

Projective Manifolds Containing a Large Linear Subspace with Nef Normal Bundle

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1. Introduction

Let $X \subset \mathbb{P}^N$ be a smooth complex projective variety of dimension n containing a linear subspace Λ of dimension s ; denote by $N_{\Lambda/X}$ its normal bundle and by c the degree of the first Chern class of $N_{\Lambda/X}$. If $N_{\Lambda/X}$ is numerically effective (nef), then X is covered by lines; if furthermore s is sufficiently large, the restrictions imposed on X become stronger.

By [8, Thm. 2.5], if $s + c > \frac{n}{2}$ then, for large m and denoting by H the restriction to X of the hyperplane bundle, the linear system $|m(K_X + (s + 1 + c)H)|$ defines an extremal ray contraction of X that contracts Λ . If s is greater than $\frac{n}{2}$ then this contraction is a projective bundle, as shown in [28] (see also [6, Thm. 2.5]). By [14, Thm. 1.7; 31, Thm. 2.4], the same result holds if $s = \frac{n}{2}$ and $N_{\Lambda/X}$ is trivial.

The complete study of the case $s = \frac{n}{2}$ is the subject of [28]; the setup of the quoted paper is different—what is assumed is not the existence of a linear space of dimension $\frac{n}{2}$ with nef normal bundle but rather the existence of a linear space of dimension $\frac{n}{2}$ through every point of X —yet the assumptions are in fact equivalent.

The most difficult cases in [28] are manifolds of Picard number 1, which turn out to be (besides linear spaces) hyperquadrics and Grassmannians of lines. In this paper we study the next case (i.e., $n = 2s + 1$) and prove the following theorem.

THEOREM 1.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$ and containing a linear subspace Λ of dimension s such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of X is 1, then X is one of the following:*

- (1) a linear space \mathbb{P}^{2s+1} ;
- (2) a smooth hyperquadric \mathbb{Q}^{2s+1} ;
- (3) a cubic threefold in \mathbb{P}^4 ;
- (4) a complete intersection of two hyperquadrics in \mathbb{P}^5 ;
- (5) the intersection of the Grassmannian of lines $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ with three general hyperplanes; or
- (6) a hyperplane section of the Grassmannian of lines $\mathbb{G}(1, s + 2)$ in its Plücker embedding.

If the Picard number of X is greater than 1, then there is an elementary contraction $\varphi: X \rightarrow Y$ that contracts Λ and one of the following occurs:

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- (7) $\varphi: X \rightarrow Y$ is a scroll; or
 (8) Y is a smooth curve and the general fiber of φ is
 (8a) the Grassmannian of lines $\mathbb{G}(1, s+1)$,
 (8b) a smooth hyperquadric \mathbb{Q}^{2s} , or
 (8c) a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$.

The outline of the paper is as follows. First of all, we use the theory of uniform vector bundles on the projective space, together with some standard exact sequences, to classify all possible normal bundles $N_{\Lambda/X}$. Then we consider separately the case of Picard number greater than 1 and the case of Picard number 1; in fact, the ideas and the proofs are very different.

If the Picard number is greater than 1, we combine the ideas and techniques of [8] with those of [9] to show that a dominating family of lines on X of anticanonical degree $\geq \frac{n+1}{2}$ is extremal; that is, the numerical class of a line spans a Mori extremal ray of $\text{NE}(X)$. The contraction of this ray is the morphism $\varphi: X \rightarrow Y$ appearing in the second part of the statement of Theorem 1.1. The general fiber F of φ is then a manifold covered by linear spaces of dimension $\geq \frac{\dim F}{2}$, and this leads to its classification.

If the Picard number is 1, the main idea is to study the manifold \tilde{X} obtained by blowing up X along Λ ; we prove that \tilde{X} is a Fano manifold, and then we study its “other” extremal contraction. As a first application of this construction, in Section 5 we show how to use it to complete [28, Main Thm.].

In the setup of Theorem 1.1, the hardest case corresponds to the normal bundle $N_{\Lambda/X} \simeq T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}$, which gives rise to case (6). In this case we need to use the blow-up construction twice: first we blow up X along Λ and show that there is a special one-parameter family Σ of linear spaces to which Λ belongs; then we blow up X along Σ and, after studying this blow-up, are able to describe completely the variety.

2. Background Material

A smooth complex projective variety X is called *Fano* if its anticanonical bundle $-K_X$ is ample; the *index* r_X of X is the largest natural number such that $-K_X = mH$ for some (ample) divisor H on X . Since X is smooth, $\text{Pic}(X)$ is torsion free; therefore the divisor L satisfying $-K_X = r_X L$ is uniquely determined and is called the *fundamental divisor* of X . Fano manifolds with $r_X = \dim X - 1$ are called *del Pezzo* manifolds.

2.1. Extremal Contractions

Let X be a smooth projective variety of dimension n defined over the field of complex numbers. A *contraction* $\varphi: X \rightarrow Z$ is a proper surjective map with connected fibers onto a normal variety Z .

If the canonical bundle K_X is not nef, then the negative part of the cone $\text{NE}(X)$ of effective 1-cycles is locally polyhedral by the Cone Theorem. By the Contraction Theorem, to every face in this part of the cone is associated a contraction, called *extremal contraction* or *Fano–Mori contraction*.

An extremal contraction associated to a face of dimension 1 (i.e., to an extremal ray) is called an *elementary contraction*. A Cartier divisor H such that $H = \varphi^*A$ for an ample divisor A on Z is called a *supporting divisor* of the contraction φ .

DEFINITION 2.1.1. An elementary fiber type extremal contraction $\varphi: X \rightarrow Z$ is called a *scroll* (resp. a *quadric fibration*) if there exists a φ -ample line bundle $L \in \text{Pic}(X)$ such that $K_X + (\dim X - \dim Z + 1)L$ (resp. $K_X + (\dim X - \dim Z)L$) is a supporting divisor of φ .

An elementary fiber type extremal contraction $\varphi: X \rightarrow Z$ onto a smooth variety Z is called a \mathbb{P} -*bundle* (resp. *quadric bundle*) if there exists a vector bundle \mathcal{E} of rank $\dim X - \dim Z + 1$ (resp. of rank $\dim X - \dim Z + 2$) on Z such that $X \simeq \mathbb{P}_Z(\mathcal{E})$ (resp. there exists an embedding of X over Z as a divisor of $\mathbb{P}_Z(\mathcal{E})$ of relative degree 2).

Some special scroll contractions arise from projectivization of Bănică sheaves (cf. [5]). In particular, if $\varphi: X \rightarrow Z$ is a scroll such that every fiber has dimension $\leq \dim X - \dim Z + 1$, then Z is smooth and X is the projectivization of a Bănică sheaf on Z (cf. [5, Prop. 2.5]). We will call these contractions *special Bănică scrolls*.

2.2. Families of Rational Curves

Let X be a smooth projective variety of dimension n defined over the field of complex numbers.

DEFINITION 2.2.1. A *family of rational curves* is an irreducible component $V \subset \text{Ratcurves}^n(X)$ (see [23, Def. 2.11]). Given a rational curve, we will call a *family of deformations* of that curve any irreducible component of $\text{Ratcurves}^n(X)$ that contains the point parameterizing that curve.

We will say that V is *unsplit* if it is proper. We define $\text{Locus}(V)$ to be the set of points of X through which there is a curve among those parameterized by V , and we say that V is a *dominating family* if $\overline{\text{Locus}(V)} = X$. We denote by V_x the subscheme of V parameterizing rational curves passing through $x \in \text{Locus}(V)$ and by $\text{Locus}(V_x)$ the set of points of X through which there is a curve among those parameterized by V_x .

By abuse of notation, given a line bundle $L \in \text{Pic}(X)$, we will denote by $L \cdot V$ the intersection number $L \cdot C_V$, with C_V any curve among those parameterized by V .

DEFINITION 2.2.2. An unsplit dominating family V defines a relation of rational connectedness with respect to V , which we shall call *rc(V)-relation* for short, in the following way: x and y are in *rc(V)-relation* if there exists a chain of rational curves, among those parameterized by V , that joins x and y .

To the *rc(V)-relation* we can associate a fibration, at least on an open subset [10; 23, IV.4.16]; we will call it the *rc(V)-fibration*.

PROPOSITION 2.2.3 [23, IV.2.6]. *Let V be an unsplit family of rational curves on X . Then*

- (a) $\dim X - K_X \cdot V \leq \dim \text{Locus}(V) + \dim \text{Locus}(V_x) + 1$; and
- (b) $-K_X \cdot V \leq \dim \text{Locus}(V_x) + 1$.

When V is the unsplit family of deformations of a minimal extremal rational curve—that is, of a rational curve of minimal anticanonical degree in an extremal face of $\text{NE}(X)$ —Proposition 2.2.3 gives the *fiber locus inequality*.

