

Birational Invariants Defined by Lawson Homology

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1. Introduction

In this paper, all varieties are defined over \mathbb{C} . Let X be an n -dimensional projective variety. The *Lawson homology* $L_p H_k(X)$ of p -cycles is defined by

$$L_p H_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad \text{for } k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology (see [F1; L1; Li1] for the quasi-projective case). For general background, the reader is referred to Lawson’s survey paper [L2].

In [FM], Friedlander and Mazur showed that there are natural transformations, called *cycle class maps*,

$$\Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X).$$

DEFINITION 1.

$$\begin{aligned} L_p H_k(X)_{\text{hom}} &:= \ker\{\Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X)\}; \\ T_p H_k(X) &:= \text{Image}\{\Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X)\}; \\ T_p H_k(X, \mathbb{Q}) &:= T_p H_k(X) \otimes \mathbb{Q}. \end{aligned}$$

The *Griffiths group* of codimension q -cycles is defined to

$$\text{Griff}^q(X) := \mathcal{Z}^q(X)_{\text{hom}} / \mathcal{Z}^q(X)_{\text{alg}}$$

It was proved by Friedlander [F1] that, for any smooth projective variety X , $L_p H_{2p}(X) \cong \mathcal{Z}_p(X) / \mathcal{Z}_p(X)_{\text{alg}}$. Therefore

$$L_p H_{2p}(X)_{\text{hom}} \cong \text{Griff}_p(X),$$

where $\text{Griff}_p(X) := \text{Griff}^{n-p}(X)$.

It was shown in [FM, Sec. 7] that the subspaces $T_p H_k(X, \mathbb{Q})$ form a decreasing filtration,

$$\cdots \subseteq T_p H_k(X, \mathbb{Q}) \subseteq T_{p-1} H_k(X, \mathbb{Q}) \subseteq \cdots \subseteq T_0 H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Q}),$$

and that $T_p H_k(X, \mathbb{Q})$ vanishes if $2p > k$.

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DEFINITION 2 [FM]. Denote by $G_p H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q})$ the \mathbb{Q} -vector subspace of $H_k(X, \mathbb{Q})$ generated by the images of mappings $H_k(Y, \mathbb{Q}) \rightarrow H_k(X, \mathbb{Q})$ induced from all morphisms $Y \rightarrow X$ of varieties of dimension $\leq k - p$.

The subspaces $G_p H_k(X, \mathbb{Q})$ also form a decreasing filtration (called *geometric filtration*):

$$\cdots \subseteq G_p H_k(X, \mathbb{Q}) \subseteq G_{p-1} H_k(X, \mathbb{Q}) \subseteq \cdots \subseteq G_0 H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q}).$$

If X is smooth, then the weak Lefschetz theorem implies that

$$G_0 H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Q}).$$

Since $H_k(Y, \mathbb{Q})$ vanishes for k greater than twice the dimension of Y , it follows that $G_p H_k(X, \mathbb{Q})$ vanishes if $2p > k$.

The first main result in this paper is as follows.

THEOREM 1.1. *If X is a smooth, n -dimensional projective variety, then $L_1 H_k(X)_{\text{hom}}$ and $L_{n-2} H_k(X)_{\text{hom}}$ are smooth, birational invariants for X . More precisely, if $\varphi: X \rightarrow X'$ is a birational map between smooth projective manifolds X and X' , then φ induces isomorphisms $L_1 H_k(X)_{\text{hom}} \cong L_1 H_k(X')_{\text{hom}}$ for $k \geq 2$ and $L_{n-2} H_k(X)_{\text{hom}} \cong L_{n-2} H_k(X')_{\text{hom}}$ for $k \geq 2(n-2)$. In particular, $L_1 H_k(X)_{\text{hom}} = 0$ and $L_{n-2} H_k(X)_{\text{hom}} = 0$ for any smooth rational variety.*

COROLLARY 1.2. *Let X be a smooth rational projective variety with $\dim(X) \leq 4$; then $\Phi_{p,k}: L_p H_k(X) \rightarrow H_k(X)$ is injective for all $k \geq 2p \geq 0$.*

REMARK 1.3. Corollary 1.2 has been proved before in dimension ≤ 2 by Friedlander [F1]. In dimensions 3 and 4, Voineagu [V] has independently proved this result by a different method.

REMARK 1.4. In general, for $2 \leq p \leq n - 3$, $L_p H_k(X)_{\text{hom}}$ is *not* a birational invariant for the smooth projective variety X . This follows from the blowup formula in Lawson homology (see Corollary 1.2 and Remark 1.3).

REMARK 1.5. If $p = 0, n - 1, n$, then $L_p H_k(X)_{\text{hom}} = 0$ for all $k \geq 2p$. In these cases, the statement in the theorem is trivial. The case for $p = 0$ follows from the Dold-Thom theorem [DT]. The case for $p = n - 1$ is due to Friedlander [F1], and the case for $p = n$ is from the definition. In particular, these invariants are trivial for smooth projective varieties with dimension ≤ 2 .

Our second main result is the following theorem.

THEOREM 1.6 (Lawson homology for a blowup). *Let X be smooth projective manifold and let $Y \subset X$ be a smooth subvariety of codimension r . Let $\sigma: \tilde{X}_Y \rightarrow X$ be the blowup of X along Y , let $\pi: D = \sigma^{-1}(Y) \rightarrow Y$ be the natural map, and let $i: D = \sigma^{-1}(Y) \rightarrow \tilde{X}_Y$ be the exceptional divisor of the blowup. Then, for each p and k with $k \geq 2p \geq 0$, we have the following isomorphism:*

$$I_{p,k}: \left\{ \bigoplus_{1 \leq j \leq r-1} L_{p-j} H_{k-2j}(Y) \right\} \oplus L_p H_k(X) \cong L_p H_k(\tilde{X}_Y).$$

As applications, we have two corollaries.

COROLLARY 1.7. *For each $n \geq 5$, there exists a rational manifold X of $\dim(X) = n$ such that*

$$\dim_{\mathbb{Q}}\{\text{Griff}_p(X) \otimes \mathbb{Q}\} = \infty, \quad 2 \leq p \leq n - 3.$$

COROLLARY 1.8. *For any integer $p > 1$ and $k \geq 0$, there exists rational projective manifold X such that $L_p H_{k+2p}(X) \otimes \mathbb{Q}$ is an infinite-dimensional vector space over \mathbb{Q} .*

The following results have been proved by Friedlander and Mazur.

PROPOSITION 1.9 [FM]. *Let X be any projective variety.*

- (i) *For nonnegative integers p and k , $T_p H_k(X, \mathbb{Q}) \subseteq G_p H_k(X, \mathbb{Q})$.*
- (ii) *If $k = 2p$, then $T_p H_{2p}(X, \mathbb{Q}) = G_p H_{2p}(X, \mathbb{Q})$.*

QUESTION 1.10 [FM; L2]. Does one have equality in Proposition 1.9 when X is a smooth projective variety?

Friedlander [F2] proved the following result.

PROPOSITION 1.11 [F2]. *Let X be a smooth projective variety of dimension n . Assume that the Grothendieck Standard Conjecture B [Gro] is valid for a resolution of singularities of each irreducible subvariety of $Y \subset X$ of dimension $k - p$. Then*

$$T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q}).$$

REMARK 1.12 [Lew, Sec. 15.32]. The Grothendieck Standard Conjecture B is known to hold for a smooth projective variety X in the following cases:

- (i) $\dim X \leq 2$;
- (ii) flag manifolds X ;
- (iii) smooth complete intersections X ;
- (iv) abelian varieties [Lie].

REMARK 1.13. The Friedlander–Mazur conjecture remains open for general threefolds. The reason is that, even though Friedlander’s result (Proposition 1.11) and the Grothendieck Standard Conjecture B both hold for 2-dimensional smooth varieties, they do not give information about $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ for $k - p \geq 3$. In particular, we don’t know if $T_1 H_4(X, \mathbb{Q}) = G_1 H_4(X, \mathbb{Q}) (= H_4(X, \mathbb{Q}))$ for X with $\dim X = 3$.

The methods employed in the proof of Theorem 1.1 can be used with the blowup formula to prove the following results.

PROPOSITION 1.14. *Let X be a smooth projective variety of dimension n . If $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ holds for X with $p = 1$ (resp. $p = n - 2$) and k arbitrary, then $T_p H_k(X', \mathbb{Q}) = G_p H_k(X', \mathbb{Q})$ holds also for any smooth projective variety X' that is birationally equivalent to X with $p = 1$ (resp. $p = n - 2$).*

In particular, for a smooth projective variety with $\dim(X) \leq 4$, the assertion that $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ holds for all $k \geq 2p \geq 0$ is a birational invariant statement.

PROPOSITION 1.15. *For any smooth projective variety X ,*

$$T_p H_{2p+1}(X, \mathbb{Q}) = G_p H_{2p+1}(X, \mathbb{Q}).$$

The next two corollaries follow from this proposition.

COROLLARY 1.16. *Let X be a smooth, n -dimensional projective variety with $H^{2,0}(X) = 0$. Then $T_{n-2} H_k(X, \mathbb{Q}) = G_{n-2} H_k(X, \mathbb{Q})$ for $k \geq 2n - 4$. In particular, this equality holds for X a complete intersection of dimension ≥ 2 , for any product of a smooth projective curve with a complete intersection of dimension ≥ 2 , et cetera.*

REMARK 1.17. The condition $H^{2,0}(X) = 0$ in Corollary 1.16 is used only to prove

$$T_{n-2} H_{2n-2}(X, \mathbb{Q}) = G_{n-2} H_{2n-2}(X, \mathbb{Q}).$$

In the following corollary we use the Künneth formula in homology with rational coefficient.

COROLLARY 1.18. *Let X be the product of a smooth projective curve and a smooth, simply connected projective variety Y with $\dim Y = n - 1$. Then $T_{n-2} H_k(X, \mathbb{Q}) = G_{n-2} H_k(X, \mathbb{Q})$ for any $k \geq 2(n - 2) \geq 0$. In particular, the Friedlander–Mazur conjecture holds for the product of a smooth projective curve and a smooth simply connected projective surface.*

CONJECTURE 1.19 (Suslin conjecture for Lawson homology with coefficient A ; [FHW, Sec. 7]). *For any abelian group A and smooth quasi-projective variety X of dimension n , the map $L_p H_k(X, A) \rightarrow H_k^{\text{BM}}(X, A)$ is an isomorphism for $k \geq n + p$ and a monomorphism for $k = n + p - 1$.*

As an application of the method used in the proof of Proposition 1.15, we have the following result.

