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Automorphisms of the Graph of Free Splittings

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In this paper we consider the graph $\mathcal{G}_n$ of free splittings of the free group $\mathbb{F}_n$ of rank $n \geq 3$. Loosely speaking, $\mathcal{G}_n$ is the graph whose vertices are nontrivial free splittings of $\mathbb{F}_n$ up to conjugacy and where two vertices are adjacent if they are represented by free splittings admitting a common refinement. The group $\mathrm{Out}(\mathbb{F}_n)$ of outer automorphisms of $\mathbb{F}_n$ acts simplicially on $\mathcal{G}_n$. Denoting by $\mathrm{Aut}(\mathcal{G}_n)$ the group of simplicial automorphisms of the free splitting graph, we prove the following statement.

Theorem 1. *The natural map* $\mathrm{Out}(\mathbb{F}_n) \to \mathrm{Aut}(\mathcal{G}_n)$ *is an isomorphism for* $n \geq 3$.

We briefly sketch the proof of Theorem 1. We identify $\mathcal{G}_n$ with the 1-skeleton of the *sphere complex* $\mathbb{S}_n$ and observe that every automorphism of $\mathcal{G}_n$ extends uniquely to an automorphism of $\mathbb{S}_n$. It is due to Hatcher [6] that the sphere complex contains an embedded copy of the *spine* K_n of Culler–Vogtmann space. We prove that the latter is invariant under $\mathrm{Aut}(\mathbb{S}_n)$ and that the restriction homomorphism $\mathrm{Aut}(\mathbb{S}_n) \to \mathrm{Aut}(K_n)$ is injective. The claim of Theorem 1 then follows from a result of Bridson and Vogtmann [1] which asserts that $\mathrm{Out}(\mathbb{F}_n)$ is the full automorphism group of K_n.

Before concluding this introduction we would like to point out that recently Martino and Francaviglia [12] have proved that $\mathrm{Out}(\mathbb{F}_n)$ is also the full isometry group of Culler–Vogtmann space when the latter is endowed with the Lipschitz metric. While one could claim that Theorem 1 is the analogue of Ivanov's theorem on the isometries of the curve complex [8], the result of Martino and Francaviglia is the analogue of Royden's theorem on the isometries of Teichmüller space [13].

We are grateful to our motherland for the beauty of its villages. Once this is said, we would like to thank the referee for providing a careful and useful report.

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Fixing from now on $n \geq 3$, let $M_n = \#^n(\mathbb{S}^1 \times \mathbb{S}^2)$ be the connected sum of n copies of $\mathbb{S}^1 \times \mathbb{S}^2$. Observe that $\pi_1(M_n)$ is isomorphic to $\mathbb{F}_n$ and that by choosing a

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basepoint and then conveniently forgetting it afterwards we can, once and for all, identify

$$\pi_1(M_n) \simeq \mathbb{F}_n \tag{1.1}$$

up to conjugacy. We denote by $\mathrm{Map}(M_n)$ the mapping class group of M_n (i.e., the group of isotopy classes of self-diffeomorphisms of M_n); observe that (1.1) induces a homomorphism

$$\mathrm{Map}(M_n) \to \mathrm{Out}(\mathbb{F}_n). \tag{1.2}$$

By work of Laudenbach [11, III.4.3], the homomorphism (1.2) is surjective and has finite kernel.

REMARK. Although it will be of no importance for us, it should be remarked that the kernel of (1.2) is generated by Dehn twists along essential embedded 2-dimensional spheres. Notice that since

$$\pi_1(\mathrm{Diff}_0(\mathbb{S}^2)) = \pi_1(\mathrm{SO}_3) = \mathbb{Z}/2\mathbb{Z},$$

any such Dehn twist has order 2. See [11, III.4.2] for a description of such a Dehn twist.

