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A Schoenflies Extension Theorem for a Class of Locally Bi-Lipschitz Homeomorphisms

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1. Introduction

1.1. Embeddings of Collars

In point-set topology, the Schoenflies theorem [W, Thm. III.5.9] is a stronger form of the well-known Jordan curve theorem; it states that *every simple closed curve separates the sphere* $\mathbb{S}^2$ *into two domains, each of which is homeomorphic to* $\mathbb{B}^2$, *the open unit disc.* The same statement does not hold in higher dimensions, since the Alexander horned sphere [A] provides a counterexample in $\mathbb{R}^3$. Despite this, Brown [B] proved that for each $n \in \mathbb{N}$, every embedding of $\mathbb{S}^{n-1} \times (-\varepsilon, \varepsilon)$ into $\mathbb{R}^n$ extends to an embedding of $\mathbb{B}^n$ into $\mathbb{R}^n$.

Similar extension problems arise by varying the regularity of the embeddings. Toward this end, we prove a Schoenflies-type theorem for a new class of homeomorphisms. Their regularity is given in terms of Sobolev spaces and Lipschitz continuity.

To begin, recall that a homeomorphism $f: \Omega \to \Omega'$ is *locally bi-Lipschitz* if, for each $z \in \Omega$, there exist a neighborhood O of z and $L \geq 1$ such that the inequality

$$L^{-1}|x - y| \leq |f(x) - f(y)| \leq L|x - y| \tag{1.1}$$

holds for all $x, y \in O$. Recall also that for $p \geq 1$ and $k \in \mathbb{N}$, the Sobolev space $W^{k,p}_{\mathrm{loc}}(\Omega; \Omega')$ consists of maps $f: \Omega \to \Omega'$, where each component f_i lies in $L^p_{\mathrm{loc}}(\Omega)$ and has weak derivatives of orders up to k in $L^p_{\mathrm{loc}}(\Omega)$.

Definition 1.1. Let $f: \Omega \to \Omega'$ be a locally bi-Lipschitz homeomorphism. For $p \in [1, \infty)$, we say that f is of *class* LW^p_2 if $f \in W^{2,p}_{\mathrm{loc}}(\Omega; \Omega')$ and $f^{-1} \in W^{2,p}_{\mathrm{loc}}(\Omega'; \Omega)$. If K and K' are closed sets, then a homeomorphism $f: K \to K'$ is of class LW^p_2 when the restriction of f to the interior of K is of class LW^p_2.

Instead of product sets of the form $\mathbb{S}^{n-1} \times (-\varepsilon, \varepsilon)$, we will consider domains in $\mathbb{R}^n$ of a similar topological type.

Definition 1.2. A bounded domain D in $\mathbb{R}^n_*$ is *Jordan* if its boundary ∂D is homeomorphic to $\mathbb{S}^{n-1}$. A *collared domain* (or *collar*) is a domain in $\mathbb{R}^n$ of the form $D_2 \setminus \bar{D}_1$, where D_1 and D_2 are Jordan domains with $\bar{D}_1 \subset D_2$.

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We now state the extension theorem for homeomorphisms of class LW_2^p between collared domains.

THEOREM 1.3. *Let D_1 and D_2 be Jordan domains in $\mathbb{R}^n$ such that $\bar{D}_1 \subset D_2$, let B_1 and B_2 be balls such that $\bar{B}_1 \subset B_2$, and let $p \in [1,n)$.*

If $f: \bar{D}_2 \setminus D_1 \to \bar{B}_2 \setminus B_1$ is a homeomorphism of class LW_2^p such that $f(\partial D_i) = \partial B_i$ holds for $i = 1,2$, then there exists a homeomorphism $F: \bar{D}_2 \to \bar{B}_2$ of class LW_2^p and a neighborhood N of ∂D_2 such that $F|(N \cap \bar{D}_2) = f|(N \cap \bar{D}_2)$.

The proof is an adaptation of Gehring's argument [Ge, Thm. 2′] from the class of quasiconformal homeomorphisms to the class LW_2^p. For the locally bi-Lipschitz class, the extension theorem was known to Sullivan [S] and later proved by Tukia and Väisälä [TV, Thm. 5.10]. For more about quasiconformal homeomorphisms, see [V].

As in Gehring's case, Theorem 1.3 is not quantitative. His extension depends on the distortion (resp. Lipschitz constants) of g as well as on the configurations of the collars $D_2 \setminus \bar{D}_1$ and $B_2 \setminus \bar{B}_1$. In addition, our modification of his extension also depends explicitly on the parameters p and n.

1.2. Motivations, Smoothness, and Sharpness

The motivation for Theorem 1.3 comes from the study of Lipschitz manifolds. Specifically, Heinonen and Keith [HKe] showed that *if an n-dimensional Lipschitz manifold ($n \neq 4$) admits an atlas with coordinate charts in the Sobolev class $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, then it admits a smooth structure.*

On the other hand, there are 10-dimensional Lipschitz manifolds without smooth structures [K]. This leads to the following question.

QUESTION 1.4. For $n \neq 4$, does there exist a $p \in [1,2)$ such that every n-dimensional Lipschitz manifold admits an atlas of charts in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$?

Sullivan [S] showed that *every n-dimensional topological manifold ($n \neq 4$) admits a Lipschitz structure*. A key step in the proof is to show that bi-Lipschitz homeomorphisms satisfy a Schoenflies-type extension theorem. One may inquire whether this direction of proof would also lead to the desired Sobolev regularity. Theorem 1.3 would be a first step in this direction. For more about Lipschitz structures on manifolds, see [LV].

It is worth noting that Theorem 1.3 is not generally true for $p > n$. Recall that for any domain Ω in $\mathbb{R}^n$, Morrey's inequality [EG, Thm. 4.5.3.3] gives $W^{2,p}(\Omega) \hookrightarrow C^{1,1-n/p}(\Omega)$, so homeomorphisms of class LW_2^p are necessarily C^1-diffeomorphisms.

Indeed, every C^∞-diffeomorphism $\varphi: \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ admits a radial extension

$$\bar{\varphi}(x) := |x|\varphi\left(\frac{x}{|x|}\right),$$

that is, a C^∞-diffeomorphism between round annuli. The validity of Theorem 1.3 for $p > n$ would therefore imply that every such φ extends to a C^1-diffeomorphism of $\bar{\mathbb{B}}^n$ onto itself. However, for $n = 7$ this conclusion is impossible.

Recall that every such φ also determines a C^∞-smooth, n-dimensional manifold M_φ^n that is homeomorphic to $\mathbb{S}^n$ [M, Construction (C)]. Indeed, M_φ^n is the quotient of two copies of $\mathbb{R}^n$ under the relation $x \sim \varphi^*(x)$ on $\mathbb{R}^n \setminus \{0\}$, where

$$\varphi^*(x) := \frac{1}{|x|}\varphi\left(\frac{x}{|x|}\right). \tag{1.2}$$

If φ is the identity map on $\mathbb{S}^{n-1}$, then φ^* is the inversion map $x \mapsto |x|^{-2}x$ and M_φ^n is precisely $\mathbb{S}^n$. By using invariants from differential topology, Milnor proved the following theorem about such manifolds.

Theorem 1.5 [M, Thm. 3]. *There exist C^∞-smooth manifolds of the form M_φ^7 that are homeomorphic, but not C^∞-diffeomorphic, to $\mathbb{S}^7$.*

Such manifolds are better known as *exotic spheres.* The next lemma, an analogue of [Hi, Thm. 8.2.1], relates exotic spheres to extension theorems.