THEOREM 1.20. *If the Suslin conjecture for Lawson homology with coefficient \mathbb{Z} holds, then the topological filtration is the same as the geometric filtration for a smooth projective variety.*

REMARK 1.21. This result was given (without proof) in Walker’s paper [Wa, Sec. 2].

The main tools used to prove the main result are: the long exact localization sequence given by Lima-Filho in [Li1], the explicit formula for the Lawson homology of codimension-1 cycles on a smooth projective manifold given by Friedlander in [F1], and the Hironaka desingularization theorem [Hi]. Using the blowup formula for Lawson homology and diagram chases, we obtain birational invariant statements for the topological and geometric filtrations.

2. Some Fundamental Materials in Lawson Homology

First recall that, for a morphism $f : U \rightarrow V$ between projective varieties, there exist induced homomorphisms

$$f_* : L_p H_k(U) \rightarrow L_p H_k(V)$$

for all $k \geq 2p \geq 0$; furthermore, if $g : V \rightarrow W$ is another morphism between projective varieties, then

$$(g \circ f)_* = g_* \circ f_*$$

It has also been shown by Peters [P] that, if U and V are smooth and projective, then there are Gysin “wrong way” homomorphisms $f^* : L_p H_k(V) \rightarrow L_{p-c} H_{k-2c}(U)$, where $c = \dim(V) - \dim(U)$. If $g : V \rightarrow W$ is another morphism between smooth projective varieties, then

$$(g \circ f)^* = f^* \circ g^*$$

Recall also that there is a long exact sequence (cf. [FGa; Li1])

$$\dots \rightarrow L_p H_k(U - V) \rightarrow L_p H_k(U) \rightarrow L_p H_k(V) \rightarrow L_p H_{k-1}(U - V) \rightarrow \dots,$$

where U is quasi-projective and $U - V$ is any algebraic closed subset in U .

Let X be a smooth projective variety and let $i_0 : Y \hookrightarrow X$ be a smooth subvariety of codimension $r \geq 2$. Let $\sigma : \tilde{X}_Y \rightarrow X$ be the blowup of X along Y , let $\pi : D = \sigma^{-1}(Y) \rightarrow Y$ be the natural map, and let $i : D = \sigma^{-1}(Y) \hookrightarrow \tilde{X}_Y$ be the exceptional divisor of the blowup. Set $U := X - Y \cong \tilde{X}_Y - D$. Denote by j_0 the inclusion $U \subset X$ and by j the inclusion $U \subset \tilde{X}_Y$. Note that $\pi : D = \sigma^{-1}(Y) \rightarrow Y$ makes D into a projective bundle of rank $r - 1$, given precisely by $D = \mathbb{P}(N_{Y/X})$, and we have (cf. [Vo, p. 271])

$$\mathcal{O}_{\tilde{X}_Y}(D)|_D = \mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1).$$

Denote by h the class of $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1)$ in $\text{Pic}(D)$. We have $h = -D|_D$ and $-h = i^* i_* : L_q H_m(D) \rightarrow L_{q-1} H_{m-2}(D)$ for $0 \leq 2q \leq m$ [FGa, Thm. 2.4; P, Lemma 11]. The last equality can be equivalently regarded as a Lefschetz operator

$$-h = i^* i_* : L_q H_m(D) \rightarrow L_{q-1} H_{m-2}(D), \quad 0 \leq 2q \leq m. \tag{1}$$

The proof of the main result is based on the following lemma.

LEMMA 2.1. *For each $p \geq 0$, we have the following commutative diagram:*

$$\begin{array}{ccccccccc} \dots & \longrightarrow & L_p H_k(D) & \xrightarrow{i_*} & L_p H_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_p H_k(U) & \xrightarrow{\delta_*} & L_p H_{k-1}(D) & \longrightarrow & \dots \\ & & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \downarrow \pi_* & & \\ \dots & \longrightarrow & L_p H_k(Y) & \xrightarrow{(i_0)_*} & L_p H_k(X) & \xrightarrow{j_0^*} & L_p H_k(U) & \xrightarrow{(\delta_0)_*} & L_p H_{k-1}(Y) & \longrightarrow & \dots \end{array}$$

Proof. The lemma follows from the corresponding commutative diagram of fibration sequences of p -cycles. More precisely, to show the first square, we begin from the following commutative diagram:

$$\begin{array}{ccc}
 D & \xrightarrow{i} & \tilde{X}_Y \\
 \downarrow \pi & & \downarrow \sigma \\
 Y & \xrightarrow{i_0} & X.
 \end{array}$$

From this, we obtain the corresponding commutative diagram of p -cycles,

$$\begin{array}{ccc}
 \mathcal{Z}_p(D) & \xrightarrow{i_*} & \mathcal{Z}_p(\tilde{X}_Y) \\
 \downarrow \pi_* & & \downarrow \sigma_* \\
 \mathcal{Z}_p(Y) & \xrightarrow{(i_0)_*} & \mathcal{Z}_p(X).
 \end{array}$$

Since Y is a smooth projective variety, it follows that \tilde{X}_Y and D are also smooth projective varieties; hence we have the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{Z}_p(\tilde{X}_Y) & \longrightarrow & \mathcal{Z}_p(\tilde{X}_Y)/\mathcal{Z}_p(D) \\
 \downarrow \sigma_* & & \downarrow \cong \\
 \mathcal{Z}_p(X) & \longrightarrow & \mathcal{Z}_p(X)/\mathcal{Z}_p(Y).
 \end{array}$$

We thus obtain the following commutative diagram of the fibration sequences of p -cycles:

$$\begin{array}{ccccc}
 \mathcal{Z}_p(D) & \xrightarrow{i_*} & \mathcal{Z}_p(\tilde{X}_Y) & \longrightarrow & \mathcal{Z}_p(\tilde{X}_Y)/\mathcal{Z}_p(D) \\
 \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong \\
 \mathcal{Z}_p(Y) & \xrightarrow{(i_0)_*} & \mathcal{Z}_p(X) & \longrightarrow & \mathcal{Z}_p(X)/\mathcal{Z}_p(Y);
 \end{array}$$

that the rows are fibration sequences is due to Lima-Filho [Li1].

By taking the homotopy groups of these fibration sequences, we get the long exact sequences of commutative diagram given in the lemma. □

PROPOSITION 2.2. *If $p = 0$ then we have the commutative diagram*

$$\begin{array}{cccccccc}
 \cdots & \longrightarrow & H_k(D) & \xrightarrow{i_*} & H_k(\tilde{X}_Y) & \xrightarrow{j^*} & H_k^{\text{BM}}(U) & \xrightarrow{\delta_*} & H_{k-1}(D) & \longrightarrow & \cdots \\
 & & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \downarrow \pi_* & & \\
 \cdots & \longrightarrow & H_k(Y) & \xrightarrow{(i_0)_*} & H_k(X) & \xrightarrow{j_0^*} & H_k^{\text{BM}}(U) & \xrightarrow{(\delta_0)_*} & H_{k-1}(Y) & \longrightarrow & \cdots
 \end{array}$$

Moreover, if $x \in H_k(D)$ maps to zero under π_* and i_* , then $x = 0 \in H_k(D)$.

Proof. The first statement follows directly from Lemma 2.1 (with $p = 0$) and the Dold–Thom theorem. For the second statement, assume $i_*(x) = 0$ and $\pi_*(x) = 0$. Then there exists an element $y \in H_{k+1}^{\text{BM}}(U)$ such that the image of y under the boundary map $(\delta_0)_*: H_{k+1}^{\text{BM}}(U) \rightarrow H_k(Y)$ is 0 by the given condition. Hence there exists an element $z \in H_{k+1}(X)$ such that $(j_0)^*(z) = y$. Now the surjectivity of the map $\sigma_*: H_{k+1}(\tilde{X}_Y) \rightarrow H_{k+1}(X)$ implies that there is an element $\tilde{z} \in H_{k+1}(\tilde{X}_Y)$ such that $j^*(\tilde{z}) = y$. Therefore, $x = 0 \in H_k(D)$. □

COROLLARY 2.3. *If $p = n - 2$ then we have the commutative diagram*

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & L_{n-2}H_k(D) & \xrightarrow{i_*} & L_{n-2}H_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_{n-2}H_k(U) & \xrightarrow{\delta_*} & L_{n-2}H_{k-1}(D) & \longrightarrow & \cdots \\
 & & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \downarrow \pi_* & & \\
 \cdots & \longrightarrow & L_{n-2}H_k(Y) & \xrightarrow{(i_0)_*} & L_{n-2}H_k(X) & \xrightarrow{j_0^*} & L_{n-2}H_k(U) & \xrightarrow{(\delta_0)_*} & L_{n-2}H_{k-1}(Y) & \longrightarrow & \cdots
 \end{array}$$

LEMMA 2.4. *For each p , we have the following commutative diagram:*

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & L_p H_k(D) & \xrightarrow{i_*} & L_p H_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_p H_k(U) & \xrightarrow{\delta_*} & L_p H_{k-1}(D) & \longrightarrow & \cdots \\
 & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & & \\
 \cdots & \longrightarrow & H_k(D) & \xrightarrow{i_*} & H_k(\tilde{X}_Y) & \xrightarrow{j^*} & H_k^{BM}(U) & \xrightarrow{\delta_*} & H_{k-1}(D) & \longrightarrow & \cdots
 \end{array}$$

In particular, this statement holds for $p = 1, n - 2$.

Proof. See [Li1] and also [FM]. □

LEMMA 2.5. *For each p , we have the following commutative diagram:*

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & L_p H_k(Y) & \xrightarrow{(i_0)_*} & L_p H_k(X) & \xrightarrow{j^*} & L_p H_k(U) & \xrightarrow{(\delta_0)_*} & L_p H_{k-1}(Y) & \longrightarrow & \cdots \\
 & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & & \\
 \cdots & \longrightarrow & H_k(Y) & \xrightarrow{(i_0)_*} & H_k(X) & \xrightarrow{j^*} & H_k^{BM}(U) & \xrightarrow{(\delta_0)_*} & H_{k-1}(Y) & \longrightarrow & \cdots
 \end{array}$$

In particular, it is true for $p = 1, n - 2$.