Recall that an embedded 2-sphere in a 3-manifold is *essential* if it does not bound a ball. Two essential embedded 2-spheres S, S' in a 3-manifold are *parallel* if they are isotopic. It is due to Laudenbach [10] that S and S' are parallel if and only if they are homotopic to each other. If two parallel essential embedded 2-spheres $S, S' \subset M_n$ are parallel then they bound a submanifold homeomorphic to $\mathbb{S}^2 \times (0, 1)$.

By a *system of spheres* in M_n we mean a collection of pairwise disjoint, non-parallel, essential embedded 2-spheres. A system of spheres is *maximal* if it is not properly contained in another system of spheres. Before moving on to more interesting topics, we recall a few useful facts as follows.

- If $\Sigma \subset M$ is a maximal system of spheres, then every component of $M \setminus \Sigma$ is homeomorphic to a 3-sphere with three balls removed. In particular, Σ has $3n - 3$ components.
- If S is a connected component of a maximal system of spheres Σ, then all components of $M \setminus (\Sigma \setminus S)$ but one are homeomorphic to a 3-sphere with three balls removed. The remaining component is either homeomorphic to a 3-sphere with four balls removed or to $\mathbb{S}^1 \times \mathbb{S}^2$ with one ball removed.
- If U is homeomorphic to a 3-sphere with four balls removed, then there are exactly three isotopy classes of embedded spheres in U that are neither isotopic to a component of ∂U nor bound balls. Namely, every such sphere S separates two of the components of ∂U from the other two, and the so obtained decomposition of the set of components of ∂U determines S up to isotopy.
- If U is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ with one ball removed, then there is single isotopy class of embedded 2-spheres in U that are neither isotopic to a component of ∂U nor bound balls.

- If S is a nonseparating component of a maximal system of spheres Σ, then there exists $\Sigma' \subset \Sigma$ with $S \subset \Sigma'$ and $M \setminus \Sigma'$ a 3-sphere with $2n$ balls removed.
- If $\Sigma \subset M$ is a sphere system and U is a component of $M \setminus \Sigma$ with $\pi_1(U) \neq 1$, then U contains a sphere that does not separate U and hence, a fortiori, does not separate M.

The facts just listed follow easily from the existence and uniqueness theorem for prime decompositions of 3-manifolds. See [7] for standard notions of 3-dimensional topology and [10; 11] for a treatment of the relation between homotopy and isotopy of embedded 2-spheres in 3-manifolds.

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Given an essential embedded 2-sphere S in M_n, we denote its isotopy class by $[S]$.

The *sphere complex* $\mathbb{S}_n$ associated to M_n is the simplicial complex whose vertices are isotopy classes of essential embedded 2-spheres in M_n and where $k+1$ distinct vertices $[S_0], \dots, [S_k]$ span a k-simplex if there is a system of spheres $S'_0 \cup \dots \cup S'_k$ with $S'_i \in [S_i]$. By definition, the mapping class group of M_n acts simplicially on the sphere complex $\mathbb{S}_n$. This yields the homomorphism

$$\mathrm{Map}(M_n) \to \mathrm{Aut}(\mathbb{S}_n). \tag{2.1}$$

Given a simplicial complex X, we denote by $\mathrm{Aut}(X)$ the group of simplicial automorphisms of X.

It also follows from work of Laudenbach [10; 11] (see also [6]) that the kernels of the homomorphisms (1.2) and (2.1) are equal. In particular, the action of $\mathrm{Map}(M)$ on $\mathbb{S}_n$ induces the following simplicial action:

$$\mathrm{Out}(\mathbb{F}_n) \curvearrowright \mathbb{S}_n. \tag{2.2}$$

We will observe that the 1-skeleton $\mathbb{S}_n^{(1)}$ of $\mathbb{S}_n$ is equivariantly isomorphic to the free splitting graph $\mathcal{G}_n$, which we now define.

By a *free splitting* of the free group $\mathbb{F}_n$ we mean an isomorphism between $\mathbb{F}_n$ and the fundamental group of a graph of groups with trivial edge groups. Two free splittings are said to be *equivalent* if there is an $\mathbb{F}_n$-equivariant isometry between the corresponding Bass–Serre trees. A free splitting of $\mathbb{F}_n$ is a *refinement* of another free splitting if there is a $\mathbb{F}_n$-equivariant edge-collapsing map from the Bass–Serre tree of the first splitting to the Bass–Serre tree of the second.