PROPOSITION 2.2.4 [20, Thm. 0.4; 30, Thm. 1.1]. *Let φ be a Fano–Mori contraction of X . Denote by E the exceptional locus of φ and by F an irreducible component of a nontrivial fiber of φ . Then*

$$\dim E + \dim F \geq \dim X + l - 1,$$

where $l := \min\{-K_X \cdot C \mid C \text{ is a rational curve in } F\}$. If φ is the contraction of an extremal ray R , then $l(R) := l$ is called the length of the ray.

DEFINITION 2.2.5. Let V be an unsplit family of rational curves on X and let $Z \subset X$. We denote by $\text{Locus}(V)_Z$ the set of points $x \in X$ such that there exists a curve C in V with $C \cap Z \neq \emptyset$ and $x \in C$.

We will use some properties of $\text{Locus}(V)_Z$, which are summarized in the following lemma.

LEMMA 2.2.6 [11, Sec. 2; 8, Proof of Lemma 1.4.5]. *Let $Z \subset X$ be a closed subset and V an unsplit family. Assume that curves contained in Z are numerically independent from curves in V and that $Z \cap \text{Locus}(V) \neq \emptyset$. Then*

$$\dim \text{Locus}(V)_Z \geq \dim Z - K_X \cdot V - 1.$$

If σ is an extremal face of $\text{NE}(X)$, if F is a fiber of the contraction associated to σ , and if V is an unsplit family that is numerically independent from curves and whose numerical class is in σ , then

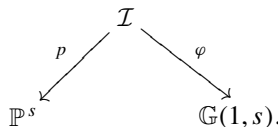
$$\text{NE}(\text{Locus}(V)_F, X) = \langle \sigma, [V] \rangle.$$

In other words, the numerical class in X of a curve in $\text{Locus}(V)_F$ is in the subcone of $\text{NE}(X)$ generated by σ and $[V]$.

2.3. Some Extremal Contractions Related to Grassmannians

We will now present some examples of Fano manifolds admitting a projective bundle structure and another extremal contraction φ whose target is a Grassmannian of lines. We will use these descriptions later in our proofs.

EXAMPLE 2.3.1. Let $\mathbb{G}(1, s)$ be the Grassmannian of lines in \mathbb{P}^s and denote by \mathcal{I} the incidence variety. Consider the incidence diagram



Then p and φ are projective bundles; more precisely, $\mathcal{I} = \mathbb{P}_{\mathbb{P}^s}(p_*\varphi^*\mathcal{O}_{\mathbb{G}(1,s)}(1)) = \mathbb{P}_{\mathbb{P}^s}(\Omega_{\mathbb{P}^s}(2))$ and $\mathcal{I} = \mathbb{P}_{\mathbb{G}(1,s)}(\varphi_*p^*\mathcal{O}_{\mathbb{P}^s}(1)) = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{Q})$, where \mathcal{Q} is the universal quotient bundle on $\mathbb{G}(1, s)$.

EXAMPLE 2.3.2. As in the previous example, let $\mathbb{G}(1, s)$ be the Grassmannian of lines in \mathbb{P}^s and let \mathcal{I} denote the incidence variety. Consider the following diagram, which is obtained by the preceding incidence diagram:

$$\begin{array}{ccccc}
 & & \mathcal{I} \times \mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{G}(1, s) \\
 & \swarrow p & & \searrow q & \\
 \mathbb{P}^s \times \mathbb{P}^1 & & & & \mathbb{G}(1, s) \times \mathbb{P}^1 \xrightarrow{g} \mathbb{G}(1, s)
 \end{array}$$

The composition $\varphi = g \circ q$ gives a morphism $\varphi: \mathcal{I} \times \mathbb{P}^1 \rightarrow \mathbb{G}(1, s)$ whose fibers are smooth two-dimensional quadrics. Let \mathcal{H} be $p^*\mathcal{O}_{\mathbb{P}^s \times \mathbb{P}^1}(1, 1)$ and put $\mathcal{E} := \varphi_*\mathcal{H}$. We have

$$\mathcal{E} = \varphi_*\mathcal{H} = g_*(q_*\mathcal{H}) = g_*(\mathcal{O}_{\mathbb{G}(1,s) \times \mathbb{P}^1}(1, 0) \otimes g^*\mathcal{Q}) = \mathcal{Q}^{\oplus 2}.$$

The product $\mathcal{I} \times \mathbb{P}^1 = \mathbb{P}_{\mathbb{P}^s \times \mathbb{P}^1}(p_1^*\Omega_{\mathbb{P}^s}(2))$, where p_1 denotes the projection onto \mathbb{P}^s , embeds in $\mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{E})$ as a divisor of relative degree 2; that is, it belongs to a linear system $|2\mathcal{H} - \varphi^*L|$ for some line bundle L in $\text{Pic}(\mathbb{G}(1, s))$. The discriminant divisor of the quadric bundle is in the linear system $|2 \det \mathcal{E} - 4L|$ and it is trivial, since every fiber of φ is smooth. It follows that $L = \mathcal{O}_{\mathbb{G}(1,s)}(1)$.

EXAMPLE 2.3.3. Let $\mathbb{G}(1, s + 1)$ be the Grassmannian of lines in \mathbb{P}^{s+1} and let $\mathbb{G}(1, H) \subset \mathbb{G}(1, s + 1)$ be a sub-Grassmannian corresponding to the lines of \mathbb{P}^{s+1} contained in a fixed hyperplane H .

Consider the rational map $\psi: \mathbb{G}(1, s + 1) \dashrightarrow H$, which associates to a line l the point of intersection of l with H . This map is not defined precisely along the points of $\mathbb{G}(1, s + 1)$ representing the lines contained in H (i.e., along the sub-Grassmannian $\mathbb{G}(1, H)$).

Consider the resolution of ψ obtained by blowing up $\mathbb{G}(1, s + 1)$ along $\mathbb{G}(1, H)$:

$$\begin{array}{ccc}
 \tilde{\mathbb{G}}(1, s + 1) & \xrightarrow{\varphi} & \mathbb{G}(1, s + 1) \\
 \downarrow p & \swarrow \psi & \\
 H & &
 \end{array}$$

The contraction p is a \mathbb{P}^s -bundle over H whose fibers are the strict transforms of linear subspaces $\mathbb{P}^s \subset \mathbb{G}(1, s + 1)$ corresponding to stars of lines with center in H ; namely, $\tilde{\mathbb{G}}(1, s + 1) = \mathbb{P}_H(p_*\varphi^*\mathcal{O}_{\mathbb{G}(1,s+1)}(1)) = \mathbb{P}_H(\Omega_{\mathbb{P}^s}(2) \oplus \mathcal{O}_{\mathbb{P}^s}(1))$.

3. Manifolds with Picard Number Greater Than 1

In this section we are going to show that, if the Picard number of X is greater than 1 and if X is covered by linear spaces of dimension $s \geq [n/2]$, then either

$\text{Pic}(X) \simeq \mathbb{Z}$ or there is an elementary Mori contraction to a positive-dimensional variety whose general fiber is covered by linear spaces; in this last case we will then get the description of the general fiber by Corollary 5.3.

Let X be a smooth complex projective variety, let V be an unsplit dominating family of rational curves for X , and let $q: X \dashrightarrow Y$ be the $\text{rc}(V)$ -fibration. Let B be the indeterminacy locus of q ; notice that $\dim B \leq \dim X - 2$, as X is smooth. Moreover, by [9, Prop. 1], B is the union of all $\text{rc}(V)$ -equivalence classes of dimension greater than $\dim X - \dim Y$.

LEMMA 3.1. *Let V be an unsplit dominating family of rational curves on a smooth projective variety X . Let B be the indeterminacy locus of the $\text{rc}(V)$ -fibration $q: X \dashrightarrow Y$, let D be very ample on $q(X \setminus B)$, and let $\hat{D} := \overline{q^{-1}D}$. Then:*

- (a) $\hat{D} \cdot V = 0$;
- (b) if $C \not\subset B$ is a curve whose numerical class is not proportional to $[V]$, then $\hat{D} \cdot C > 0$;
- (c) if $[V]$ does not span an extremal ray of $\text{NE}(X)$, then there exists a curve $C \subset B$ whose class is not proportional to $[V]$ and such that $\hat{D} \cdot C \leq 0$.

Proof. A general cycle of V is contained in a fiber of q disjoint from \hat{D} , so $\hat{D} \cdot V = 0$. If C is as in (b), then $q(C)$ is a curve in Y and the result follows from the projection formula.