Lemma 1.6. *Let $\varphi\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be a C^∞-diffeomorphism and let $\bar\varphi\colon \bar{\mathbb{B}}^n\setminus\{0\} \to \bar{\mathbb{B}}^n\setminus\{0\}$ be its radial (diffeomorphic) extension. If there exists a C^1-diffeomorphism $\Phi\colon \bar{\mathbb{B}}^n \to \bar{\mathbb{B}}^n$ that agrees with $\bar\varphi$ on a neighborhood of $\mathbb{S}^{n-1}$ in $\bar{\mathbb{B}}^n$, then M_φ^n is C^1-diffeomorphic to $\mathbb{S}^n$.*

Proof. Let φ^* be the diffeomorphism defined in (1.2). By construction, there is an atlas of charts $\{M_i\}_{i=1}^2$ for M_φ^n with homeomorphisms $\psi_i\colon M_i \to \mathbb{R}^n$ that satisfy $\psi_1 \circ \psi_2^{-1} = \varphi^*$.

Let $\pi_1, \pi_2\colon \mathbb{R}^n \to \mathbb{S}^n$ be stereographic projections relative to the "north" and "south" poles on $\mathbb{S}^n$, respectively, so $\pi_2^{-1} \circ \pi_1 = \mathrm{id}^* = (\mathrm{id}^*)^{-1}$. Observe that

$$((\mathrm{id}^*)^{-1} \circ \varphi^*)(x) = \frac{\varphi^*(x)}{|\varphi^*(x)|^2} = |x|\varphi\left(\frac{x}{|x|}\right) = \bar\varphi(x)$$

holds for all $x \in \mathbb{R}^n \setminus \{0\}$. It follows that

$$x \mapsto \begin{cases} (\pi_1^{-1} \circ \psi_1)(x) & \text{if } x \in M_1, \\ (\pi_2^{-1} \circ \Phi \circ \psi_2)(x) & \text{if } x \in M_2 \end{cases}$$

is a C^1-diffeomorphism of M_φ^n onto $\mathbb{S}^n$. □

By [Hi, Thm. 2.2.10], if two C^∞-smooth manifolds are C^1-diffeomorphic then they are C^∞-diffeomorphic. It follows that there exist C^1-diffeomorphisms of collars in $\mathbb{R}^7$ that do not admit diffeomorphic extensions of class LW_2^p for any $p > 7$.

The next result follows from the inclusion $W_{\mathrm{loc}}^{2,p}(\Omega; \Omega') \subseteq W_{\mathrm{loc}}^{2,q}(\Omega; \Omega')$ for $q \le p$.

Corollary 1.7. *Let $n = 7$. For $p > n$, there exist collars Ω, Ω' in $\mathbb{R}^n$ and homeomorphisms $\varphi\colon \Omega \to \Omega'$ of class LW_2^p that admit homeomorphic extensions of class LW_2^q $(1 \le q < n)$ but not of class LW_2^p.*

Since the preceding discussion relies crucially on Sobolev embedding theorems, it leaves open the borderline case $p = n$.

QUESTION 1.8. Is Theorem 1.3 true for the case $p = n$?

For $p > n$, the main obstruction to an extension theorem is the existence of exotic n-spheres. It is known that no exotic spheres exist for $n = 1, 2, 3, 5, 6$ [KM], and the case $n = 1$ can be done by hand. It would be interesting to determine whether other geometric obstructions arise.

QUESTION 1.9. For $n = 2, 3, 5, 6$, is Theorem 1.3 true for all $p \geq 1$?

The outline of the paper is as follows. In Section 2 we review basic facts about Lipschitz mappings, Sobolev spaces, and the class LW_2^p. In Section 3 we prove extension theorems in the setting of doubly punctured domains. Section 4 addresses the case of homeomorphisms between collars by employing suitable generalizations of inversion maps and reducing to previous cases.

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2. Notation and Basic Facts

For $A \subset \mathbb{R}^n$, we write A^c for the complement of A in $\mathbb{R}^n$. The open unit ball in $\mathbb{R}^n$ is denoted $\mathbb{B}^n$; if the dimension is understood, we will write $\mathbb{B}$ for $\mathbb{B}^n$.

We write $A \lesssim B$ for inequalities of the form $A \leq kB$, where k is a fixed dimensional constant and does not depend on A or B.

For domains Ω and Ω' in $\mathbb{R}^n$, recall that a map $f : \Omega \to \Omega'$ is *Lipschitz* whenever

$$L(f) := \sup\left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in \Omega,\ x \neq y \right\} < \infty.$$

The map f is *locally Lipschitz* if every point in Ω has a neighborhood on which f is Lipschitz. A homeomorphism $f : \Omega \to \Omega'$ is *bi-Lipschitz* (resp. *locally bi-Lipschitz*) if f and f^{-1} are both Lipschitz (resp. locally Lipschitz); compare inequality (1.1).

The following lemmas about bi-Lipschitz maps are used in Section 2. The first is a special case of [TV, Lemma 2.17]; the second one is elementary, so we omit the proof.

LEMMA 2.1 (Tukia–Väisälä). *Let O and O' be open connected sets in $\mathbb{R}^n$ and let K be a compact subset of O. If $f : O \to O'$ is locally bi-Lipschitz then $f|K$ is bi-Lipschitz, where $L((f|K)^{-1})$ depends only on O, K, and $L(f)$.*

LEMMA 2.2. *For $i = 1, 2$, let $h_i : \Omega_i \to \mathbb{R}^n$ be locally bi-Lipschitz embeddings such that $h_1(\Omega_1 \setminus \Omega_2) \cap h_2(\Omega_2 \setminus \Omega_1) = \emptyset$. If $h_1 = h_2$ holds on all of $\Omega_1 \cap \Omega_2$, then*

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in \Omega_1, \\ h_2(x) & \text{if } x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

is also a locally bi-Lipschitz embedding.

For $f \in W^{2,p}(\Omega; \Omega')$, we will use the Hilbert–Schmidt norm for the weak derivatives $Df(x) := [\partial_j f_i(x)]_{i,1=1}^n$ and $D^2 f(x) := [\partial_k \partial_j f_i(x)]_{i,j,k=1}^n$. That is,

$$|Df(x)| := \left[\sum_{i,j=1}^n |\partial_j f_i(x)|^2 \right]^{1/2}, \qquad |D^2 f(x)| := \left[\sum_{i,j,k=1}^n |\partial_k \partial_j f_i(x)|^2 \right]^{1/2}.$$

In what follows, we will use basic facts about Sobolev spaces, such as the change of variables formula [Z, Thm. 2.2.2] and that Lipschitz functions on Ω are characterized by the class $W^{1,\infty}(\Omega)$ [EG, Thm. 4.2.3.5]. Our next lemma gives a gluing procedure for Sobolev functions.

LEMMA 2.3. *For $i = 1, 2$, let O_i be a domain in $\mathbb{R}^n$ and let $f_i \in W^{1,p}_{\text{loc}}(O_i)$. If $f_1 = f_2$ holds a.e. on $O_1 \cap O_2$, then $\chi_{O_1} f_1 + \chi_{O_2 \setminus O_1} f_2 \in W^{1,p}_{\text{loc}}(O_1 \cup O_2)$.*

Proof. Let O be a bounded domain in $\mathbb{R}^n$ such that $\bar{O} \subset O_1 \cup O_2$. For each $x \in O$, there exists an $r > 0$ such that $B(x, r)$ lies entirely in O_1 or in O_2. Since $\bar{O}$ is compact, there exist an $N \in \mathbb{N}$ and a collection of balls $\{B(x_i, r_i)\}_{i=1}^N$ whose union covers O.

