar. Replace f_k by the parameterization $\frac{1}{t} f_k\left(\frac{1}{t}\right)$. Again a direct computation shows that the composition of τ with the projection on T_u is

$$Q \mapsto dx\,dy \sum_{i \neq j} s_{ij}\alpha_{1,i}\alpha_{2,j} + 2x\,dx\,dy \sum_{i \neq j} s_{ij}\alpha_{1,i}^2\alpha_{2,j} + 2y\,dx\,dy \sum_{i \neq j} s_{ij}\alpha_{1,i}\alpha_{2,j}^2, \quad (17)$$

where $s_{ji} = s_{ij}$ and where x and y are local coordinates around u such that $C_1 : y = 0$ and $C_2 : x = 0$.

Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
0 \longrightarrow & T & \longrightarrow & H^0(C, \mathrm{Sym}^2(\Omega^1_C) \otimes K_C^{\otimes 2}) & \twoheadrightarrow & H^0(C_1, K_{C_1}^{\otimes 2}(2)) \oplus H^0(C_2, K_{C_1}^{\otimes 2}(2)) \cong H^0(\mathcal{F}_C) \\
& \tau\uparrow & & \mu\uparrow & \nearrow_{\nu} & \\
0 \longrightarrow & \ker(\nu) & \longrightarrow & \mathcal{I}_2(K_C) & &
\end{array} \quad (18)$$

We asked Maple to compute the rank of the map τ for $11 \leq g \leq 18$ (see the script in the Appendix). Taking into account diagram (18), we obtain the following results.

PROPOSITION 11. *Let C be a general binary curve of genus g. Then the maps τ and μ have maximal rank for $g \leq 18$: they are injective for $g \leq 17$ and surjective for $g = 18$.*

COROLLARY 12. *The map μ is injective for the general curve of genus $g \leq 17$ and is surjective for $g = 18$.*

5. The Induction Step

In this section we prove our main result—namely, the surjectivity of the second Gaussian map μ for the general curve of genus ≥ 18.

Let $C \subset \mathbb{P}^{g-1}$ be a nodal canonical curve and let $p \in C$ be a node. Let $\tilde{C} \to C$ be the partial normalization of C at p, and let $p_1, p_2 \in \tilde{C}$ be the points over p. Note that the projection from p maps C to the canonical model of $\tilde{C}$ in $\mathbb{P}^{g-2}$; we will assume that this induces an isomorphism of $\tilde{C}$ to its canonical model. Consider the following diagrams.

$$\begin{array}{ccccc}
0 \longrightarrow & H^0(\mathcal{F}_{\tilde{C}}) & \hookrightarrow & H^0(\mathcal{F}_C) & \twoheadrightarrow \mathcal{O}_{2p_1} \oplus \mathcal{O}_{2p_2} \\
& \tilde{\nu}\uparrow & & \nu\uparrow & \nearrow_{\chi} \\
0 \longrightarrow & \mathcal{I}_2(K_{\tilde{C}}) & \hookrightarrow & \mathcal{I}_2(K_C) &
\end{array}
\qquad
\begin{array}{ccccc}
0 \longrightarrow & \tilde{T} & \hookrightarrow & T & \twoheadrightarrow T_p \\
& \tilde{\tau}\uparrow & & \tau\uparrow & \nearrow_{\tau_p} \\
0 \longrightarrow & \ker(\tilde{\nu}) & \hookrightarrow & \ker(\nu) &
\end{array} \quad (19)$$

Here $\tilde{T}$ is the torsion subsheaf of $\mathrm{Sym}^2(\Omega^1_{\tilde{C}})$, ν, τ are the maps of diagram (18) for the curve C, and $\tilde{\nu}, \tilde{\tau}$ are the corresponding ones for $\tilde{C}$. The diagrams (19) are commutative and the horizontal sequences are exact, so the next lemma is clear.

LEMMA 13. *If $\tilde{\nu}$ and χ (resp., $\tilde{\tau}$ and τ_p) are surjective, then ν (resp., τ) is also surjective.*

We apply this lemma to prove our next theorem.

