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Equisingularity of Families of Hypersurfaces and Applications to Mappings

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1. Introduction

Notions of equisingularity for varieties date back many years, the modern era being started effectively by Zariski. Much work has been done in this area; see [2] for some recent significant and interesting results. A classical theorem of Teissier and of Briançon and Speder gives conditions for equisingularity of a family of complex hypersurfaces such that each member has an isolated singularity. In this case the family is called *Whitney equisingular* if the singular set of the variety formed by the family is a stratum in a Whitney stratification.

For any isolated hypersurface singularity we may associate a μ^**-sequence*: The intersection of the Milnor fibre of the singularity and a generic i-plane passing through the singularity is homotopically equivalent to a wedge of spheres, the number of which is denoted μ^i. This is a sequence of analytic invariants.

The result of Briançon–Speder–Teissier is that the μ^*-sequence is constant in the family if and only if the family is Whitney equisingular.

A natural and long-standing question is: What happens in the nonisolated case? More precisely, we assume that we can stratify a family so that outside the parameter axis of the family we have a Whitney stratification and seek conditions that give an equivalence between a collection of topological invariants and Whitney equisingularity of the parameter axis. In some sense this was answered in [31] using the multiplicity of polar invariants. However, in many situations the number of invariants is very large. We would like a small number of topological or algebraic invariants, defined in a simple manner, that control and are controlled by the equisingularity of the family.

An important theorem of Gaffney and Gassler [12, Thm. 6.3] gives a partial result. They define the sequence χ^* as the Euler characteristic of the Milnor fibres that occur for the family. This is an obvious generalization of the μ^*-sequence, since the Euler characteristic of the Milnor fibre is determined by the Milnor number in the isolated singularity case. The constancy of this sequence does not seem to be sufficient to ensure Whitney equisingularity. Thus they define another sequence, called the relative polar multiplicities and denoted m_* (see Section 3 for a precise definition). In the case of isolated singularities, the constancy of μ^* in the family implies the constancy of m_* in the family.

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The Gaffney–Gassler theorem is that Whitney equisingularity of a family implies that the sequences (m_*, χ^*) are constant in the family. The aim of this paper is to give further conditions to ensure a converse, which is done in Theorem 4.9. Better than that, the number of invariants is reduced considerably in Theorem 5.9 to a certain selection from m_* and χ^*.

The key condition in Theorem 4.9 is that the complex links of strata in the family (outside the parameter axis) have nontrivial homology. There are plenty of examples of spaces with this condition. In fact, in the applications we have in mind there is a plethora of examples. Many more need to be found, though.

In Section 2 we describe the basic notation used and make precise the definition of equisingularity. Section 3 defines the relative polar invariant and Euler characteristic sequences via the method of blowing up of ideals. Two further sequences, the Lê numbers of Massey (denoted λ^*) and Damon's higher multiplicities (denoted μ^*), are defined in Section 4. As one can see from the notation, the latter is a generalization of the usual Milnor number. In fact, the sequence is equivalent to m_*; only the indexing is different. The λ^* sequence is closely related to the χ^* sequence—in a family, constancy of one implies constancy of the other. In contrast to m_* and Damon's μ^*, however, λ^* and χ^* are not equal even after reindexing. Theorem 4.9 gives a partial converse to the Gaffney–Gassler theorem; in other words, it gives the conditions under which (m_*, χ^*) constant implies Whitney equisingularity.

The main theorem, Theorem 5.9, is given in Section 5. It gives conditions for equivalence of different sequences and Whitney equisingularity. The theorem also shows that one needs only a selection of invariants from two sequences; we do not require every element from both sequences.

Section 6 gives applications of the main theorem to families of maps with isolated instabilities such that the discriminant is a hypersurface. In this case we have a large supply of hypersurfaces that satisfy an important condition of the main theorem: the complex link of strata are not homologically trivial. The equisingularity of a family of maps—rather than merely equisingularity of their discriminants—is considered for corank-1 multi-germs $f : (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$. Ultimately, we can produce theorems concerning topological triviality of families of maps.