Proof. See [Li1] and also [FM]. □

REMARK 2.6. The smoothness of X and Y is not necessary in Lemma 2.5.

REMARK 2.7. All the preceding commutative diagrams of long exact sequences remain commutative and exact when tensored with \mathbb{Q} . We will use these lemmas and corollaries with rational coefficients.

3. Lawson Homology for Blowups

As an application of Lemma 2.1, we give an explicit formula for a blowup in Lawson homology. Since it may have some independent interest, we devote a separate section to it. First, we want to revise the projective bundle theorem given by Friedlander and Gabber [FGa, Prop. 2.5]. It is convenient to extend the definition of Lawson homology by setting

$$L_p H_k(X) = L_0 H_k(X) \quad \text{if } p < 0.$$

Now we have the following “revised” projective bundle theorem.

PROPOSITION 3.1. *Let E be an algebraic vector bundle of rank r over a smooth projective variety Y . Then, for each $p \geq 0$, we have*

$$L_p H_k(\mathbb{P}(E)) \cong \bigoplus_{j=0}^{r-1} L_{p-j} H_{k-2j}(Y),$$

where $\mathbb{P}(E)$ is the projectivization of the vector bundle E .

REMARK 3.2. The difference between this and the projective bundle theorem of [FGa] is that here we place no restriction on p .

Proof of Proposition 3.1. For $p \geq r - 1$, this is exactly the projective bundle theorem given in [FGa]. If $p < r - 1$, then we can use the same method of [FGa] (i.e., the localization sequence and the naturality of Φ) to reduce to the case in which E is trivial. From

$$\mathcal{Z}_0(\mathbb{P}^{r-1} \times Y) \rightarrow \mathcal{Z}_0(\mathbb{P}^r \times Y) \rightarrow \mathcal{Z}_0(\mathbb{C}^r \times Y)$$

we obtain the long exact localization sequence given at the beginning of Section 2:

$$\begin{aligned} \cdots \rightarrow L_0 H_k(\mathbb{P}^{r-1} \times Y) \rightarrow L_0 H_k(\mathbb{P}^r \times Y) \\ \rightarrow L_0 H_k(\mathbb{C}^r \times Y) \rightarrow L_0 H_{k-1}(\mathbb{P}^{r-1} \times Y) \rightarrow \cdots \end{aligned}$$

From this, together with the Künneth formula for $\mathbb{P}^r \times Y$, we have the following isomorphism:

$$H_{k-2r}(Y) \cong L_0 H_k(\mathbb{C}^r \times Y) \cong H_k^{\text{BM}}(\mathbb{C}^r \times Y). \tag{*}$$

Note that

$$H_{k-2r}(Y) \cong L_{p-r} H_{k-2r}(Y) \quad \text{if } p \leq r. \tag{**}$$

All the remaining arguments are the same as those in [FGa, Prop. 2.5], as we review next.

We want to use induction on r . For $r - 1 = p$, the conclusion holds. From the commutative diagram of abelian groups of cycles, we have

$$\begin{array}{ccc} \left\{ \bigoplus_{j=0}^p \mathcal{Z}_{p-j}(X) \right\} \oplus \left\{ \bigoplus_{j=p+1}^{r-1} \mathcal{Z}_0(X \times \mathbb{C}^{j-p}) \right\} & \longrightarrow & \left\{ \bigoplus_{j=0}^p \mathcal{Z}_{p-j}(X) \right\} \oplus \left\{ \bigoplus_{j=p+1}^r \mathcal{Z}_0(X \times \mathbb{C}^{j-p}) \right\} \\ \downarrow & & \downarrow \\ \mathcal{Z}_p(X \times \mathbb{P}^{r-1}) & \longrightarrow & \mathcal{Z}_p(X \times \mathbb{P}^r). \end{array}$$

We obtain the commutative diagram of fibration sequences:

$$\begin{array}{ccc} \left\{ \bigoplus_{j=0}^p \mathcal{Z}_{p-j}(X) \right\} \oplus \left\{ \bigoplus_{j=p+1}^{r-1} \mathcal{Z}_{p-j}(X) \right\} & \longrightarrow & \left\{ \bigoplus_{j=0}^p \mathcal{Z}_{p-j}(X) \right\} \oplus \left\{ \bigoplus_{j=p+1}^r \mathcal{Z}_{p-j}(X) \right\} \\ \downarrow & & \downarrow \\ \mathcal{Z}_p(X \times \mathbb{P}^{r-1}) & \longrightarrow & \mathcal{Z}_p(X \times \mathbb{P}^r) \\ & & \longrightarrow \mathcal{Z}_0(X \times \mathbb{C}^{r-p}) \\ & & \downarrow \\ & & \mathcal{Z}_p(X \times \mathbb{C}^r), \end{array}$$

where $\mathcal{Z}_{p-j}(X) := \mathcal{Z}_0(X \times \mathbb{C}^{j-p})$ for $p - j < 0$.

The first vertical arrow is a homotopy equivalence by induction; the last one is a homotopy equivalence by complex suspension theorem [L1]. Hence, by the five lemma, we obtain the homotopy equivalence of the middle vertical arrow.

The proof is completed by combining this with statements (*) and (**). □

REMARK 3.3. The isomorphism

$$\psi : \bigoplus_{j=0}^{r-1} L_{p-j}H_{k-2j}(Y) \xrightarrow{\cong} L_p H_k(\mathbb{P}(E))$$

in Proposition 3.1 is given explicitly by

$$\psi(u_0, u_1, \dots, u_{r-1}) = \sum_{j=0}^{r-1} h^j \pi^* u_j,$$

where h is the Lefschetz hyperplane operator

$$h : L_q H_m(\mathbb{P}(E)) \rightarrow L_{q-1} H_{m-2}(\mathbb{P}(E))$$

defined in (1). For $p \geq r - 1$, this explicit formula has been proved in [FGa, Prop. 2.5]. In the remaining cases, h is the Lefschetz hyperplane operator $h : H_m(\mathbb{P}(E)) \rightarrow H_{m-2}(\mathbb{P}(E))$ defined in (1).

In the notation of Section 2, we have the following result.

THEOREM 3.4 (Lawson homology for a blowup). *Let X be smooth projective manifold and $Y \subset X$ a smooth subvariety of codimension r . Let $\sigma : \tilde{X}_Y \rightarrow X$ be the blowup of X along Y , let $\pi : D = \sigma^{-1}(Y) \rightarrow Y$ be the natural map, and let $i : D = \sigma^{-1}(Y) \rightarrow \tilde{X}_Y$ be the exceptional divisor of the blowup. Then, for each p and k with $k \geq 2p \geq 0$, we have the isomorphism*

$$I_{p,k} : \left\{ \bigoplus_{1 \leq j \leq r-1} L_{p-j}H_{k-2j}(Y) \right\} \oplus L_p H_k(X) \xrightarrow{\cong} L_p H_k(\tilde{X}_Y)$$

given by

$$I_{p,k}(u_1, \dots, u_{r-1}, u) = \sum_{j=1}^{r-1} i_* h^j \pi^* u_j + \sigma^* u.$$

Proof. We use certain ideas of the proof of Chow groups for blowups. Let $U := \tilde{X}_Y - D = X - Y$. By Lemma 2.1 and our definitions of the maps i, π , and σ , we have the following commutative diagram of the long exact localization sequences:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & L_p H_k(D) & \xrightarrow{i_*} & L_p H_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_p H_k(U) & \xrightarrow{\delta_*} & L_p H_{k-1}(D) & \longrightarrow & \cdots \\ & & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \downarrow \pi_* & & \\ \cdots & \longrightarrow & L_p H_k(Y) & \xrightarrow{(i_0)_*} & L_p H_k(X) & \xrightarrow{j_0^*} & L_p H_k(U) & \xrightarrow{(\delta_0)_*} & L_p H_{k-1}(Y) & \longrightarrow & \cdots \end{array} \tag{2}$$

From this and the surjectivity of j^* , we have

$$L_p H_{2p}(\tilde{X}_Y) = \sigma^* L_p H_{2p}(X) + i_* L_p H_{2p}(D).$$

By the “revised” projective bundle theorem (Proposition 3.1), for any $p \geq 0$ there is an isomorphism

$$L_p H_k(D) \cong \bigoplus_{j=0}^{r-1} h^j \pi^* L_{p-j} H_{k-2j}(Y), \quad 0 \leq 2p \leq k.$$

Hence we see that

$$L_p H_{2p}(\tilde{X}_Y) = \sigma^* L_p H_{2p}(X) + \sum_{j=0}^{r-1} i_* h^j \pi^* L_{p-j} H_{2p-2j}(Y). \quad (3)$$

But clearly, by Lemma 2.1 and the projective bundle theorem, if $u \in L_p H_k(Y)$ then

$$\sigma_*(i_* h^{r-1} \pi^*(u)) = (i_0)_*(u).$$

Since σ is a birational morphism, it has degree 1. As a direct corollary of the projection formula (cf. [P, Lemma 11(c)]), we have $\sigma_*(\sigma^*a) = a$ for any $a \in L_p H_k(X)$. Now

$$\sigma_*(\sigma^*((i_0)_*u)) = (i_0)_*u, \quad u \in L_p H_k(Y).$$

Thus we obtain the relations

$$i_* h^{r-1} \pi^* u - \sigma^*((i_0)_*u) =: v \in \ker \sigma_*, \quad u \in L_p H_k(Y).$$

Since $j^* = (j_0)^* \sigma_*$ in (2), we get $j^*(v) = 0$. From the exactness of the upper row in (2), we get

$$v \in \sum_{j=1}^{r-1} i_* h^j L_{p-j} H_{k-2j}(Y). \quad (4)$$

The equality (3) and the relation (4) together imply immediately that the map $I_{p,2p}$ is surjective for the case $k = 2p$.