THEOREM 14. *If $C = C_1 \cup C_2$ is a general binary curve of genus $g \geq 18$, then μ is surjective for C.*

Proof. The case $g = 18$ has already been addressed by direct computation (cf. Proposition 11). We therefore proceed by induction on g: the commutativity of diagram (18) and Lemma 13 show that it is enough to prove the surjectivity of χ and τ_p, where p is a node of C. We will do this for $p = u$ the unitary point.

In this situation, the map ν is the one $\nu_1 \oplus \nu_2$ considered in Section 3. Hence $\chi = \chi_1 \oplus \chi_2$, where χ_k is the composition of ν_k with the restriction to $\mathcal{O}_{2p_k}$, $k = 1, 2$. In local coordinates, $\chi_k(Q)$ is the pair formed by the constant term and the coefficient of the degree-1 term of the Taylor expansion around p of the polynomial $\nu_k(Q)$. In Section 3 we computed ν_k using a local coordinate t on C_k. In this coordinate, the point $p = [1, \dots, 1]$ corresponds to $t = \infty$. So if $Q \in \mathcal{I}_2(K_C)$ is of the form (6), with the s_{ij} satisfying (13), then $\chi_k(Q)$ is the pair of coefficients of the highest degrees $2g - 6$ and $2g - 7$ of the polynomial $\nu_k(Q)$—that is, of the polynomial $R_k(t)$ given in (14). We denote these coefficients by $R_{k,2g-6}$ and $R_{k,2g-7}$, which are linear polynomials in the s_{ij}. We will now compute them.

We fix the index k and then omit it. By expanding A^2 in (14), one sees that the coefficient of s_{ij} in R_{2g-6} is

$$4p_{2;i,j} + \sum_{i_1 \neq i,j} \alpha_{i_1}^2 = 4p_{2;i,j} + n_2 - (\alpha_i^2 + \alpha_j^2),$$

where $n_2 = \sum_{m=1}^{g} \alpha_m^2$ is independent of i, j and $p_{2;i,j}$ is the coefficient of s_{ij} in $P_{k,g-4}$ (cf. (7)). By (10), this means that

$$R_{2g-6} = 4P_{g-4} + n_2 P_{g-2} - \sum_{i<j} (\alpha_i^2 + \alpha_j^2) s_{ij} = -\sum_{i<j} (\alpha_i^2 + \alpha_j^2) s_{ij}.$$

Similarly, one sees that the coefficient of $s_{i,j}$ in R_{2g-7} is *twice*

$$4p_{3;i,j} - \sum_{\substack{i_1 \neq i_2 \\ \text{both} \neq i,j}} \alpha_{i_1}^2 \alpha_{i_2} = 4p_{3;i,j} - n_3 + n_2 p_{1;i,j} - c_1(\alpha_i^2 + \alpha_j^2) - (\alpha_i^3 + \alpha_j^3) - q_{3;i,j},$$

where $n_3 = -\sum_{m=1}^{g} \alpha_m^3$ is independent of i, j. Therefore, taking into account (10) and (11), one has

$$R_{2g-7} = -2c_1 R_{2g-6} - 2 \sum_{i<j} (\alpha_i^3 + \alpha_j^3) s_{ij}.$$

Form the matrix $Y := Y(\alpha_{k,i})_{1 \leq k \leq 2, 1 \leq i < j \leq g}$ of size $(2g + 1) \times \binom{g}{2}$ obtained by concatenating vertically the matrix Z in the proof of Proposition 7(b) and the matrix of size $4 \times \binom{g}{2}$ whose rows are $(\alpha_{k,i}^h + \alpha_{k,j}^h)_{1 \leq i < j \leq g}$ with $1 \leq k \leq 2$ and $2 \leq h \leq 3$. In order to prove that χ is surjective, we must first prove that there is a minor of order $2g + 1$ of Y that is not identically zero. We will do this for the minor $N := N(\alpha_{k,i})_{1 \leq k \leq 2, 1 \leq i < j \leq g}$ determined by the first $2g + 1$ columns indexed by $(1, i)$, $(2, j)$, and $(3, \ell)$, where $2 \leq i \leq g$, $3 \leq j \leq g$, and $4 \leq \ell \leq 7$.

