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Polyhedral Divisors of Cox Rings

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1. Introduction

Let Z be a $\mathbb{Q}$-factorial projective variety defined over the field of complex numbers such that its divisor class group $\mathrm{Cl}(Z)$ is a lattice that is a free abelian, finitely generated group. We consider the Cox ring of Z,

$$\mathrm{Cox}(Z) = \bigoplus_{D \in \mathrm{Cl}(Z)} \Gamma(Z, \mathcal{O}(D)),$$

with multiplicative structure defined by a choice of divisors whose classes form a basis of $\mathrm{Cl}(Z)$. Our standing assumption in this paper is the finite generation of the $\mathbb{C}$-algebra $\mathrm{Cox}(Z)$. We will call such Z a Mori dream space (or MDS), as it was baptized by Hu and Keel in [HuKe]. We note that a somewhat more general definition of MDS, without the $\mathbb{Q}$ factoriality assumption, was developed by Artebani, Hausen, and Laface [AHL, Thm. 2.3]. However, $\mathbb{Q}$-factoriality of Z is a part of our setup in the present paper.

The $\mathrm{Cl}(Z)$-grading of $\mathrm{Cox}(Z)$ yields an algebraic action of the associated torus $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(Z), \mathbb{C}^*) \cong (\mathbb{C}^*)^{\mathrm{rk}(\mathrm{Cl}(Z))}$ on the affine variety $\mathrm{Spec}(\mathrm{Cox}(Z))$. The variety Z is a GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(Z))$ by the action of this torus. More precisely, a choice of an ample divisor on Z determines an open subset of $\mathrm{Spec}(\mathrm{Cox}((Z))$ such that Z is a good geometric quotient of this set; see [HuKe, Prop. 2.9].

Affine varieties with an algebraic torus action were dealt with by Altmann and Hausen [AlH1], who introduced the notion of polyhedral divisors, or *p-divisors*. Every normal, affine variety X with an algebraic torus action can be described in terms of a polyhedral divisor $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ over its Chow quotient Y [AlH1, Thm. 3.4]. Alternatively, such a p-divisor can be interpreted as a convex, fanwise linear (i.e., piecewise linear and homogeneous, defined on a cone) map from the character lattice M of the torus to $\mathrm{CaDiv}_{\mathbb{Q}}(Y)$; see Section 2.1 for more details. Note that, by abuse of notation, we use the word "Chow quotient" for the normalization of the distinguished component of the inverse limit of the GIT quotients of X (cf. [AlH1, Sec. 6; Hu]).

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We apply this formalism to treat the case of $X = \operatorname{Spec}(\operatorname{Cox}(Z))$ for Z as above. Although in general the structure of the Chow quotient Y is rather obscure, our main result (Theorem 11) asserts that the associated p-divisor is supported on a finite number of exceptional divisors D_i with polyhedral coefficients Δ_i described clearly in terms of stabilized multiplicities with respect to these divisors:

$$\Delta_i = \{C \in \mathrm{Cl}^*(Z)_{\mathbb{Q}} \mid C \geq -\mathrm{mult}^{\mathrm{st}}_{D_i}\} + \text{shift}.$$

Thus, polyhedral divisors provide an alternative view of the stabilized base point loci and the asymptotic order of the vanishing of linear series on Z, as defined by Ein, Lazarsfeld, Mustaţă, Nakamaye, and Popa [E+].

The composition of the p-divisor associated to $\operatorname{Cox}(Z)$, treated as a fanwise linear map $\mathcal{D}\colon M_{\mathbb{Q}} = \mathrm{Cl}_{\mathbb{Q}}(Z) \supset \mathrm{Eff}(Z) \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$, with the divisor class map $\mathrm{CaDiv}_{\mathbb{Q}}(Y) \to \mathrm{Pic}_{\mathbb{Q}}(Y)$ (dividing by $\mathbb{Q}$-principal divisors) maps the cone of effective divisors on Z, denoted by $\mathrm{Eff}(Z)$, to the cone $\mathrm{Nef}(Y)$ of nef (in this case also semiample) divisors on Y. In Corollary 12 we show that it is a composition of two other maps $\mathrm{Cl}_{\mathbb{Q}}(Z) \supset \mathrm{Eff}(Z) \to \mathrm{Cl}_{\mathbb{Q}}(Z) = \mathrm{Pic}_{\mathbb{Q}}(Z) \to \mathrm{Pic}_{\mathbb{Q}}(Y)$. First, one performs a retraction of $\mathrm{Eff}(Z)$ to the cone of movable divisors $\mathrm{Mov}(Z)$ that is a union of cones $\mathrm{Nef}(Z_i)$, where the Z_i are different GIT quotients of $\operatorname{Cox}(Z)$. Second, the chambers $\mathrm{Nef}(Z_i)$ are mapped to faces of $\mathrm{Nef}(Y)$ by pulling the divisors back along the natural morphisms $Y \to Z_i$.

Our starting point, however, is the toric case where both the Chow quotient of $\operatorname{Cox}(Z)$ and the p-divisor can be described explicitly. We discuss this in Section 3 right after the introductory Section 2, in which we recall the language of p-divisors. The main toric result, Theorem 7, is obtained by explicit methods. In the subsequent Section 4, we rephrase it by using dual polyhedra and the associated fanwise linear functions. These easy observations lead us to the relation to multiplicities of divisors in base point loci of linear systems, which forms the core of the proof of Theorem 11. This is contained in Section 5, where we also recall the basic information about MDS.

Finally, in Section 6 we discuss the surface case and provide some further examples. If $Z = S$ with $\dim S = 2$, then the Chow limit Y coincides with S. So the p-divisor defines a retraction $\mathrm{Eff}(S)$ to $\mathrm{Nef}(S)$ that reflects the Zariski decomposition on S. It is linear on the Zariski chambers, as defined in [BKS]. The coefficients of the p-divisor on an MDS surface are presented in Theorem 13. For a del Pezzo surface S they look particularly nice, as in the following result.

COROLLARY 14. *If S is a del Pezzo surface with exceptional curves $E_i \subseteq S$, then the p-divisor encoding* $\operatorname{Cox}(S)$ *equals* $\mathcal{D} = \mathrm{id}_{\mathrm{Cl}(S)} + \sum_i(\overline{0E_i} + \mathrm{Nef}(S)) \otimes E_i$.

2. The Language of Polyhedral Divisors

2.1. Definition of Polyhedral Divisors

We start by recalling the basic notions of [AlH1]. Let T be an affine torus over a field of complex numbers $\mathbb{C}$. It gives rise to the mutually dual free abelian groups, or lattices, $M := \mathrm{Hom}_{\mathrm{algGrp}}(T, \mathbb{C}^*)$ and $N := \mathrm{Hom}_{\mathrm{algGrp}}(\mathbb{C}^*, T)$. The pairing of dual lattices (or, also, dual vector spaces) will be denoted by $\langle\cdot,\cdot\rangle$. Via $T = \operatorname{Spec}\mathbb{C}[M] = N \otimes_{\mathbb{Z}} \mathbb{C}^*$, the torus can be recovered from these lattices. Denote by $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ the corresponding vector spaces over $\mathbb{Q}$ (the same notation will be used whenever we extend a lattice to a $\mathbb{Q}$-vector space).

Definition 1. If $\sigma \subseteq N_{\mathbb{Q}}$ is a polyhedral cone, then we denote by $\operatorname{Pol}(N_{\mathbb{Q}},\sigma)$ the Grothendieck group of the semigroup

$$\operatorname{Pol}^+(N_{\mathbb{Q}},\sigma) := \{\Delta \subseteq N_{\mathbb{Q}} \mid \Delta = \sigma + [\text{compact polytope}]\}$$

with respect to Minkowski addition. Via $a \mapsto a + \sigma$, the latter contains $N_{\mathbb{Q}}$. Moreover, $\operatorname{tail}(\Delta) := \sigma$ is called the *tail cone* of the elements of $\operatorname{Pol}(N_{\mathbb{Q}},\sigma)$.

Let Y be a normal and semiprojective (i.e., $Y \to Y_0$ is projective over an affine Y_0) $\mathbb{C}$-variety. By $\operatorname{CaDiv}(Y)$ and $\operatorname{Div}(Y)$ we denote the group of Cartier and Weil divisors on Y with linear equivalence groups by $\operatorname{Pic}(Y)$ and $\operatorname{Cl}(Y)$, respectively. A $\mathbb{Q}$-Cartier divisor on Y is called *semiample* if a multiple of it becomes base point free.

Definition 2. An element $\mathcal{D} = \sum_i \Delta_i \otimes D_i \in \operatorname{Pol}(N_{\mathbb{Q}},\sigma) \otimes_{\mathbb{Z}} \operatorname{CaDiv}(Y)$ with effective divisors D_i and $\Delta_i \in \operatorname{Pol}^+(N_{\mathbb{Q}},\sigma)$ is called a *polyhedral divisor* on (Y,N) with tail cone σ. Moreover, it is called *semiample* if the evaluations $\mathcal{D}(u) := \sum_i \min\langle\Delta_i,u\rangle D_i$ are semiample for $u \in \sigma^\vee \cap M$ and big for $u \in \operatorname{int}\sigma^\vee \cap M$.