To prove the injectivity for the case that $k = 2p$, we consider

$$(u_1, u_2, \dots, u_{r-1}, u) \in \ker I_{p,2p}.$$

Applying σ_* , we find that $u = 0$. Note that $i^* i_* = -h$. Applying i^* to the equality

$$\sum_{j=1}^{r-1} i_* h^j \pi^* u_j = 0,$$

we get

$$\sum_{j=1}^{r-1} h^{j+1} \pi^* u_j = 0 \in L_{p-1} H_{k-2}(D).$$

The isomorphism in Proposition 3.1 implies that $u_j = 0$ for $1 \leq j \leq r - 1$. This completes the proof for the case $k = 2p$.

From this and (2), we have

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & L_p H_{2p+1}(D) & \xrightarrow{i_*} & L_p H_{2p+1}(\tilde{X}_Y) & \xrightarrow{j^*} & L_p H_{2p+1}(U) \xrightarrow{\delta_*} 0 \\
 & & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong \\
 \cdots & \longrightarrow & L_p H_{2p+1}(Y) & \xrightarrow{(i_0)_*} & L_p H_{2p+1}(X) & \xrightarrow{j_0^*} & L_p H_{2p+1}(U) \xrightarrow{(\delta_0)_*} 0.
 \end{array} \tag{5}$$

Now the situation for $k = 2p + 1$ is the same as that in the case $k = 2p$. From (5) and the “revised” projective bundle theorem, we have

$$L_p H_{2p+1}(\tilde{X}_Y) = \sigma^* L_p H_{2p+1}(X) + \sum_{j=0}^{r-1} i_* h^j \pi^* L_{p-j} H_{2p+1-2j}(Y). \tag{6}$$

From (4) and (6) we obtain the surjectivity of $I_{p,2p+1}$ for the case that $k = 2p + 1$.

To prove the injectivity, consider $(u_1, u_2, \dots, u_{r-1}, u) \in \ker I_{p,2p+1}$. Applying σ_* , we find that $u = 0$. Note that $i^* i_* = -h$. By applying i^* to the equality

$$\sum_{j=1}^{r-1} i_* h^j \pi^* u_j = 0,$$

we get

$$\sum_{j=1}^{r-1} h^{j+1} \pi^* u_j = 0 \in L_{p-1} H_{k-2}(D).$$

The isomorphism in Proposition 3.1 again implies that $u_j = 0$ for $1 \leq j \leq r - 1$. This completes the proof for the case $k = 2p + 1$.

Now, for $k \geq 2p + 2$, we reach the same situation as in the case that $k = 2p$ or $k = 2p + 1$. More precisely, we give the complete argument by using mathematical induction.

Suppose that we have

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & L_p H_{2p+m}(D) & \xrightarrow{i_*} & L_p H_{2p+m}(\tilde{X}_Y) & \xrightarrow{j^*} & L_p H_{2p+m}(U) \xrightarrow{\delta_*} 0 \\
 & & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong \\
 \cdots & \longrightarrow & L_p H_{2p+m}(Y) & \xrightarrow{(i_0)_*} & L_p H_{2p+m}(X) & \xrightarrow{j_0^*} & L_p H_{2p+m}(U) \xrightarrow{(\delta_0)_*} 0
 \end{array} \tag{7}$$

for some integer $m \geq 0$. We want to prove that $I_{p,2p+m}$ is an isomorphism and

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & L_p H_{2p+m+1}(D) & \xrightarrow{i_*} & L_p H_{2p+m+1}(\tilde{X}_Y) & \xrightarrow{j^*} & L_p H_{2p+m+1}(U) \xrightarrow{\delta_*} 0 \\
 & & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong \\
 \cdots & \longrightarrow & L_p H_{2p+m+1}(Y) & \xrightarrow{(i_0)_*} & L_p H_{2p+m+1}(X) & \xrightarrow{j_0^*} & L_p H_{2p+m+1}(U) \xrightarrow{(\delta_0)_*} 0.
 \end{array} \tag{8}$$

Once this step is done, the proof of Theorem 3.4 will be complete.

From the assumption (7), we have

$$L_p H_{2p+m}(\tilde{X}_Y) = \sigma^* L_p H_{2p+m}(X) + \sum_{j=0}^{r-1} i_* h^j \pi^* L_{p-j} H_{2p+m-2j}(Y). \quad (9)$$

From (4) for $k = 2p + m$ and (9), we obtain the surjectivity of $I_{p,2p+m}$ for the case that $k = 2p + m$.

To prove the injectivity, consider $(u_1, u_2, \dots, u_{r-1}, u) \in \ker I_{p,2p+m}$. Applying σ_* , we find that $u = 0$. Note that $i^* i_* = -h$. By applying i^* to the equality

$$\sum_{j=1}^{r-1} i_* h^j \pi^* u_j = 0,$$

we get

$$\sum_{j=1}^{r-1} h^{j+1} \pi^* u_j = 0 \in L_{p-1} H_{k-2}(D).$$

The isomorphism in Proposition 3.1 once again implies that $u_j = 0$ for $1 \leq j \leq r - 1$. This completes the proof for the case $k = 2p + m$. Now (7) automatically reduces to (8), and this completes the proof of the theorem. \square

As an application, this result gives many examples of smooth projective manifolds (even rational ones) for which the Griffiths group of p -cycles is infinitely generated (even modulo torsion) for $p \geq 2$. Recall that the Griffiths group $\text{Griff}_p(X)$ is defined as the p -cycles homologically equivalent to zero modulo the subgroup of p -cycles algebraically equivalent to zero.

EXAMPLE. Recall from [F1] that $\text{Griff}_2(\tilde{X}_Y) \cong L_2 H_4(\tilde{X}_Y)_{\text{hom}}$. For $X = \mathbb{P}^5, Y \subset \mathbb{P}^4$ the general hypersurface of degree 5, we obtain an infinite-dimensional \mathbb{Q} -vector space $\text{Griff}_2(\tilde{X}_Y) \otimes \mathbb{Q}$ from the fact $\dim_{\mathbb{Q}}(\text{Griff}_1(Y) \otimes \mathbb{Q}) = \infty$ (cf. [C]). This space gives the example mentioned in Remark 1.1.

From the blowup formula for Lawson homology and Clemens’s result [C], we have our next corollary. This result is probably known to experts in this field, but I cannot find an explicit statement in the literature.

COROLLARY 3.5. *For each $n \geq 5$, there exists a rational manifold X with $\dim(X) = n$ such that*

$$\dim_{\mathbb{Q}}\{\text{Griff}_p(X) \otimes \mathbb{Q}\} = \infty, \quad 2 \leq p \leq n - 3.$$

Proof. Note that $\text{Griff}_p(X) \cong L_p H_{2p}(X)_{\text{hom}}$ for any smooth projective variety X . Now the remaining argument follows directly from Theorem 3.4 and the result of Clemens [C]. \square

More generally, from the blowup formula for Lawson homology and a result given in [Hu], we have the following.

COROLLARY 3.6. *For any integers $p > 1$ and $k \geq 0$, there exists a rational projective manifold X such that $L_p H_{k+2p}(X) \otimes \mathbb{Q}$ is an infinite-dimensional vector space over \mathbb{Q} .*

Proof. This follows from the blowup formula for Lawson homology and [Hu, Thm. 1.4]. For example, if $p = 2$ and $k = 1$, we can find a rational projective manifold X with $\dim(X) = 6$ such that $L_2 H_5(X) \otimes \mathbb{Q}$ is an infinite-dimensional \mathbb{Q} -vector space. □

4. Proof of the First Main Theorem

Now we begin the proof of our main results. We first address a special case that involves only one blowup along a smooth submanifold of codimension ≥ 2 . Then we use Proposition 4.4 to obtain general cases.

The following result of Friedlander will be used several times in the proof of Theorem 1.1.

THEOREM 4.1 [F1]. *Let X be any smooth projective variety of dimension n . Then we have the following isomorphisms:*

$$\begin{aligned} L_{n-1} H_{2n}(X) &\cong \mathbb{Z}, \\ L_{n-1} H_{2n-1}(X) &\cong H_{2n-1}(X, \mathbb{Z}), \\ L_{n-1} H_{2n-2}(X) &\cong H_{n-1, n-1}(X, \mathbb{Z}) = \text{NS}(X), \\ L_{n-1} H_k(X) &= 0 \quad \text{for } k > 2n. \end{aligned}$$

Here $\text{NS}(X)$ is the Néron–Severi group of X .

REMARK 4.2. In what follows we adopt the notational convention $H_k(X) = H_k(X, \mathbb{Z})$.

Now we give a proof of our main Theorem 1.1. It is reproduced here for the reader’s convenience.

THEOREM 4.3 (Theorem 1.1). *If X is a smooth, n -dimensional projective variety, then $L_1 H_k(X)_{\text{hom}}$ and $L_{n-2} H_k(X)_{\text{hom}}$ are smooth, birational invariants for X . More precisely, if $\varphi: X \rightarrow X'$ is a birational map between smooth projective manifolds X and X' , then φ induces isomorphisms $L_1 H_k(X)_{\text{hom}} \cong L_1 H_k(X')_{\text{hom}}$ for $k \geq 2$ and $L_{n-2} H_k(X)_{\text{hom}} \cong L_{n-2} H_k(X')_{\text{hom}}$ for $k \geq 2(n-2)$. In particular, $L_1 H_k(X)_{\text{hom}} = 0$ and $L_{n-2} H_k(X)_{\text{hom}} = 0$ for any smooth rational variety.*

Proof. There are two parts of the proof of the main theorem: $p = 1$ and $p = n - 2$.

Part I: $p = 1$.

Case A: $\sigma_: L_1 H_k(\tilde{X}_Y)_{\text{hom}} \rightarrow L_1 H_k(X)_{\text{hom}}$ is injective.* We will use the commutative diagrams in Lemmas 2.1 and 2.5.