This minor is nonzero for $g = 7$, which we verified using Maple (see the script in the Appendix). Then we proceed by induction on g and assume $g \geq 8$. The argument here is the same as the one in the proof of Proposition 7(b). Look at N as a polynomial in $\alpha_{1,g}$ and $\alpha_{2,g}$: the monomial $\alpha_{1,g}^{g-2}\alpha_{2,g}^{g-3}$ appears in N with coefficient $(\alpha_{2,2} - \alpha_{2,1})N(\alpha_{k,i})_{1\leq k\leq 2, 1\leq i<j\leq g-1}$; this coefficient is nonzero by induction, proving that χ is surjective.

It remains to show that τ_p is surjective. This could be seen with a quick monodromy argument, but we prefer to present an argument in the same style as the ones made so far.

Recall that $\ker(\nu)$ is defined in $\mathcal{I}_2(K_C)$ by the vanishing of the polynomials $R_k(t)$, $k = 1, 2$, whose coefficients of degree $\leq 2g - 8$ are polynomials in the $\alpha_{k,i}$ of degree ≥ 4. By the description of the torsion at the unitary point given in (17), we need to show the rank maximality of the matrix $Y' = Y'(\alpha_{k,i})_{1\leq k\leq 2, 1\leq i<j\leq g}$ of size $(2g + 4) \times \binom{g}{2}$ obtained by concatenating vertically the above matrix Y and the matrix of size $3 \times \binom{g}{2}$ whose rows are $(\alpha_{1,i}\alpha_{2,j} + \alpha_{1,j}\alpha_{2,i})_{1\leq i<j\leq g}$, $(\alpha_{1,i}^2\alpha_{2,j} + \alpha_{1,j}^2\alpha_{2,i})_{1\leq i<j\leq g}$, and $(\alpha_{1,i}\alpha_{2,j}^2 + \alpha_{1,j}\alpha_{2,i}^2)_{1\leq i<j\leq g}$. We claim that the minor $N' = N'(\alpha_{k,i})_{1\leq k\leq 2, 1\leq i<j\leq g}$ of Y' determined by the first $2g + 4$ columns—indexed by $(1, i)$, $(2, j)$, and $(3, \ell)$, where $2 \leq i \leq g$, $3 \leq j \leq g$, and $4 \leq \ell \leq 10$—is nonzero for $g \geq 10$. We verify the case $g = 10$ with Maple (see the script in the Appendix), and the induction is the same as before because the monomial $\alpha_{1,g}^{g-2}\alpha_{2,g}^{g-3}$ appears in N' again with coefficient $(\alpha_{2,2} - \alpha_{2,1})N'(\alpha_{k,i})_{1\leq k\leq 2, 1\leq i<j\leq g-1}$. This concludes the proof that τ_p is surjective and hence the proof of the theorem. □

COROLLARY 15. *The second Gaussian map μ is surjective for the general curve of genus $g \geq 18$.*