Note that the membership $u \in \sigma^\vee := \{u \in M_{\mathbb{Q}} \mid \langle\sigma,u\rangle \geq 0\}$ guarantees that $\min\langle\Delta_i,u\rangle > -\infty$ and therefore $\mathcal{D}$ defines a function $\sigma^\vee \to \operatorname{CaDiv}_{\mathbb{Q}}(Y)$ that we will denote by the same name. Sometimes, by abuse, we will refer to $\mathcal{D}$ as a function defined on the whole lattice M or space $M_{\mathbb{Q}}$. In such a case, for $u \notin \sigma^\vee$ we have $\min\langle\Delta_i,u\rangle = -\infty$ and thus, although $-\infty$ as a Cartier divisor coefficient does not make sense, we get as a reasonable conclusion that $\Gamma(Y,\mathcal{O}_Y(\mathcal{D}(u))) = 0$.

The common tail cone σ of the coefficients Δ_i will be denoted by $\operatorname{tail}(\mathcal{D})$. Semiample polyhedral divisors will be called *p-divisors* for short. Their positivity assumptions imply that $\mathcal{D}(u) + \mathcal{D}(u') \leq \mathcal{D}(u+u')$; hence $\mathcal{O}_Y(\mathcal{D}) := \bigoplus_{u\in\sigma^\vee\cap M} \mathcal{O}_Y(\mathcal{D}(u))$ becomes a sheaf of rings, and we can define $X := X(\mathcal{D}) := \operatorname{Spec}\Gamma(Y,\mathcal{O}(\mathcal{D}))$ over Y_0.

This space does not change if $\mathcal{D}$ is pulled back via a birational modification $Y' \to Y$ or if $\mathcal{D}$ is altered by a polyhedral *principal* divisor—the latter means an image under $N \otimes_{\mathbb{Z}} \mathbb{C}(Y)^* \to \operatorname{Pol}(N_{\mathbb{Q}},\sigma) \otimes_{\mathbb{Z}} \operatorname{CaDiv}(Y)$. Polyhedral divisors that differ by (chains of) those operations only are called *equivalent*. Note that this implies that one can always ask for a smooth Y.

Theorem 3 [AlH1, Thm. 3.1, Thm. 3.4, Cor. 8.12]. *The map $\mathcal{D} \mapsto X(\mathcal{D})$ yields a bijection between equivalence classes of p-divisors and normal, affine $\mathbb{C}$-varieties with an effective T-action.*

Remark. The T-action on X corresponds to the M-valued grading of $\Gamma(Y,\mathcal{O}(\mathcal{D}))$. In this context, $\operatorname{tail}(\mathcal{D})^\vee$ becomes the cone generated by the weights. Note also that the knowledge of $\mathcal{D} \in \operatorname{Pol}(N_{\mathbb{Q}},\sigma) \otimes_{\mathbb{Z}} \operatorname{CaDiv}(Y)$ is equivalent to the knowledge of $\mathcal{D}$ as the above fanwise linear (cf. Section 4.4) function $\sigma^\vee \cap M \to \operatorname{CaDiv}(Y)$, $u \mapsto \mathcal{D}(u)$.

2.2. Morphisms between Polyhedral Divisors

The construction of $X(\mathcal{D})$ is functorial: up to the aforementioned equivalences of p-divisors, a map $(Y', N', \mathcal{D}') \to (Y, N, \mathcal{D})$ consists of a morphism $\psi\colon Y' \to Y$ such that the support $\bigcup_i D_i$ of $\mathcal{D}$ does not contain $\psi(Y')$ and a linear map $F\colon N' \to N$ with

$$\sum_i (F(\Delta'_i) + \operatorname{tail}\mathcal{D}) \otimes D'_i =: F_*(\mathcal{D}') \subseteq \psi^*(\mathcal{D}) := \sum_i \Delta_i \otimes \psi^*(D_i)$$

inside $\operatorname{Pol}(N_{\mathbb{Q}}, \operatorname{tail}\mathcal{D}) \otimes_{\mathbb{Z}} \operatorname{CaDiv}(Y')$. The inclusion is understood as a relation between the coefficients of the same divisors. In particular, we ask for $F(\operatorname{tail}\mathcal{D}') \subseteq \operatorname{tail}\mathcal{D}$.

THEOREM 4 [AlH1, Cor. 8.14]. *A map $(Y', N', \mathcal{D}') \to (Y, N, \mathcal{D})$ with dominant $\psi\colon Y' \to Y$ gives rise to an equivariant, dominant map $X(\mathcal{D}') \to X(\mathcal{D})$, and eventually this leads to an equivalence of categories.*

2.3. Polyhedral Divisors Encode Toric Degenerations

The representation or encoding of a multigraded algebra as a p-divisor has many advantages. First, although one misses direct information about generators and syzygies, one should notice that this construction does entail being of finite type. This is because only semiample divisors are used to produce the homogeneous parts of the algebra.

However, the main advantage of a p-divisor is that it is possible to read off equivariant and geometric properties of the associated affine T-variety X. This becomes possible because X is the contraction of $\tilde{X} := \tilde{X}(\mathcal{D}) := \operatorname{Spec}_Y \mathcal{O}(\mathcal{D})$, and this space is a degenerate toric fibration over Y. That is, there is a flat map $\tilde{X} \to Y$ where the general fiber is the toric variety $\mathbb{TV}(\operatorname{tail}(\mathcal{D}), N) := \operatorname{Spec}\mathbb{C}[\operatorname{tail}(\mathcal{D})^\vee \cap M]$. Moreover, the divisors D_i and their polyhedral coefficients Δ_i provide the information about the location and the quality of the degeneration, respectively:

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y_0. \end{array}$$

Special fibers over $y \in Y$ can be reducible; their components are in a one-to-one correspondence with the vertices of the polyhedron $\Delta_y := \sum_{D_i \ni y} \Delta_i$.

Thus, also the configuration of T-orbits and their closures are directly encoded in the presentation of X as a polyhedral divisor $\mathcal{D}$. The orbits in $\tilde{X}$ correspond to pairs (y, F) with $y \in Y$ and faces $F \leq \Delta_y$. Moreover, as is known from the toric case, mutual inclusions among orbit closures correspond to opposite inclusions of the corresponding faces. The orbit structure of X may be obtained from that of $\tilde{X}$ by keeping track of when certain orbits from $\tilde{X}$ will be identified in X. This happens in relation to the different contractions of Y provided by the semiample divisors $\mathcal{D}(u)$.

As an example of how to use this information, see Hausen's [H2] description of those open subsets $U \subseteq X$ providing a complete quotient U/T.

3. The Toric Situation

3.1. Restriction to Subtorus Actions

If $T \subseteq (\mathbb{C}^*)^n$ occurs as a subtorus induced by a surjective map $\deg\colon \mathbb{Z}^n \twoheadrightarrow M$ (corresponding to the choice of degrees $\deg x_i \in M$), then every affine toric variety $\mathbb{TV}(\delta)$ with $\delta \subseteq \mathbb{Q}^n$ inherits a T-action. By [AlH1, Sec. 11], the associated p-divisor $\mathcal{D}(\delta)$ can be obtained as follows. Defining $M_Y := \ker(\deg)$, we have two mutually dual exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] & \mathbb{Z}^n \arrow[r, "\pi"] & N_Y \arrow[r] \arrow[l, bend left, "s"] & 0; \end{tikzcd}$$

$$\begin{tikzcd} 0 & M \arrow[l] & \mathbb{Z}^n \arrow[l, "\deg"'] \arrow[r, bend right, "s^*"'] & M_Y \arrow[l] & 0. \arrow[l] \end{tikzcd}$$

Here by s we denote a section of π. Then $\mathcal{D}(\delta)$ lives on the toric variety $Y := \mathbb{TV}(\Sigma) \supseteq N_Y \otimes_{\mathbb{Z}} \mathbb{C}^* =: T_Y$, where Σ denotes the fan in N_Y that is the coarsest common refinement of the image under π of all faces of δ. As a function, $\mathcal{D}(\delta)$ is given by

$$\mathcal{D}(\delta)(u) = s^*(\deg^{-1}(u) \cap \delta^\vee),$$

where the right-hand side is a polyhedron in M_Y whose normal fan is refined by Σ. Thus, it encodes a semiample, T_Y-invariant divisor on Y. This implies that

$$\mathcal{D}(\delta) = \sum_{a \in \Sigma(1)} \Delta_a \otimes \overline{\mathrm{orb}}(a) \quad \text{with} \quad \Delta_a = (\pi^{-1}(a) \cap \delta) - s(a) \subseteq N_{\mathbb{Q}}.$$

Here $a \in \Sigma(1)$ are primitive lattice elements of rays in Σ and $\overline{\mathrm{orb}}(a)$ are their associated T_Y-invariant divisors. The relation between these two representations of $\mathcal{D}$ has been proved by [AlH2, Prop. 8.5] and, in a broader context, by [CM].

3.2. The Polyhedral Coefficients

We will now present a method to describe the coefficients Δ_a with inequalities. This observation is as trivial as it is useful.