In what follows we will pass freely between free splittings, the associated graph of group decompositions, and the associated Bass–Serre trees. Similarly, we will say for instance that a Bass–Serre tree is a refinement of some other Bass–Serre tree. We trust that this will not cause any confusion.

The *free splitting graph* $\mathcal{G}_n$ of $\mathbb{F}_n$ is the simplicial graph whose vertices are equivalence classes of free splittings of $\mathbb{F}_n$ whose corresponding graph of groups have a single edge. Two vertices of $\mathcal{G}_n$ are *adjacent* if they are represented by free splittings that have a common refinement. Equivalently, there exists a free splitting of $\mathbb{F}_n$ with two edges such that the first splitting is obtained by collapsing one edge of this graph of groups and the second by collapsing the other edge. Observe that $\mathrm{Out}(\mathbb{F}_n)$ acts on $\mathcal{G}_n$ by simplicial automorphisms.

REMARK. As the referee kindly pointed out, in the literature the name *free splitting graph* is sometimes reserved for the graph whose vertices correspond to the graph of group decompositions of $\mathbb{F}_n$ with a single edge that, moreover, has trivial edge group (true free splittings). We are allowing for free HNN extensions as well (cf. [9]).

LEMMA 2. *There is a simplicial isomorphism $\mathbb{S}_n^{(1)} \to \mathcal{G}_n$ conjugating the standard actions of* $\mathrm{Out}(\mathbb{F}_n)$.

Lemma 2 is surely known to all experts in the field; we sketch a proof for completeness. We also refer to [11, IV] for a complete treatment of the relation between embedded spheres in 3-manifolds and free splittings of their fundamental groups; see also [4; 5].

Sketch of Proof. To an essential embedded sphere S in M_n we associate its dual tree—that is, the Bass–Serre tree of the graph of groups decomposition of $\pi_1(M_n) \simeq \mathbb{F}_n$ given by the Seifert–van Kampen theorem. Isotopic spheres yield equivalent free splittings and hence we obtain a vertex of $\mathcal{G}_n$ for every vertex of $\mathbb{S}_n$. The dual tree to the union of two disjoint embedded spheres S, S' is the Bass–Serre tree of a free splitting of $\mathbb{F}_n \simeq \pi_1(M_n)$, which is clearly a refinement of the Bass–Serre trees associated to S and S'. In other words, the map between vertices extends to a map

$$\Phi \colon \mathbb{S}_n^{(1)} \to \mathcal{G}_n,$$

which clearly conjugates the actions of $\mathrm{Out}(\mathbb{F}_n)$ on $\mathbb{S}_n^{(1)}$ and $\mathcal{G}_n$. The map Φ is surjective by the work of Stallings [14] and injective by the work of Laudenbach [11, IV, Thm. 3.1]. □

In light of Lemma 2, the claim of Theorem 1 will follow once we prove that $\mathrm{Out}(\mathbb{F}_n)$ is the full automorphism group of $\mathbb{S}_n^{(1)}$. The first step in this direction is to observe that every automorphism of $\mathbb{S}_n^{(1)}$ is induced by an automorphism of the whole sphere complex.

LEMMA 3. *$\mathbb{S}_n$ is a flag complex; in particular, every automorphism of $\mathbb{S}_n^{(1)}$ is the restriction of a unique automorphism of $\mathbb{S}_n$.*

Recall that a simplicial complex is *flag* if every complete subgraph on $r+1$ vertices contained in the 1-skeleton is the 1-skeleton of an r-simplex.

Lemma 3 has also been established in [4, Thm. 3.3]. Again, we supply a proof here for completeness.

Proof of Lemma 3. By the very definition of simplicial complex as a subset of the power set of the set of vertices, a simplex in a simplicial complex is uniquely determined by its set of vertices. Hence, it is clear that every automorphism of the 1-skeleton of a flag simplicial complex extends uniquely to an automorphism of the whole complex. Therefore, it suffices to prove the first claim of the lemma.