Let $\{\varphi_i\}_{i=1}^N$ be a smooth partition of unity that is subordinate to the cover $\{B(x_i, r_i)\}_{i=1}^N$. For each $i = 1, 2, \dots, N$, one of $f_1\varphi_i$ or $f_2\varphi_i$ is well-defined and lies in $W^{1,p}(O)$; call it ψ_i. We now observe that $\psi := \sum_{i=1}^N \psi_i$ also lies in $W^{1,p}(O)$, and by construction it agrees with $\chi_{O_1} f_1 + \chi_{O_2 \setminus O_1} f_2$. □

It is a fact that the class LW_2^p is preserved under composition. We now state this as a lemma that follows directly from the product rule [EG, Thm. 4.2.2.4] and the change of variables formula [Z, Thm. 2.2.2].

LEMMA 2.4. *Let $p \geq 1$. If $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$ are homeomorphisms of class LW_2^p, then so is $h := g \circ f$. In addition, for a.e. $x \in \Omega$ and for all $i, j, k \in \{1, \dots, n\}$, the weak derivatives satisfy*

$$\begin{aligned} \partial_j h_i(x) &= \sum_{l=1}^n \partial_l g_i(f(x)) \partial_j f_l(x) \\ \partial_{kj}^2 h_i(x) &= \sum_{l=1}^n \left[\partial_l g_i(f(x)) \partial_{kj}^2 f_l(x) + \sum_{m=1}^n \partial_{ml}^2 g_i(f(x)) \partial_k f_m(x) \partial_j f_l(x) \right]. \end{aligned} \tag{2.1}$$

REMARK 2.5. Linear maps (homeomorphisms) such as dilation and translation are clearly of class LW_2^p. So if $g : \Omega \to \Omega'$ is any homeomorphism of class LW_2^p, then by Lemma 2.4 its composition with such linear maps is also of class LW_2^p. In what follows, we will implicitly use this fact to obtain convenient geometrical configurations.

3. Extensions for Homeomorphisms of Class LW_2^p between Doubly Punctured Domains

First we formulate the extension theorem in a different geometric configuration.

THEOREM 3.1. *Let $p \geq 1$, let E_1 and E_2 be Jordan domains such that $\overline{E_1} \cap \overline{E_2} = \emptyset$, and let B_1 and B_2 be balls so that $\overline{B_1} \cap \overline{B_2} = \emptyset$.*

If $g\colon (E_2 \cup E_1)^c \to (B_1 \cup B_2)^c$ is a homeomorphism of class LW_2^p such that $g(\partial E_i) = \partial B_i$ holds for $i = 1, 2$, then there exist a homeomorphism $G\colon E_2^c \to B_2^c$ of class LW_2^p and a neighborhood N of ∂E_2 such that $g|(N \cap E_2^c) = G|(N \cap E_2^c)$.

Following the outline of [Ge, Sec. 3], we begin with a special case.

LEMMA 3.2. *Theorem 3.1 holds under the additional assumption that*

$$g|B^c = \mathrm{id}|B^c, \tag{3.1}$$

where B is an open ball that contains $\overline{E_1}$ and $\overline{E_2}$.

Proof.

Step 1. By composing with linear maps, we may assume that $B = \mathbb{B}$ and that there exist $a, b \in \mathbb{R}$ such that $a < b$, $\bar{B}_1 \subset \{x_n < a\}$, and $\bar{B}_2 \subset \{x_n > b\}$.

Put $c = (b - a)/2$. Define an odd, $C^{1,1}$-smooth function $s_0\colon \mathbb{R} \to [-1, 1]$ by

$$s_0(t) := \begin{cases} 1 - (t - c)^2/c^2 & \text{if } 0 \leq t \leq c, \\ 1 & \text{if } t > c. \end{cases}$$

Using the auxiliary function $s\colon \mathbb{R} \to [0, 3]$ given by

$$s(t) := \frac{3}{2}\left(s_0\left(t - \frac{a+b}{2}\right) + 1\right),$$

we define a bi-Lipschitz homeomorphism $S\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$S(x) = x - s(x_n)e_1 \tag{3.2}$$

(see Figure 1). It is clear that S is of class LW^p and satisfies the a.e. estimate

$$|D^2 S| \leq 2c^{-2}. \tag{3.3}$$

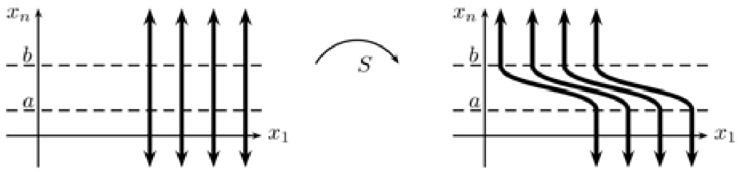

Figure 1 For $\mathbb{R}^2$, level curves for the map S

Step 2. For $k \in \mathbb{Z}$, put $\tau_k(x) = x + 3ke_1$ and consider the sets

$$\Omega := \left(\bigcup_{k=0}^{\infty} \tau_k(E_1) \cup \tau_k(E_2)\right)^c \quad \text{and} \quad \Omega' := \left(\bigcup_{k=0}^{\infty} \tau_k(B_1) \cup \tau_k(B_2)\right)^c.$$

We now modify g into a new homeomorphism $g_*\colon \Omega \to \Omega'$ as follows:

$$g_*(x) := \begin{cases} (\tau_k \circ g \circ \tau_{-k})(x) & \text{if } x \in \Omega \cap \tau_k(\mathbb{B}) \text{ for some } k \geq 0, \\ x & \text{if } x \in \Omega \setminus \bigcup_{k=0}^{\infty} \tau_k(\mathbb{B}). \end{cases} \tag{3.4}$$

By our hypotheses, there exists an $r \in (0,1)$ such that $E_1 \cup E_2 \subset B(0,r)$ and $g|\mathbb{B} \setminus B(0,r) = \mathrm{id}$. If we put $\Omega_1 := \tau_k(\mathbb{B}) \cap \Omega$ and $\Omega_2 := \Omega \setminus \bigcup_{l=0}^{\infty} \tau_k(\overline{B(0,r)})$ for each $k \in \mathbb{N}$, then Lemma 2.2 implies that g_* is locally bi-Lipschitz.

Similarly, for any bounded domain O in Ω that meets $\tau_k(\partial\mathbb{B})$, put $O_1 := O \cap \Omega$ and $O_2 := O \setminus \tau_k(\overline{B(0,r)})$. For $f_1 := D(\tau_k \circ g \circ \tau_{-k})$ and $f_2 := D(\mathrm{id})$, Lemma 2.3 implies that $g_* \in W^{2,p}(O)$ and therefore $g_* \in W^{2,p}_{\mathrm{loc}}(\Omega; \Omega')$. By symmetry, the same is true of g_*^{-1} and so g_* is of class LW_2^p.