LEMMA 5. *In the situation of Section 3.1, the polyhedral coefficients Δ_a are cut out by the inequalities $\langle \cdot, \deg(r)\rangle \geq -\langle s(a), r\rangle$ for $r \in \delta^\vee$ (or generators of $\delta^\vee$).*

Proof.

$$\begin{aligned} x \in \Delta_a &\iff i(x) + s(a) \in \pi^{-1}(a) \cap \delta \iff i(x) + s(a) \in \delta \\ &\iff \langle x, \deg(r)\rangle + \langle s(a), r\rangle = \langle i(x) + s(a), r\rangle \geq 0 \end{aligned}$$

for all $r \in \delta^\vee$. □

3.3. Toric Cox Rings

Let $\mathcal{F}$ be a simplicial fan in some lattice N_Z. Identifying again its 1-dimensional rays $\mathcal{F}(1) = \{a^1, \dots, a^n\}$ with the first lattice points sitting on them, we assume that $\mathcal{F}(1)$ generates N_Z. We would like to apply Lemma 5 to understand the Cox ring of the $\mathbb{Q}$-factorial toric variety $Z := \mathbb{TV}(\mathcal{F})$. As a ring, it is simply $\operatorname{Cox}(Z) = \mathbb{C}[x_a \mid a \in \mathcal{F}(1)]$; but by setting $M := \operatorname{Cl}(Z)$, it is then the M-grading that makes it interesting. The exact sequences from Section 3.1 become

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Cl}(Z)^* \arrow[r, "i"] & \operatorname{Div}^*_{\mathrm{eq}} Z \arrow[r, "\pi"] & N_Z \arrow[l, bend left, "s"] \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 & \operatorname{Cl}(Z) \arrow[l] & \operatorname{Div}_{\mathrm{eq}} Z \arrow[l, "p"'] & M_Z \arrow[l, "\operatorname{div}"'] & 0. \arrow[l] \end{tikzcd}$$

Here we have denoted by $\operatorname{Div}_{\mathrm{eq}} Z \cong \mathbb{Z}^n$ the group of T_Z-equivariant divisors; the rays a^i are the images of the unit vectors e^i. Note that the torus T acting on $\operatorname{Cox}(Z)$ is the Picard torus $T = \operatorname{Hom}(\operatorname{Cl}(Z), \mathbb{C}^*)$. The degree cone of $\operatorname{Cox}(Z)$ is the cone of effective divisors $\operatorname{Eff}(Z) \subseteq \operatorname{Cl}_{\mathbb{Q}}(Z)$. Hence, the tail of the p-divisor $\mathcal{D}_{\mathrm{Cox}}$ will be the dualized cone $\operatorname{Eff}(Z)^\vee \subseteq \operatorname{Cl}_{\mathbb{Q}}(Z)^*$.

According to Section 3.1, $\mathcal{D}_{\mathrm{Cox}}$ lives on $Y := \mathbb{TV}(\Sigma)$ for Σ the coarsest fan in $N_Y = N_Z$ containing all possible cones generated by subsets of $\mathcal{F}(1)$. In particular, Σ is a subdivision of $\mathcal{F}$ (i.e., $\Sigma \le \mathcal{F}$); in other words, there is a proper map $\psi : Y \to Z$ that becomes an isomorphism if it is restricted on the tori $T_Y = T_Z$. In the surface case we have $\Sigma = \mathcal{F}$; hence $Y = Z$ and $\psi = \mathrm{id}$. Finally, the choice of the section s will not affect the upcoming result.

3.4. The Splitting of $\mathcal{D}_{\mathrm{Cox}}$

Although p-divisors on Y may be altered by so-called principal p-divisors coming from $N \otimes_{\mathbb{Z}} \mathbb{C}(Y)^* = \operatorname{Hom}(M, \mathbb{C}(Y)^*)$, this does not mean that $\mathcal{D}$ is determined by an element of $\operatorname{Pol}(N_{\mathbb{Q}}, \sigma) \otimes_{\mathbb{Z}} \operatorname{Pic}_{\mathbb{Q}}(Y)$. However, elements of the group $N \otimes_{\mathbb{Z}} \operatorname{Pic}^{\mathbb{Q}}(Y) = \operatorname{Hom}(M, \operatorname{Pic}^{\mathbb{Q}}(Y))$ with $\operatorname{Pic}^{\mathbb{Q}}(Y) := \operatorname{CaDiv}_{\mathbb{Q}}(Y)/\operatorname{PDiv}(Y) \neq \operatorname{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ denoting the $\mathbb{Q}$-Cartier divisors modulo principal divisors do indeed give a correct description of an equivalent class of a polyhedral divisor. In particular, it makes sense to add those elements to already existing p-divisors.

DEFINITION 6. In the case of Section 3.3, the pull-back map

$$M = \operatorname{Cl}(Z) \subseteq \operatorname{Pic}^{\mathbb{Q}}(Z) \to \operatorname{Pic}^{\mathbb{Q}}(Y)$$

defines an element $\psi^* \in \operatorname{Hom}(M, \operatorname{Pic}^{\mathbb{Q}}(Y)) = N \otimes_{\mathbb{Z}} \operatorname{Pic}^{\mathbb{Q}}(Y)$ that gives rise to a splitting $\mathcal{D}_{\mathrm{Cox}} = \psi^* + \mathcal{D}'_{\mathrm{Cox}}$ with some correction term $\mathcal{D}'_{\mathrm{Cox}}$.

REMARK. Note that, although $\operatorname{Pic}^{\mathbb{Q}}(Y) \neq \operatorname{Pic}_{\mathbb{Q}}(Y)$, we nevertheless have a map $\operatorname{Pic}^{\mathbb{Q}}(Y) = \operatorname{CaDiv}_{\mathbb{Q}}(Y)/\operatorname{PDiv}(Y) \to \operatorname{CaDiv}_{\mathbb{Q}}(Y)/\operatorname{PDiv}_{\mathbb{Q}}(Y) = \operatorname{Pic}_{\mathbb{Q}}(Y)$ and therefore $\mathcal{D}_{\mathrm{Cox}}$ determines a map $\operatorname{Cl}_{\mathbb{Q}}(Z) \supset \operatorname{Eff}(Z) \to \operatorname{Nef}(Y) \subset \operatorname{Pic}_{\mathbb{Q}}(Y)$.

The splitting of $\mathcal{D}_{\mathrm{Cox}}$ into ψ^* and a correction term is then quite natural. Since, on the one hand, a p-divisor $\mathcal{D}$ encodes the ring $\bigoplus_{u \in M} \Gamma(Y, \mathcal{D}(u))$ and, on the other hand, $\operatorname{Cox}(Z) = \bigoplus_{u \in M} \Gamma(Z, u) = \bigoplus_{u \in M} \Gamma(Y, \psi^* u)$, one is tempted to say that $\mathcal{D} = \psi^*$. However, because $\operatorname{tail}(\mathcal{D})^\vee = \operatorname{Eff}(Z) \supseteq \operatorname{Nef}(Z)$, it is generally the case that $\psi^*(\operatorname{tail}(\mathcal{D})^\vee) \not\subseteq \operatorname{Nef}(Y)$; that is, ψ^* is not a p-divisor. Thus, all $u \in \operatorname{tail}(\mathcal{D})^\vee \cap M$ leading to non-semiample divisors must be processed.

3.5. The Polyhedral Divisor of Toric Cox Rings

If $E \subseteq Y = \mathbb{TV}(\Sigma)$ and $P \subseteq Z = \mathbb{TV}(\mathcal{F})$ are toric prime divisors, then there exist associated rays $a(E) \in \Sigma(1) \subseteq N_{YZ} := N_Y = N_Z$ and $a(P) \in \mathcal{F}(1) \subseteq N_{YZ}$, respectively. Recall that we identify a ray with its integral, primitive generator. In particular, each $a(E)$ sits in a unique minimal cone $C_E \in \mathcal{F}$; hence there are unique $\lambda_E(P) \in \mathbb{Q}_{>0}$ such that $a(E) = \sum_{a(P)\in C_E} \lambda_E(P)a(P)$. (Remember, $\mathcal{F}$ is a simplicial fan.) Set $\lambda_E(P) := 0$ for $a(P) \notin C_E$.

REMARK. Note that $\lambda_E(P) > 0$ if and only if $a(P) \in C_E$ if and only if $\psi(E) \subseteq P$. In nontoric terms, these coefficients can be expressed as $\lambda_E(P) = \mathrm{mult}_E(\psi^*P)$. If E is not contracted, we may identify $E \subseteq Y$ with its divisorial image $\psi(E) \subseteq Z$; then $\lambda_E(P) = 1$ if $E = P$ and $\lambda_E(P) = 0$ if $E \neq P$. In dimension 2, this is always the case (because $Y = Z$).