Note that in the applications we treat the case of multi-germs. Despite not requiring much more work and their great importance, particularly in the study of images of maps, these have often been ignored in the past.

A number of remarks concerning the work of others and areas of possible research are made in the final section.

2. Equisingularity and Basic Definitions

In this section we give some notation and basic definitions related to equisingularity for the sets and the complex analytic maps that concern us.

Standard definitions from singularity theory, such as finite $\mathcal{A}$-determinacy, can be found in [8] and [32]. The zero set of a map F will be denoted $V(F)$, and its singular set (i.e., the points in the domain where the rank of the differential is less than the codomain) will be denoted by $\Sigma(F)$. A differentiable map is called *corank 1* if its differential has corank at most 1 at all points. Note that, for convenience, this includes the case of nonsingular maps.

Often we shall need to move from a germ and choose a representative, or a smaller neighborhood, et cetera. This is entirely standard and is obvious when it occurs and so no explicit mention will be made of the details, which would distract from the exposition.

DEFINITION 2.1. Let X be complex analytic set and Y a subset of X. We say that X is *Whitney equisingular along* Y if Y is a stratum of some Whitney stratification of X.

This notion has been the subject of considerable investigation; see [13] for a survey from ten years ago and [2] for more recent developments in the hypersurface case. In the more general case of maps, Gaffney made many of the fundamental definitions for the study of the equisingularity; see [9]. His work has been continued by him and others (see [10; 17; 19]).

The famous example of Briançon and Speder [6] shows that, even in the hypersurface case, the notion of equisingularity can be a delicate one.

EXAMPLE 2.2 [6]. Let $f(x, y, z, t) = z^5 + ty^6z + y^7x + x^{15}$. This is a family of quasihomogeneous hypersurface singularities indexed by t such that $f_t\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ has an isolated singularity at $(0, 0, 0)$ and the Milnor number is constant for all t.

Since $\Sigma(f)$ is a manifold, the obvious stratification of $f^{-1}(0)$ consists of the manifolds $f^{-1}(0)\backslash\Sigma(f)$ and $\Sigma(f)$. However, this stratification is not Whitney equisingular along $\Sigma(f)$ because the Whitney conditions fail at $(0, 0, 0, 0)$. What is most interesting is that the family is still topologically trivial.

This example shows that the Milnor number is insufficient to achieve a Whitney stratification; the Briançon–Speder–Teissier theorem tells us we need to look at generic slices of the hypersurfaces. Precise conditions to achieve topological triviality are still the subject of current research.

3. Polar Invariants via Blowing Up

We shall consider what are called *polar invariants,* which are very important in the study of equisingularity; see for example [31]. In this section we will consider them as arising from the method of blowing up ideals and in the next from the viewpoint of sheaf theory.

Let $f\colon (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic function, and denote the Jacobian ideal by $J(f)$:

$$J(f) = \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_N}\right)$$

for coordinates $z_0, \dots, z_N$ in $\mathbb{C}^{N+1}$.

DEFINITION 3.1. The *blowup of* $\mathbb{C}^{N+1}$ *along the Jacobian ideal,* denoted $Bl_{J(f)}\mathbb{C}^{N+1}$, is the closure in $\mathbb{C}^{N+1} \times \mathbb{P}^N$ of the graph of the map

$$\mathbb{C}^{N+1}\backslash V(J(f)) \to \mathbb{P}^N, \quad x \mapsto \left(\frac{\partial f}{\partial z_0}(x) : \cdots : \frac{\partial f}{\partial z_N}(x)\right),$$

where $V(J(f))$ is the zero set of $J(f)$.

A hyperplane h in $\mathbb{P}^N$ can be pulled back by the natural projection $p\colon \mathbb{C}^{N+1} \times \mathbb{P}^N \to \mathbb{P}^N$ to a Cartier divisor H on $Bl_{J(f)}\mathbb{C}^{N+1}$ (provided $Bl_{J(f)}\mathbb{C}^{N+1}$ is not contained in the product of $\mathbb{C}^{N+1}$ and h). We call this a *hyperplane on* $Bl_{J(f)}\mathbb{C}^{N+1}$.