We shall argue by induction. For complete graphs with two vertices, there is nothing to prove. So suppose that the claim has been proved for all complete graphs with $k-1 \geq 2$ vertices, and let $v_1, \ldots, v_k$ be vertices in $\mathbb{S}_n^{(1)}$ spanning a complete graph. Applying the induction assumption three times (namely, to the complete subgraphs spanned by $\{v_1, \ldots, v_{k-1}\}$, $\{v_1, \ldots, v_{k-2}, v_k\}$, and $\{v_1, \ldots, v_{k-3}, v_{k-1}, v_k\}$) and then isotopying spheres to avoid redundancies, we can find spheres

$$S_1, \ldots, S_{k-1}, S_k, S_k' \subset M_n$$

that satisfy the following conditions:

(1) $S_1, \ldots, S_{k-1}$ represent $v_1, \ldots, v_{k-1}$ (respectively), and both S_k and S_k' represent v_k;
(2) $S_i \cap S_j = \emptyset$ for $i, j = 1, \ldots, k-1$ with $i \neq j$;
(3) $S_i \cap S_k = \emptyset$ for $i = 1, \ldots, k-2$; and
(4) $S_i \cap S_k' = \emptyset$ for $i = 1, \ldots, k-3$ and $i = k-1$.

Choose a maximal system of spheres Σ with

$$S_1, \ldots, S_{k-1} \subset \Sigma.$$

By [6, Prop. 1.1] we can assume that the sphere S_k is in *normal form* with respect to Σ. This means that S_k is either contained in Σ or meets Σ transversally and that, in the latter case, the closure P of any component of $S_k \setminus \Sigma$ satisfies:

- P meets any component of Σ in at most one circle; and
- P is not a disk that is isotopic, relative to its boundary, to a disk in Σ.

Similarly, we assume that S_k' is also in normal form with respect to Σ.

By assumption, the spheres S_k and S_k' are isotopic. By [6, Prop. 1.2], there is a homotopy $(S(t))_{t\in[0,1]}$ by immersed spheres with $S(0) = S_k$ and $S(1) = S_k'$ satisfying:

- if $S(0) \subset \Sigma$, then $S(t) \subset \Sigma$ for all t and hence $S(1) = S(0)$; and
- if $S(0) \not\subset \Sigma$, then $S(t)$ is transverse to Σ for all t.

If we are in the first case then $S_1, \ldots, S_k \subset \Sigma$ are pairwise disjoint and we are done. For the second case, recall that $S(1) = S_k'$ does not intersect S_{k-1}. Since transversality is preserved through the homotopy, it follows that $S(0) = S_k$ does not intersect S_{k-1} either. We deduce from conditions (1) and (2) that the spheres $S_1, \ldots, S_k$ are pairwise disjoint. This concludes the proof of the induction step, thus showing that $\mathbb{S}_n$ is a flag complex. □

3

We now briefly recall the definition of Culler–Vogtmann space CV_n, also called *outer space*. A point in CV_n is an equivalence class of marked metric graphs X

with $\pi_1(X) = \mathbb{F}_n$, of total length 1, without vertices of valence 1 and without separating edges. Two such marked graphs are *equivalent* if they are isometric via an isometry in the correct homotopy class. Two marked metric graphs X, Y are *close* in CV_n if, for some L close to 1, there are L-Lipschitz maps $X \to Y$ and $Y \to X$ in the correct homotopy classes. See [2] for details.

Following Hatcher [6], we denote by $\mathbb{O}_n$ the similarly defined space where one allows the graphs to have separating edges (but no vertices of valence 1). Observe that

$$\mathrm{CV}_n \subset \mathbb{O}_n.$$

We are tempted to refer to $\mathbb{O}_n$ as *hairy outer space*. (In the literature, there does not seem to be complete agreement on the name for CV_n: sometimes it is referred to as *reduced outer space* and $\mathbb{O}_n$ simply as *outer space*. We find our terminology more descriptive.)