THEOREM 7. *$\mathcal{D}'_{\mathrm{Cox}} = \sum_E \Delta_E \otimes E$, where $E \subseteq Y$ runs through the toric prime divisors and $\Delta_E \subseteq \mathrm{Cl}(Z)^*_{\mathbb{Q}}$ is the polyhedron cut out by the inequalities $\langle \cdot, [P]\rangle \geq -\lambda_E(P)$ for toric prime divisors P. In particular, $\Delta_E \supseteq \mathrm{tail}\,\mathcal{D}_{\mathrm{Cox}}$.*

Proof. In the first exact sequence of Section 3.3, we add the cosection $t\colon \mathbb{Z}^n \to \mathrm{Cl}(Z)^*$ induced from s. Then the maps satisfy $it + s\pi = \mathrm{id}_{\mathbb{Z}^n}$, $ti = \mathrm{id}_{\mathrm{Cl}^*}$, and $\pi s = \mathrm{id}_{N_{YZ}}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Cl}(Z)^* \arrow[r, "i"] & (\mathbb{Z}^n = \mathrm{Div}^*_{\mathrm{eq}} Z) \arrow[r, "\pi"] \arrow[l, bend left, "t"] & N_{YZ} \arrow[r] \arrow[l, bend left, "s"] & 0. \end{tikzcd}$$

Denote by $\{e(P)\} \subseteq \mathbb{Z}^n = \mathrm{Div}^*_{\mathrm{eq}} Z$ the dual basis with respect to that of the toric prime divisors of Z. In particular, $\pi(e(P)) = a(P) \in N_{YZ}$. This notion can be extended to the prime divisors on Y via $e(E) := \sum_{a(P)\in C_E} \lambda_E(P)e(P)$; we keep the property $\pi e = a$.

If $E \subseteq Y$ is a toric prime divisor (corresponding to the ray $a(E) \in \Sigma(1) \subseteq N_{YZ}$) then, by Lemma 5, the true coefficient Δ^{Cox}_E is given by the inequalities $\langle \cdot, [P]\rangle \geq -\langle s(a(E)), P\rangle$, where the latter just means the Pth entry of $-s(a(E)) \in \mathbb{Z}^n$. On the other hand, the claimed inequalities for Δ_E of $\mathcal{D}'_{\mathrm{Cox}}$ are $\langle \cdot, [P]\rangle \geq -\lambda_E(P) = -\langle e(E), P\rangle$. Thus, it remains to show that $b(E) := e(E) - s(a(E)) \in \mathbb{Z}^n$ is contained in $\mathrm{Cl}(Z)^* \subseteq \mathbb{Z}^n$ and satisfies $d := \sum_E b(E) \otimes [E] = \psi^* \in \mathrm{Cl}(Z)^* \otimes \mathrm{Cl}(Y)$.

The first claim follows from $b(E) = e(E) - s(a(E)) = e(E) - s\pi(e(E)) = it(e(E))$. Moreover,

$$d = \sum_E it(e(E)) \otimes [E] = ((it) \otimes \mathrm{cl}_Y) \circ \left(\sum_E e(E) \otimes E \in \mathrm{Div}^*_{\mathrm{eq}} Z \otimes_{\mathbb{Z}} \mathrm{Div}_{\mathrm{eq}} Y \right),$$

where cl denotes the canonical map $\mathrm{Div} \to \mathrm{Cl}$. On the other hand, since for a toric prime divisor $P \subseteq Z$ we have $\psi^*P = \sum_E \lambda_E(P)E$, it follows that $\psi^* = \sum_{E,P} \lambda_E(P)e(P) \otimes E = \sum_E e(E) \otimes E$; that is, $d = ((it) \otimes \mathrm{cl}_Y) \circ \psi^*$. Restricted via i to $\mathrm{Cl}(Z)^*$, this yields $((iti) \otimes \mathrm{cl}_Y) \circ \psi^* = (i \otimes \mathrm{cl}_Y) \circ \psi^* = (\mathrm{cl}^*_Z \otimes \mathrm{cl}_Y) \circ \psi^* = \psi^*_{\mathrm{Cl}}$. □

See Section 6.3 for an example.

4. Duality of Polyhedra

4.1. Cones over Polyhedra

Dualization of polyhedral cones via $\sigma^\vee := \{x \mid \langle \sigma, x\rangle \geq 0\}$ is a straightforward generalization of the dualization of vector spaces. One has the basic relations $(\sigma^\vee)^\vee = \sigma$ and $(\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee$. Moreover, via $\tau \mapsto \tau' := \tau^\perp \cap \sigma^\vee$ (for $\tau \leq \sigma$ and where the RHS $\leq \sigma^\vee$), it provides a bijection of faces. For the convenience of the reader, we will recall how this theory can be further extended to the set of polyhedra containing the origin.

Let V be a finite-dimensional $\mathbb{Q}$-vector space and let $\Delta \subseteq V$ be a polyhedron containing 0. Then we define

$$\nabla := \Delta^\vee := \{x \in V^* \mid \langle \Delta, x\rangle \geq -1\}.$$

This construction can be understood by the ordinary duality notion of cones; it just requires a definition of the cone $C(\Delta)$ spanned over a polyhedron Δ located in an affine hyperplane $V \times \{1\} \subset V \times \mathbb{Q}$. Namely, we set

$$C(\Delta) := \overline{\mathbb{Q}_{\geq 0} \cdot (\Delta, 1)} = \mathbb{Q}_{>0} \cdot (\Delta, 1) \sqcup (\operatorname{tail}(\Delta), 0) \subseteq V \oplus \mathbb{Q}.$$

The polyhedron Δ can be recovered as the cross section $\Delta = C(\Delta) \cap (V \times \{1\})$. Then we verify that $C(\nabla) = C(\Delta)^\vee$; hence $\nabla^\vee = (\Delta^\vee)^\vee = \Delta$ and $(\Delta_1 \cap \Delta_2)^\vee = \operatorname{conv}(\Delta_1^\vee \cup \Delta_2^\vee)$. Note that $\Delta_1 + \Delta_2 \subseteq 2\operatorname{conv}(\Delta_1 \cup \Delta_2) \subseteq 2(\Delta_1 + \Delta_2)$ and that, in general, $C(\Delta_1 + \Delta_2) \neq C(\operatorname{conv}(\Delta_1 \cup \Delta_2)) = C(\Delta_1) + C(\Delta_2)$.

4.2. Heads and Tails

Inside V there are two cones associated to Δ. One is the already mentioned $\operatorname{tail}(\Delta) = C(\Delta) \cap (\underline{0}, 1)^\perp$; since $0 \in \Delta$, we have $\operatorname{tail}(\Delta) \subseteq \Delta$. The other is $\operatorname{head}(\Delta) := \mathbb{Q}_{\geq 0}\Delta \supseteq \Delta$.

If Δ is itself already a polyhedral cone, then both cones coincide and are equal to Δ. In general, polyhedral duality interchanges both constructions—that is, $\operatorname{tail}(\nabla) = \operatorname{head}(\Delta)^\vee$ and $\operatorname{head}(\nabla) = \operatorname{tail}(\Delta)^\vee$. Indeed, $x \in \operatorname{tail}\Delta^\vee$ if and only if $\Delta^\vee + \mathbb{Q}_{\geq 0}x \subseteq \Delta^\vee$ if and only if $\langle x, \Delta\rangle \geq 0$ if and only if $\langle x, \mathbb{Q}_{\geq 0}\Delta\rangle \geq 0$. This duality is even more transparent if we note that

$$\operatorname{head}(\Delta) = \bigcup_{t\to\infty} t \cdot \Delta \quad \text{and} \quad \operatorname{tail}(\Delta) = \bigcap_{t \to 0} t \cdot \Delta.$$

4.3. Face Duality

Via application of C, the nonempty faces $F \leq \Delta$ correspond bijectively to the faces of $C(\Delta)$ not contained in $\operatorname{tail}(\Delta) \leq C(\Delta)$. The inverse map is the intersection with $V \times \{1\}$. Since the dual face $(\operatorname{tail}\Delta)' \leq C(\Delta)^\vee = C(\nabla)$ contains $(\underline{0}, 1)$, it is not contained in $\operatorname{tail}\nabla$ and corresponds to the minimal face of ∇ that contains 0. Thus, restricting the duality $\operatorname{faces}(C\Delta) \leftrightarrow \operatorname{faces}(C\nabla)$ to those faces with $\not\subseteq (\operatorname{tail}\Delta)$ and $\not\supseteq (\operatorname{tail}\nabla)'$ on the left-hand side (and doing similarly on the right), we obtain an order- and dimension-reversing bijection

$$\{\text{faces } F \leq \Delta \mid 0 \notin F\} \leftrightarrow \{\text{faces } F' \leq \nabla \mid 0 \notin F'\}.$$

The remainder of the bijection $\text{faces}(C\Delta) \leftrightarrow \text{faces}(C\nabla)$ translate into

$$\{\text{faces } F \leq \Delta \mid 0 \in F\} = \text{faces}(\text{head}\,\Delta) \leftrightarrow \text{faces}((\text{head}\,\Delta)^\vee) = \text{faces}(\text{tail}\,\nabla)$$

and, analogously, $\text{faces}(\text{tail}\,\Delta) \leftrightarrow \{\nabla\text{-faces containing } 0\}$.

4.4. Fanwise Linear Functions

A rational (or real) function is called *fanwise linear* if it is linear on the closed cones of a fan (and hence is continuous on the support of the fan). This is equivalent to being piecewise (affine) linear and homogeneous; that is, $f(t \cdot v) = t \cdot f(v)$ for $t \in \mathbb{Q}_{\geq 0}$. For a polyhedron $\Delta \subseteq V$, we define the fanwise linear function $\min(\Delta)\colon V^* \to \mathbb{Q} \cup \{-\infty\}$ by setting $\min(\Delta)(v) = \min\langle \Delta, v\rangle$. In particular, $\min(\Delta)^{-1}(\mathbb{Q}) = (\text{tail}\,\Delta)^\vee$. If additionally $0 \in \Delta$, then $\min(\Delta)\colon V^* \to \mathbb{Q}_{\leq 0} \cup \{-\infty\}$ with $\min(\Delta)^{-1}(\mathbb{Q}_{\leq 0}) = \text{head}(\nabla)$. Moreover, $\min(\Delta)^{-1}(0) = \text{tail}(\nabla)$.