Finally, if $[V]$ does not span an extremal ray, then either \hat{D} is not nef or \hat{D} is nef but $\hat{D}^\perp \cap \text{NE}(X) \supsetneq [V]$. In both cases there exists a curve $C \subset X$ whose class is not proportional to $[V]$ such that $\hat{D} \cdot C \leq 0$. Such a curve must be contained in B by [9, Proof of Prop. 1]. □

LEMMA 3.2. *Let X be a manifold that admits an unsplit dominating family of rational curves V . Assume there exists an extremal face $\Sigma \subseteq \overline{\text{NE}(X)}_{K_X < 0}$ such that $[V] \subset \Sigma$. Then either $[V]$ spans an extremal ray or there exists an extremal ray in Σ whose exceptional locus is contained in the indeterminacy locus B of the $\text{rc}(V)$ -fibration. In particular, this ray is associated with a small contraction.*

Proof. Let τ be a minimal subface of Σ containing $[V]$. If $\dim \tau = 1$, then $[V]$ spans an extremal ray.

Assume that $\dim \tau \geq 2$. Let \hat{D} be as in Lemma 3.1. Since $\hat{D} \cdot V = 0$, it follows that either \hat{D} is zero on every extremal ray of σ or is negative on at least one ray. In both cases, by Lemma 3.1(b) there is at least one ray whose exceptional locus is contained in B , and the assertion follows because $\dim B \leq \dim X - 2$. □

The following is a slight improvement of [8, Thm. 2.5] (cf. [6, Thm. 2.4], where the case $-K_X \cdot V \geq \frac{n+3}{2}$ is treated).

THEOREM 3.3. *Let (X, H) be a polarized n -fold with a dominating family of rational curves V such that $H \cdot V = 1$. If $-K_X \cdot V \geq \frac{n+1}{2}$, then $[V]$ spans an extremal ray of $\text{NE}(X)$.*

Proof. Denote by m the positive integer $-K_X \cdot V$ and by L the adjoint divisor $K_X + mH$.

Case 1: L is nef. Denote by $q: X \dashrightarrow Y$ the $\text{rc}(V)$ -fibration and by B its indeterminacy locus. Assume that $[V]$ does not span an extremal ray in $\text{NE}(X)$. This implies that L defines an extremal face Σ of dimension ≥ 2 containing $[V]$.

By Lemma 3.2 there exists an extremal ray $R \in \Sigma$ whose associated contraction φ is small; moreover, since $L \cdot R = 0$, the length of this extremal ray is greater than or equal to m . If F is a nontrivial fiber of φ , then by Proposition 2.2.4 we have $\dim F \geq m + 1$.

Let x be a point in F ; $\text{Locus}(V_x)$ meets F but, since $[V]$ is independent from R , the intersection must be zero-dimensional. This implies that

$$\dim \text{Locus}(V_x) \leq n - m - 1 \leq \frac{n - 3}{2},$$

contradicting part (b) of Proposition 2.2.3.

Case 2: L is not nef. This assumption yields the existence of an extremal ray R such that $L \cdot R < 0$. Notice that R has length $\geq m + 1$, so every nontrivial fiber of the associated contraction has dimension $\geq m$ by Proposition 2.2.4.

By Lemma 2.2.6 we have

$$\dim X \geq \dim \text{Locus}(V)_F \geq -K_X \cdot V + \dim F - 1 \geq m + m - 1 \geq n;$$

hence $\text{Locus}(V)_F = X$. We can apply the second part of Lemma 2.2.6 to get $\text{NE}(X) = \langle [V], R \rangle$ and we are done. \square

REMARK 3.4. By combining the ideas and techniques of [8] with those of [9], it is actually possible to prove the statement of Theorem 3.3 under the weaker assumption that $-K_X \cdot V \geq \frac{n-1}{2}$. However, the proof becomes very long and complicated and so, since it is not necessary for our main theorem, we will present it elsewhere [24].

THEOREM 3.5. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n covered by linear spaces of dimension $s \geq [n/2]$; then there is an elementary Mori contraction $\varphi: X \rightarrow Y$ that contracts lines in the corresponding covering family. Moreover, either $\text{Pic}(X) \simeq \mathbb{Z}$ and Y is a point, or, denoting by F a general fiber of φ , one of the following occurs: $\text{Pic}(F) \simeq \mathbb{Z}$ or $n = 2s + 1$ and $F \simeq \mathbb{P}^s \times \mathbb{P}^s$.*

Proof. Let l be a general line in a general linear space; by the assumptions there is a dominating family of lines in X containing l . By adjunction we have $-K_X \cdot l \geq s + 1$; hence, by Theorem 3.3, the numerical class of l spans an extremal ray of X .

Let $\varphi: X \rightarrow Y$ be the contraction of this extremal ray and let F be a general fiber of φ . Then F has dimension $\leq 2s$ and, by adjunction, is a Fano manifold of index $\geq s + 1$; hence either $F \simeq \mathbb{P}^s \times \mathbb{P}^s$ or $\rho_F = 1$ by [29, Thm. B]. \square

EXAMPLE 3.6. We show with an example that the last case of Theorem 3.5 is effective; the idea on which it is based was suggested by Wiśniewski for [25, Exm. 7.2].

Let C' be a smooth curve with a free \mathbb{Z}_2 -action, so that the action induces an étale covering $\pi: C' \rightarrow C$ of degree 2. Let G be $\mathbb{P}^s \times \mathbb{P}^s$ and take, on G , the \mathbb{Z}_2 -action that exchanges the factors.

Let $X' := G \times C'$ and denote by X the quotient of X' by the product action of \mathbb{Z}_2 ; the action is free and so X is smooth. By the universal property of group actions there exists a morphism $\varphi: X \rightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc} X' & \longrightarrow & C' \\ \pi' \downarrow & & \downarrow \pi \\ X & \xrightarrow{\varphi} & C. \end{array}$$

The map $\varphi: X \rightarrow C$ is an extremal contraction, and every fiber is a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$. We will now show that φ is elementary.

Let l be a line in G and consider the product $l \times C' \subset G \times C' = X'$: it is a flat family of rational curves. Let c be a point of C and let $\{c'_1, c'_2\}$ be $\pi^{-1}(c)$. Finally, set $l'_i := l \times \{c'_i\}$ and consider the following restriction of the previous diagram:

$$\begin{array}{ccc} G \times \{c'_1, c'_2\} & \longrightarrow & \{c'_1, c'_2\} \\ \pi' \downarrow & & \downarrow \pi \\ \varphi^{-1}(c) \simeq \mathbb{P}^s \times \mathbb{P}^s & \xrightarrow{\varphi} & c. \end{array}$$

Since the product action identifies $G \times \{c'_1\}$ with $G \times \{c'_2\}$ exchanging the factors, it follows that $l_1 = \pi'(l'_1)$ is a line in a fiber of the projection of $\varphi^{-1}(c)$ onto the first factor and that $l_2 = \pi'(l'_2)$ is a line in a fiber of the projection of $\varphi^{-1}(c)$ onto the second factor. Hence lines in the two factors are algebraically and thus numerically equivalent.

REMARK 3.7. In Example 3.6, X has an unsplit dominating family of rational curves V such that V_x has dimension $\frac{\dim X - 3}{2}$ and is reducible for every x . This should be compared with [21, Thm. 5.1], in which it is proved that if $\dim V_x \geq \frac{\dim X - 1}{2}$ then V_x is irreducible.

4. A General Construction

In this section we will present a blow-up construction and show how to apply it to manifolds of Picard number 1 containing a linear space with nef normal bundle. The construction in the following proposition was inspired by the graduate thesis [26] supervised by the second-named author.

PROPOSITION 4.1. *Let $X \subset \mathbb{P}^N$ be a Fano manifold of dimension n , index r_X , and Picard number 1, covered by lines and containing a smooth subvariety Σ of dimension s , that is the intersection of its linear span with X ; that is, $\Sigma = X \cap \langle \Sigma \rangle$ with $[n/2] \leq s \leq n - 2$. Let $\sigma: \tilde{X} \rightarrow X$ be the blow-up of X along Σ and let $E = \mathbb{P}_\Sigma(N_{\Sigma/X}^*)$ be the exceptional divisor of σ . Then $\text{NE}(\tilde{X}) = \langle [C_\sigma], [\ell] \rangle$, where C_σ is a minimal curve contracted by σ and ℓ is the strict transform of a line meeting Σ at one point.*

If $r_X \geq [n/2] + 1$ then \tilde{X} is a Fano manifold, the length of the ray $\mathbb{R}_+[\ell]$ is $r_X - n + s + 1$, and the extremal contraction $\varphi: \tilde{X} \rightarrow Y$ associated to $\mathbb{R}_+[\ell]$ is

the morphism defined by the linear system $|m(\sigma^*\mathcal{O}_X(1) - E)|$ for $m \gg 0$. If $r_X \geq [(n + 1)/2] + 1$, then also E is a Fano manifold; moreover, for any positive m , the restriction to E of the morphism given by $|m(\sigma^*\mathcal{O}_X(1) - E)|$ is the morphism given by the linear system $|m\xi_{N^*_{\Sigma/X}(1)}|$.