Appendix: Maple Script for Computations

Listed here is the Maple script we run. We explain it afterwards, for which purpose we have added line numbers at each five lines.

```
alpha[1]:=[3,12,21,29,37,41,43,46,54,62,65,72,81,85,89,94,97,105]:
alpha[2]:=[6,18,24,36,39,42,45,52,60,63,71,80,84,86,91,96,104,108]:
for g from 4 to 18 do
listsij:=[seq(seq(s[i,j],j=i+1..g),i=1..g)]:
for k from 1 to 2 do
   A[k]:=mul(t-alpha[k][i],i=1..g):
   R[k]:=add(add(s[i,j]*(A[k]^2)/((t-alpha[k][i])^2*(t-alpha[k][j])^2),
                  j=i+1..g),i=1..g):
end do:
Z:=linalg[matrix]([seq([seq(seq(add(alpha[1][i]^m*alpha[1][j]^(h-m),m=0..h),
                      j=i+1..g),i=1..g)],h=0..g-2),
                      seq([seq(seq(add(alpha[2][i]^m*alpha[2][j]^(h-m),m=0..h),
                      j=i+1..g),i=1..g)],h=1..g-2)]):
Zref:=Gausselim(Z,'r0') mod 109:
printf("For g=%2d, one has dim I2(K)=%3d, ",g,nops(listsij)-r0):
EqsKerNu:=[seq(seq(primpart(coeff(R[k],t,n)),n=0..2*g-6),k=1..2)]:
```

```
   K:=Gausselim(linalg[stackmatrix](Zref,
                linalg[genmatrix](EqsKerNu,listsij)),'r1') mod 109:
   printf("dim Ker(nu)=%2d, corank(nu)=%d, ",nops(listsij)-r1,4*g-10-r1+r0):
20 for k from 1 to 2 do for i from 1 to g do
     phi1[k,i]:=diff(A[k]/(t-alpha[k][i]),t):  phi2[k,i]:=diff(phi1[k,i],t):
     for h from 1 to g do
       phi1e[k,i,h]:=eval(phi1[k,i],t=alpha[k][h]):
       phi2e[k,i,h]:=eval(phi2[k,i],t=alpha[k][h]):
25 end do:  end do:  end do:
   for h from 1 to g do
     tors[h,1]:=add(add(s[i,j]*(phi1e[1,i,h]*phi1e[2,j,h]
                               +phi1e[1,j,h]*phi1e[2,i,h]),j=i+1..g),i=1..g):
     tors[h,2]:=add(add(s[i,j]*(phi2e[1,i,h]*phi1e[2,j,h]
30                             +phi2e[1,j,h]*phi1e[2,i,h]),j=i+1..g),i=1..g):
     tors[h,3]:=add(add(s[i,j]*(phi1e[1,i,h]*phi2e[2,j,h]
                               +phi1e[1,j,h]*phi2e[2,i,h]),j=i+1..g),i=1..g):
   end do:
   tors[0,1]:=add(add(s[i,j]*(alpha[1][i]*alpha[2][j]
35                           +alpha[1][j]*alpha[2][i]),j=i+1..g),i=1..g):
   tors[0,2]:=add(add(s[i,j]*(alpha[1][i]^2*alpha[2][j]
                             +alpha[1][j]^2*alpha[2][i]),j=i+1..g),i=1..g):
   tors[0,3]:=add(add(s[i,j]*(alpha[1][i]*alpha[2][j]^2
                             +alpha[1][j]*alpha[2][i]^2),j=i+1..g),i=1..g):
40 EqsKerTau:=[seq(seq(primpart(tors[h,l]),l=1..3),h=0..g)]:
   Gausselim(linalg[stackmatrix](K,linalg[genmatrix](EqsKerTau,listsij)),'r2') mod 109:
   printf("dim ker(tau)=%d, corank(tau)=%2d\n",nops(listsij)-r2,3*g+3-r2+r1):
   if g=7 then
     N:=linalg[det](linalg[stackmatrix](linalg[delcols](Z,16..21),
45                 linalg[matrix]([seq(seq([seq(seq(alpha[k][i]^h+alpha[k][j]^h,
                                 j=i+1..7),i=1..3)],h=2..3),k=1..2)]))):
     printf("For g= 7, the minor N is congruent to %d (mod 5)\n",N mod 5):
   elif g=10 then
     N2:=linalg[det](linalg[stackmatrix](linalg[delcols](Z,25..45),
50       linalg[matrix]([seq(seq([seq(seq(alpha[k][i]^h+alpha[k][j]^h,
                                       j=i+1..10),i=1..3)],h=2..3),k=1..2)]),
         linalg[matrix]([[seq(seq(alpha[1][i]*alpha[2][j]
                                 +alpha[1][j]*alpha[2][i],j=i+1..10),i=1..3)],
                        [seq(seq(alpha[1][i]^2*alpha[2][j]
55                               +alpha[1][j]^2*alpha[2][i],j=i+1..10),i=1..3)],
                        [seq(seq(alpha[1][i]*alpha[2][j]^2
                                 +alpha[1][j]*alpha[2][i]^2,j=i+1..10),i=1..3)]]))):
     printf("For g=10, the minor N' is congruent to %d (mod 23)\n",N2 mod 23):
   end if:
   end do:
```