Let $a \in L_1 H_k(\tilde{X}_Y)_{\text{hom}}$ be such that $\sigma_*(a) = 0$. By Lemma 2.1, we have $j^*(a) = 0 \in L_1 H_k(U)$ and hence there exists an element $b \in L_1 H_k(D)$ such

that $i_*(b) = a$. Set $\tilde{b} = \pi_*(b)$. By the commutative diagram in Lemma 2.1 again, we have $(i_0)_*(\tilde{b}) = 0 \in L_1H_k(X)$. By the exactness of the rows in the commutative diagram, there exists an element $\tilde{c} \in L_1H_{k+1}(U)$ such that the image of \tilde{c} under the boundary map $(\delta_0)_*: L_1H_{k+1}(U) \rightarrow L_1H_k(Y)$ is \tilde{b} . Note that δ_* is the other boundary map $\delta_*: L_1H_{k+1}(U) \rightarrow L_1H_k(D)$. Therefore, $\pi_*(b - \delta_*(\tilde{c})) = 0 \in L_1H_k(Y)$ and $j_*(b - \delta_*(\tilde{c})) = a$. Now by the “revised” projective bundle theorem and the Dold–Thom theorem [DT], we have

$$\begin{aligned} L_1H_k(D) &\cong L_1H_k(Y) \oplus L_0H_{k-2}(Y) \oplus H_{k-4}(Y) \oplus \cdots \\ &\cong L_1H_k(Y) \oplus H_{k-2}(Y) \oplus H_{k-4}(Y) \oplus \cdots . \end{aligned}$$

We know that $b - \delta_*(\tilde{c}) \in H_{k-2}(Y) \oplus H_{k-4}(Y) \oplus \cdots$. By the explicit formula of the cohomology (and homology) for a blowup [GHa], it follows that each map $H_{k-2*}(Y) \rightarrow H_k(\tilde{X}_Y)$ is injective. Hence a must be zero in $L_1H_k(\tilde{X}_Y)$. This is the injectivity of σ_* .

Case B: $\sigma_*: L_1H_k(\tilde{X}_Y)_{\text{hom}} \rightarrow L_1H_k(X)_{\text{hom}}$ is surjective. Let $a \in L_1H_k(X)_{\text{hom}}$. From the surjectivity of the map $\sigma_*: L_1H_k(\tilde{X}_Y) \rightarrow L_1H_k(X)$, we know there exists an element $\tilde{a} \in L_1H_k(\tilde{X}_Y)$ such that $\sigma_*(\tilde{a}) = a$. Set $\tilde{b} = \Phi_{1,k}(\tilde{a})$. By the commutative diagram in Lemma 2.1 we have $j^*(\tilde{b}) = 0 \in H_k^{\text{BM}}(U)$. From the exactness of the rows of the diagram in Lemma 2.1, we have an element $\tilde{c} \in H_k(D)$ such that $i_*(\tilde{c}) = \tilde{b}$. Set $c = \pi_*(\tilde{c})$. Then $(i_0)_*(c) = 0$ by the assumption on a and the commutativity of the diagram in Lemma 2.1. Using the exactness of rows in Lemma 2.1 again, we can find an element $d \in H_{k+1}^{\text{BM}}(U)$ such that $(\delta_0)_*(d) = c$. Hence $i_*(\tilde{c} - \delta_*(d)) = \tilde{b} \in H_k(\tilde{X}_Y)$ and $\pi_*(\tilde{c} - \delta_*(d)) = 0$. Now we need to use the formula $L_1H_k(D) \cong L_1H_k(Y) \oplus H_{k-2}(Y) \oplus H_{k-4}(Y) \oplus \cdots$ again. From this we can find an element $e \in L_1H_k(D)$ such that $\Phi_{1,k}(e) = \tilde{c} - \delta(d)$. Obviously, $\Phi_{1,k}(\tilde{a} - i_*(e)) = 0$ and $\sigma_*(\tilde{a} - i_*(e)) = a$ as we want. This completes the proof of Part I.

Part II: $p = n - 2$.

Case I: σ_* is injective. The injectivity of

$$j_0^*: L_{n-2}H_k(X)_{\text{hom}} \rightarrow L_{n-2}H_k(U)_{\text{hom}}$$

is trivial because $\dim(Y) \leq n - 2$, where $j_0: U \rightarrow X$ is the inclusion. In fact, if $\dim(Y) < n - 2$, then $j_0^*: L_{n-2}H_k(X) \rightarrow L_{n-2}H_k(U)$ is an isomorphism and so is $j_0^*: L_{n-2}H_k(X)_{\text{hom}} \rightarrow L_{n-2}H_k(U)_{\text{hom}}$. If $\dim(Y) = n - 2$ then, for $k \geq 2(n - 2) + 1$, the injectivity of j_0^* follows from the commutative diagram in Lemma 2.5 and the vanishing of $L_{n-2}H_k(Y)$ and $H_k(Y)$; for $k = 2(n - 2)$, the injectivity of j_0^* is from the commutative diagram in Lemma 2.5 and the nontriviality of $(i_0)_*: H_{2(n-2)}(Y) \rightarrow H_{2(n-2)}(X)$, since Y is a Kähler submanifold of X with complex dimension $n - 2$.

Now we need to prove that $j^*: L_{n-2}H_k(\tilde{X}_Y)_{\text{hom}} \rightarrow L_{n-2}H_k(U)_{\text{hom}}$ is injective, where $j: U \rightarrow \tilde{X}_Y$ is the inclusion. Let $a \in L_{n-2}H_k(\tilde{X}_Y)_{\text{hom}}$ be such that $j^*(a) = 0 \in L_{n-2}H_k(U)_{\text{hom}}$; then there exists an element $b \in L_{n-2}H_k(D)$ such that $i_*(b) = a$. Now, by the commutative diagram in Corollary 2.3, we have

$j_0^*(\sigma_*(a)) = 0$. Set $a' \equiv \sigma_*(a)$. From the exactness of the localization sequence in the bottom row of Corollary 2.3, there is an element $b' \in L_{n-2}H_k(Y)$ such that $(i_0)_*(b') = a'$.

Claim: In the commutative diagram in Corollary 2.3, there exists an element $c' \in L_{n-2}H_{k+1}(U)$ such that $(\delta_0)_*(c') = b'$ under the map $(\delta_0)_*: L_{n-2}H_{k+1}(U) \rightarrow L_{n-2}H_k(Y)$ and $\delta_*(c') = b$ under the map $\delta_*: L_{n-2}H_{k+1}(U) \rightarrow L_{n-2}H_k(D)$.

Proof of Claim. Since $\Phi_{n-2,k}: L_{n-2}H_k(Y) \cong H_k(Y)$ (note: $k \geq 2(n-2) \geq \dim(Y)$), we use the same notation b' for its image in $H_k(Y)$ since $L_{n-2}H_k(Y) \rightarrow H_k(Y)$ is injective for all $k \geq 2(n-2)$. At the beginning of the proof of the injectivity of the main theorem, we showed that $j_0^*: L_{n-2}H_k(X)_{\text{hom}} \rightarrow L_{n-2}H_k(U)_{\text{hom}}$ is injective. That is to say, $(i_0)_*(b') = 0 \in L_{n-2}H_k(X)_{\text{hom}}$. Hence there exists an element $c \in L_{n-2}H_{k+1}(U)$ whose image is b' under the boundary map

$$(\delta_0)_*: L_{n-2}H_{k+1}(U) \rightarrow L_{n-2}H_k(Y).$$

Let \tilde{b} be the image of c under the map $L_{n-2}H_{k+1}(U) \rightarrow L_{n-2}H_k(D)$. Now $\pi_*(\tilde{b} - b) = 0 \in L_{n-2}H_k(Y)$ and $i_*(\Phi_{n-2,k}(\tilde{b} - b)) = 0 \in H_k(\tilde{X}_Y)$ by Proposition 2.1, so we have $\Phi_{n-2,k}(\tilde{b} - b) = 0$. Since $\Phi_{n-2,k}$ is injective on $L_{n-2}H_k(D)$ (see Theorem 4.1), we get $\tilde{b} - b = 0$. This c satisfies both conditions of the claim.

Now everything is clear. The element a comes from the element c in $L_{n-2}H_{k+1}(U)$. By the exactness of the localization sequence in the upper row of Lemma 2.1, we get $a = 0 \in L_{n-2}H_k(\tilde{X}_Y)$. This completes the proof of the injectivity.

Case 2: σ_ is surjective.* Similarly to the injectivity, the surjectivity of

$$j_0^*: L_{n-2}H_k(X)_{\text{hom}} \rightarrow L_{n-2}H_k(U)_{\text{hom}}$$

is trivial because $\dim(Y) \leq n-2$, where $j_0: U \rightarrow X$ is the inclusion. In fact, if $\dim(Y) < n-2$, then $j_0^*: L_{n-2}H_k(X) \rightarrow L_{n-2}H_k(U)$ is an isomorphism and so is $j_0^*: L_{n-2}H_k(X)_{\text{hom}} \rightarrow L_{n-2}H_k(U)_{\text{hom}}$. If $\dim(Y) = n-2$, then the surjectivity of j_0^* follows from the commutative diagram in Lemma 2.5 and the isomorphism

$$\Phi_{n-2,2(n-2)}: L_{n-2}H_{2(n-2)}(Y) \cong H_{2(n-2)}(Y) \cong \mathbb{Z}.$$

We need only show that $j^*: L_{n-2}H_k(\tilde{X}_Y)_{\text{hom}} \cong L_{n-2}H_k(U)_{\text{hom}}$, where $j: U \rightarrow \tilde{X}_Y$ is the inclusion. There are a few cases.

(a) $k = 2(n-2)$: The map $j^*: L_{n-2}H_k(\tilde{X}_Y) \rightarrow L_{n-2}H_k(U)$ is a surjective map. Hence the induced map j^* on $L_{n-2}H_k(\tilde{X}_Y)_{\text{hom}}$ is also surjective by trivial reasoning.