Let $b\colon \mathbb{C}^{N+1} \times \mathbb{P}^N \to \mathbb{C}^{N+1}$ be the other natural projection. For suitably generic hyperplanes $h_1, \dots, h_k$ in $\mathbb{P}^N$, the multiplicity at the origin of $b(H_1 \cap \cdots \cap H_k \cap Bl_{J(f)}\mathbb{C}^{N+1})$ is a well-defined invariant of f; see [12].

DEFINITION 3.2. For $1 \le k \le N$, the kth *relative polar multiplicity of* f is the multiplicity of the scheme $b_*(H_1 \cap \cdots \cap H_k \cap Bl_{J(f)}\mathbb{C}^{N+1})$ at the origin. It is denoted by $m_k(f)$.

From this we can define a sequence of invariants $m_*(f)$. Full details of the preceding construction and proofs of the various assertions can be found in [12], where the authors also show that the situation can be generalized to ideals other than the Jacobian.

We can now define another, perhaps more familiar, sequence of invariants; these have a topological nature.

DEFINITION 3.3 [13, p. 238]. Let $f\colon (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic function and let $L^i \subseteq \mathbb{C}^{N+1}$ be a generic i-dimensional linear subspace. Denote the reduced Euler characteristic of the Milnor fibre of $f|L^i$ by $\tilde{\chi}^i(f)$.

From this we can define a sequence

$$\tilde{\chi}^*(f) := (\tilde{\chi}^2(f), \dots, \tilde{\chi}^{N+1}(f)).$$

In the case of an isolated singularity, this (effectively) reduces to the standard μ^*-sequence in equisingularity theory.

REMARK 3.4. It transpires that the number $\tilde{\chi}^1(f)$ is not needed in the theory in [12] and so is omitted. This is because, in a family of hypersurfaces, $\tilde{\chi}^2(f)$ will be the Euler characteristic of the Milnor fibre of a plane curve singularity; hence constancy of this implies constancy of the multiplicity of the singularity, which implies the constancy of $\tilde{\chi}^1(f)$.

Massey (see e.g. [27, p. 73]) shows how one can calculate the reduced Euler characteristic in practice: it is equal to the alternating sum of the Lê numbers. His definition of Lê numbers involves taking certain hyperplanes. The precise conditions needed on these hyperplanes are not important here; what is important is that they need not be generic. (The lack of genericity means that we can, in practice, calculate the Lê numbers.)

The generic Lê numbers—that is, those formed by taking generic hyperplanes—can be defined using the blowing up setup as follows. Denote the exceptional divisor of the blowup by E. Then, Gaffney and Gassler [12] define the kth Lê number, $\lambda_k(f)$, to be the multiplicity of

$$b_*(H_1 \cdots H_{k-1} \cdot E \cdot Bl_{J(f)}\mathbb{C}^{N+1}),$$

where $\cdot$ denotes intersection product and $1 \le k \le N$. That these numbers coincide with Massey's definition is shown in [27, Thm. II.1.26]. Note, however, that this defines the number by codimension whereas Massey defines the Lê numbers by dimension. To avoid confusion in this paper, for our invariants we will generally use superscripts to denote dimension and subscripts to denote codimension. Hence, in the Gaffney–Gassler notation of [12], $\lambda_i(f)$ is Massey's $\lambda^{N-i+1}(f)$ (which we shall define in the next section).

The significance of the invariants $m_*(f)$ and $\tilde{\chi}^*(f)$ is made clear in [12].

Theorem 3.5 [12, Thm. 6.3]. *Suppose that we have a family of maps f_t: $(\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$. Let F: $(\mathbb{C}^{N+1} \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be given by $\bar{F}(x,t) = f_t(x)$, so that $F(x,t) = (\bar{F}(x,t,),t)$ is a 1-parameter unfolding.*

If $V(\bar{F})$ admits a Whitney stratification with $T = (\{0\} \times \mathbb{C}, 0) \subset (\mathbb{C}^{N+1} \times \mathbb{C}, 0)$ as a stratum, then the map $t \mapsto (m_*(f_t), \tilde{\chi}^*(f_t))$ is constant on T.