LEMMA 8. *If Δ and ∇ are mutually dual polyhedra containing 0, then*

$$\min(\Delta)(v) = \frac{-1}{\max\{t \in \mathbb{Q} \mid tv \in \nabla\}}.$$

Equivalently, the homogeneous, continuous function $\min(\Delta)\colon \text{head}(\nabla) \to \mathbb{Q}_{\leq 0}$ *is characterized by the property that* $\min(\Delta) \equiv -1$ *on* $\partial\nabla \cap \text{int}(\text{head}\,\nabla)$, *where* ∂ *and* int *denote (respectively) the relative boundary and interior of the cone. In particular,* $\min(\Delta)$ *is equal to* -1 *on all nonzero vertices of* ∇.

Proof. Let us consider $v \in \partial\nabla \cap \text{int}(\text{head}\,\nabla)$; then $t \cdot v \notin \nabla$ for every $t > 1$. Moreover, by definition, $\langle v, \Delta\rangle \geq -1$ and so $\min(\Delta)(v) \geq -1$. On the other hand, if $0 > \lambda > -1$ is such that $\langle u, v\rangle \geq \lambda$ for all $u \in \Delta$, then $\langle u, |\lambda|^{-1}v\rangle \geq -1$. Hence, by definition of duality of polyhedra, $|\lambda|^{-1}v$ is in ∇—contradicting the assumption. $\square$

Conversely, let $f\colon \beta \to \mathbb{Q}_{\geq 0}$ be a fanwise linear function defined on a rational, convex polyhedral cone $\beta \subseteq V^*$. We assume that f is also concave; that is, $f(v_1 + v_2) \leq f(v_1) + f(v_2)$. Defining

$$\nabla_f := \text{conv}\{f(v)^{-1} \cdot v \mid v \in \beta\} \quad \text{with } \ 0^{-1} \cdot v := \mathbb{Q}_{\geq 0} \cdot v$$

yields a polyhedron with $\text{head}(\nabla_f) = \beta$ and $\text{tail}(\nabla_f) = f^{-1}(0)$.

LEMMA 9. *Let Δ_f be a polyhedron dual to ∇_f as defined previously. Then, over the cone $\beta \subseteq V^*$,*

$$\min(\Delta_f) = -f.$$

Proof. Let us set $g(v) = (\sup\{t \mid tv \in \nabla_f\})^{-1}$. Clearly, both f and g vanish exactly on $\text{tail}(\nabla_f) \subset \sigma$, so we can assume that v is chosen such that both are nonzero. By definition of ∇_f we have $f(v)^{-1} \cdot v \in \nabla_f$; hence $f(v)^{-1} \leq \sup\{t \mid tv \in \nabla_f\}$ and thus $g(v) \leq f(v)$. Now suppose that $t \cdot v \in \nabla_f$. Then, by definition of ∇_f,

$$t \cdot v = \sum_i a_i f(v_i)^{-1} \cdot v_i$$

for some $v_i \in \sigma$ and positive numbers a_i such that $\sum_i a_i = 1$. Applying the function f to both sides of the equality and using its homogenity and convexity, we get

$$t \cdot f(v) \le \sum_i a_i f(v_i)^{-1} \cdot f(v_i)$$

and so $t \cdot f(v) \le 1$. Thus $\sup\{t \mid tv \in \nabla_f\} \le f(v)^{-1}$; hence $g(v)^{-1} \le f(v)^{-1}$ and thus $g(v) \ge f(v)$. Since $g = -\min(\Delta_f)$, this concludes the proof. □

REMARK. It is possible to weaken the assumption of fanwise linearity to homogeneity of f (see the preceding proof). Then ∇_f and Δ_f still become well-defined, mutually dual convex bodies, although they lose their polyhedral structure.

4.5. Dualized Cox Coefficients

The duality described in Section 4.1 allows a nicer description of the polyhedral coefficients $\Delta_E \subseteq \mathrm{Cl}_{\mathbb{Q}}(Y)^*$ from Theorem 7. Since they contain the origin, it makes sense to define their duals $\nabla_E := \Delta_E^\vee \subseteq \mathrm{Cl}_{\mathbb{Q}}(Z)$. It follows that

$$\begin{aligned}\nabla_E &= \mathrm{conv}\{0, [P]/\lambda_E(P) \mid \psi(E) \subseteq P \subseteq Z\} + \textstyle\sum_{P \not\supseteq \psi(E)} \mathbb{Q}_{\ge 0} \cdot [P] \\ &= \mathrm{conv}\{[P]/\lambda_E(P) \mid P \subseteq Z\} \subseteq \mathrm{head}\, \nabla_E = \mathrm{Eff}(Z),\end{aligned}$$

where $v/0 := \mathbb{Q}_{\ge 0} \cdot v$, P runs through the toric prime divisors of Z, and $\lambda_E(P) = \mathrm{mult}_E(\psi^* P)$. Using these polyhedra, we obtain that $\mathcal{D}'_{\mathrm{Cox}} = \sum_E \nabla_E^\vee \otimes E$ and that $\mathcal{D}'_{\mathrm{Cox}}(u)$ contains E with multiplicity

$$\min\langle \Delta_E, u\rangle = \frac{-1}{\max\{\lambda \in \mathbb{Q} \mid \lambda u \in \nabla_E\}} \in \mathbb{Q}_{\le 0} \cup \{-\infty\}.$$

5. Mori Dream Spaces and Their Cox Polyhedral Divisors

5.1. Mori Dream Spaces

Mori dream spaces (MDS) were introduced in [HuKe]. Recall that Z is a $\mathbb{Q}$-factorial variety, $\mathrm{Cl}(Z)$ is a lattice, and $\mathrm{Cox}(Z)$ is finitely generated.

The birational geometry of Z is finite. In other words, Z has finitely many small (i.e., isomorphic in codimension 1) $\mathbb{Q}$-factorial modifications Z_i (set $Z_0 := Z$); we will call them SQM models of Z. The varieties Z_i are exactly the $\mathbb{Q}$-factorial GIT quotients of $\mathrm{Cox}(Z)$ by the Picard torus arising from linearizations of the trivial bundle depending on the choice of a character of the torus; see [HuKe]. All models Z_i share the same Cox ring and can be distinguished by pure combinatorics (cf. [H1]). In particular, by strict transforms we can identify $\mathrm{Div}(Z_i)$ and $\mathrm{Cl}(Z_i)$ with $\mathrm{Div}(Z)$ and $\mathrm{Cl}(Z)$, respectively. The same holds true for the cones $\mathrm{Eff}(Z_i) = \mathrm{Eff}(Z)$ and $\mathrm{Mov}(Z_i) = \mathrm{Mov}(Z)$. However, the cones $\mathrm{Nef}(Z_i)$ are different: $\mathrm{int}\,\mathrm{Nef}(Z_i) \cap \mathrm{int}\,\mathrm{Nef}(Z_j) = \emptyset$ if $Z_i \ne Z_j$, and we have the decomposition $\mathrm{Mov}(Z) = \bigcup_i \mathrm{Nef}(Z_i)$ [HuKe, 1.11(3)]. This chamber decomposition is polyhedral and coincides with that of the stability with respect to the Picard torus (cf. [HuKe, 2.3; DHu]). Finally, perhaps the most striking feature of Mori dream spaces is that nefness implies semiampleness.

5.2. *The Chow Limit*

Let Y be the Chow quotient of $\mathrm{Cox}(Z)$ by the Picard torus—that is, by abuse of notation, the normalized component of the inverse limit of the models (GIT quotients) Z_i that is birational to the original Z. In particular, we have birational morphisms $\psi_i : Y \to Z_i$.

Note that Y carries the following two types of exceptional divisors.

(i) An irreducible divisor $E \subseteq Y$ is said to be *of the first kind* if it is a component of the exceptional locus of a morphism $\psi_i : Y \to Z_i$. Note that, since Z_i is $\mathbb{Q}$-factorial, the exceptional locus of ψ_i is of pure codimension 1. Moreover, since the Z_i are isomorphic outside codimension 2, the set of exceptional divisors is the same for all ψ_i.

(ii) We say that an irreducible divisor E is an exceptional divisor *of the second kind* if it is a strict transform to Y of a (divisorial) component of an exceptional locus of a birational morphism (divisorial contraction) of a Z_i. In other words (cf. [HuKe, 1.11(5)]), E is a strict transform of a nonmovable divisor from Z.

5.3. *Stabilized Multiplicities*

Let $\psi : Y \to Z$ be a proper, birational morphism and $E \subseteq Y$ a prime divisor. Then, in the toric case of Sections 3.5 and 4.5, the multiplicities $\lambda_E(P) = \mathrm{mult}_E(\psi^*P)$ of a divisor ψ^*P in the general point of E in Z.