(b) $k = 2(n-2) + 1$: By the commutative diagram in Lemma 2.4 and since $\Phi_{n-2,2(n-2)}: L_{n-2}H_{2(n-2)}(D) \rightarrow H_{2(n-2)}(D)$ is injective, it follows for $a \in L_{n-2}H_{2(n-2)+1}(U)_{\text{hom}}$ that the image of a under the boundary map

$$\delta_*: L_{n-2}H_{2(n-2)+1}(U) \rightarrow L_{n-2}H_{2n}(D)$$

must be zero. Hence a comes from an element $b \in L_{n-2}H_{2(n-2)+1}(\tilde{X}_Y)$. If $\tilde{b} := \Phi_{n-2,2(n-2)+1}(b) \neq 0$, then there exists a $c \in L_{n-2}H_{2(n-2)+1}(D)$ such

that $b - i_*(c) \in L_{n-2}H_{2(n-2)+1}(\tilde{X}_Y)_{\text{hom}}$ and $j^*(b - i_*(c)) = a$. In fact, since $j^*(\bar{b}) = 0$, there exists a $\bar{c} \in H_{2(n-2)+1}(D)$ such that $(i_0)_*(\bar{c}) = \bar{b}$. Note that $\Phi_{n-2,2(n-2)+1}: L_{n-2}H_{2(n-2)+1}(D) \rightarrow H_{2(n-2)+1}(D)$ is an isomorphism by Theorem 4.1; hence there exists a $c \in L_{n-2}H_{2(n-2)+1}(D)$ such that $\Phi_{n-2,2(n-2)+1}(c) = \bar{c}$. This shows the surjectivity in this case.

(c) $k \geq 2(n-2)+2$: In this last case, the surjectivity of $j^*: L_{n-2}H_k(\tilde{X}_Y)_{\text{hom}} \rightarrow L_{n-2}H_k(U)_{\text{hom}}$ is from the commutative diagram in Lemma 2.4 and the surjectivity of the map $\Phi_{n-2,k}: L_{n-2}H_k(D) \rightarrow H_k(D)$ (see Theorem 4.1). In fact, if $a \in L_{n-2}H_k(U)_{\text{hom}}$ then, by the exactness of rows in the commutative diagram in Lemma 2.4, there is an element $b \in L_{n-2}H_k(\tilde{X}_Y)$ such that $j^*(b) = a$. Set $\bar{b} = \Phi_{n-2,k}(b)$. Since $j^*(\bar{b}) = 0 \in H_k^{\text{BM}}(U)$, there exists a $\bar{c} \in H_k(D)$ such that $i_*(\bar{c}) = \bar{b}$. Now $\Phi_{n-2,k}: L_{n-2}H_k(D) \cong H_k(D)$ (see Theorem 4.1), and there exists a $c \in L_{n-2}H_k(D)$ such that $\Phi_{n-2,k}(c) = \bar{c}$. The commutative diagram in Lemma 2.4 implies that $\Phi_{n-2,k}(b - i_*(c)) = 0$; that is, $b - i_*(c) \in L_{n-2}H_k(\tilde{X}_Y)_{\text{hom}}$. The exactness of the upper row in Lemma 2.4 gives $j^*(b - i_*(c)) = a$. This completes the surjectivity in this case.

This completes the proof for a blowup along a smooth subvariety Y of codimension ≥ 2 in X .

Note that $\varphi: X \rightarrow X'$ is birational between projective manifolds. We complete the proof of the birational invariance of $L_{n-2}H_k(X)_{\text{hom}}$ for any smooth X by applying the following proposition. □

PROPOSITION 4.4. *If we know the birational invariance under one blowup of Lawson homology groups $L_p H_k(X)_{\text{hom}}$ for a smooth projective variety X , where p and k are given as in Theorem 1.1, then we can deduce the birational invariance of $L_p H_k(X)_{\text{hom}}$ for any birational transformation.*

Proof. We need to use the Hironaka desingularization theorem together with the functoriality properties described in Section 2.

Let $\varphi: X \dashrightarrow X'$ be a birational map. Then, by the desingularization theorem (cf. [Hi]), there exist

$$\hat{\varphi}: \hat{X} \rightarrow X' \quad \text{and} \quad \tau: \hat{X} \rightarrow X,$$

where $\hat{\varphi}$ is a morphism and τ is the composition of a sequence of blowups along smooth centers. By using the desingularization theorem once again, we have

$$\psi: \hat{X}' \rightarrow \hat{X} \quad \text{and} \quad \tau': \hat{X}' \rightarrow X',$$

where ψ is a morphism and τ' is the composition of a sequence of blowups along smooth centers. Furthermore, ψ is the quasi-inverse of $\hat{\varphi}$ in the sense that $\tau' = \hat{\varphi} \circ \psi$.

Now we can define the homomorphism $\varphi_*: L_p H_k(X)_{\text{hom}} \rightarrow L_p H_k(X')_{\text{hom}}$ as $\hat{\varphi}_*$ by using that τ_* is an isomorphism from $L_p H_k(\hat{X})_{\text{hom}}$ to $L_p H_k(X)_{\text{hom}}$, as we proved in the first step. Now we prove that φ_* is an isomorphism of abelian groups.

Note that, since τ'_* is an isomorphism, we see that $\hat{\varphi}_*$ is surjective because $\tau'_* = \hat{\varphi}_* \circ \psi_*$ is surjective. Thus we have proved the surjectivity of φ_* for birational

maps. From this, now we prove the injectivity of φ_* . Note that, by definition, $\varphi_* = \hat{\varphi}_*$. Since the surjectivity holds for any birational map by the previous step, ψ_* is surjective. Hence it suffices to show that $\hat{\varphi}_* \circ \psi_*$ is injective. This is true because $\hat{\varphi}_* \circ \psi_* = \tau'_*$ is an isomorphism. \square

REMARK 4.5. Griffiths [G] showed the nontriviality of the Griffiths group of 1-cycles of general quintic hypersurfaces in \mathbb{P}^4 , and Friedlander [F1] showed that $L_1H_2(X)_{\text{hom}} \cong \text{Griff}_1(X)$ for any smooth projective variety X . Hence, in general, this is a *nontrivial* birational invariant even for projective threefolds.

5. The Geometric and Topological Filtration

In this section, we prove Propositions 1.14 and 1.15 and Theorem 1.20. The proof of Proposition 1.14 consists of a diagram chase by using the results presented in Section 2.

PROPOSITION 5.1 (Proposition 1.14). *Let X be a smooth projective variety of dimension n . If $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ holds for X with $p = 1$ (resp. $p = n - 2$) and k arbitrary, then $T_p H_k(X', \mathbb{Q}) = G_p H_k(X', \mathbb{Q})$ holds also for any smooth projective variety X' that is birationally equivalent to X with $p = 1$ (resp. $p = n - 2$). In particular, for a smooth projective variety with $\dim(X) \leq 4$, the assertion that $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ holds for all $k \geq 2p \geq 0$ is a birational invariant statement.*

Proof.

Part I: $p = n - 2$. There are two cases.

Case 1: If $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$, then $T_p H_k(\tilde{X}_Y, \mathbb{Q}) = G_p H_k(\tilde{X}_Y, \mathbb{Q})$. The injectivity of $T_p H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow G_p H_k(\tilde{X}_Y, \mathbb{Q})$ was proved by Friedlander and Mazur in [FM]; hence we need only show the surjectivity. Note that the case $k = 2p + 1$ holds for any smooth projective variety (Proposition 1.15). We only need to consider the cases where $k \geq 2p + 2$. In these cases, $k - p \geq p + 2 = n$ by definition of the geometric filtrations, so we have $G_p H_k(\tilde{X}, \mathbb{Q}) = H_k(\tilde{X}_Y, \mathbb{Q})$ and $G_p H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Q})$.

Since $\sigma_* : L_{n-2}H_k(\tilde{X}_Y) \otimes \mathbb{Q} \rightarrow L_{n-2}H_k(X) \otimes \mathbb{Q}$ is surjective, it follows from Proposition 2.2 and the commutative diagram

$$\begin{CD}
 L_{n-2}H_k(\tilde{X}_Y) \otimes \mathbb{Q} @>\sigma_*>> L_{n-2}H_k(X) \otimes \mathbb{Q} \\
 @V\Phi_{n-2,k}VV @VV\Phi_{n-2,k}V \\
 H_k(\tilde{X}_Y, \mathbb{Q}) @>\sigma_*>> H_k(X, \mathbb{Q})
 \end{CD}$$

that $T_p H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow G_p H_k(\tilde{X}_Y, \mathbb{Q})$ is surjective (by using a diagram chase).

Case 2: If $T_p H_k(\tilde{X}_Y, \mathbb{Q}) = G_p H_k(\tilde{X}_Y, \mathbb{Q})$, then $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$. This part is relatively easy; it follows from Lemma 2.4 and the blowup formula for

singular homology [GHa]. This completes the proof for a blowup along a smooth codimension ≥ 2 subvariety Y in X . This completes the proof of Part I.

Part II: $p = 1$. The injectivity of the map

$$T_1H_k(W, \mathbb{Q}) \rightarrow G_1H_k(W, \mathbb{Q})$$

has been proved for any smooth projective variety W by Friedlander and Mazur in [FM]. We need only show the surjectivity under certain assumptions.

Similar to the case $p = n - 2$, one can show the following two cases by using Lemma 2.4 together with the blowup formula for Lawson homology (Theorem 1.6) and the singular homology.

1. If $T_1H_k(X, \mathbb{Q}) = G_1H_k(X, \mathbb{Q})$, then $T_1H_k(\tilde{X}_Y, \mathbb{Q}) = G_1H_k(\tilde{X}_Y, \mathbb{Q})$.
2. If $T_1H_k(\tilde{X}_Y, \mathbb{Q}) = G_1H_k(\tilde{X}_Y, \mathbb{Q})$, then $T_1H_k(X, \mathbb{Q}) = G_1H_k(X, \mathbb{Q})$.

This completes the proof for one blowup along a smooth codimension ≥ 2 subvariety Y in X . The birational invariant statement follows from Proposition 4.4. This completes the proof of Part II. □

REMARK 5.2. From the proof of the Proposition 1.14, we can draw the following conclusions.

(i) If

$$T_rH_k(Y, \mathbb{Q}) = G_rH_k(Y, \mathbb{Q})$$

for all k is true for algebraic r -cycles with $r \geq p$ for $\dim(Y) = n$, then

$$T_{p-1}H_k(X, \mathbb{Q}) = G_{p-1}H_k(X, \mathbb{Q}) \quad \text{for all } k$$

is a birationally invariant statement for smooth projective varieties X with $\dim(X) \leq n + 2$.