The main aim of this paper is to investigate extra conditions upon $V(\bar{F})$ which imply that the converse holds. Note that in [12] the authors do prove a partial converse in their Theorem 6.2 by showing that the smooth part of $V(\bar{F})$, the smooth part of its critical locus Σ, and the components of the singular locus of codimension 1 in Σ are all Whitney regular over the parameter axis.

Note that, if f_t is a family of isolated hypersurface singularities, then $\mu^*(f_t)$ constant is equivalent to $\tilde{\chi}^*(f_t)$ constant and these imply that $m_*(f_t)$ is constant. Hence, in this particular case we know by the Briançon–Speder–Teissier result that $(m_*(f_t), \tilde{\chi}^*(f_t))$ constant does imply that there is a stratification such that T is a Whitney stratum.

Definition 3.6. Given a family of maps, we wish to make precise what it means for an invariant of the members to be *constant in the family.* We shall take this to mean that there is an open contractible neighborhood of the origin in the parameter space over which the invariant is constant for elements of the family. This definition saves us from constantly referring to the neighborhood.

4. Polar Invariants via Sheaf Theory

Using intersection theory and sheaf theory, Massey has given a different interpretation of the blowing up we have just seen. The material in this section comes mostly from [25; 26; 27]. The book [27] in particular contains useful appendices on analytic cycles, intersection theory, and vanishing cycles for sheaves.

Suppose that $\mathbf{F}^\bullet$ is a complex of constructible sheaves on an analytic space X and that $f: X \to \mathbb{C}$ is a complex analytic function. Then we denote the *vanishing* *cycles of* $\mathbf{F}^\bullet$ by $\phi_f \mathbf{F}^\bullet$. See [27, Apx. B] for a full definition and important properties of this complex.

Goresky and MacPherson [14] developed a theory of Morse data on stratified spaces with respect to constructible sheaves. We recall their definition of a nondegenerate conormal vector. Let p be a point in the stratum S of X and let $T^*_S M$ denote the set of all covectors $\omega \in T^*_p M$ such that $\omega(T_p S) = 0$.

DEFINITION 4.1 [14, p. 160]. A plane $Q \subseteq T_p(M)$ is called a *generalized tangent space* if $Q = \lim_{i\to\infty} T_{q_i} S_\alpha$, where $S \subset \overline{S_\alpha}$ and q_i is a sequence of points in S_α converging to p.

Also, the set of *nondegenerate normal covectors* is the set

$$C_S := \{\omega \in T^*_S M \mid \omega(Q) \neq 0 \text{ for any generalized tangent space } Q \neq T_p(S)\}.$$

DEFINITION 4.2. Let $X \subset \mathbb{C}^N$ be a complex analytic space with a Whitney stratification $\{S_\alpha\}$ such that the strata are connected. Let $\mathbf{F}^\bullet$ be a complex of sheaves that is constructible with respect to this stratification.

Let x be a point in the d-dimensional stratum S_α. Let M be a normal slice to S_α at x and let $L\colon (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ be a linear map such that $d_p L$ is a nondegenerate covector.

Then, the *characteristic normal Morse data* for the pair $(S_\alpha, \mathbf{F}^\bullet)$ is

$$m(S_\alpha, \mathbf{F}^\bullet) = (-1)^{N-1}\chi(\phi_{L|X}\mathbf{F}^\bullet)_x = (-1)^{N-d-1}\chi(\phi_{L|M\cap X}\mathbf{F}^\bullet_{|M\cap X})_x,$$

where χ denotes the Euler characteristic of the sheaf (at the point $x \in S_\alpha$).

When $\mathbf{F}^\bullet$ is the constant sheaf $\mathbb{C}^\bullet_X$ we can write $m(S_\alpha)$ and call it simply the *characteristic normal Morse data of the stratum.* In this case,

$$m(S_\alpha) = (-1)^{N-d}\chi(B_\varepsilon(x) \cap X \cap M, B_\varepsilon(x) \cap X \cap M \cap L^{-1}(\eta)),$$

where $B_\varepsilon(x)$ is a sufficiently small open ball of radius ε centered at x and $\eta \neq 0$ is also sufficiently small. Note that Massey uses a different notation: our $m(S_\alpha, \mathbf{F}^\bullet)$ is his $m_\alpha(\mathbf{F}^\bullet)$; and our $m(S_\alpha)$ is his m_α.