In [E+, Sec. 2] there is a stable version of these multiplicities. At least for big divisors P, one defines $\mathrm{mult}_E^{\mathrm{st}}(\psi^*P)$ either as the E-multiplicity of the stable base locus of P or, by [E+, Lemma 3.3], as

$$\mathrm{mult}_E^{\mathrm{st}}(\psi^*[P]) := \inf_{D \in |P|_{\mathbb{Q}}} \mathrm{mult}_E(\psi^*D) \le \mathrm{mult}_E(\psi^*P).$$

Here $D \in |P|_{\mathbb{Q}}$ means that D is an (effective) $\mathbb{Q}$-divisor with $mD \in |mP|$ for $m \gg 0$. Finally, it follows from [E+, Thm. D] that, for a Mori dream space Z, the stable multiplicity function $\mathrm{mult}_E^{\mathrm{st}} := \mathrm{mult}_E^{\mathrm{st}} \circ \psi^*$ can be extended to a concave, fanwise linear function on $\mathrm{Eff}(Z) \subseteq \mathrm{Cl}(Z)_{\mathbb{Q}}$. We have the following immediate consequence of Lemma 9.

Corollary 10. *Let Z be an MDS and $\psi : Y \to Z$ the birational morphism from the Chow quotient of* $\mathrm{Cox}(Z)$. *Let $E \subseteq Y$ a prime divisor. Then*

$$\nabla_E := \mathrm{conv}\left\{ \frac{[P]}{\mathrm{mult}_E^{\mathrm{st}}\, \psi^*[P]} \;\middle|\; [P] \in \mathrm{Eff}\, Z \right\} \subseteq \mathrm{Cl}(Z)_{\mathbb{Q}}$$

and

$$\Delta_E := \{C \in \mathrm{Cl}^*(Z)_{\mathbb{Q}} \mid C \ge -\mathrm{mult}_E^{\mathrm{st}}\}$$

are mutually dual polyhedra with $\min(\Delta_E) = -\mathrm{mult}_E^{\mathrm{st}}$. *Moreover, if Z is toric then they coincide with those from Section 4.5.*

5.4. The Cox Polyhedral Divisor of a Mori Dream Space

Now we are able to present the p-divisor $\mathcal{D}_{\mathrm{Cox}}$ describing the Cox ring of an MDS. As in Definition 6, we split $\mathcal{D}_{\mathrm{Cox}} = \psi^* + \mathcal{D}'_{\mathrm{Cox}}$.

THEOREM 11. *The part $\mathcal{D}'_{\mathrm{Cox}}$ of the p-divisor of the Cox ring of a MDS equals*

$$\mathcal{D}'_{\mathrm{Cox}} = \sum_{E \subset Y} \Delta_E \otimes E,$$

where the coefficients Δ_E are defined in Corollary 10 and the sum is formally taken over all divisors $E \subset Y$. However, if E is not one of the finitely many exceptional divisors as described in (i) *and* (ii) *of Section 5.2, then the corresponding coefficient is trivial*; *that is, $\Delta_E = \operatorname{tail} \mathcal{D} = \operatorname{Eff}(Z)^\vee \subseteq \operatorname{Cl}(Z)^*_{\mathbb{Q}}$ anyway.*

Proof. We will treat all SQM models Z_i on equal footing; that is, we consider $\mathcal{D}_i := \psi_i^* + \mathcal{D}'_i$ with $\mathcal{D}'_i := \sum_{E \subset Y} \Delta^i_E \otimes E$ and $\Delta^i_E := \{C \in \operatorname{Cl}^*(Z_i)_{\mathbb{Q}} \mid \langle C, [P] \rangle \geq -\operatorname{mult}_E \psi_i^* P\}$. Since the divisors on Z_i are identified, via the strict transform, with those on Z, we can compare the $\mathcal{D}_i$ as functions $\mathcal{D}_i : \operatorname{Div}(Z) = \operatorname{Div}(Z_i) \to \operatorname{CaDiv}_{\mathbb{Q}}(Y)$. Taking, as we did in Corollary 10, the function $\operatorname{mult}^{\mathrm{st}}_E \circ \psi_i^*$ for the fanwise linear map f in Section 4.4, we obtain that $\mathcal{D}_i(D) = \psi_i^*(D) - \sum_{E \subset Y} \operatorname{mult}^{\mathrm{st}}_E \psi_i^*(D) \cdot E$ for $D \in \operatorname{Div} Z$.

We claim that $\mathcal{D}_i(D) = \mathcal{D}_j(D)$. Indeed, since the multiplicities of D along divisors E contained in Z (isomorphic in codimension 1 to Z_i and Z_j) are the same, we conclude that the difference $\mathcal{D}_i(D) - \mathcal{D}_j(D)$ is supported on divisors contracted by ψ. More precisely, we have

$$\begin{aligned}\mathcal{D}_i(D) - \mathcal{D}_j(D) = {} & \Bigg(\psi_i^*(D) - \sum_{E \subset \operatorname{Exc}(\psi)} \operatorname{mult}^{\mathrm{st}}_E \psi_i^*(D) \cdot E\Bigg) \\ & - \Bigg(\psi_j^*(D) - \sum_{E \subset \operatorname{Exc}(\psi)} \operatorname{mult}^{\mathrm{st}}_E \psi_j^*(D) \cdot E\Bigg).\end{aligned}$$

But $\psi_i^*(D) - \sum_{E \subset \operatorname{Exc}(\psi_i)} \operatorname{mult}_E \psi_i^*(D) \cdot E$ is the strict transform of the $\mathbb{Q}$-Cartier divisor D from Z_i to Y via the birational mapping $\psi_i : Y \to Z_i$; hence, again by isomorphism in codimension 1, it is the same for $\psi_j : Y \to Z_j$. Thus, passing to the limit from mult_E to $\operatorname{mult}^{\mathrm{st}}_E$, we get the conclusion of our claim.

Let us recall that, by [HuKe, Prop. 1.11(5)], every big divisor $D \in \operatorname{Div} Z$ (possibly replaced by its multiple) admits a canonical splitting $D = \operatorname{mov}(D) + \operatorname{fix}(D)$ into the stable movable and fixed part, respectively. Moreover, there is an SQM model Z_i such that $\operatorname{mov}(D) \in \operatorname{Nef}(Z_i)$; in other words, $\operatorname{mov}(D)$ is semiample on Z_i. Thus, the linear system $|\operatorname{mov}(D)|$ can be assumed base point free so that it defines a contraction of Z_i such that the support of $\operatorname{fix}(D)$ is in the exceptional locus of the contraction. If $E_\nu \subseteq Z_i$ denote divisors contracted by $|\operatorname{mov}(D)|$ then, by definition, $\operatorname{fix}(D) = \sum_\nu \operatorname{mult}^{\mathrm{st}}_{E_\nu}(D) \cdot E_\nu$. We note that we can write $\operatorname{mult}^{\mathrm{st}}_{E_\nu}(D) = \operatorname{mult}_{E_\nu}(D)$ because $|\operatorname{mov}(D)|$ is base point free and $D \in |D| = |\operatorname{mov}(D)|$ can be chosen general. Thus,

$$\begin{aligned}\psi_i^*(D) &= \psi_i^*(\mathrm{mov}(D)) + \sum_\nu \mathrm{mult}^{\mathrm{st}}_{E_\nu}(D)\cdot\psi_i^*(E_\nu)\\ &= \psi_i^*(\mathrm{mov}(D)) + \sum_\nu \mathrm{mult}^{\mathrm{st}}_{E_\nu}(D)\cdot\Bigg(\hat{E}_\nu + \sum_{E\subset \mathrm{Exc}(\psi_i)} \mathrm{mult}_E(\psi_i^* E_\nu)\cdot E\Bigg)\end{aligned}$$

where $\hat{E}_\nu \subseteq Y$ denotes the strict transform via ψ_i^* of E_ν (i.e., becoming an exceptional divisor of the second kind) and where the second summation is restricted to exceptional divisors of the first kind. In particular, $\psi_i^*(D) - \psi_i^*(\mathrm{mov}(D))$ is supported exclusively on exceptional divisors (of both kinds). On the other hand, as the pull back of a semiample divisor, $\psi_i^*(\mathrm{mov}(D))$ contains no exceptional components when D is general in its linear system. Thus,

$$\psi_i^*(D) = \psi_i^*(\mathrm{mov}\, D) + \sum_{E\subset Y} \mathrm{mult}^{\mathrm{st}}_E(\psi_i^* D)\cdot E;$$

therefore, if $\mathcal{D}(D)$ denotes the mutually equal $\mathcal{D}_i(D)$, we obtain that $\mathcal{D}(D) = \psi_i^*(\mathrm{mov}(D))$ and that $\mathcal{D}(D)$ inherits the semiampleness from $\mathrm{mov}(D)$ on Z_i.