(ii) If

$$T_rH_k(Y, \mathbb{Q}) = G_rH_k(Y, \mathbb{Q})$$

for all k is true for r -algebraic cycles with $r \leq p$ for $\dim(Y) = n$, then

$$T_{p+1}H_k(X, \mathbb{Q}) = G_{p+1}H_k(X, \mathbb{Q}) \quad \text{for all } k$$

is a birationally invariant statement for smooth projective varieties X with $\dim(X) \leq n + 2$.

Now we give the proof of Proposition 1.15. First we revise a result of Friedlander (cf. Theorem 4.1) as follows.

PROPOSITION 5.3. *For any irreducible projective variety Y of dimension n , we have*

$$\begin{aligned} L_{n-1}H_{2n}(Y) &\cong \mathbb{Z}, \\ L_{n-1}H_{2n-1}(Y) &\cong H_{2n-1}(Y, \mathbb{Z}), \\ L_{n-1}H_{2n-2}(Y) &\rightarrow H_{2n-2}(Y, \mathbb{Z}) \quad \text{is injective,} \\ L_{n-1}H_k(Y) &= 0 \quad \text{for } k > 2n. \end{aligned}$$

Proof. The proof is based on Friedlander’s result on the smooth projective variety.

Set $S = \text{sing}(Y)$, the set of singular points. Then S is the union of proper irreducible subvarieties. Set $S = (\bigcup_i S_i) \cup S'$, where $\dim(S_i) = n - 1$ and S' is the union of subvarieties with dimension $\leq n - 2$. Let $V = Y - S$ be the smooth open part of Y . According to Hironaka [Hi], we can find \tilde{Y} such that \tilde{Y} is a smooth compactification of V . Let $D = \tilde{Y} - V$, where D is a divisor on \tilde{Y} with normal crossing. Denote by $i_0: S \hookrightarrow Y$ and $i: D \hookrightarrow \tilde{Y}$ the inclusions of closed sets; denote by $j_0: V \hookrightarrow Y$ and $j: V \hookrightarrow \tilde{Y}$ the inclusions of open sets.

There are three cases need to be proved: $k \geq 2n$; $k = 2n - 1$; and $k = 2n - 2$.

Case 1: $k \geq 2n$. This follows from the localization long exact sequence in Lawson homology and the singular homology.

Case 2: $k = 2n - 1$. This case follows from applying Lemma 2.5 to the pairs (Y, S) and (\tilde{Y}, D) for $p = n - 1$ together with the five lemma.

Case 3: $k = 2n - 2$. This case follows from the five lemma, Lemma 2.5, and the fact that the homology class of an algebraic subvariety is nontrivial in the homology of the Kähler manifold [GHa, p. 110]. The last fact still holds when “Kähler manifold” is replaced with “complex projective algebraic variety” because the latter can be embedded into a complex projective space, which is a Kähler manifold.

To see this, apply the five lemma to the pair (\tilde{Y}, D) in the commutative diagram of Lemma 2.5 for the case $p = n - 1$. We obtain the injectivity of the map $L_{n-1}H_{2n-2}(V) \rightarrow H_{2n-2}^{\text{BM}}(V)$ because $L_{n-1}H_{2n-2}(\tilde{Y}) \rightarrow H_{2n-2}(\tilde{Y})$ is injective and $L_{n-1}H_{2n-2}(D) \rightarrow H_{2n-2}(D)$ is surjective (in fact, it is also injective). Now, applying Lemma 2.5 to the pair (Y, S) for the case $p = n - 1$ yields the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & L_{n-1}H_{2n-2}(S) & \xrightarrow{(i_0)_*} & L_{n-1}H_{2n-2}(Y) & \xrightarrow{j^*} & L_{n-1}H_{2n-2}(V) & \longrightarrow & 0 \\
 & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \\
 \cdots & \longrightarrow & H_{2n-2}(S) & \xrightarrow{(i_0)_*} & H_{2n-2}(Y) & \xrightarrow{j^*} & H_{2n-2}^{\text{BM}}(V) & \longrightarrow & H_{2n-3}(S).
 \end{array}$$

For $a \in L_{n-1}H_{2n-2}(Y)$ such that $\Phi_{p,k}(a) = 0$, set $b = j^*(a)$. Since the map $L_{n-1}H_{2n-2}(V) \rightarrow H_{2n-2}^{\text{BM}}(V)$ is injective, we get $b = 0$. Since the first row of the preceding diagram is exact, there exists an element $c \in L_{n-1}H_{2n-2}(S)$ such that $(i_0)_*(c) = a$. Set $\bar{c} = \Phi_{p,k}(c)$, and note that $L_{n-1}H_{2n-2}(S) \rightarrow H_{2n-2}(S)$ is an isomorphism owing to the dimension of S . Moreover, $(i_0)_*(\bar{c}) = 0$ by assumption. So we get $\bar{c} = 0$ and then $c = 0$. This implies that $a = 0$ and hence the injectivity of $L_{n-1}H_{2n-2}(Y) \rightarrow H_{2n-2}(Y)$. □

PROPOSITION 5.4 (Proposition 1.15). *For any smooth projective variety X ,*

$$T_p H_{2p+1}(X, \mathbb{Q}) = G_p H_{2p+1}(X, \mathbb{Q}).$$

Proof. For any smooth projective variety X , the injectivity of $T_p H_{2p+1}(X, \mathbb{Q}) \rightarrow G_p H_{2p+1}(X, \mathbb{Q})$ was proved in [FM, Sec. 7]; hence we need only show the

surjectivity of $T_p H_{2p+1}(X, \mathbb{Q}) \rightarrow G_p H_{2p+1}(X, \mathbb{Q})$. For any subvariety $i: Y \subset X$, we denote by $V =: X - Y$ the complement of Y in X . We have the following commutative diagram of the long exact sequences (cf. Lemma 2.5):

$$\begin{CD} \cdots @>>> L_p H_{2p+1}(Y) @>>> L_p H_{2p+1}(X) @>>> L_p H_{2p+1}(V) @>>> L_p H_{2p}(Y) @>>> \cdots \\ @. @V \Phi_{p,2p+1} VV @V \Phi_{p,2p+1} VV @V \Phi_{p,2p+1} VV @V \Phi_{p,2p} VV @. \\ \cdots @>>> H_{2p+1}(Y) @>>> H_{2p+1}(X) @>>> H_{2p+1}^{\text{BM}}(V) @>>> H_{2p}(Y) @>>> \cdots \end{CD}$$

Obviously, this commutative diagram holds when tensored with \mathbb{Q} . In what follows we consider only the commutative diagrams with \mathbb{Q} -coefficient.

Now let $a \in G_p H_{2p+1}(X, \mathbb{Q})$. By definition, we can assume that a lies in the image of the map $i_*: H_{2p+1}(Y, \mathbb{Q}) \rightarrow H_{2p+1}(X, \mathbb{Q})$ for some subvariety $Y \subset X$ with dimension $\dim Y = (2p + 1) - p = p + 1$. Hence there exists an element $b \in H_{2p+1}(Y, \mathbb{Q})$ such that $i_*(b) = a$. By Proposition 5.3, we know that $\Phi_{p,2p+1}: L_p H_{2p+1}(Y) \otimes \mathbb{Q} \rightarrow H_{2p+1}(Y, \mathbb{Q})$ is an isomorphism. Hence there exists an element $\tilde{b} \in L_p H_{2p+1}(Y) \otimes \mathbb{Q}$ such that $\Phi_{p,2p+1}(\tilde{b}) = b$. Set $\tilde{a} = i_*(\tilde{b})$. Then \tilde{a} maps to a under the map $L_p H_{2p+1}(X) \otimes \mathbb{Q} \rightarrow H_{2p+1}(X, \mathbb{Q})$. By the definition of the topological filtration, $a \in T_p H_{2p+1}(X, \mathbb{Q})$. This completes the proof of surjectivity of $T_p H_{2p+1}(X, \mathbb{Q}) \rightarrow G_p H_{2p+1}(X, \mathbb{Q})$. \square

REMARK 5.5. In the proof of the surjectivity of Proposition 1.15, the assumption of smoothness is not necessary. More precisely, for any irreducible projective variety X , the image of the natural transformation $\Phi_{p,2p+1} \otimes \mathbb{Q}: L_p H_{2p+1}(X, \mathbb{Q}) \rightarrow H_{2p+1}(X, \mathbb{Q})$ contains $G_p H_{2p+1}(X, \mathbb{Q})$.

REMARK 5.6. Independently, Warkner [Wa, Prop. 2.5] has recently also obtained this result.

COROLLARY 5.7 (Corollary 1.16). *Let X be a smooth, n -dimensional projective variety with $H^{2,0}(X) = 0$. Then $T_{n-2} H_k(X, \mathbb{Q}) = G_{n-2} H_k(X, \mathbb{Q})$ for $k \geq 2n - 4$. In particular, this equality holds for X a complete intersection of dimension ≥ 2 , for any product of a smooth projective curve with a complete intersection of dimension ≥ 2 , et cetera.*

Proof. By Propositions 1.9 and 1.15, we need only prove the cases $k \geq 2n - 2$.