Step 3. Consider the bi-Lipschitz homeomorphism given by

$$G_* := \tau_1 \circ g_*^{-1} \circ S \circ g_* \tag{3.5}$$

(see Figure 2). By Lemma 2.4, it is also of class LW_2^p. We now define $G\colon E_2^c \to B_2^c$ as

$$G(x) := \begin{cases} G_*(x) & \text{if } x \in \Omega, \\ \tau_1(x) & \text{if } x \in \bigcup_{k=0}^{\infty} \tau_k(E_1), \\ x & \text{if } x \in \bigcup_{k=1}^{\infty} \tau_k(E_2). \end{cases} \tag{3.6}$$

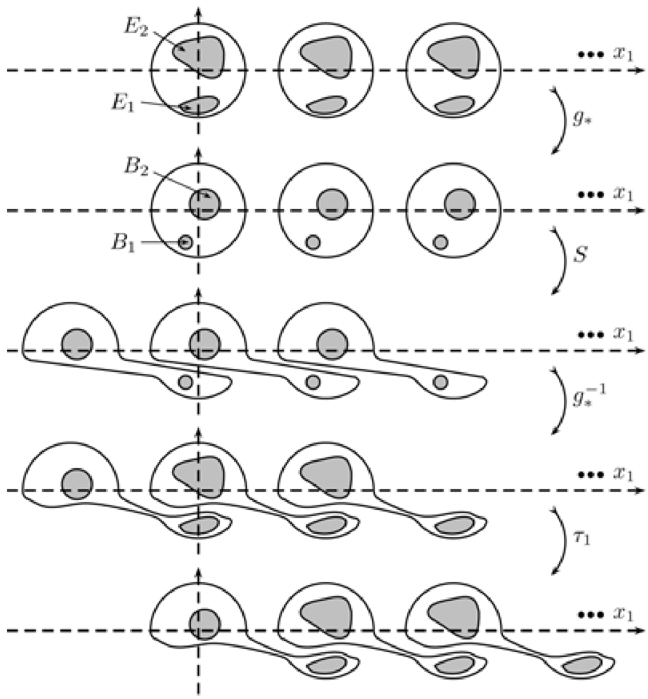

Figure 2 A schematic of the mapping G_*

By the same argument as [Ge, pp. 153–154], the map G is a homeomorphism. We also note that G is "periodic" in the sense that, for each $k \in \mathbb{N}$,

$$(\tau_k \circ G \circ \tau_{-k})|\tau_k(\bar{\mathbb{B}} \setminus E_2) = G|\tau_k(\bar{\mathbb{B}} \setminus E_2). \tag{3.7}$$

To see that G extends g, consider the set $\sigma_{ab} := g_*^{-1}(\{a \le x_n \le b\})$. Its complement $\mathbb{R}^n \setminus \sigma_{ab}$ consists of two (connected) components. Let σ_b be the component containing the vector e_n, let σ_a be the component containing $-e_n$, and consider the open set $N := \mathbb{B} \cap \sigma_b$. By assumption, $\bar{B}_2$ lies in $\mathbb{B} \cap \{x_n > b\}$, so $\bar{E}_2$ lies in N. From before, we have $g_* = g$ on $\mathbb{B}$ and $S = \tau_{-1}$ on $\{x_n > b\}$, which imply that

$$(S \circ g_*)(N) = (\tau_{-1} \circ g)(N) = \tau_{-1}(\mathbb{B} \cap \{x_n > b\}) \subset \tau_{-1}(\mathbb{B}).$$

By hypothesis we have $g_*^{-1} = \mathrm{id}$ on $\tau_{-1}(\mathbb{B})$ and hence on $(S \circ g)(N)$. It follows that

$$G|N = G_*|N = (\tau_1 \circ g_*^{-1} \circ S \circ g_*)|N = (\tau_1 \circ \mathrm{id} \circ \tau_{-1} \circ g)|N = g|N.$$

As a result, G agrees with g on $N \cap E_2^c$.

Finally, $G = \mathrm{id}$ holds on $\sigma_b \setminus E_2$ and $G = \tau_1$ holds on σ_a. Using these domains for Ω_1 and $\mathbb{R}^n \setminus \bigcup_{k=0}^{\infty} \tau_k(\bar{\mathbb{B}})$ for Ω_2, Lemma 2.2 implies that G is locally bi-Lipschitz. With the same choice of domains, Lemma 2.3 implies that also $G \in W^{2,p}_{\mathrm{loc}}(E_2^c; B_2^c)$. For the case of G^{-1}, note that the inverse is given by

$$G^{-1}(x) = \begin{cases} G_*^{-1}(x) & \text{if } x \in \Omega \setminus \tau_{-1}(B_2), \\ \tau_{-1}(x) & \text{if } x \in \bigcup_{k=0}^{\infty} \tau_k(E_1), \\ x & \text{if } x \in \bigcup_{k=1}^{\infty} \tau_k(E_2). \end{cases} \tag{3.8}$$

Arguing similarly with $g_*(N)$ for N, it follows that $G^{-1} \in W^{2,p}_{\mathrm{loc}}(B_2^c; E_2^c)$, which proves the lemma. □

We now observe that Lemma 3.2 holds even when B_1 and B_2 are not balls. In the preceding proof it is enough that, up to rotation, there is a slab $\{c_1 < x_n < c_2\}$ that separates B_1 from B_2. This result, which we state as Lemma 3.3, is used in Section 4.

LEMMA 3.3. *Let $p \ge 1$ and let E_1, E_2, C_1, and C_2 be Jordan domains such that $\overline{E_1} \cap \overline{E_2} = \emptyset$ and $\overline{C_1} \cap \overline{C_2} = \emptyset$. Suppose $g\colon (E_1 \cup E_2)^c \to (C_1 \cup C_2)^c$ is a homeomorphism of class LW_2^p such that*

(1) *$g(\partial E_i) = \partial B_i$ holds for $i = 1, 2$,*
(2) *there exists a ball B containing $\overline{E_1}$ and $\overline{E_2}$ such that $g|B^c = \mathrm{id}|B^c$, and*
(3) *there exist a rotation $\Theta\colon \mathbb{R}^n \to \mathbb{R}^n$ and numbers $c_1, c_2 \in \mathbb{R}$ $(c_1 < c_2)$ such that $\Theta(C_1) \subset \{x_n < c_1\}$ and $\Theta(C_2) \subset \{x_n > c_2\}$.*

Then there exist a homeomorphism $G\colon E_2^c \to C_2^c$ of class LW_2^p and a neighborhood N of ∂E_2 such that $g|(N \cap E_2^c) = G|(N \cap E_2^c)$.

Although the regularity of the extension G is local in nature, it nonetheless enjoys certain uniform properties. We summarize them in the next lemma.

LEMMA 3.4. *Let E_1, E_2, C_1, C_2, B, and g be as in Lemma 3.3. If G is the extension of g as defined in* (3.6), *then*:

(1) $DG \in L^\infty(E_2^c)$ *and* $DG^{-1} \in L^\infty(C_2^c)$; *and*

(2) *the restriction $G|B^c$ is a bi-Lipschitz homeomorphism.*

Proof. From Lemma 3.3, the map G is already locally bi-Lipschitz. To prove part (1), we give a uniform bound for $L(G|K)$ over all compact subsets K of B^c. Let $B = \mathbb{B}$ and let S and g_* be as defined in the proof of Lemma 3.2.

Again, let $\sigma_{ab} := g_*^{-1}(\{a \le x_n \le b\})$ and let σ_b and σ_a be the (connected) components of $\mathbb{R}^n \setminus \sigma_{ab}$ containing the vectors e_n and $-e_n$, respectively. By equation (3.2), we have $S|\{x_n < a\} = \mathrm{id}$ and $S|\{x_n > b\} = \tau_{-1}$, which imply (respectively) the bounds $L(G|\mathbb{B}^c \cap \sigma_a) \le 1$ and $L(G|\mathbb{B}^c \cap \sigma_b) \le 1$.