Proof. Since the Picard number of X is 1 and since X is covered by lines, it follows from [3, Prop. 1.1] that X is rationally connected with respect to a dominating family of lines.

Consider the rational map $X \dashrightarrow \tilde{Y}$ defined by the linear system $|\mathcal{O}_X(1) \otimes \mathcal{I}_\Sigma|$ of hyperplanes containing Σ . Let $\tilde{\varphi}: \tilde{X} \rightarrow \tilde{Y}$ be the resolution of this map. Then the morphism $\tilde{\varphi}$ is defined by the linear system $|\mathcal{H} - E|$, where \mathcal{H} denotes the pull-back $\sigma^*\mathcal{O}_X(1)$. Let $l \subset X$ be a line meeting Σ but not contained in it; notice that such a line exists because X is rationally connected with respect to a family of lines.

Since $\Sigma = X \cap \langle \Sigma \rangle$, the intersection $l \cap \Sigma$ consists of one point; hence the morphism $\tilde{\varphi}$ contracts l , the strict transform of l . Therefore, denoting by C_σ a rational curve of minimal degree contracted by σ , we obtain $\text{NE}(\tilde{X}) = \langle [C_\sigma], [\ell] \rangle$. The contraction associated to the ray $R = \mathbb{R}_+[\ell]$ is therefore given by the Stein factorization of $\tilde{\varphi}$; that is, it is defined by the linear system $|m(\mathcal{H} - E)|$ for $m \gg 0$.

By the canonical bundle formula for blow-ups we have

$$-K_{\tilde{X}} = -\sigma^*K_X - (n - s - 1)E = r_X\mathcal{H} - (n - s - 1)E. \tag{4.1.1}$$

Clearly, $-K_{\tilde{X}} \cdot C_\sigma > 0$.

If $r_X \geq [\frac{n}{2}] + 1$, we get $-K_{\tilde{X}} \cdot \ell = r_X - n + s + 1 > 0$. By the Kleiman criterion it follows that $-K_{\tilde{X}}$ is ample, so \tilde{X} is a Fano manifold. We also get that the length of the ray contracted by φ is $r_X - n + s + 1$.

Assume now that $r_X \geq [\frac{n+1}{2}] + 1$. From (4.1.1) it follows that the line bundle

$$-K_{\tilde{X}} - E = r_X\mathcal{H} - (n - s)E$$

is ample on \tilde{X} , since $(-K_{\tilde{X}} - E) \cdot C_\sigma = n - s$ and $(-K_{\tilde{X}} - E) \cdot \ell = r_X + s - n > 0$. Therefore its restriction to E , which by adjunction is $-K_E$, is ample, too; hence E is a Fano manifold.

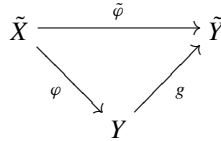
Let m be a positive integer and denote by D_m the divisor $-K_{\tilde{X}} - E + m(\mathcal{H} - E)$. Then D_m , as the sum of an ample line bundle and a nef one, is ample on \tilde{X} and so $h^1(m\mathcal{H} - (m + 1)E) = h^1(K_{\tilde{X}} + D_m) = 0$ by the Kodaira Vanishing Theorem. It follows that the morphism

$$H^0(\tilde{X}, m(\mathcal{H} - E)) \longrightarrow H^0(E, m(\mathcal{H} - E)|_E) = H^0(E, m\xi_{N^*_{\Sigma/X}(1)})$$

is surjective, proving the last claim. □

CLAIM 4.2. *In the setting of Proposition 4.1, assume that $r_X \geq [(n + 1)/2] + 1$, that $|\xi_{N^*_{\Sigma/X}(1)}|$ gives a morphism $\varphi_E: E \rightarrow T$ onto a normal variety, and that φ is of fiber type. Then we can assume that φ is defined by the linear system $|\mathcal{H} - E|$.*

Proof. Let $\tilde{\varphi}: \tilde{X} \rightarrow \tilde{Y}$ be the morphism defined by the linear system $|\mathcal{H} - E|$. The fibers of $\tilde{\varphi}$ are connected and so, in the Stein factorization of $\tilde{\varphi}$, the finite morphism g is the normalization



(cf. [13, 1.13]). The divisor E is $\tilde{\varphi}$ -ample, so the restriction of $\tilde{\varphi}$ to E is onto \tilde{Y} . Since this restriction is φ_E by the last claim of Proposition 4.1, we have $\tilde{Y} = T$. Hence \tilde{Y} is normal and g is an isomorphism. \square

The following lemma shows that we can apply our construction to manifolds of Picard number 1 containing a large linear space whose normal bundle is numerically effective.

LEMMA 4.3. *Let $X \subset \mathbb{P}^N$ be a smooth variety, of dimension n and Picard number 1, that contains a linear space Λ of dimension s . Assume that the normal bundle $N_{\Lambda/X}$ of Λ in X is nef, and denote by c the nonnegative integer such that $\det N_{\Lambda/X} = \mathcal{O}_{\Lambda}(c)$. Then X is a Fano manifold of index $r_X = s + 1 + c$ that is covered by lines.*

Proof. By the adjunction formula we have

$$K_{\Lambda} = (K_X + \det N_{\Lambda/X})|_{\Lambda}.$$

Hence

$$(-K_X)|_{\Lambda} = \mathcal{O}_{\Lambda}(s + 1 + c),$$

from which we can derive

$$-K_X = \mathcal{O}_X(s + 1 + c). \tag{4.3.1}$$

Let l be a general line in Λ . From the nefness of $N_{\Lambda/X}$ and the exact sequence

$$0 \longrightarrow N_{l/\Lambda} = \mathcal{O}_{\Lambda}(1)^{\oplus(s-1)} \longrightarrow N_{l/X} \longrightarrow (N_{\Lambda/X})|_l \longrightarrow 0,$$

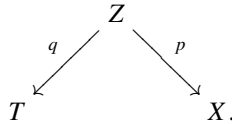
we have that $N_{l/X}$ is nef. Therefore, l is a free rational curve in X (see [23, Def. II.3.1]), which is thus covered by lines by [23, Prop. II.3.10]. \square

5. Projective n -Folds Covered by Linear Subspaces of Dimension $\geq n/2$

In this section we will prove that, if a smooth complex projective variety $X \subset \mathbb{P}^N$ of Picard number 1 and dimension $2s$ contains a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1)$, then X is the Grassmannian of lines in \mathbb{P}^{s+1} . This result, as explained in Corollary 5.3, completes [28, Main Thm.], which classified smooth projective varieties of dimension n covered by linear subspaces of dimension $\geq \frac{n}{2}$.

LEMMA 5.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety containing a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle $N_{\Lambda/X}$ is globally generated and such that $h^1(N_{\Lambda/X}) = 0$. Then X is covered by linear subspaces of dimension s .*

Proof. The Hilbert scheme of s -planes in X is smooth at the point λ corresponding to Λ . Let T be the unique irreducible component of the Hilbert scheme containing λ , and let Z be the universal family. Then we have the following diagram:



Let z be a point in $\Lambda' := q^{-1}(\lambda)$; we consider the differential of p at that point,

$$d_z p: T_z Z \longrightarrow T_{p(z)} X;$$

this map is the identity when restricted to $T_z \Lambda'$. Recalling that $T_\lambda T \simeq H^0(N_{\Lambda/X})$ and considering the exact sequence of the normal bundle of Λ in X , we get the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_z \Lambda' & \longrightarrow & T_z Z & \longrightarrow & H^0(N_{\Lambda/X}) \longrightarrow 0 \\
 & & \downarrow Id & & \downarrow d_z p & & \downarrow ev \\
 0 & \longrightarrow & T_{p(z)} \Lambda & \longrightarrow & T_{p(z)} X & \longrightarrow & (N_{\Lambda/X})_{p(z)} \longrightarrow 0,
 \end{array}$$

which shows that $d_z p$ is surjective (since ev is surjective by the spannedness of $N_{\Lambda/X}$). Hence p is smooth at z . □

PROPOSITION 5.2. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number 1 and dimension $2s$ containing a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1)$. Then X is the Grassmannian of lines $\mathbb{G}(1, s + 1)$.*

Proof. Observe first that, by Lemma 4.3, X is a Fano manifold of index $r_X = s + 1 + c = s + 2$ that is covered by lines.