By the assumption and Poincaré duality, for $k = 2n - 2$ we have

$$H_{2n-2}(X, \mathbb{Q}) \cong H_2(X, \mathbb{Q}) = H_{1,1}(X, \mathbb{Q}).$$

Therefore, $G_{n-2} H_{2n-2}(X, \mathbb{Q}) \cong H_{1,1}(X, \mathbb{Q})$ and, by the commutative diagram

$$\begin{CD} L_{n-1} H_{2n-2}(X) \otimes \mathbb{Q} @>^s>> L_{n-2} H_{2n-2}(X) \otimes \mathbb{Q} \\ @V \Phi_{n-1,2n-2} VV @V \Phi_{n-2,2n-2} VV \\ H_{2n-2}(X, \mathbb{Q}) @>^{\cong}>> H_{2n-2}(X, \mathbb{Q}) \end{CD}$$

[FM, Prop. 6.3], we have the surjectivity of $L_{n-2}H_{2n-2}(X) \otimes \mathbb{Q} \rightarrow H_{2n-2}(X, \mathbb{Q})$ given the surjectivity of $L_{n-1}H_{2n-2}(X) \otimes \mathbb{Q} \rightarrow H_{2n-2}(X, \mathbb{Q})$. Now we need to show the cases when $k \geq 2n - 1$. Again we use a commutative diagram,

$$\begin{CD} L_{n-1}H_k(X) \otimes \mathbb{Q} @>{s}>> L_{n-2}H_k(X) \otimes \mathbb{Q} \\ @VV{\Phi_{n-1,k}}V @VV{\Phi_{n-2,k}}V \\ H_k(X, \mathbb{Q}) @>{=}>> H_k(X, \mathbb{Q}) \end{CD}$$

[FM, Prop. 6.3], to obtain the surjectivity of $\Phi_{n-2,k}: L_{n-2}H_k(X) \otimes \mathbb{Q} \rightarrow H_k(X, \mathbb{Q})$ from the surjectivity of $\Phi_{n-1,k}: L_{n-1}H_k(X) \otimes \mathbb{Q} \rightarrow H_k(X, \mathbb{Q})$. The latter is guaranteed by Theorem 4.1. □

COROLLARY 5.8 (Corollary 1.18). *Let X be the product of a smooth projective curve and a smooth, simply connected projective variety Y with $\dim Y = n - 1$. Then $T_{n-2}H_k(X, \mathbb{Q}) = G_{n-2}H_k(X, \mathbb{Q})$ for any $k \geq 2(n - 2) \geq 0$. In particular, the Friedlander–Mazur conjecture holds for the product of a smooth projective curve and a smooth simply connected projective surface.*

Proof. Suppose $X = C \times Y$, where C is a smooth projective curve and Y is a smooth projective variety of dimension $n - 1$. By the proof of Corollary 1.16, we need only consider the surjectivity of $L_{n-2}H_{2n-2}(X) \otimes \mathbb{Q} \rightarrow H_{2n-2}(X, \mathbb{Q})$. Now the Künneth formula for the rational homology of $H_{2n-2}(C \times Y, \mathbb{Q})$, together with Theorem 4.1 for Y and C , gives the surjectivity in this case. □

Now we give the proof of Theorem 1.20.

THEOREM 5.9 (Theorem 1.20). *If the Suslin conjecture for Lawson homology with coefficient \mathbb{Z} holds, then the topological filtration is the same as the geometric filtration for a smooth projective variety.*

Proof. By Propositions 1.9 and 1.15, we only need to show that $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ for $k \geq 2p + 2$. By the definition of geometric filtration, an element $a \in G_p H_k(X, \mathbb{Q})$ comes from the linear combinations of the images of elements $b_j \in H_k(Y_j, \mathbb{Q})$ for subvarieties Y_j of $\dim Y_j \leq k - p$ (equivalently, $\dim Y_j = k - p$). From the commutative diagram

$$\begin{CD} L_p H_k(Y) \otimes \mathbb{Q} @>{i_* \otimes \mathbb{Q}}>> L_p H_k(X) \otimes \mathbb{Q} \\ @VV{\Phi_{p,k} \otimes \mathbb{Q}}V @VV{\Phi_{p,k} \otimes \mathbb{Q}}V \\ H_k(Y, \mathbb{Q}) @>{i_* \otimes \mathbb{Q}}>> H_k(X, \mathbb{Q}) \end{CD}$$

it is enough to show that $\Phi_{p,k}: L_p H_k(Y) \rightarrow H_k(Y)$ is surjective for any irreducible subvariety $Y \subset X$ with $\dim(Y) = k - p$. From the Suslin conjecture, this map $\Phi_{p,k}$ is surjective for any smooth variety Y with $\dim(Y) = k - p$. Hence it is enough to show that $\Phi_{p,k}$ is also surjective for any singular irreducible variety Y (under the assumption that the Suslin conjecture for Lawson homology with coefficient \mathbb{Z} holds).

We will use induction to prove the following lemma.

LEMMA 5.10. *If the Suslin conjecture for Lawson homology with coefficient \mathbb{Z} holds for every smooth projective variety, then the map $L_p H_k(Y) \rightarrow H_k^{\text{BM}}(Y)$ is an isomorphism for $k \geq m + p$ and a monomorphism for $k = m + p - 1$ for every (possibly singular) quasi-projective variety Y , where $m = \dim(Y)$.*

Proof. Supposing that Y is an irreducible quasi-projective variety with $\dim(Y) = m$, we shall prove the lemma by induction on the dimension of Y . The statement is trivial if $m = \dim(Y) = 0$.

Let W be an irreducible quasi-projective variety with $\dim(W) = n < m$. Then, by the induction assumption, we have

$$\begin{aligned} L_p H_{n+p-1}(W) &\rightarrow H_{n+p-1}(W) \text{ is injective,} \\ L_p H_{n+q}(W) &\cong H_{n+q}(W) \text{ for } q \geq p. \end{aligned}$$

Denote by \tilde{Y} a projective closure of Y and $S = \text{sing}(\tilde{Y})$ the singular point set of \tilde{Y} . Set $U = \tilde{Y} - S$. Let $\sigma: \tilde{Y} \rightarrow \tilde{Y}$ be a desingularization of \tilde{Y} , and set $D := \tilde{Y} - U$. The existence of a smooth \tilde{Y} is guaranteed by Hironaka [Hi]. Then D is the union of irreducible varieties with dimension $\leq m - 1$.

By Lemma 2.5, we have the commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & L_p H_k(Z) & \longrightarrow & L_p H_k(V) & \longrightarrow & L_p H_k(U) & \longrightarrow & L_p H_{k-1}(Z) & \longrightarrow & \cdots \\ & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & & \\ \cdots & \longrightarrow & H_k(Z) & \longrightarrow & H_k(V) & \longrightarrow & H_k^{\text{BM}}(U) & \longrightarrow & L_p H_{k-1}(Z) & \longrightarrow & \cdots, \end{array} \tag{10}$$

where $U \subset V$ are quasi-projective varieties of $\dim(V) = \dim(U) = m$ and $Z = V - U$ is a closed subvariety of V .

Claim: *By inductive assumption, the commutative diagram (10), and the five lemma, there is an equivalence between*

$$\begin{aligned} L_p H_{m+p-1}(U) &\rightarrow H_{m+p-1}(U) \text{ is injective,} \\ L_p H_{m+q}(U) &\cong H_{m+q}(U) \text{ for } q \geq p \end{aligned}$$

and

$$\begin{aligned} L_p H_{m+p-1}(V) &\rightarrow H_{m+p-1}(V) \text{ is injective,} \\ L_p H_{m+q}(V) &\cong H_{m+q}(V) \text{ for } q \geq p. \end{aligned}$$

Using the Claim a finite number of times beginning from $V = \tilde{Y}$, we obtain the result for any quasi-projective variety U and hence for \tilde{Y} , since S is the union of irreducible varieties of lower dimensions. Using the Claim once again yields the statement for Y because $\tilde{Y} - Y$ is also the union of irreducible varieties of lower dimensions. This completes the proof of Lemma 5.10. □

By Lemma 5.10, the Suslin conjecture holds for all singular varieties if it holds for all smooth projective varieties. This completes the proof of Theorem 1.20. □

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References

- [C] H. Clemens, *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, Inst. Hautes Études Sci. Publ. Math. 58 (1983), 19–38.
- [DT] A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. (2) 67 (1958) 239–281.
- [F1] E. Friedlander, *Algebraic cycles, Chow varieties, and Lawson homology*, Compositio Math. 77 (1991), 55–93.
- [F2] ———, *Filtrations on algebraic cycles and homology*, Ann. Sci. École Norm. Sup. (4) 28 (1995), 317–343.
- [FGa] E. Friedlander and O. Gabber, *Cycle spaces and intersection theory*, Topological methods in modern mathematics (Stony Brook, 1991), pp. 325–370, Publish or Perish, Houston, TX, 1993.
- [FHW] E. Friedlander, C. Haesemeyer, and M. Walker, *Techniques, computations, and conjectures for semi-topological K-theory*, Math. Ann. 330 (2004), 759–807.
- [FL] E. Friedlander and B. Lawson Jr., *A theory of algebraic cocycles*, Ann. of Math. (2) 136 (1992), 361–428.
- [FM] E. Friedlander and B. Mazur, *Filtrations on the homology of algebraic varieties. With an appendix by Daniel Quillen*, Mem. Amer. Math. Soc. 110 (1994).
- [G] P. Griffiths, *On the periods of certain rational integrals I, II*, Ann. of Math. (2) 90 (1969), 460–495, 496–541.
- [GHa] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1994.
- [Gro] A. Grothendieck, *Standard conjectures on algebraic cycles*, Algebraic geometry (Bombay, 1968), pp. 193–199, Oxford Univ. Press, London, 1969.
- [Hi] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203, 205–326.
- [Hu] W. Hu, *Generalized Abel–Jacobi map on Lawson homology*, Amer. J. Math. 131 (2009), 1241–1260.
- [L1] H. B. Lawson Jr., *Algebraic cycles and homotopy theory*, Ann. of Math. (2) 129 (1989), 253–291.
- [L2] ———, *Spaces of algebraic cycles*, Surveys in differential geometry (Cambridge, 1993), vol. 2, pp. 137–213, International Press, Cambridge, MA, 1995.
- [Lew] J. D. Lewis, *A survey of the Hodge conjecture*, 2nd ed., CRM Monogr. Ser., 10, Amer. Math. Soc., Providence, RI, 1999.
- [Lie] D. I. Lieberman, *Numerical and homological equivalence of algebraic cycles on Hodge manifolds*, Amer. J. Math. 90 (1968), 366–374.
- [Li1] P. Lima-Filho, *Lawson homology for quasiprojective varieties*, Compositio Math. 84 (1992), 1–23.
- [Li2] ———, *On the generalized cycle map*, J. Differential Geom. 38 (1993), 105–129.
- [P] C. Peters, *Lawson homology for varieties with small Chow groups and the induced filtration on the Griffiths groups*, Math. Z. 234 (2000), 209–223.
- [V] M. Voineagu, *Semi-topological K-theory for certain projective varieties*, J. Pure Appl. Algebra 212 (2008), 1960–1983.

- [Vo] C. Voisin, *Hodge theory and complex algebraic geometry, II* (L. Schneps, transl.), Cambridge Stud. Adv. Math., 77, Cambridge Univ. Press, Cambridge, 2003.
- [Wa] M. E. Walker, *The morphic Abel–Jacobi map*, Compositio Math. 143 (2007), 909–944.

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