It remains to estimate $L(G|\mathbb{B}^c \cap \sigma_{ab})$. For each $k \in \mathbb{N}$, the set $\sigma_{ab}^k := \sigma_{ab} \cap \tau_k(\bar{\mathbb{B}})$ is compact and so, by Lemma 2.1, the restriction $G|\sigma_{ab}^k$ is bi-Lipschitz. Equation (3.7) then implies that $L(G|\sigma_{ab}^k) = L(G|\sigma_{ab}^1)$ holds for each $k \in \mathbb{N}$.

The remaining set $\sigma_{ab} \setminus \bigcup_{k=0}^{\infty} \tau_k(\mathbb{B})$ consists of infinitely many components, of which one is an unbounded subset U of $\{x_1 < 0\}$ and the others are translates of a compact subset K_0 of $\sigma_{ab} \cap B(0,3)$. Since $g|U = \mathrm{id}$, it follows that

$$G|\sigma = (\tau_1 \circ g_*^{-1} \circ S \circ g_*)|\sigma = (\tau_1 \circ S)|\sigma,$$

from which we derive that $L(G|\sigma) \le L(S)$. By the "periodicity" of G (equation (3.7)), for all $k \in \mathbb{N}$ we also have $L(G|\tau_k(K_0) = L(G|K_0)$. Part (1) of the lemma now follows from [EG, Thm. 4.2.3.5] and the preceding estimates, where

$$\|DG\|_{L^\infty(E_2^c)} \le \max\{1, L(G|K_0), L(G|\sigma_{ab}^1), L(S)\}.$$

Using the explicit formula in (3.8), the case of G^{-1} follows similarly.

To prove part (2) of the lemma, let ℓ be any line segment that does not intersect $\mathbb{B}$. The restriction $G|\ell$ is bi-Lipschitz with $L(G|\ell) \le C$. Since $\partial\mathbb{B}$ is compact, it follows from Lemma 2.1 that the restriction $G|\partial\mathbb{B}$ is bi-Lipschitz.

Let x_1 and x_2 be arbitrary points in $\mathbb{B}^c$ and let ℓ be the line segment in $\mathbb{R}^n$ that joins x_1 to x_2. If ℓ crosses through $\mathbb{B}$ then let y_1 and y_2 be points on $\ell \cap \partial\mathbb{B}$, where $|x_1 - y_1| < |x_1 - y_2|$. Since ℓ is a geodesic, we have the identity

$$|x_1 - x_2| = |x_1 - y_1| + |y_1 - y_2| + |y_2 - x_2|.$$

The triangle inequality then implies that

$$\begin{aligned} |G(x_1) - G(x_2)| &\le |G(x_1) - G(y_1)| + |G(y_1) - G(y_2)| + |G(y_2) - G(x_2)| \\ &\le C(|x_1 - y_1| + |y_2 - x_2|) + L(G|\partial\mathbb{B})|y_1 - y_2| \\ &\le (C + L(G|\partial\mathbb{B}))(|x_1 - y_1| + |y_1 - y_2| + |y_2 - x_2|) \\ &= (C + L(G|\partial\mathbb{B}))|x_1 - x_2|. \end{aligned}$$

Again the argument is symmetric for G^{-1}, so this proves the lemma. □

Theorem 3.1 now follows easily from Lemma 3.2, and a more general version of the theorem follows from Lemma 3.3. As in [Ge, Lemma 2], one takes compositions with the extension, its inverse, and a radial stretch map.

Proof of Theorem 3.1. By composing g with linear maps, we may assume that E_1, E_2, B_1, and B_2 are subsets of $\mathbb{B}$, that $0 \in E_2$, and that $\mathbb{B}^c \subset g(\mathbb{B}^c)$. Choose $r_1, r_2 \in (0,1)$ such that $B(0,r_1) \subset E_2$ and $E_1 \cup E_2 \subset B(0,r_2)$.

Let $\rho\colon [0,\infty) \to [0,\infty)$ be a smooth increasing function such that $\rho([0,r_1]) = [0,r_2]$ and $\rho([1,\infty)) = [1,\infty)$. Define a homemorphism $R\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$R(x) := \begin{cases} \rho(|x|) \cdot |x|^{-1}x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{3.9}$$

Clearly, R is of class LW_2^p, is bi-Lipschitz, and maps $B(0,r_1)$ onto $B(0,r_2)$.

Putting $E_1' := (g \circ R)(E_1)$ and $E_2' := ((g \circ R)(E_2^c))^c$, Lemma 2.4 implies that

$$h := g \circ R \circ g^{-1}\colon (E_1' \cup E_2')^c \to (B_1 \cup B_2)^c$$

is also a homeomorphism of class LW_2^p. Since $R|\mathbb{B}^c = \mathrm{id}|\mathbb{B}^c$, we further obtain

$$h|\mathbb{B}^c = (g \circ R \circ g^{-1})|\mathbb{B}^c = \mathrm{id}|\mathbb{B}^c. \tag{3.10}$$

So with E_1' and E_2' in place of E_1 and E_2, respectively, h satisfies equation (3.1) and the other hypotheses of Lemma 3.2. As a result, there exist a homeomorphism $H\colon (E_2')^c \to B_2^c$ of class LW_2^p and a neighborhood N' of $\partial E_2'$ such that

$$h|(N' \cap (E_2')^c) = H|(N' \cap (E_2')^c).$$

Let $G := H \circ g \circ R^{-1}$. The open set

$$N := (R \circ g^{-1})(N' \setminus (\bar{B}_1 \cup \bar{B}_2))$$

contains ∂E_2 and, by Lemma 2.4, the map G is of class LW_2^p. Moreover, for each $x \in N \setminus E_2$, there is a $y \in N' \setminus D_2'$ such that $x = (R \circ g^{-1})(y)$ and therefore

$$\begin{aligned} G(x) &= (H \circ g \circ R^{-1})((R \circ g^{-1})(y)) = H(y) \\ &= h(y) = (g \circ R \circ g^{-1})((g \circ R^{-1})(x)) = g(x). \end{aligned}$$

Thus we obtain $g = G$ on $N \cap E_2^c$, as desired. □

4. Extensions of Homeomorphisms of Class LW_2^p between Collars

4.1. Generalized Inversions

To pass to the configurations of domains in Theorem 1.3, we will use *generalized inversions.* For fixed $a, r > 0$, these are homeomorphisms $I_{a,r}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ of the form

$$I_{a,r}(x) := r^{a+1}|x|^{-(a+1)}x.$$

Indeed, the inverse map satisfies $(I_{a,r})^{-1} = I_{1/a,r}$ as well as the estimate

$$|x|^{a+1} = (r^{1/a+1}|I_{a,r}(x)|^{-1/a})^{a+1} \approx |I_{a,r}(x)|^{-(1/a+1)}. \tag{4.1}$$

For derivatives of $I_{a,r}$, an elementary computation gives

$$|D^k I_{a,r}(x)| \lesssim r^{a+1}|x|^{-(a+k)}; \tag{4.2}$$

similarly, for the Jacobian determinant $JI_{a,r} := |\det(DI_{a,r})|$ we have

$$JI_{a,r}(x) \le nr^{n(a+1)}|x|^{-n(a+1)} \approx |I_{a,r}(x)|^{n(a+1)/a}. \tag{4.3}$$

If $a = 1$ then $I_{1,r}$ is conformal and maps spheres to spheres. In general, the map $I_{a,r}$ possesses weaker properties that are sufficient for our purposes. For instance, it preserves radial rays: sets of the form $\{\lambda x : \lambda > 0\}$ for some $x \in \mathbb{R}^n \setminus \{0\}$.