Let $\sigma: \tilde{X} \rightarrow X$ be the blow-up of X along Λ . Denote by $E = \mathbb{P}(N_{\Lambda/X}^*)$ the exceptional divisor and by \mathcal{H} the pull-back $\sigma^* \mathcal{O}_X(1)$. By Proposition 4.1, \tilde{X} is a Fano manifold with a contraction $\varphi: \tilde{X} \rightarrow Y$ whose restriction to E is the map associated on E to the linear system $|m\xi_{N_{\Lambda/X}^*(1)}| = |m\xi_{\Omega(2)}|$. This map is, up to a Veronese embedding of the target, the \mathbb{P}^1 -bundle over $\mathbb{G}(1, s)$ given by the projectivization of the universal quotient bundle \mathcal{Q} over $\mathbb{G}(1, s)$, as shown in Example 2.3.1.

Moreover, by Proposition 4.1, $NE(\tilde{X}) = \langle [C_\sigma], [\ell] \rangle$; here C_σ is a rational curve of minimal degree contracted by σ , ℓ is the strict transform of a line meeting Λ at one point, and the length of the extremal ray generated by $[\ell]$ is $r_X - n + s + 1 = 3$. By Proposition 2.2.4, every nontrivial fiber of the contraction φ has dimension ≥ 2 . Since $E \cdot \ell = 1$, we have that E meets every nontrivial fiber of φ . Because $\varphi|_E$ is equidimensional with one-dimensional fibers, it follows that φ cannot have fibers of dimension > 2 , for otherwise their intersection with E would be a fiber of dimension ≥ 2 of $\varphi|_E$.

Therefore, every nontrivial fiber of φ has dimension 2 and so, by Proposition 2.2.4, φ is of fiber type. By Claim 4.2 we can assume that φ is defined by the linear system $|\mathcal{H} - E|$.

Let F be a general fiber of φ . By adjunction, we have

$$-K_F = (-K_{\tilde{X}})|_F = ((s + 2)\mathcal{H} - (s - 1)E)|_F = 3\mathcal{H}|_F;$$

hence $(F, \mathcal{H}|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. The line bundle $2\mathcal{H} - E$ is ample and

$$(2\mathcal{H} - E)|_F \simeq \mathcal{O}_{\mathbb{P}^2}(1);$$

thus we can apply [19, Lemma 2.12] to obtain that φ is a projective bundle over $\mathbb{G}(1, s)$.

Let $\mathcal{E} := \varphi_*\mathcal{H}$. Then the inclusion $E = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{Q}) \hookrightarrow \tilde{X} = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{E})$ gives the following exact sequence of vector bundles over $\mathbb{G}(1, s)$:

$$0 \longrightarrow L \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

We can use the canonical bundle formula for projective bundles to compute that $\det \mathcal{E} = \mathcal{O}_{\mathbb{G}(1,s)}(2)$; so, recalling that $\det \mathcal{Q} = \mathcal{O}_{\mathbb{G}(1,s)}(1)$, we have $L = \mathcal{O}_{\mathbb{G}(1,s)}(1)$. Since $h^1(\mathcal{Q}^*(1)) = h^1(\mathcal{Q}) = 0$, the sequence splits and $\tilde{X} = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{Q} \oplus \mathcal{O}_{\mathbb{G}(1,s)}(1))$.

We have thus proved that the existence in X of a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1)$ completely determines X . Since the Grassmannian of lines $\mathbb{G}(1, s + 1)$ contains such a linear space (take a linear space corresponding to the lines passing through a fixed point), the proposition is proved. \square

COROLLARY 5.3 (cf. [28, Main Thm.]). *Let $X \subset \mathbb{P}^N$ be a smooth complex variety of dimension $n \geq 2$ covered by linear subspaces of dimensions $s \geq \frac{n}{2}$. Then X is one of the following:*

- (1) a \mathbb{P}^r -bundle over a smooth variety ($r \geq s$);
- (2) a smooth hyperquadric \mathbb{Q}^{2s} ; or
- (3) the Grassmannian of lines $\mathbb{G}(1, s + 1)$.

Proof. In [28], Sato first showed that the normal bundle of a general linear subspace is one of the following:

- (i) $\mathcal{O}_{\mathbb{P}^s}^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(n-s-a)}$;
- (ii) $\Omega_{\mathbb{P}^s}(2)$; or
- (iii) $T_{\mathbb{P}^s}(-1)$.

Then he showed that X is a \mathbb{P}^r -bundle over a smooth variety ($r \geq s$) in case (i) and that X is a smooth hyperquadric in case (ii). As for case (iii), Sato showed that X is the Grassmannian of lines in \mathbb{P}^{s+1} if s is even or if one assumes that all the linear subspaces of the covering family have normal bundle $T_{\mathbb{P}^s}(-1)$.

Thus to prove the statement of the corollary it is enough to show that, in case (iii), X is the Grassmannian of lines in \mathbb{P}^{s+1} . This will follow from Proposition 5.2 once we prove that, if X is as in case (iii), then its Picard number is 1. So assume that this is not the case. By Theorem 3.5 there is an elementary contraction that contracts a covering family of linear subspaces of dimension s . A general fiber F has dimension at most $2s - 1$ and is covered by linear spaces of dimension s . Applying

[32, Cor. I.2.20] as in [17, Thm. 2], we derive that F is a projective space; hence the normal bundle of a general s -plane cannot be $T_{\mathbb{P}^s}(-1)$. \square

COROLLARY 5.4. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s$ containing a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1)$. Then X is the Grassmannian of lines $\mathbb{G}(1, s + 1)$.*

Proof. By Lemma 5.1, through every point of X there is a linear subspace of dimension s . By Corollary 5.3, X is a \mathbb{P}^r -bundle over a smooth variety, a smooth hyperquadric \mathbb{Q}^{2s} , or the Grassmannian of lines $\mathbb{G}(1, s + 1)$. However, the first two cases are ruled out because the corresponding manifolds do not contain a linear subspace with normal bundle $T_{\mathbb{P}^s}(-1)$. \square

We can now prove the part of Theorem 1.1 regarding manifolds with Picard number greater than 1.

COROLLARY 5.5. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$ and Picard number greater than 1, containing a linear subspace Λ of dimension s , whose normal bundle $N_{\Lambda/X}$ is nef. Then there is an elementary contraction $\varphi: X \rightarrow Y$ that contracts Λ and one of the following occurs:*

- (a) φ is a scroll; or
- (b) Y is a smooth curve and the general fiber of φ is
 - the Grassmannian of lines $\mathbb{G}(1, s + 1)$,
 - a smooth hyperquadric \mathbb{Q}^{2s} , or
 - a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$.

Proof. Combine Theorem 3.5 with Corollary 5.3. \square

REMARK 5.6. If φ is a scroll and $\dim F \geq s + 1$, then X has a projective bundle structure over Y by [14, Thm. 1.7]. By [7, Conj. 14.1.10], this should be the case also if $\dim F = s$.

6. Normal Bundles

Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$ and containing a linear subspace Λ of dimension s whose normal bundle $N_{\Lambda/X}$ is numerically effective. In this section we will start the proof of the first part of Theorem 1.1. We shall give the list of all possible normal bundles of the linear subspace Λ , show that X is covered by linear subspaces of dimension s , and settle the case of decomposable normal bundles.

PROPOSITION 6.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$ and containing a linear subspace Λ of dimension s whose normal bundle $N_{\Lambda/X}$ is nef. Then $N_{\Lambda/X}$ is one of the following:*

- (1) $\Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}$;
- (2) $\Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1)$;

- (3) $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda(1)$;
- (4) $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda$; or
- (5) $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{\oplus(s+1-c)}$.

Moreover, through every point of X there is a linear subspace of dimension s .

Proof. From the exact sequence

$$0 \longrightarrow N_{\Lambda/X} \longrightarrow N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_\Lambda(1)^{\oplus(N-s)} \longrightarrow (N_{X/\mathbb{P}^N})|_\Lambda \longrightarrow 0$$

and the nefness of $N_{\Lambda/X}$, we get that the splitting of $N_{\Lambda/X}$ on lines in Λ is of type $(0, \dots, 0, 1, \dots, 1)$ and hence uniform. By the classification of uniform vector bundles of rank $s + 1$ on \mathbb{P}^s given in [16] and [4] and taking into account the splitting type, we have that $N_{\Lambda/X}$ is one of the bundles listed in the statement. Since all these bundles are generated by global sections and have $h^1(N_{\Lambda/X}) = 0$, the last assertion follows from Lemma 5.1. □

QUESTION 6.2. Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$ such that, through every point of X , there is a linear subspace of dimension s . It is possible to prove, as in [28], that the general linear subspace has a normal bundle that is spanned at the general point. Is it true that there exists a linear subspace Λ with nef normal bundle?

We next recall a general construction (see [1, Proof of 0.7]).