Eventually, since $|\mathrm{mov}(D)| = |D|$, the natural inclusion map $\iota_i\colon \Gamma(Y,\mathcal{D}(D)) = \Gamma(Y, \psi_i^*(\mathrm{mov}\, D)) \to \Gamma(Y,\psi_i^*(D)) = \Gamma(Z,D)$ becomes an isomorphism. Since the maps $D\mapsto \mathcal{D}(D)$ and $D\mapsto \psi_i^*(D) - \sum_E \mathrm{mult}^{\mathrm{st}}_E(\psi_i^* D)\cdot E$ are both piecewise linear, this extends to the whole effective cone being the closure of the cone of big divisors (cf. [La, Thm. 2.2.26]). In particular, $\mathcal{D}$ is a decent p-divisor and $\Gamma(Y,\mathcal{D}(D)) \to \Gamma(Z,D)$ is an isomorphism for every $D\in \mathrm{Eff}(Z)\cap \mathrm{Cl}(Z)$; hence

$$\bigoplus_{D\in \mathrm{Cl}(Z)} \Gamma(Z,D) = \bigoplus_{D\in\mathrm{Cl}(Z)} \Gamma(Y,\mathcal{D}(D))$$

gives a presentation of $\mathrm{Cox}(Z)$ as a p-divisor. □

The arguments in the proof of Theorem 11 yield the following observation (cf. the remark following Definition 6).

COROLLARY 12. *The fanwise linear map* $\mathcal{D}_{\mathrm{Cox}}\colon \mathrm{Eff}(Z)\to \mathrm{Nef}(Y)$ *associated to the p-divisor* $\mathcal{D}_{\mathrm{Cox}}$ *is a composition of a fanwise linear retraction* $\mathrm{Eff}(Z)\to \mathrm{Mov}(Z)$ *and a fanwise linear map* $\mathrm{Mov}(Z)\to\mathrm{Nef}(Y)$ *whose restriction to the cone* $\mathrm{Nef}(Z_i)$, *for every SQM model* Z_i, *coincides with the pull-back map* $\psi_i^*\colon \mathrm{Nef}(Z_i)\to\mathrm{Nef}(Y)$.

5.5. Example: Blowing Up Two Points in $\mathbb{P}^3$

This is perhaps the simplest 3-dimensional example to illustrate Corollary 12. Let Z be the blow-up of $\mathbb{P}^3$ in two points, say x_1 and x_2, with exceptional divisors denoted by E_1 and E_2. The strict transform of a general plane, a plane passing through each of these points, and a plane passing through both of them defines divisors whose classes span $\mathrm{Mov}(Z)$. The rational maps defined by these divisors are onto $\mathbb{P}^3$, $\mathbb{P}^2$, and $\mathbb{P}^1$, respectively. The flop along the strict transform of the line passing through x_1 and x_2 yields another SQM model, let us call it Z_1. The variety Y results from blowing up this strict transform.

Now Figure 1 presents sections of cones in spaces of divisor classes. The 3-dimensional cone $\mathrm{Eff}(Z)$ presented on the left-hand side gets retracted to $\mathrm{Mov}(Z)$;

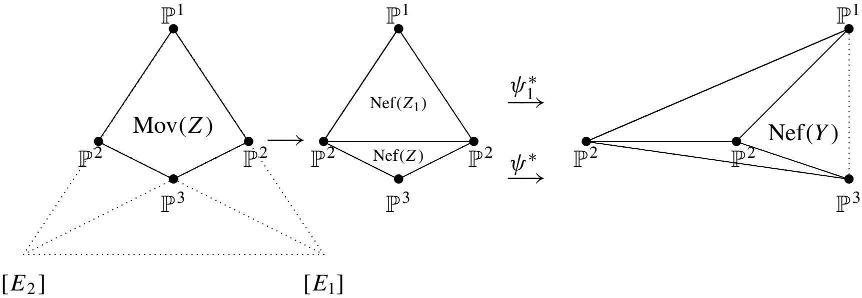

Figure 1

the regions on which the retraction is linear are denoted by dotted line segments. Next $\mathrm{Mov}(Z) = \mathrm{Nef}(Z) \cup \mathrm{Nef}(Z_1)$ is mapped linearly on each Nef cone to two 3-dimensional faces of the 4-dimensional cone $\mathrm{Nef}(Y)$.

We note that only two of the four faces of the tetrahedron representing the section of the 4-dimensional cone $\mathrm{Nef}(Y)$ are associated to SQM models of Z. The other two faces represent contractions of Y to $\mathbb{P}^3$ blown up at one point (x_1 or x_2) and then along the strict transform of the line passing through x_1 and x_2. This is equivalent to blowing up the line first and then blowing up the fiber of the exceptional divisor above x_1 or x_2. In particular, the dotted edge of the tetrahedron represents the contraction of Y to $\mathbb{P}^3$ blown up along the line passing through x_1 and x_2.

6. Surfaces

6.1. Specializing the General Result

The case of ($\mathbb{Q}$-factorial Mori dream) surfaces $Z = S$ is special for two reasons. First, it does not require the pull back to the Chow quotient (i.e., $Y = Z = S$ with $\psi = \psi_i = \mathrm{id}$), and $\mathcal{D}\colon \mathrm{Eff}(S) \to \mathrm{Nef}(S)$ simply reflects the Zariski decomposition. Indeed, given any effective divisor D on S, we can write it uniquely as the sum $D \equiv P + \sum_i a_i E_i$; here $P \in \mathrm{Nef}(S)$, the E_i are exceptional curves (if there are any) such that $(P \cdot E_i) = 0$, and the coefficients $a_i = \mathrm{mult}^{\mathrm{st}}_{E_i} D$. Thus $P = \mathcal{D}(D)$.

Second, the $\mathbb{Q}$-valued intersection product, denoted simply by a dot, allows one to identify vector spaces $\mathrm{Cl}(S)^*_{\mathbb{Q}} = \mathrm{Cl}(S)_{\mathbb{Q}}$ with $\langle C_1, C_2\rangle = (C_1 \cdot C_2)$. In particular, the polyhedral coefficients Δ_E will be contained in $\mathrm{Cl}(S)_{\mathbb{Q}}$ now and have $\mathrm{Nef}(S) = \mathrm{Eff}(S)^\vee$ as their common tail cone. If S is smooth, then we even know that $\mathrm{Cl}(S)^* = \mathrm{Cl}(S)$. In general, this equation has to be replaced by $\mathrm{Cl}(S)^* = \{D \in \mathrm{Cl}(S)_{\mathbb{Q}} \mid \langle D, \mathrm{Cl}(S)\rangle \subset \mathbb{Z}\}$. Finally, we recognize the (finitely many) exceptional divisors $E_i \subseteq S$ by their negative self-intersection numbers (E_i^2).

THEOREM 13. *Let S be a Mori dream surface with exceptional divisors $E_i \subset S$. Then $\mathcal{D}'_{\mathrm{Cox}} = \sum_i \Delta_i \otimes E_i$ with*

$$\Delta_i = \{D \in \mathrm{Eff}(S) \mid (D \cdot E_i) \geq -1 \text{ and } (D \cdot E_j) \geq 0 \text{ for } j \neq i\},$$

and the dual coefficients equal $\nabla_i = \overline{0E_i} + \sum_{j\neq i} \mathbb{Q}_{\geq 0}[E_j] + \mathrm{Nef}(S)$. (By $\overline{0E_i}$ we denote the line segment connecting 0 and $[E_i]$ inside $\mathrm{Cl}_{\mathbb{Q}}(S)$.)

Proof. This is a reformulation of Theorem 11. Note that Δ_i and ∇_i are dual with respect to the intersection product. On the other hand, by Lemma 8 the function defined by ∇_i is just $-\mathrm{mult}^{\mathrm{st}}_{E_i}$. $\square$

6.2. *del Pezzo Surfaces*

Let $S = S_d$ be a smooth del Pezzo surface of degree $d = K_S^2$. By definition, $-K_S$ is ample. Any such S_d is known to be $\mathbb{P}^1 \times \mathbb{P}^1$ $(d = 8)$ or a blow-up of $\mathbb{P}^2$ at $r := 9 - d$ general points. It is known that, for $d \le 7$, the cone $\mathrm{Eff}(S)$ is generated by a finite number of (-1)-curves. In fact, any nonmovable curve on such S is a (-1)-curve. In this special case, the polyhedral coefficients from Theorem 13 become especially easy.

Corollary 14. *If S is a del Pezzo surface and E is a (-1)-curve, then the only vertices of Δ_E are 0 and $[E]$. In particular, the polyhedral coefficients of $\mathcal{D}'_{\mathrm{Cox}}$ are as follows*: $\Delta_E = \mathrm{conv}\{0, [E]\} + \mathrm{Nef}(S) = \overline{0[E]} + \mathrm{Nef}(S)$.