DEFINITION 4.3. The space $B_\varepsilon(x) \cap X \cap M \cap L^{-1}(\eta)$ in the pair above is called the *complex link of the stratum* S_α.

For complete intersections, the number $m(S_\alpha)$ is very important.

REMARK 4.4. If X is a complete intersection, then the complex link of a stratum is homotopically equivalent to a wedge of spheres; see [14, p. 187; 22]. Provided S_α is not a "top", nonsingular stratum of X (i.e., a stratum of maximal dimension), it follows that $m(S_\alpha)$ is just the number of these spheres. See, for example, [25, Exm. 6.5].

Thus, in the case of complete intersections, $m(S_\alpha) \geq 0$.

In the general case, Massey calls strata *visible* if they have the property that $m(S_\alpha) \neq 0$. More important to us are the cases in which this number is positive. The latter property will be a vital assumption in later theorems and their applications and so, to preclude "empty theorems", we must produce a significant set of examples for which this holds.

EXAMPLE 4.5. Let S_α be a component of the top strata of X. That is, S_α is open in the nonsingular part of X. Then, since the normal slice reduces the normal data to a point, the complex link of the stratum is empty and the homology of the normal Morse data is just the homology of a point. Hence, $m(S_\alpha) = 1$.

We now come to some interesting cases that not only supply plenty of examples but also are useful in applications (see Section 6).

EXAMPLE 4.6 [15, Thm. 7.3]. Suppose that $F\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$, $n \geq p$, is a stable, corank-1 map such that $n < p$. If we stratify the image of F by stable type, then $m(S_\alpha) = 1$ for all strata S_α. (Stratification by stable type is described in detail in [9, Sec. 6] and in Section 6 of this paper.)

EXAMPLE 4.7 [7, p. 33]. Suppose that $F\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$ is a stable multigerm in Mather's nice dimensions (see [8; 28]). If we stratify the discriminant of F by stable type, then $m(S_\alpha) = 1$.

This is because of the same reasoning that is behind the previous example. Namely, the complex link is actually homotopically equivalent to the stabilization of an $\mathcal{A}_e$-codimension-1 germ.

EXAMPLE 4.8. Let X be a hypersurface with an isolated singularity at x. For the trivial stratification $\{X \backslash \{x\}, \{x\}\}$ we have $m(\{x\}) = \mu(X \cap H)$, where H is a generic hyperplane.

We can now state a generalization of the Briançon–Speder–Teissier theorem that is a partial converse to Gaffney and Gassler's theorem.

THEOREM 4.9. *Suppose that f_t is a family as in Theorem 3.5. Suppose further that f_t is reduced and $X \backslash T$ is Whitney stratified so that the characteristic normal Morse data $m(S_\alpha)$ is nonzero (and hence positive) for all $S_\alpha \subseteq X \backslash T$.*

Then, the following statements are equivalent.

(i) *$(m_1(f_t), \dots, m_N(f_t), \tilde{\chi}^2(f_t), \dots, \tilde{\chi}^{N+1}(f_t))$ is constant in the family.*
(ii) *The stratum T is Whitney equisingular over all the strata S_α.*

Proof. (ii) $\Rightarrow$ (i): This is [12, Thm. 6.3], stated here as Theorem 3.5.

(i) $\Rightarrow$ (ii): This follows from the argument of [13, Thm. 6.5]. Consider the stratification of $X \backslash T$. The set $\overline{S_\alpha}$ is a complex analytic set and thus we can take a Whitney stratification $\{R_\beta\}$ of X such that $\overline{S_\alpha}$ is a union of strata.

For each S_α there exists a unique β such that $\overline{S_\alpha} = \overline{R_\beta}$. By [27, Part III, Chap. 3], the exceptional divisor of the blowup of the Jacobian ideal of F is, as a cycle, the sum of the projectivization of the conormal of strata in the stratification $\{R_\beta\}$, where each stratum has a multiplicity equal to $m(R_\beta)$. Thus, since $m(R_\beta) = m(S_\alpha) \neq 0$, the closure of R_β is the image of a component of the exceptional divisor of the blowup.