The group $\mathrm{Out}(\mathbb{F}_n)$ acts on $\mathbb{O}_n$ by changing the marking. This action preserves CV_n as a subset of $\mathbb{O}_n$. The interest of the space $\mathbb{O}_n$ in our setting is that, by work of Hatcher, it is $\mathrm{Out}(\mathbb{F}_n)$-equivariantly homeomorphic to a subset of the sphere complex $\mathbb{S}_n$. We now describe this homeomorphism.

From now on we interpret points in the sphere complex $\mathbb{S}_n$ as weighted sphere systems in M; to avoid redundancies we assume without further mention that all weights are positive. As in [6], let $\mathbb{S}_n^\infty$ be the subcomplex of $\mathbb{S}_n$ consisting of those elements $\sum_i a_i S_i \in \mathbb{S}_n$ such that $M \setminus \bigcup_i S_i$ has at least one nonsimply connected component.

To a point $\sum_i a_i S_i \in \mathbb{S}_n \setminus \mathbb{S}_n^\infty$ we associate the dual graph to $\bigcup_i S_i$ and declare the edge corresponding to S_i to have length a_i. This yields a map

$$\mathbb{S}_n \setminus \mathbb{S}_n^\infty \to \mathbb{O}_n.$$

Hatcher [6, Apx.] shows the following.

PROPOSITION 4 (Hatcher). *The map $\mathbb{S}_n \setminus \mathbb{S}_n^\infty \to \mathbb{O}_n$ is an $\mathrm{Out}(\mathbb{F}_n)$-equivariant homeomorphism.*

Besides introducing CV_n, in [2] Culler and Vogtmann define what is called the *spine K_n* of CV_n. Considering CV_n as a subset of $\mathbb{S}_n$, the spine K_n is the maximal simplicial subcomplex in the first barycentric subdivision of $\mathbb{S}_n$ that is contained in CV_n; this means that every simplex is contained in a closed simplex in CV_n.

By construction, (2.2) induces an action $\mathrm{Out}(\mathbb{F}_n) \curvearrowright K_n$ by simplicial automorphisms and hence a homomorphism

$$\mathrm{Out}(\mathbb{F}_n) \to \mathrm{Aut}(K_n). \tag{3.1}$$

The key ingredient in the proof of Theorem 1 in the next section is the following result of Bridson and Vogtmann [1].

THEOREM 5 (Bridson–Vogtmann). *For $n \geq 3$, the homomorphism* (3.1) *is an isomorphism.*

Observe that, for $n = 2$, $\mathrm{Aut}(K_2)$ is uncountable.

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In this section we prove Theorem 1, whose statement we now recall.

THEOREM 1. *The natural map* $\mathrm{Out}(\mathbb{F}_n) \to \mathrm{Aut}(\mathcal{G}_n)$ *is an isomorphism for* $n \geq 3$.

To begin we recall that, by Lemma 2, the free splitting graph $\mathcal{G}_n$ is $\mathrm{Out}(\mathbb{F}_n)$-equivariantly isomorphic to the 1-skeleton $\mathbb{S}_n^{(1)}$ of the sphere complex $\mathbb{S}_n$. By Lemma 3, $\mathbb{S}_n$ is flag and hence the claim of Theorem 1 will follow once we prove that the simplicial action $\mathrm{Out}(F_n) \curvearrowright \mathbb{S}_n$ in (2.2) induces the isomorphism

$$\mathrm{Out}(\mathbb{F}_n) \to \mathrm{Aut}(\mathbb{S}_n). \tag{4.1}$$

We start by proving that $\mathbb{S}_n^\infty$ is invariant under $\mathrm{Aut}(\mathbb{S}_n)$.

LEMMA 6. *Every automorphism of* $\mathbb{S}_n$ *preserves the subcomplex* $\mathbb{S}_n^\infty$.