LEMMA 6.3. Let $\Lambda \subset X \subset \mathbb{P}^N$ be a linear space contained in a smooth projective variety and such that $N_{\Lambda/X} \simeq N' \oplus \mathcal{O}_\Lambda(1)$ for some vector bundle N' over Λ . Then there exists a smooth hyperplane section X' of X that contains Λ and such that $N_{\Lambda/X'} \simeq N'$.

Proof. The existence of the smooth hyperplane section follows from [7, Cor. 1.7.5]. Notice that, to apply this result, since $N_{\Lambda/X}^*(1) \simeq N'^*(1) \oplus \mathcal{O}_\Lambda$ we do not need assumptions on $\dim \Lambda$. Then, by the exact sequence

$$0 \longrightarrow N_{\Lambda/X'} \longrightarrow N_{\Lambda/X} \simeq N' \oplus \mathcal{O}_\Lambda(1) \longrightarrow \mathcal{O}_\Lambda(1) \longrightarrow 0,$$

we obtain the statement on the normal bundle. □

PROPOSITION 6.4. Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $n \geq 4$ and containing a linear space Λ of dimension s with $\lfloor \frac{n}{2} \rfloor \leq s \leq n - 2$. Assume that the normal bundle $N_{\Lambda/X}$ is trivial. Then the Picard number of X is at least 2.

Proof. Assume by contradiction that $\rho_X = 1$. Then, by Lemma 4.3, X is a Fano manifold of index $r_X = s + 1$ that is covered by lines. By the first part of Proposition 4.1, the blow-up of X along Λ , which we will denote by \tilde{X} , is a Fano manifold with $\rho_{\tilde{X}} = 2$ and whose “other” contraction $\varphi: \tilde{X} \rightarrow Y$ is given by the linear system $|m(\mathcal{H} - E)|$, where $\mathcal{H} := \sigma^*\mathcal{O}_X(1)$. The restriction of $m(\mathcal{H} - E)$ to $E = \Lambda \times \mathbb{P}^{n-s-1}$ is $m\xi_{N_{\Lambda/X}^*}(1) = \mathcal{O}_E(m, 1)$. In particular, no curves of E are contracted by φ .

The extremal ray associated with φ is generated by the class $[\ell]$ of the strict transform of a line meeting Λ at one point, so $E \cdot \ell = 1$. Since E has positive

intersection number with curves contracted by φ , it follows that every nontrivial fiber of φ has dimension 1.

By [30, Thm. 1.2], φ is either a conic bundle or a blow-up of a smooth subvariety of codimension 2; in both cases, Y is smooth. Assume that φ is a conic bundle. Then the finite morphism $\varphi|_E: E \rightarrow Y$ is either birational (if φ has no reducible fibers) or of degree 2; since $\rho_Y = 1$, in both cases we should have $\rho_E = 1$. This is clear in the first case, and in the second it follows from [12]. So we get a contradiction.

If φ is a blow-up of a smooth codimension-2 subvariety, then the divisor $D = (m + s + 1)(\mathcal{H} - E) = \varphi^* \mathcal{O}_Y(m + s + 2)$ is nef and big on \tilde{X} for $m > 0$. Moreover, the length of φ is 1 and so, by Proposition 4.1, we get $r_X = n - s$; hence $n = 2s + 1$. This implies that $m\mathcal{H} - (m + 1)E = K_{\tilde{X}} + D$, so $h^1(m\mathcal{H} - (m + 1)E) = 0$ by the Kawamata–Viehweg Vanishing Theorem. It follows that the morphism

$$H^0(\tilde{X}, m(\mathcal{H} - E)) \longrightarrow H^0(E, m(\mathcal{H} - E)|_E) = H^0(E, m\xi_{N_{\Sigma/X}^*(1)})$$

is surjective, so the restriction to E of φ is the morphism given by the linear system $|m\xi_{N_{\Sigma/X}^*(1)}| = |\mathcal{O}_E(m, 1)|$. In particular, the image of E via φ is a smooth divisor isomorphic to $\mathbb{P}^s \times \mathbb{P}^s$. But this is impossible, by Lefschetz’s theorem on hyperplane sections, because $\rho_Y = 1$. □

REMARK 6.5. If $n = 3$ then, by Lemma 4.3, X is a Fano manifold of index 2 that is covered by lines; hence X is a del Pezzo manifold. By the classification in [19, Thm. 8.11], a del Pezzo threefold of Picard number 1 with very ample fundamental divisor is a cubic hypersurface in \mathbb{P}^4 , the complete intersection of two hyperquadrics in \mathbb{P}^5 , or a linear section of $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ with three general hyperplanes.

Proposition 6.4 allows us to prove that if the normal bundle of Λ is decomposable, the Picard number of X is 1, and $s \geq 2$, then X is a linear space.

COROLLARY 6.6. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n and Picard number 1 that contains a linear space Λ of dimension $s \geq 2$ with $\lfloor \frac{n}{2} \rfloor \leq s \leq n - 2$. Assume that the normal bundle $N_{\Lambda/X}$ is $\mathcal{O}_{\Lambda}(1)^{\oplus c} \oplus \mathcal{O}_{\Lambda}^{\oplus(n-s-c)}$. Then $c = n - s$ and X is a linear space.*

Proof. By Proposition 6.4 we can assume that $c > 0$ and so, by Lemma 6.3, we can find a smooth hyperplane section X' of X containing Λ . Then, as in [17, Thm. 2], we apply [32, Cor. I.2.20]; this yields that X' is a linear space, so we conclude that X is a linear space, too. □

7. Manifolds with Picard Number 1

In this section we will consider projective manifolds of dimension $2s + 1$ and Picard number 1 containing a linear subspace of dimension s with numerically effective normal bundle. We shall prove the following result.

THEOREM 7.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$ and Picard number 1 that contains a linear subspace Λ of dimension s such that its normal bundle $N_{\Lambda/X}$ is nef. Then X is one of the following:*

- a linear space \mathbb{P}^{2s+1} ;
- a smooth hyperquadric \mathbb{Q}^{2s+1} ;
- a cubic threefold in \mathbb{P}^4 ;
- a complete intersection of two hyperquadrics in \mathbb{P}^5 ;
- the intersection of the Grassmannian of lines $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ with three general hyperplanes; or
- a hyperplane section of the Grassmannian of lines $\mathbb{G}(1, s + 2)$ in its Plücker embedding.

Proof. First of all notice that, by Lemma 4.3, X is a Fano manifold of index $r_X = s + 1 + c$. Moreover, all possible nef normal bundles $N_{\Lambda/X}$ are listed in Proposition 6.1.

When $N_{\Lambda/X} \simeq \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}$, X is a del Pezzo manifold with very ample fundamental divisor and hence, by the classification in [19, Thm. 8.11], is of degree ≥ 3 . Recalling that the Picard number of X is 1 and that X contains lines, by the same classification we have that the degree of X is at most 5.

A del Pezzo manifold of degree 3 and Picard number 1 is a cubic hypersurface in \mathbb{P}^{n+1} . On the other hand, by the exact sequence of normal bundles

$$0 \longrightarrow N_{\Lambda/X} \longrightarrow \mathcal{O}_{\Lambda}(1)^{\oplus(s+2)} \longrightarrow \mathcal{O}_{\Lambda}(3) \longrightarrow 0,$$

it follows that we cannot have $N_{\Lambda/X} \simeq \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}$ unless $s = 1$. Again by [19, Thm. 8.11], a del Pezzo manifold of degree 4 is the complete intersection of two quadric hypersurfaces.

Let us show also that this case is possible only if $s = 1$. We owe this remark and its proof to Andrea Luigi Tironi.

Let \mathcal{Q} and \mathcal{Q}' be hyperquadrics such that $X = \mathcal{Q} \cap \mathcal{Q}'$, and let \mathcal{F} be the pencil generated by \mathcal{Q} and \mathcal{Q}' . By [27, Prop. 2.1], the general quadric in \mathcal{F} is smooth; hence we can assume that \mathcal{Q} is smooth.

By [7, Cor. 1.7.5] there is a smooth hyperplane section \mathcal{Q}_H of \mathcal{Q} containing Λ . By the exact sequence of normal bundles

$$0 \longrightarrow N_{\Lambda/\mathcal{Q}_H} \longrightarrow N_{\Lambda/\mathcal{Q}} \longrightarrow \mathcal{O}_{\Lambda}(1) \longrightarrow 0$$

and recalling that $N_{\Lambda/\mathcal{Q}_H} \simeq \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1)$, we have $N_{\Lambda/\mathcal{Q}} \simeq \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1)^{\oplus 2}$. Therefore, the exact sequence

$$0 \longrightarrow N_{\Lambda/X} \longrightarrow N_{\Lambda/\mathcal{Q}} \longrightarrow (N_{X/\mathcal{Q}})|_{\Lambda} \longrightarrow 0$$

becomes

$$0 \longrightarrow \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda} \longrightarrow \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1)^{\oplus 2} \longrightarrow \mathcal{O}_{\Lambda}(2) \longrightarrow 0.$$

A computation of the total Chern class shows that this is possible only if $s = 1$. Again by [19, Thm. 8.11], a del Pezzo manifold of degree 5 is a linear section of $\mathbb{G}(1, 4)$.