So by [13, Thm. 6.5] we know that R_β satisfies the Whitney condition along T. Actually, more than this is true because, in their proof, the authors use Teissier's Theorem V.1.2 in [31], which states that the nonsingular part of the closure of R_β is Whitney over T. Since $\overline{S_\alpha} = \overline{R_\beta}$, we deduce that S_α is Whitney over T. □

Now recall the definition of *polar varieties* as described in [25]. Let M be the affine space $\mathbb{C}^{N+1}$, and let $\underline{z} = (z_0, z_1, \dots, z_N)$ denote a choice of coordinates for M. Define $L^i_{\underline{z}} \colon M \to \mathbb{C}^i$ by $L^i_{\underline{z}}(z) = (z_0, \dots, z_{i-1})$.

Let Y be an analytic subset of M and let $p \in Y$.

DEFINITION 4.10. Suppose that $\dim_{\mathbb{C}} \Sigma(L^{i+1}_{\underline{z}}|_{Y\setminus\Sigma Y}) \geq i$. Then the ith *absolute polar variety* with respect to the coordinates $\underline{z}$ at the point p, denoted $\Gamma^i_{\underline{z}}(Y)$, is

$$\Gamma^i_{\underline{z}}(Y) = \operatorname{closure}(\Sigma(L^{i+1}_{\underline{z}}|_{Y\setminus\Sigma Y})),$$

where $\Sigma(f)$ denotes the critical set of the map f and ΣY denotes the singular set of the set Y. If the dimension condition does not hold, then we define $\Gamma^i_{\underline{z}}(Y)$ to be the empty set.

If the coordinates are chosen to be sufficiently general, then we get the (generic) absolute polar varieties of [31] and [24], which allows us to drop the $\underline{z}$ and write $\Gamma^i(Y)$.

The following definition arises from Section 7 and Theorem 0.5 of [25]. The characteristic polar cycle of a complex of sheaves is defined there in a different way, but is shown to be equal to the following in "good" situations.

DEFINITION 4.11. Suppose that $\mathbf{F}^\bullet$ is a constructible sheaf with respect to the Whitney stratification $\{S_\alpha\}$ of $X \subset M$, where X is a complex analytic set.

The kth *characteristic polar cycle of* $\mathbf{F}^\bullet$ (at p) is the cycle

$$\Lambda^k(\mathbf{F}^\bullet)_p = \sum_{S_\alpha} m(S_\alpha, \mathbf{F}^\bullet)\Gamma^k(\overline{S_\alpha}),$$

where the sum is over all S_α such that $p \in \overline{S_\alpha}$.

Since the coordinates are generic, there is a well-defined multiplicity for $\Gamma^k(\overline{S_\alpha})$ and so we define the multiplicity of $\Lambda^k(\mathbf{F}^\bullet)$ at p, denoted by $\lambda^k_p(\mathbf{F}^\bullet)$, to be

$$\lambda^k_p(\mathbf{F}^\bullet) := \operatorname{mult}_p(\Lambda^k(\mathbf{F}^\bullet)_p) = \sum_{S_\alpha} m(S_\alpha, \mathbf{F}^\bullet) \operatorname{mult}_p \Gamma^k(\overline{S_\alpha}).$$

EXAMPLE 4.12. Let $f \colon (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic map and let $\mathbf{F}^\bullet$ be the constant sheaf $\mathbb{C}^\bullet_{V(f)}$ on the hypersurface $X = V(f) \subset \mathbb{C}^{N+1}$. Then

$$\lambda^k_0(\mathbb{C}^\bullet_{V(f)}) = \sum_{S_\alpha} m(S_\alpha) \operatorname{mult}_0 \Gamma^k(\overline{S_\alpha}).$$

This example is very important: we will see in Lemma 4.16 that we can relate these invariants to the relative polar multiplicities of f defined earlier.

We shall drop the reference to p in $\lambda^k_p(\mathbf{F}^\bullet)$ because generally p will be the origin.