Proof. We first observe that every simplex in $\mathbb{S}_n^\infty$ is contained in a codimension-1 simplex that is also contained in $\mathbb{S}_n^\infty$. To see this, let $v_0, \ldots, v_k$ be vertices of $\mathbb{S}^n$ spanning a k-simplex in $\mathbb{S}_n^\infty$. We represent these vertices by pairwise disjoint embedded spheres $S_0, \ldots, S_k$. By the definition of $\mathbb{S}_n^\infty$ there is a component U of $M \setminus \bigcup_i S_i$ that is not simply connected. As remarked previously, this implies that U contains a nonseparating sphere S. Let α be an embedded curve in U intersecting S exactly once, V a closed regular neighborhood of $S \cup \alpha$, and S' the boundary of V. Clearly S' is an essential embedded sphere, for otherwise $M \setminus V$ would be a 3-ball and hence $M_n = \mathbb{S}^1 \times \mathbb{S}^2 = M_1$, contradicting our assumption that $n \geq 3$. Cutting V open along S, we obtain a 3-sphere with three balls removed. In particular, every embedded sphere in V disjoint from S is parallel to one of S or S'. Let Σ be a maximal sphere system containing $S_0, \ldots, S_k, S, S'$. The simplex in $\mathbb{S}_n$ determined by $\Sigma \setminus S$ has codimension 1 and is contained in $\mathbb{S}_n^\infty$.

The upshot of this observation is that the claim of Lemma 6 follows once we show that codimension-1 simplices contained in $\mathbb{S}_n^\infty$ can be characterized in terms of simplicial data. Namely, we claim that a codimension-1 simplex is contained in $\mathbb{S}_n^\infty$ if and only if it is contained in a unique top-dimensional simplex. Suppose that a system of spheres Σ determines a codimension-1 simplex $[\Sigma]$, and consider $M \setminus \Sigma$. As already mentioned, all components of $M \setminus \Sigma$ but one are homeomorphic to a 3-sphere with three balls removed. The remaining component, call it U, is either a 3-sphere with four balls removed or $\mathbb{S}^1 \times \mathbb{S}^2$ with one ball removed. In the first case, $[\Sigma]$ is contained in $\mathbb{S}_n \setminus \mathbb{S}_n^\infty$ and, since U contains three different essential spheres, the simplex $[\Sigma]$ is a face of three distinct maximal simplices. In the second case, if U is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ with one ball removed, then $[\Sigma] \subset \mathbb{S}_n^\infty$ and U contains a unique embedded sphere that does not bound a ball and is not parallel to ∂U. We deduce that $[\Sigma]$ is a face of a unique maximal simplex.

This concludes the proof of Lemma 6. □

It follows from Lemma 6 that $\mathrm{Aut}(\mathbb{S}_n)$ preserves the hairy Culler–Vogtmann space $\mathbb{O}_n = \mathbb{S}_n \setminus \mathbb{S}_n^\infty$. We now prove that $\mathrm{Aut}(\mathbb{S}_n)$ preserves the spine K_n of $\mathrm{CV}_n \subset \mathbb{O}_n$ as well.

LEMMA 7. *Every simplicial automorphism of $\mathbb{O}_n$ preserves the spine K_n of* CV_n.

Abusing terminology, we will say from now on that a simplex σ is contained in, for instance, CV_n if the associated open simplex is.

Proof of Lemma 7. Since the spine of CV_n is defined simplicially, it suffices to show that every automorphism of $\mathbb{O}_n$ preserves CV_n itself. As in the proof of Lemma 6, it suffices to characterize combinatorially the maximal simplices (whose interior is) contained in CV_n; equivalently, we characterize those in $\mathbb{O}_n \setminus \mathrm{CV}_n$.

We claim that a top-dimensional simplex σ is contained in $\mathbb{O}_n \setminus \mathrm{CV}_n$ if and only if the following condition is satisfied:

($*$) σ has a codimension-1 face τ such that, if $\eta \subset \sigma$ is a face contained in $\mathbb{S}_n \setminus \mathbb{S}_n^\infty$, then $\eta \cap \tau \subset \mathbb{S}_n \setminus \mathbb{S}_n^\infty$.