If $N_{\Lambda/X} \simeq \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1)$, then X is a smooth hyperquadric by the Kobayashi–Ochiai theorem [22]. If $N_{\Lambda/X} \simeq T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}(1)$ then, by Lemma 6.3, there exists a smooth hyperplane section X' of X containing Λ . Moreover, the normal bundle of Λ in X' is $T_{\Lambda}(-1)$; hence, by Corollary 5.4, X' is the Grassmannian of lines

$\mathbb{G}(1, s + 1)$. But by [18, Cor. 1.3, Prop. 2.1], $\mathbb{G}(1, s + 1)$ cannot be a hyperplane section of another manifold unless $s = 2$. In this case X' is a four-dimensional hyperquadric and so X is a five-dimensional hyperquadric. Note that, since $T_{\mathbb{P}^2}(-1) \simeq \Omega_{\mathbb{P}^2}(2)$, this was already part of the previous case.

As for the remaining possibility, if the normal bundle is decomposable then X is a linear space by Corollary 6.6. The more difficult case, $N_{\Lambda/X} \simeq T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}$, is settled in the next subsection. □

7.1. Normal Bundle Isomorphic to $T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}$

We start by proving that Λ belongs to a special one-dimensional family of linear subspaces of X .

PROPOSITION 7.1.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number 1 and dimension $2s + 1$ that contains a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1) \oplus \mathcal{O}_{\mathbb{P}^s}$. Then there is a subvariety $\Sigma \subset X$ such that $(\Sigma, (\mathcal{O}_X(1)|_{\Sigma}) \simeq (\mathbb{P}^1 \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^s}(1, 1))$ and such that Σ contains Λ as a fiber of the first projection. Moreover, $\Sigma = \langle \Sigma \rangle \cap X$.*

Proof. By Lemma 4.3, X is a Fano manifold of index $s + 2$ that is covered by lines. Let $\sigma : \tilde{X} \rightarrow X$ be the blow-up of X along Λ , and denote by $E = \mathbb{P}(N_{\Lambda/X}^*)$ the exceptional divisor. By Proposition 4.1, \tilde{X} is a Fano manifold with a contraction $\varphi : \tilde{X} \rightarrow Y$ whose restriction to E is the map associated on E to the linear system $|m\xi_{N_{\Lambda/X}^*(1)}| = |m\xi_{\Omega(2) \oplus \mathcal{O}(1)}|$ —that is, up to a Veronese embedding of the target, the blow-up of $\mathbb{G}(1, s + 1)$ along a sub-Grassmannian $\mathbb{G}(1, s)$ as shown in Example 2.3.3. By Proposition 4.1 we also have that the extremal ray associated with φ has length 2 and is generated by the class $[\ell]$ of the strict transform of a line $l \subset X$ meeting Λ at one point. Let \mathcal{H} be the pull-back $\sigma^*\mathcal{O}_X(1)$, and let $A \in \text{Pic}(Y)$ be an ample line bundle. Then, for some t , $K_{\tilde{X}} + 2(\mathcal{H} + t\varphi^*A)$ is a supporting divisor for φ .

Since $E \cdot \ell = 1$, we have that E meets every nontrivial fiber of φ . Because $\varphi|_E$ is equidimensional with one-dimensional fibers, it follows that φ cannot have fibers of dimension > 2 , for otherwise their intersection with E would be a fiber of $\varphi|_E$ of dimension ≥ 2 . Therefore, every nontrivial fiber of φ has dimension ≤ 2 .

We claim that φ is of fiber type. Assume by contradiction that φ is birational; then it is equidimensional by Proposition 2.2.4. We can apply [2, Thm. 4.1] to get that Y is smooth and that φ is the blow-up of a smooth, codimension-3 center T . Since E meets every nontrivial fiber of φ , we have $T \simeq \mathbb{G}(1, s)$.

So Y contains $\varphi(E) \simeq \mathbb{G}(1, s + 1)$ as an effective divisor. However, since $\rho_Y = 1$, this implies that $\mathbb{G}(1, s + 1)$ is ample in Y ; it then follows from [18, Cor. 1.3, Prop. 2.1] that $s = 2$. Therefore, Y is a projective space or a smooth hyperquadric and $T \simeq \mathbb{P}^2$. Using the two different blow-up structures of \tilde{X} , we can write

$$4\mathcal{H} - 2E = -K_{\tilde{X}} = -\varphi^*K_Y - 2\text{Exc}(\varphi).$$

Hence the index of Y is even, so $Y \simeq \mathbb{P}^5$. But the blow-up of \mathbb{P}^5 along \mathbb{P}^2 has one fiber type contraction and so this case cannot happen, either. It follows that

φ is of fiber type. By Claim 4.2, we can assume that φ is defined by the linear system $|\mathcal{H} - E|$.

Since E meets every fiber of φ , we have $Y = \mathbb{G}(1, s + 1)$. The restriction of φ to E is birational, so the general fiber of φ has dimension 1. As already noted, any fiber of φ has dimension ≤ 2 ; hence φ is a special Bănică scroll and so $\tilde{X} = \mathbb{P}_{\mathbb{G}(1, s+1)}(\mathcal{E})$, where $\mathcal{E} := \varphi_*\mathcal{H}$.

Now combining the canonical bundle formula for \tilde{X} as a blow-up with the canonical bundle formula for \tilde{X} as a Bănică scroll, we get

$$\begin{aligned} -s(\mathcal{H} - E) &= K_{\tilde{X}} + 2\mathcal{H} = \varphi^*(K_{\mathbb{G}(1, s+1)} + \det \mathcal{E}) \\ &= \varphi^*\mathcal{O}_{\mathbb{G}(1, s+1)}(-s - 2 + \deg \det \mathcal{E}). \end{aligned}$$

Therefore, $\det \mathcal{E} = \mathcal{O}_{\mathbb{G}(1, s+1)}(2)$.

Denote by Λ_0 the section of $\sigma: E \rightarrow \Lambda$ that corresponds to the surjection $\Omega_{\mathbb{P}^s}(1) \oplus \mathcal{O}_{\mathbb{P}^s} \rightarrow \mathcal{O}_{\mathbb{P}^s}$. The restriction of $\xi_{N_{\Lambda/X}^*}$ to Λ_0 is $\mathcal{O}_{\Lambda_0}(1)$, so Λ_0 is mapped to a linear subspace Λ_1 of $\mathbb{G}(1, s + 1)$. Let l be any line in Λ_1 ; the line in Λ_0 mapped to l is a section corresponding to a surjection $\mathcal{E}|_l \rightarrow \mathcal{O}_l(1)$, so $\mathcal{E}|_l \simeq \mathcal{O}_l(1)^{\oplus 2}$. By [15, Thm.] the restriction of \mathcal{E} to Λ_1 is decomposable: $\mathcal{E}|_{\Lambda_1} \simeq \mathcal{O}_{\Lambda_1}(1)^{\oplus 2}$. As a result, $(\mathbb{P}_{\Lambda_1}(\mathcal{E}|_{\Lambda_1}), \mathcal{H}) \simeq (\mathbb{P}^1 \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^s}(1, 1))$, and $\Sigma := \sigma(\mathbb{P}_{\Lambda_1}(\mathcal{E}|_{\Lambda_1}))$ is a subvariety such that $(\Sigma, \mathcal{O}_X(1)) \simeq (\mathbb{P}^1 \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^s}(1, 1))$ and such that Σ contains Λ as a fiber of the first projection.

For the last assertion, observe that Σ is the base locus of the linear subsystem of $|\mathcal{O}_X(1) \otimes \mathcal{I}_\Lambda|$ given by the pull-back of the linear system $|\mathcal{O}_{\mathbb{G}(1, s+1)}(1) \otimes \mathcal{I}_{\Lambda_1}|$. \square

Now we will determine the normal bundle in X of the subvariety Σ constructed in Proposition 7.1.1.

PROPOSITION 7.1.2. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number 1 and dimension $2s + 1$ that contains a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda$. Let $\Sigma \subset X$ be as in Proposition 7.1.1. Then $N_{\Sigma/X} \simeq p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*T_{\mathbb{P}^s}(-1)$, where p_1 and p_2 denote the projections of $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^s$ onto the factors.*

Proof. By Lemma 4.3, X is a Fano manifold of index $s + 2$ covered by lines. Let $\sigma: \tilde{X} \rightarrow X$ be the blow-up of X along Σ , and denote by $E = \mathbb{P}_\Sigma(N_{\Sigma/X}^*)$ the exceptional divisor. By Proposition 4.1, E is a Fano manifold. By adjunction,

$$\det N_{\Sigma/X} = K_\Sigma - (K_X)|_\Sigma = \mathcal{O}_\Sigma(s, 1).$$

Let $p: E \rightarrow \mathbb{P}^s$ be the composition of the bundle projection with p_2 ; the fiber F_x of p over a point $x \in \mathbb{P}^s$ is $\mathbb{P}_{l_x}((N_{\Sigma/X}^*)|_{l_x})$, where l_x is the fiber of p_2 over x .