In lines 1–2, we define the $\alpha_{k,i}$ that will be used. We chose them randomly. In line 3 we start the main loop, which runs for $4 \leq g \leq 18$. In line 4, we collect the unknowns $\{s_{i,j}\}_{1 \leq i < j \leq g}$ in the list `listsij`: there are $\binom{g}{2}$ of them. In lines 6–8 we define the polynomials $A_k(t)$ and $R_k(t)$ (cf. (5) and (14)).

In lines 10–13 we define the matrix `Z` associated to the linear system (13), whose solutions give us the quadrics in $\mathcal{I}_2(K_C)$ (cf. the proof of Proposition 7). In line 14, Maple computes the rank `r0` of `Z` via Gaussian elimination, calculating modulo 109 to speed up computations. The resulting matrix in row echelon form is called `Zref`. In line with Proposition 7(b), Maple finds $\texttt{r0} = 2g - 3$ for each $g = 4, \ldots, 18$. In line 15, Maple prints out the genus g and $\dim(\mathcal{I}_2(K_C)) = \binom{g}{2} - \texttt{r0} = \binom{g-2}{2}$.

In line 16, we collect in EqsKerNu the list of equations that determine $\ker(\nu)$ (cf. the definition (14) of ν in Section 3). In lines 17–18, Maple computes the rank r1 of the linear system EqsKerNu $\cap$ ker(Zref), again via Gaussian elimination modulo 109, and the resulting row echelon matrix is called K. Maple finds that $\texttt{r1} = \binom{g}{2}$ for $4 \le g \le 10$ and that $\texttt{r1} = 6g - 13$ for $11 \le g \le 18$. Therefore, the rank of ν is $\texttt{r1} - \texttt{r0} = \binom{g-2}{2}$ for $4 \le g \le 10$ and is $4g - 10$ for $11 \le g \le 18$. This proves Proposition 9.

In line 19, Maple prints out the dimension of $\ker(\nu)$ and the corank of ν; that is, $4g - 10 - \texttt{r1} + \texttt{r0}$.

In lines 20–25, we define the first and second derivatives phi1 and phi2 of the $\phi_{k,i}$ (cf. (15)). We then define their evaluations phi1e and phi2e at the coordinate point p_h. In lines 26–33, we use these evaluations to compute the torsion at p_h, $h = 1, \dots, g$ (cf. (16)). In lines 34–39 we compute the torsion at the unit point u (cf. (17)).

In lines 40 and 41, we collect in EqsKerTau the equations that determine $\ker(\tau)$ and Maple computes the rank r2 of EqsKerTau $\cap$ ker(K) via Gaussian elimination modulo 109 as before. Maple finds that $\texttt{r2} = \binom{g}{2}$ for $4 \le g \le 17$ and that $\texttt{r2} = 152$ for $g = 18$. Hence the rank of τ is $\texttt{r2} - \texttt{r1} = (g^2 - 13g + 26)/2$ for $11 \le g \le 17$ and is 57 for $g = 18$. This proves Proposition 11.

In line 42, Maple prints out the the dimension of $\ker(\tau)$ and the corank of τ; that is, $3g + 3 - \texttt{r2} + \texttt{r1}$.

Finally, in lines 43–59, Maple computes the minors N (when $g = 7$) and N' (when $g = 10$), which are needed in the proof of Theorem 14, and prints out that $N \bmod 5 = 4$ and $N' \bmod 23 = 16$.

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