We first prove that every top-dimensional simplex $\sigma \subset \mathbb{O}_n \setminus \mathrm{CV}_n$ satisfies ($*$). Let σ be represented by a maximal sphere system Σ. The assumption $\sigma \subset \mathbb{O}_n \setminus \mathrm{CV}_n$ implies that Σ has a component S that separates M. Let τ be the codimension-1 face of σ determined by $\Sigma \setminus S$, and suppose that $\eta \subset \sigma$ is a face contained in $\mathbb{S}_n \setminus \mathbb{S}_n^\infty$. Denote by Σ' the subsystem of Σ corresponding to η. Since S is separating and since all components of $M \setminus \Sigma'$ are simply connected, it follows from the Seifert–van Kampen theorem that every component of $M \setminus (\Sigma' \setminus S)$ is simply connected as well. In other words, the simplex $\eta \cap \tau$ is contained in $\mathbb{S}_n \setminus \mathbb{S}_n^\infty$ as claimed.

Suppose now that the top-dimensional simplex σ is contained in CV_n; we shall prove that ($*$) is not satisfied for any codimension-1 face $\tau \subset \sigma$. Continuing with the same notation, let $S_1 \subset \Sigma$ be the sphere such that $\Sigma \setminus S_1$ represents τ. Since S_1 is (by assumption) nonseparating, we can find other $n-1$ components $S_2, \dots, S_n$ of Σ such that $M \setminus \bigcup_i S_i$ is homeomorphic to a 3-sphere with $2n$ balls removed. The simplex η associated to the system $S_1 \cup \cdots \cup S_n$ is contained in $\mathbb{S}_n \setminus \mathbb{S}_n^\infty$. On the other hand, the complement of $S_2 \cup \cdots \cup S_n$ is not simply connected. Hence, the simplex $\eta \cap \tau$ is not contained in $\mathbb{S}_n \setminus \mathbb{S}_n^\infty$. This proves that ($*$) is not satisfied for the face τ. Since τ is arbitrary, this concludes the proof of Lemma 7. □

It follows from Lemma 7 and the Bridson–Vogtmann theorem that there is a homomorphism

$$\mathrm{Aut}(\mathbb{S}_n) \to \mathrm{Aut}(K_n) \simeq \mathrm{Out}(\mathbb{F}_n). \tag{4.2}$$

As mentioned before, by Lemma 2 and Lemma 3 we have identified $\mathrm{Aut}(\mathcal{G}_n)$ with $\mathrm{Aut}(\mathbb{S}_n)$. In particular, the proof of Theorem 1 boils down to showing that the homomorphism (4.2) is injective. This is the content of the following lemma.

LEMMA 8. *The identity is the only automorphism of $\mathbb{S}_n$ acting trivially on the spine K_n.*

Proof. Recall that K_n is the maximal full subcomplex of the first barycentric subdivision of $\mathbb{S}_n$ that is contained in CV_n and is disjoint from $\mathbb{S}_n^\infty$. The interior of every simplex contained in CV_n intersects K_n. In particular, an automorphism α

of $\mathbb{S}_n$ that acts trivially on the spine K_n maps every simplex in CV_n to itself. We claim that the restriction of α to CV_n is actually the identity.

Let Σ be a sphere system in M determining a top-dimensional simplex σ in CV_n, and let S be a component of Σ. We claim that the codimension-1 face given by $\Sigma \setminus S$ is also contained in CV_n; in order to see this, it suffices to prove that it is contained in $\mathbb{S}_n \setminus \mathbb{S}_n^\infty$. If this were not the case, then the unique component U of $M \setminus (\Sigma \setminus S)$ distinct from a 3-sphere with three balls removed is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ with one ball removed. The boundary of U would then be a connected component of Σ that separates M, in contradiction to our assumption that $\sigma \subset \mathrm{CV}_n$. It follows that the automorphism α maps the codimension-1 face of σ determined by $\Sigma \setminus S$ to itself. In particular, α must fix the opposite vertex $[S]$ of σ; since $[S]$ is arbitrary, we have proved that α is the identity on σ. Hence, α is the identity on CV_n.