By adjunction, F_x is a Fano manifold; hence, recalling that $c_1((N_{\Sigma/X}^*)|_{l_x}) = -s$, we have that $(N_{\Sigma/X}^*)|_{l_x} \simeq \mathcal{O}_{l_x}(-1)^{\oplus s}$. Thus $N_{\Sigma/X} \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(-1)$ is trivial on the fibers of p_2 , so $N_{\Sigma/X} \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(-1) = p_2^*(\mathcal{F})$ for \mathcal{F} a vector bundle on \mathbb{P}^s . In particular,

$$(N_{\Sigma/X})|_\Lambda \simeq (N_{\Sigma/X} \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(-1))|_\Lambda \simeq (p_2^*(\mathcal{F}))|_\Lambda = \mathcal{F}.$$

From the exact sequence of normal bundles

$$0 \longrightarrow \mathcal{O}_\Lambda \longrightarrow T_\Lambda(-1) \oplus \mathcal{O}_\Lambda \longrightarrow (N_{\Sigma/X})|_\Lambda \longrightarrow 0,$$

it follows that $(N_{\Sigma/X})|_\Lambda$ is nef. Recalling that $c_1((N_{\Sigma/X})|_\Lambda) = 1$, we have that $(N_{\Sigma/X})|_\Lambda$ is uniform; hence either $(N_{\Sigma/X})|_\Lambda$ is decomposable or $(N_{\Sigma/X})|_\Lambda \simeq T_{\mathbb{P}^s}(-1)$. The first case is not possible because the sequence would split. Therefore, $\mathcal{F} = T_{\mathbb{P}^s}(-1)$ and $N_{\Sigma/X} \simeq p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* T_{\mathbb{P}^s}(-1)$. \square

Now we prove that the existence of a subvariety Σ as just described completely determines the manifold X .

PROPOSITION 7.1.3. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number 1 and dimension $2s + 1$ that contains a subvariety Σ such that $\Sigma = \langle \Sigma \rangle \cap X$ and $(\Sigma, (\mathcal{O}_X(1))|_\Sigma) \simeq (\mathbb{P}^1 \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^s}(1, 1))$ with $N_{\Sigma/X} \simeq p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* T_{\mathbb{P}^s}(-1)$. Let $\sigma: \tilde{X} \rightarrow X$ be the blow-up of X along Σ . Then \tilde{X} is a divisor in the linear system $|2\xi - \varphi^* \mathcal{O}_{\mathbb{G}(1,s)}(1)|$ in $\mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{O}_{\mathbb{G}(1,s)}(1) \oplus \mathcal{Q}^{\oplus 2})$, where ξ denotes the tautological line bundle and φ the bundle projection.*

Proof. By Lemma 4.3, X is a Fano manifold of index $s + 2$ covered by lines. Denote by $E = \mathbb{P}_\Sigma(N_{\Sigma/X}^*)$ the exceptional divisor; by Proposition 4.1, \tilde{X} and E are Fano manifolds. Moreover, the ray associated with the extremal contraction $\varphi: \tilde{X} \rightarrow Y$ that is different from σ has length 3, and its restriction to E is the map associated on E to the linear system $|m\xi_{N_{\Sigma/X}^*(1)}| = |m\xi_{p_2^* \Omega_{\mathbb{P}^s}(2)}|$; up to a Veronese immersion of the target, this is the map described in Example 2.3.2. Denote by \mathcal{H} the pull-back $\sigma^* \mathcal{O}_X(1)$; we can take $K_{\tilde{X}} + 3\mathcal{H}$ as a supporting divisor for φ .

Since $E \cdot \ell = 1$ we have that E meets every nontrivial fiber of φ . Since $\varphi|_E$ is equidimensional with two-dimensional fibers, it follows that φ cannot have fibers of dimension > 3 ; otherwise, their intersection with E would be a fiber of $\varphi|_E$ of dimension ≥ 3 . Therefore, every nontrivial fiber of φ has dimension ≤ 3 .

We claim that φ is of fiber type. Assume by contradiction that φ is birational; then, by Proposition 2.2.4, it is equidimensional. We can apply [2, Thm. 4.1] to get that Y is smooth and that φ is the blow-up of Y along a smooth center. Since $E \cdot \ell = 1$, the intersection of E with a fiber of φ is a \mathbb{P}^2 ; this contradicts the fact that fibers of $\varphi|_E$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

It follows that φ is of fiber type. By Claim 4.2, we can assume that φ is defined by the linear system $|\mathcal{H} - E|$. Since E meets every fiber of φ , we have $Y = \mathbb{G}(1, s)$. The contraction φ is supported by $K_{\tilde{X}} + 3\mathcal{H}$; it is elementary and equidimensional with three-dimensional fibers, so it is a quadric bundle.

Let $\mathcal{E} := \varphi_* \mathcal{H}$; then \tilde{X} embeds in $P := \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{E})$ as a divisor of relative degree 2. Let $\mathcal{E}' := \varphi_*(\mathcal{H}|_E)$ and observe that, as shown in Example 2.3.2, $\mathcal{E}' \simeq \mathcal{Q}^{\oplus 2}$. The vector bundle \mathcal{E} has \mathcal{E}' as a quotient. Indeed, if $x \in \mathbb{G}(1, s)$ is a point and we denote by F and f (respectively) the fibers of φ and $\varphi|_E$ over x , then $\mathcal{E}'_x = H^0(\mathcal{H}|_f)$ is a quotient of $\mathcal{E}_x = H^0(\mathcal{H}|_F)$.

It follows that there exists an exact sequence on $\mathbb{G}(1, s)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{G}(1,s)}(a) \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \oplus \mathcal{Q} \longrightarrow 0.$$

Call P' the projectivization of \mathcal{E}' ; since $\tilde{X}|_{P'} = E$ we have, by Example 2.3.2, that

$$\tilde{X} = 2\mathcal{H} - \varphi^* \mathcal{O}_{\mathbb{G}(1,s)}(1).$$

Combining the canonical bundle formula for P ,

$$K_P + 5\mathcal{H} = \varphi^*(K_{\mathbb{G}(1,s)} + \det \mathcal{E}),$$

with the blow-up formula giving the canonical bundle of \tilde{X} ,

$$K_{\tilde{X}} = -(s + 2)\mathcal{H} + (s - 1)E,$$

and the adjunction formula,

$$K_{\tilde{X}} = (K_P + \tilde{X})|_{\tilde{X}},$$

we obtain

$$\begin{aligned} -(s + 2)\mathcal{H} + (s - 1)E &= -5\mathcal{H} + \varphi^*(K_{\mathbb{G}(1,s)} + \det \mathcal{E}) + 2\mathcal{H} - \varphi^* \mathcal{O}_{\mathbb{G}(1,s)}(1) \\ &= -3\mathcal{H} + \varphi^* \mathcal{O}_{\mathbb{G}(1,s)}(-s - 2 + \deg \det \mathcal{E}) \\ &= -3\mathcal{H} + (-s - 2 + \deg \det \mathcal{E})(\mathcal{H} - E) \\ &= (-s - 5 + \deg \det \mathcal{E})\mathcal{H} + (s + 2 + \deg \det \mathcal{E})E. \end{aligned}$$

It follows that $\deg \det \mathcal{E} = 3$ and therefore $a = 1$. Since $h^1(\mathcal{Q}^*(1)^{\oplus 2}) = h^1(\mathcal{Q}^{\oplus 2}) = 0$, the above sequence splits and $\mathcal{E} = \mathcal{O}_{\mathbb{G}(1,s)}(1) \oplus \mathcal{Q}^{\oplus 2}$. □

COROLLARY 7.1.4. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number 1 and dimension $2s + 1$ that contains a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1) \oplus \mathcal{O}_{\mathbb{P}^s}$. Then X is a hyperplane section of the Grassmannian of lines $\mathbb{G}(1, s + 2)$.*

Proof. By Propositions 7.1.1–7.1.3, there is only one manifold that contains a linear subspace as in the statement. A smooth hyperplane section $\mathbb{G}(1, s + 2) \cap H$ of the Grassmannian of lines $\mathbb{G}(1, s + 2)$ contains such a linear space—just take the intersection of H with a linear space corresponding to lines passing through a fixed point—and so the statement follows. □

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