Another property, stated as Lemma 4.1, is used in the proof of Theorem 1.3 under the following hypotheses. To begin, write $B_1 = B(t, r_1)$ and $B_2 = B(z, r_2)$, where $\bar{B}_1 \subset B_2$. By composing with linear maps, we may assume that the following statements hold.

(H1) The x_n-coordinate axis crosses through the points t and z with $t_n \le z_n \le 0$, so the "south poles" $\tau := t - r_1\vec{e}_n$ on $\bar{B}_1$ and $\zeta := z - r_2\vec{e}_n$ on $\bar{B}_2$ satisfy $\zeta_n < \tau_n$ and $|\zeta - \tau| = \operatorname{dist}(\bar{B}_1, B_2^c)$.

(H2) There exists an $r \in (0, r_2)$ such that the sphere $\partial B(0, r)$ is tangent to both ∂B_1 and ∂B_2 with $B(0, r) \subset B_2 \setminus B_1$. In particular, this gives $r_1 < |t_n|$.

See Figure 3.

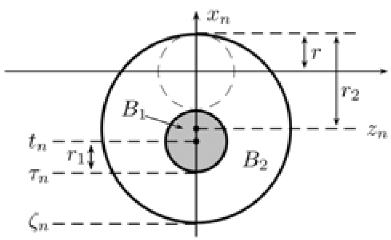

Figure 3 A possible configuration for B_1, B_2, and $B(0, r)$

LEMMA 4.1. *Let $a \in (0, 1)$. If B_1 and B_2 are balls in $\mathbb{R}^n$ with $\bar{B}_1 \subset B_2$ and that satisfy hypotheses* (H1) *and* (H2), *then there exist real numbers $c_1 < c_2$ such that $I_{a,r}(B_1) \subset \{x_n < c_1\}$ and $I_{a,r}(B_2^c) \subset \{x_n > c_2\}$.*

The proof is a computation, and the basic idea is simple. Although the bounded domains $I_{a,r}(B_1)$ and $I_{a,r}(B_2^c)$ may not be balls, the distance between them is still attained by the images of the "north" and "south" poles of B_1 and B_2, respectively.

Proof of Lemma 4.1. Once again, let τ and ζ be the "south poles" of B_1 and B_2, respectively. From (H1) and (H2) we have

$$\zeta_n = -|\zeta| < -|\tau| = \tau_n.$$

Therefore, if we put $I := I_{a,r}$ then the image points $\tau' := I(\tau)$ and $\zeta' := I(\zeta)$ satisfy

$$\tau_n' = -|\tau'| < -|\zeta'| = \zeta_n'. \tag{4.4}$$

CLAIM 4.2. *For all $y' \in I(B_1)$, we have $y'_n < \tau'_n$.*

Supposing otherwise, there exists a $y \in \partial B_1$ with $y \neq \tau$ such that y' has the same nth coordinate as τ'. Let θ be the angle between the x_n-axis and the line crossing through y' and 0. By our hypotheses, we have $t_n \leq 0$ and $0 < \theta < \pi/2$; hence $0 < \cos\theta < 1$. From $|\tau| = r_1 - t_n$ we obtain

$$|y'| = \frac{|\tau'|}{\cos\theta} = \frac{r^{a+1}|\tau|^{-a}}{\cos\theta} = \frac{r^{a+1}}{(r_1 - t_n)^a \cos\theta},$$

so from this and $|y'| = r^{a+1}|y|^{-a}$ we further obtain

$$|y| = r^{(a+1)/a}\left[\frac{r^{a+1}}{(r_1 - t_n)^a \cos\theta}\right]^{-1/a} = (\cos\theta)^{1/a}(r_1 - t_n). \tag{4.5}$$

On the other hand, I preserves radial rays and hence angles between radial rays. As a result, $y \in \partial B_1$ (and the law of cosines) implies that

$$r_1^2 = |y|^2 + t_n^2 - 2|y|t_n \cos\theta$$

and so

$$|y| = -t_n \cos\theta + \sqrt{r_1^2 - t_n^2 \sin^2\theta}.$$

From hypothesis (H2) once again, we obtain $r_1 < |\tau_n|$ and hence

$$|y| < -t_n \cos\theta + \sqrt{r_1^2 - r_1^2 \sin^2\theta} = (r_1 - t_n)\cos\theta.$$

This is in contradiction with Equation (4.5), since the inequality $\cos\theta \leq (\cos\theta)^{1/a}$ is a consequence of $a \geq 1$. The claim follows.

CLAIM 4.3. *For all $w' \in I(B_2^c)$, we have $\zeta'_n < w'_n$.*

Suppose there exists a $w \in \partial B_2$ such that $w \neq \zeta$ and $w'_n = \zeta'_n$. If α is the angle between w and the x_n-axis, then a computation similar to before gives

$$(2r_2 - r)\cos^{1/a}\alpha = |w| = (r_2 - r)\cos\alpha + \sqrt{r_2^2 - (r_2 - r)^2 \sin^2\theta}.$$

Computing further yields $\psi(a) = r_2^2$, where $\psi\colon (0,\infty) \to (0,\infty)$ is given by

$$\psi(a) := ((2r_2 - r)\cos^{1/a}\alpha - (r_2 - r)\cos\alpha)^2 + (r_2 - r)^2 \sin^2\alpha.$$

Clearly, ψ is smooth, and an elementary computation shows that it attains a minimum at a unique point in $(0, 1)$. We observe that

$$\psi(1) = r_2^2 \cos^2\alpha + (r_2 - r)^2 \sin^2\alpha < r_2^2.$$

Since $0 < \cos\alpha < 1$, we see that $\cos^{1/a}\alpha \to 0$ as $a \to 0$. It follows that

$$\lim_{a\to 0} \psi(a) = (0 + (r_2 - r)\cos\alpha)^2 + (r_2 - r)^2 \sin^2\alpha = (r_2 - r)^2 < r_2^2$$

and therefore $\psi(a) < r_2^2$ holds for all $(0, 1)$. This is a contradiction, which proves Claim 4.3. When we combine both claims and (4.4), the lemma follows. $\square$

4.2. From Doubly Punctured Domains to Collars

We now prove Theorem 1.3. The argument requires several lemmas.

LEMMA 4.4. *Let $a > 0$ and let D_1, D_2, B_1, B_2, and f be as given in Theorem 1.3. If there exists an $r > 0$ such that $\bar{B}(0,r) \subset D_2 \setminus D_1$ and $\bar{B}(0,r) \subset B_2 \setminus B_1$ and if $f(0) = 0$, then $I_{a,r} \circ f \circ I_{a,r}^{-1}$ is a homeomorphism of class LW_2^p.*

Proof. Because $\Omega := I_{a,r}(D_2 \setminus (\bar{D}_1 \cup \{0\}))$ and $I_{a,r}(B_2 \setminus (\bar{B}_1 \cup \{0\}))$ lie in $\mathbb{R}^n \setminus B(0,\varepsilon)$ for some $\varepsilon > 0$, the restricted maps $I_{a,r}^{-1}|\Omega$ and $I_{a,r}|\Omega'$ are diffeomorphisms. By Lemma 2.4, it follows that $g := I_{a,r} \circ f \circ I_{a,r}^{-1} : \Omega \to \Omega'$ is of class LW_2^p. $\square$

LEMMA 4.5. *Let E_1, E_2, C_1, C_2, B, and g be as given in Lemma 3.3, and let G be as given in* (3.6). *If $0 \in E_2$, if $0 \in C_2$, and if there exists an $r > 0$ such that $B = B(0,r)$, then for each $a > 0$ the map*

$$F(x) := \begin{cases} (I^{-1} \circ G \circ I)(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is a locally bi-Lipschitz homeomorphism.