We are now ready to prove that α fixes every vertex $[S]$ in $\mathbb{S}_n$; once we have done this, the claim of the lemma will follow. Given a vertex $[S]$, there are two possibilities. If the sphere S is nonseparating then we can extend S to a maximal sphere system Σ with no separating components. The simplex determined by Σ is contained in CV_n and hence is fixed by α. If S is separating, let U and V be the two components of $M \setminus S$. For a suitable choice of r, we identify U with the complement of a ball B in the connected sum $\#^r(\mathbb{S}^1 \times \mathbb{S}^2)$ of r copies of $\mathbb{S}^1 \times \mathbb{S}^2$. Similarly, V is the complement of a ball in the connected sum of s copies of $\mathbb{S}^1 \times \mathbb{S}^2$. The cases $r = 1$ and $s = 1$ are minimally special; they are left to the reader.

We choose a maximal sphere system Σ_U in $\#^r(\mathbb{S}^1 \times \mathbb{S}^2)$ whose dual graph has no separating edges. Choosing some sphere S' in Σ_U, we take a small regular neighborhood $\mathcal{N}(S')$ of S' in $\#^r(\mathbb{S}^1 \times \mathbb{S}^2)$. Up to isotopy we may assume that the ball B is contained in $\mathcal{N}(S')$. The collection of spheres $\partial\mathcal{N}(S') \cup \Sigma_U \setminus S'$ is contained in $U = \#^r(\mathbb{S}^1 \times \mathbb{S}^2) \setminus B$ and hence determines a sphere system Σ'_U in M. We do a similar construction for V to obtain a system Σ'_V, and we set

$$\Sigma = S \cup \Sigma'_U \cup \Sigma'_V.$$

The simplex determined by $\Sigma \setminus S$ is contained in CV_n and is thus fixed by the automorphism α. Also, it follows from our construction that S is the unique separating sphere contained in $M \setminus (\Sigma \setminus S)$. By the foregoing, all the vertices determined by any other sphere disjoint from $\Sigma \setminus S$ are fixed by α. It follows that α fixes the vertex $[S]$ as claimed. □

As mentioned previously, the proof of Lemma 8 concludes the proof of Theorem 1. □

Post scriptum. After the completion of this paper, the authors were informed by numerous experts in the field—most prominently Yael Algom-Kfir, Mladen Bestvina, and Larsen Edmund Louder—that our proof could easily be modified to avoid making use of 3-manifold topology. This suggestion was followed up by the referee, who, with infinite kindness and a patience of biblical proportions, outlined the main steps of the argument in his/her report.

The basic idea of this alternative approach is to work with the complex of all graphs of group decompositions of $\mathbb{F}_n$ with trivial edge group. This complex is $\mathrm{Out}(\mathbb{F}_n)$-equivariantly isomorphic to $\mathbb{S}_n$; compare with Lemma 2 and Proposition 4. From this point of view, $\mathbb{S}_n^\infty$ corresponds to splittings with some nontrivial vertex stabilizer in the splitting graph.

All facts in this paper can be proved directly in terms of graphs, edge collapses, and so forth. For instance, Lemma 3 follows rapidly from a result of Dunwoody [3]. Also, the manipulations in the proofs of Lemmas 6–8—for example, removing spheres and completing sphere systems—can be replaced by collapsing edges and completing graphs.

This approach to Theorem 1 bypasses completely the rather difficult results of Laudenbach [10; 11], which control the relation between homotopy and isotopy of spheres in M_n. In other words, it has the virtue of being more elementary. On the other hand, what we find remarkable is that, by interpreting free splittings as isotopy classes of spheres, one can think of free splittings of free groups almost in the same way as one thinks of elements in the curve complex of a surface: compare with [4; 5] or with the list of facts in Section 1. It is due to our own limitations that we could probably not have proved Theorem 1 without being able to visualize free splittings in this way.

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