Proof. If $D \in \Delta_E$ (i.e., if D is an effective $\mathbb{Q}$-divisor with $(D \cdot E) \ge -1$ and $(D \cdot F) \ge 0$ for (-1)-curves $F \ne E$), then we must show that $D \in \overline{0E} + \mathrm{Nef}(S)$. If D was already nef, then we are done. If not, then by rescaling we may assume that $(D \cdot E) = -1$ and then claim that $D' := D - E$ is nef. First, $(D' \cdot E) = (D \cdot E) - (E^2) = 0$. Then, if F is an arbitrary (-1)-curve different from E, we may write $D = eE + fF + P$ with $e, f \ge 0$ and P being effective without E and F contributions. Thus,

$$-1 = (D \cdot E) = -e + f(F \cdot E) + (P \cdot E) \ge -e + f(F \cdot E);$$

hence $e - 1 \ge f(F \cdot E) \ge 0$. This implies that D' is effective and, moreover, that

$$(D' \cdot F) \ge (e-1)(E \cdot F) - f \ge f(E \cdot F)^2 - f = f((E \cdot F)^2 - 1).$$

If $(E \cdot F) \ne 0$ then we obtain $(D' \cdot F) \ge 0$; in the opposite case of $(E \cdot F) = 0$, we simply conclude via $(D' \cdot F) = (D \cdot F) - (E \cdot F) = (D \cdot F) \ge 0$. $\square$

Remark. Let $S = S_d$ be a smooth del Pezzo surface of degree $d \le 7$ that is a blow-up of $\mathbb{P}^2$ at $r := 9 - d$ general points; by $E_1, \dots, E_r \subset S$ we denote their preimages. Then $\mathrm{Cl}(S) = \mathbb{Z}H \oplus \left(\bigoplus_{i=1}^r \mathbb{Z}E_i\right)$; hence $\mathrm{id}_{\mathrm{Cl}\, S} = [H] \otimes [H] - \sum_{i=1}^r [E_i] \otimes [E_i]$. In particular,

$$\begin{aligned}\mathcal{D}_{\mathrm{Cox}} &= ([H] + \mathrm{Nef}(S)) \otimes H \\ &\qquad + \sum_{i=1}^r (\overline{[-E_i]0} + \mathrm{Nef}(S)) \otimes E_i + \sum_{E \notin \{E_i\}} (\overline{0[E]} + \mathrm{Nef}(S)) \otimes E,\end{aligned}$$

where E in the last sum is meant to run through the (-1)-curves that are outside $\{E_1, \dots, E_r\}$.

Corollary 14 says that Zariski decomposition on a del Pezzo surface is orthogonal. That is, any effective divisor D on S can be written uniquely as the sum $D \equiv P + \sum_i a_i E_i$, where $P \in \mathrm{Nef}(S)$ and the E_i are (-1)-curves such that

$(P \cdot E_i) = 0$, $a_i = \mathrm{mult}_{E_i} D$, and $(E_i \cdot E_j) = 0$ if $i \neq j$. The last of these properties is known and follows from the fact that the birational morphism of a del Pezzo surface associated to $|mP|$, $m \gg 0$, contracts disjoint (-1)-curves E_i.

6.3. Example: Blowing Up Two Points in $\mathbb{P}^2$

The following two examples are just toric, but they nevertheless illustrate the special shape of $\mathcal{D}_{\mathrm{Cox}}$ for del Pezzo surfaces and indicate how it differs from a somewhat more general situation. First, we consider a surface S_1 that is an ordinary blowing up of $\mathbb{P}^2$ in two points; second, we present a surface S_2 that is a $\mathbb{P}^2$ with two infinitesimally near points blown up.

The toric surface S_1 is given by the fan $\Sigma_1 = \{(1,0), (1,1), (0,1), (-1,0), (-1,-1)\}$. The exceptional divisors of the blowing up are $E_1 = \overline{\mathrm{orb}}(1,1)$ and $E_2 = \overline{\mathrm{orb}}(-1,0)$ together with the strict transform $E_0 = \overline{\mathrm{orb}}(0,1)$ of the line connecting the two centers; they are the only (-1)-curves in S_1.

Let $[H]$ denote the pull back of the line in $\mathbb{P}^2$. Then $[E_0] = [H] - [E_1] - [E_2]$, and the nef cone $\mathrm{Nef}(S_1)$ is formed by the strict transforms $[A] = [H] - [E_1] = [E_0] + [E_2]$ and $[B] = [H] - [E_2] = [E_0] + [E_1]$ and by $[H] = [E_0] + [E_1] + [E_2]$ itself. The ample anticanonical bundle is $[-K] = 3[H] - [E_1] - [E_2] = [A] + [B] + [H]$; see Figure 2.

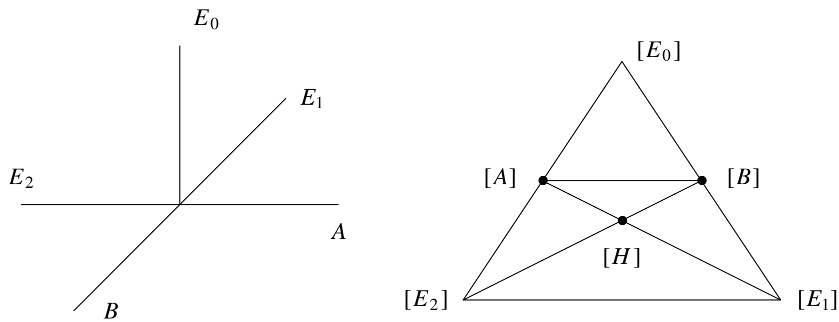

Figure 2

The classes of the E_i form a basis of $\mathrm{Cl}(S_1)$, and the associated intersection matrix is

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

This implies that $\mathrm{id}_{\mathrm{Cl}\,S_1} = [H] \otimes [E_0] + [A] \otimes [E_1] + [B] \otimes [E_2]$, and the coefficients of E_i in $\mathcal{D}'_{\mathrm{Cox}\,S_1}$ are indeed $\Delta_{E_i} = \overline{0[E_i]} + \mathrm{Nef}(S_1)$.

For the second example S_2, the left- and right-hand sides of Figure 3 depict the fan and the class group, respectively. Here E_2 is the exceptional curve of the second blow-up, E_1 is the (strict transform via the second blow-up of the) exceptional curve of the first blow-up, and E_0 is the strict transform of the line.

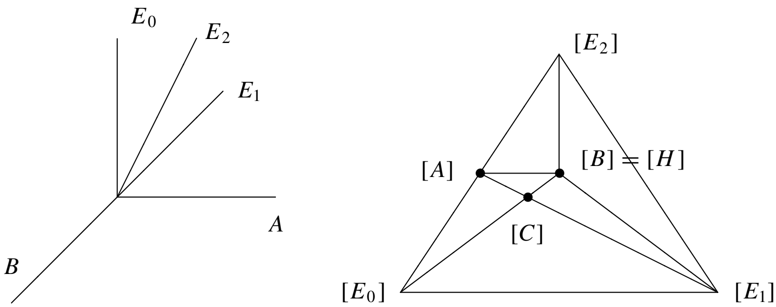

Figure 3

Using the basis $\{[E_0], [E_1], [E_2]\}$, the intersection matrix is

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

The pull back of the line is $[H] = [B] = [A] + [E_1] + [E_2]$ with $[A] = [E_2] + [E_0]$, $[B] = [E_0] + [E_1] + 2[E_2]$, and $[C] := 2[E_0] + [E_1] + 2[E_2] = 2[A] + [E_1] = [B] + [E_0]$ generating the nef cone $\mathrm{Nef}(S_2)$. This implies that $\mathrm{id}_{\mathrm{Cl}\, S_2} = [A] \otimes [E_1] + [C] \otimes [E_2] + [B] \otimes [E_0]$ and that the compact parts of the coefficients of the E_i in $\mathcal{D}'_{\mathrm{Cox}\, S_2}$ are $\Delta^{\mathrm{comp}}_{E_0} = \overline{0[E_0]}$; however,

$$\Delta^{\mathrm{comp}}_{E_1} = \mathrm{conv}\left\{0, \tfrac{1}{2}[E_1], [E_1] + [E_2]\right\} \quad \text{and}$$
$$\Delta^{\mathrm{comp}}_{E_2} = \mathrm{conv}\{0, [E_2], [E_1] + 2[E_2]\}.$$

The two surfaces are homeomorphic; in fact, there exists a deformation of S_2 to S_1. Thus we can identify respective homology classes and put them in one picture (see Figure 4). The cohomology classes $[H]$, $[E_0]$, $[E_2]$, and $[A]$ are the same for both surfaces, the class of the second blow-up we denote by $[E_1]^1$ and $[E_1]^2$, respectively. To make the picture transparent, the boundaries of Eff cones (as well as their division in Zariski chambers) are denoted by dotted line segments.

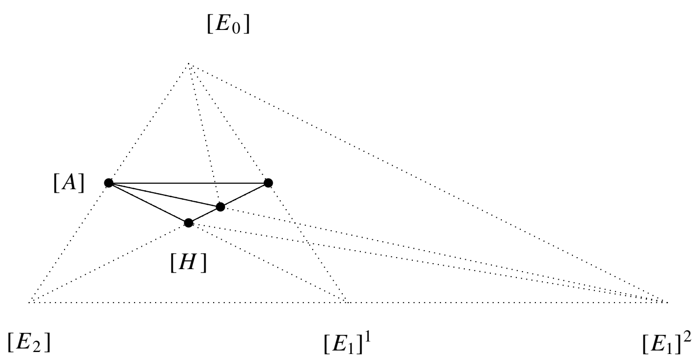

Figure 4

Figure 4 describes a typical situation: the effective cone, as the function of a deformation, is upper semicontinuous (i.e., $\mathrm{Eff}(S_2) \supset \mathrm{Eff}(S_1)$) while the nef or movable cone is lower semicontinuous (i.e., $\mathrm{Mov}(S_2) \subset \mathrm{Mov}(S_1)$).

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