Proof. Without loss of generality, let $r = 1$ and put $I = I_{a,r}$ and $b = 1/a$. By (3.6), we have $|G(x)| \to \infty$ as $|x| \to \infty$ and so F is a well-defined homeomorphism. For each $\varepsilon > 0$, put $B_\varepsilon := B(0,\varepsilon)$. The restrictions $I|B_\varepsilon^c$ and $I^{-1}|B_\varepsilon^c$ are diffeomorphisms, so $F|B_\varepsilon^c$ is already locally bi-Lipschitz for each $\varepsilon > 0$.

To show that $F|B_\varepsilon$ is bi-Lipschitz, recall that $DG \in L^\infty(E_2^c)$ follows from Lemma 3.4. So from (2.1), (4.1), and (4.2) it follows that, for a.e. $x \in I^{-1}(E_2^c)$,

$$\begin{aligned} |DF(x)| &\leq |DI^{-1}((G \circ I)(x))||DG(I(x))||DI(x)| \\ &\lesssim \frac{\|DG\|_\infty}{|(G \circ I)(x)|^{b+1}|x|^{a+1}} \approx \frac{\|DG\|_\infty |I(x)|^{b+1}}{|(G \circ I)(x)|^{b+1}}. \end{aligned}$$

Now fix $y_0 \in E_2^c$. Putting $L := L(G^{-1}|B^c)$, for all $x \in B_\varepsilon$ we have

$$|G(I(x)) - G(y_0)| \geq L^{-1}(|I(x) - y_0|) \geq L^{-1}(|I(x)| - |y_0|).$$

Applying the triangle inequality to the right-hand side, we obtain

$$|G(I(x))| \geq L^{-1}(|I(x)| - |y_0|) - |G(y_0)|;$$

taking reciprocals, we further obtain

$$\begin{aligned} \frac{|I(x)|}{|(G \circ I)(x)|} &\leq \frac{L|I(x)|}{|I(x)| - |y_0| - L|G(y_0)|} \\ &= \frac{Lr^{a+1}}{r^{a+1} - |x|^a|y_0| - |x|^a L|G(y_0)|} \to L \end{aligned} \tag{4.6}$$

as $x \to 0$. Combining the previous estimates, for sufficiently small $\varepsilon > 0$ we have that

$$|DF(x)| \lesssim \frac{\|DG\|_\infty |I(x)|^{b+1}}{|(G \circ I)(x)|^{b+1}} \lesssim (2L)^{b+1}\|DG\|_\infty < \infty$$

holds for a.e. $x \in B_\varepsilon$, and therefore $|DF| \in L^\infty_{\text{loc}}(I^{-1}(E_2^c))$. By [EG, Thm. 4.2.3.5], it follows that F is locally Lipschitz on $B(0,\varepsilon)$. By symmetry, the same holds for F^{-1} and so F is locally bi-Lipschitz on all of $I^{-1}(E_2^c)$.

In the remaining proofs, we will require explicit forms of the extensions from Lemma 3.2 and Theorem 3.1.

LEMMA 4.6. *Let E_1, E_2, C_1, C_2, g, and $B = B(0,r)$ be as given in Lemma 4.5, let G be as given in* (3.6), *and let $p \in [1,n)$. If $a < n/p - 1$, then the homeomorphism $I_{a,r}^{-1} \circ G \circ I_{a,r}$ is of class LW_2^p.*

Proof. For convenience, we retain the notation from the proof of Lemma 4.5. As before, $I|B_\varepsilon^c$ and $I^{-1}|B_\varepsilon^c$ are diffeomorphisms, so by Lemma 2.4 the map $F|B_\varepsilon^c$ is of class LW_2^p. It suffices to show that $F \in W^{2,p}_{\text{loc}}(B_\varepsilon;\mathbb{R}^n)$ and $F^{-1} \in W^{2,p}_{\text{loc}}(F(B_\varepsilon);\, B_\varepsilon)$ for each $\varepsilon > 0$.

To estimate second derivatives, we use (2.1), (4.1), (4.2), and (4.6) once again. For convenience, set $y := I(x)$ and $z := (G \circ I)(x)$. We then obtain

$$\begin{aligned}
|D^2F(x)| &= |D^2(I^{-1} \circ G \circ I)(x))| \\
&\le |D^2I^{-1}(z)||DG(y)|^2|DI(x)|^2 \\
&\quad + |DI^{-1}(z)|\big(|D^2G(y)||DI(x)|^2 + |DG(y)||D^2I(x)|\big) \\
&\lesssim \frac{\|DG\|_\infty^2}{|z|^{b+2}|x|^{2(a+1)}} + \frac{1}{|z|^{b+1}}\left(\frac{|D^2G(y)|}{|x|^{2(a+1)}} + \frac{\|DG\|_\infty}{|x|^{a+2}}\right) \\
&\lesssim \frac{|I(x)|^{2(b+1)}}{|G(I(x))|^{b+2}} + \frac{|I(x)|^{2(b+1)}|D^2G(I(x))|}{|G(I(x))|^{b+1}} + \frac{|I(x)|^{b+1}}{|G(I(x))|^{b+1}|x|} \\
&\lesssim |I(x)|^b + |I(x)|^{b+1}|D^2G(I(x))| + |x|^{-1}
\end{aligned} \tag{4.7}$$

for a.e. $x \in B_\varepsilon$. Since $p < n$ and $b = 1/a$, the function $x \mapsto |I(x)|^b = |x|^{-1}$ lies in $L^p(B_\varepsilon)$. For the remaining term, (4.1) and (4.3) imply that

$$1 = JI^{-1}(I(x))JI(x) \lesssim |I^{-1}(I(x))|^{n(a+1)}JI(x) = |I(x)|^{-n(b+1)}JI(x);$$

therefore, by a change of variables [Z, Thm. 2.2.2] and (4.3), we have

$$\begin{aligned}
\int_{B_\varepsilon} |I(x)|^{p(b+1)}|D^2G(I(x))|^p\,dx &\lesssim \int_{B_\varepsilon} \frac{|D^2G(I(x))|^p JI(x)}{|I(x)|^{(n-p)(b+1)}}\,dx \\
&= \int_{\mathbb{B}^c} \frac{|D^2G(y)|^p}{|y|^{(n-p)(b+1)}}\,dy.
\end{aligned} \tag{4.8}$$

For each $k \in \mathbb{N}$, (3.6) implies that $G|\tau_k(E_2) = \text{id}$ and $G|\tau_k(E_1) = \tau_1$; hence $D^2G|\tau_k(E_1 \cup E_2) = 0$. The rightmost integral in (4.8) can therefore be restricted to the subset

$$\Omega := \mathbb{B}^c \setminus \bigcup_{k=1}^\infty \tau_k(E_1 \cup E_2).$$

As defined in the proof of Lemma 3.2, the maps g_*, G_*, and G satisfy

$$|D^2G(y)| \lesssim |D^2g_*^{-1}((S\circ g_*)(y))| + |D^2S(g_*(y))| + |D^2g_*(y)| \tag{4.9}$$

for a.e. $y \in I^{-1}(E_2^c)$, where $\lesssim$ includes the constants $L(g_*)$, $L(g_*^{-1})$, $L(S)$, and $L(\tau_1)$. Using the second-derivative bound for S (inequality (3.3)), we obtain

$$\int_\Omega \frac{|D^2S(g_*(y))|^p}{|y|^{(n-p)(b+1)}}\,dy \le \int_\Omega \frac{2c^{2p}}{|y|^{(n-p)(b+1)}}\,dy \lesssim \int_1^\infty \frac{\rho^{n-1}}{\rho^{(n-p)(b+1)}}\,d\rho.$$

The rightmost integral is finite, since $a < n/p - 1$ implies that $b > p/(n-p)$ and

$$(n-1)-(n-p)(b+1) < (n-1)-(n-p)\Big(\frac{p}{n-p}-1\Big) = -1.$$

For the other terms of (4.9), note that (3.4) implies $D^2g_*^{-1}(z) = 0$ for a.e. $z \notin \bigcup_{k=1}^\infty \tau_k(\mathbb{B})$. Since $S\circ g_*$ is locally bi-Lipschitz, we estimate

$$\begin{aligned}\int_\Omega \frac{|D^2g_*^{-1}((S\circ g_*)(y))|^p}{|y|^{(n-p)(b+1)}}\,dy &= \sum_{k=1}^\infty \int_{\tau_k((S\circ g_*)^{-1}(\mathbb{B}))\cap\Omega} \frac{|D^2g_*^{-1}((S\circ g_*)(y))|^p}{|y|^{(n-p)(b+1)}}\,dy\\ &\approx \sum_{k=1}^\infty \int_{g_*^{-1}(\Omega)\cap\tau_k(\mathbb{B})} \frac{|D^2g_*^{-1}(z)|^p}{|(S\circ g_*)^{-1}(z)|^{(n-p)(b+1)}}\,dz.\end{aligned}$$

Equation (3.2) implies that $|S^{-1}(y)| \ge |y|$ holds for each $y\in\mathbb{R}^n$ and hence that

$$|(S\circ g_*)^{-1}(z)| \ge 3k-1 > k$$

holds for each $z \in \tau_k(\mathbb{B})$ and $k\in\mathbb{N}$. From the previous inequalities and another change of variables, we further estimate

$$\begin{aligned}\int_{g_*^{-1}(\Omega)\cap\tau_k(\mathbb{B})} \frac{|D^2g_*^{-1}(z)|^p}{|(S\circ g_*)^{-1}(z)|^{(n-p)(b+1)}}\,dz &\lesssim \int_{g_*^{-1}(\Omega)\cap\tau_k(\mathbb{B})} \frac{|D^2g_*^{-1}(z)|^p}{k^{(n-p)(b+1)}}\,dz\\ &\le \int_{\mathbb{B}\setminus(C_1\cup C_2)} \frac{|D^2g^{-1}(z)|^p}{k^{(n-p)(b+1)}}\,dz\end{aligned}$$

and so

$$\int_\Omega \frac{|D^2g_*^{-1}((S\circ g_*)(y))|^p}{|y|^{(n-p)(b+1)}}\,dy \lesssim \sum_{k=1}^\infty \frac{\|D^2g^{-1}\|_{L^p(\mathbb{B}\setminus(C_1\cup C_2))}}{k^{(n-p)(b+1)}}.$$

The summation is finite because $(n-p)(b+1) > 1$ follows from the hypothesis that $a < n/p - 1$. A similar estimate gives $|y|^{(p-n)(b+1)}|D^2g_*(y)| \in L^p(B_\varepsilon)$ and so, by (4.7)–(4.9), we obtain $|D^2F| \in L^p(B_\varepsilon)$ as desired.

The same argument, but with G^{-1} replacing G, shows that the map $F^{-1} = I^{-1}\circ G^{-1}\circ I$ also lies in $W^{2,p}_{\mathrm{loc}}(F(B_\varepsilon); B_\varepsilon)$. This proves the lemma. $\square$

Using the previous lemmas, we now prove the main theorem.

Proof of Theorem 1.3. Let $a < n/p - 1$ be given. By post-composing f with linear maps, we may assume that the balls B_1 and B_2 satisfy hypotheses (H1) and (H2) from Section 4.1; so, in particular, we have $B(0,r) \subset B_2 \setminus \bar{B}_1$. We further assume that $B(0,r) \subset D_2 \setminus \bar{D}_1$ and $f(0) = 0$. By Lemma 4.1, there exist $c_1 < c_2$ such that $B_1 \subset \{x_n < c_1\}$ and $B_2 \subset \{x_n > c_2\}$. For $I := I_{a,r}$ and $g := I \circ f \circ I^{-1}$, Lemma 4.4 implies that g is of class LW_2^p.

Put $E_1 = I(D_1)$, $E_2 := I(D_2^c)^c$, $C_1 := I(B_1)$, and $C_2 := I((B_2)^c)^c$. By Lemma 3.3 and the proof of Theorem 3.1, there exist a homeomorphism $G\colon E_2^c \to C_2^c$ of class LW_2^p and a neighborhood N' of ∂E_2 such that

$$g|(N' \cap E_2^c) = G|(N' \cap E_2^c).$$

As a result, the homeomorphism F (as defined in Lemma 4.5) and the open set $N := I^{-1}(N')$, which is a neighborhood of ∂D_2, satisfy the identity

$$f|(N \cap \bar{D}_2) = F|(N \cap \bar{D}_2).$$

Recalling the proof of Theorem 3.1, we have $G = H \circ g \circ R^{-1}$, where:

(H3) R is a diffeomorphism that agrees with the identity map on $\mathbb{B}^c$; and

(H4) H is a homeomorphism of class LW_2^p (as given by Lemma 3.3) that agrees with $h = g \circ R \circ g^{-1}$ on the open set $(g \circ R)(N')$.

Putting $H_* := I^{-1} \circ H \circ I$ and $R_* := I^{-1} \circ R \circ I$, we rewrite

$$F = I^{-1} \circ (H \circ g \circ R^{-1}) \circ I = H_* \circ f \circ R_*^{-1}.$$

From (H3) and the properties of I and I^{-1}, we see that R_*^{-1} is a diffeomorphism from $\mathbb{R}^n \setminus \{0\}$ onto itself. In particular, for each $r > 0$, the restriction $R_*^{-1}|B(0,r)^c$ is bi-Lipschitz. On the other hand, for sufficiently small $r > 0$ we have $R^{-1} \circ I = I$ on $B(0,r)$. Letting Id_n be the $n \times n$ identity matrix, we can write

$$DR_*^{-1}|B(0,r) = D(I^{-1} \circ R^{-1} \circ I)|B(0,r) = D(I^{-1} \circ I)|B(0,r) = \mathrm{Id}_n,$$

$$D^2R_*^{-1}|B(0,r) = D^2(I^{-1} \circ R^{-1} \circ I)|B(0,r) = D^2(I^{-1} \circ I)|B(0,r) = 0.$$

This implies that $R_*^{-1} \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ and, by Lemma 2.2, that R_*^{-1} is bi-Lipschitz. By symmetry the same holds for $R_* = I^{-1} \circ R \circ I$, so R_*^{-1} is of class LW_2^p.

Now (H4) and Lemma 4.6 imply that H_* is of class LW_2^p. By hypothesis, f is of class LW_2^p and so, by Lemma 2.4, F is of class LW_2^p. The theorem follows. □

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