EXAMPLE 4.13 [26, Chap. 10; 27, Apx. B]. Suppose that $f: (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ is a complex analytic map on the manifold $M = \mathbb{C}^{N+1}$. Let $\mathbf{F}^\bullet = \phi_f \mathbb{C}^\bullet_M$. Then $\lambda^i(\mathbf{F}^\bullet)$ at 0 is the ith Lê number of f at 0, $\lambda^i(f)$, as defined by Massey. (Recall that the previously given Gaffney and Gassler version of this definition was indexed by codimension whereas the indexing here is by dimension.)

Since the codimension of $J(f)$ in $\mathbb{C}^{N+1}$ is at least 2, it follows that $\lambda^N(f)$ is zero. This is because the sheaf $\phi_f \mathbb{C}^\bullet_M$ is supported only on the critical points of f.

REMARK 4.14. In the preceding example, note that Massey restricts the sheaf of vanishing cycles to its support and shifts the resulting complex to ensure that it is perverse.

Recall that if (X, x) is a complete intersection complex analytic set then the complex link of x is a wedge of spheres of real dimension $\dim_{\mathbb{C}} X - 1$.

Let $f: (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic function. If H^i is a plane of dimension i through the origin, then $V(f) \cap H^i$ is a complete intersection.

DEFINITION 4.15 (cf. [7]). The kth *higher multiplicity* is the number

$$\mu^k(f) = \dim_{\mathbb{C}} H_{k-1}(\mathcal{L}^k; \mathbb{C}),$$

where $\mathcal{L}^k$ is the complex link of $V(f) \cap H^{k+1}$ at 0 and $1 \le k \le N$.

For sufficiently general H^k, this is a well-defined invariant of $V(f)$.

These invariants are linked to the relative polar multiplicities by the following lemma, whose three parts are effectively from [25, Exm. 8.4].

LEMMA 4.16. *Let $f: (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic function, and let f_t be an analytic family of such functions.*

(i) *We have $\lambda^0(\mathbb{C}^\bullet_{V(f)}) = \mu^N(f)$.*
(ii) *The numbers $\mu^i(f_t)$ are constant in a family for all $1 \le i \le N - r$ if and only if the numbers $\lambda^k(\mathbb{C}^\bullet_{V(f_t)})$ are constant in a family for all $r + 1 \le k \le N - 1$, where r is a nonnegative integer.*
(iii) *For all $1 \le i \le N$ we have $\mu^i(f) = m_{N-i+1}(f)$.*

Proof. For parts (i) and (ii) we note Massey's statement in [25, Exm. 8.4] that

$$\begin{aligned}
\lambda^0(\mathbb{C}^\bullet_{V(f)}) &= \mu^N(f),\\
\lambda^1(\mathbb{C}^\bullet_{V(f)}) &= \mu^N(f) + \mu^{N-1}(f),\\
\lambda^2(\mathbb{C}^\bullet_{V(f)}) &= \mu^{N-1}(f) + \mu^{N-2}(f),\\
&\vdots\\
\lambda^i(\mathbb{C}^\bullet_{V(f)}) &= \mu^{N-i+1}(f) + \mu^{N-i}(f),\\
&\vdots\\
\lambda^{N-1}(\mathbb{C}^\bullet_{V(f)}) &= \mu^2(f) + \mu^1(f),\\
\lambda^N(\mathbb{C}^\bullet_{V(f)}) &= \mu^1(f) + 1.
\end{aligned}$$

Part (iii) is just the comment from the end of [25, Exm. 8.4]. See also Examples 6.5 and 6.10 of the same paper for further information. □

REMARK 4.17. Note that parts (i) and (ii) of Lemma 4.16 combine, such that the numbers $\mu^i(f_t)$ are constant in a family for all $1 \leq i \leq N$, if and only if the numbers $\lambda^k(\mathbb{C}^\bullet_{V(f_t)})$ are constant in a family for all $0 \leq k \leq N-1$.

EXAMPLE 4.18. Suppose that $f : (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ defines an isolated hypersurface singularity. Then Lemma 4.16 shows that $\mu^k(f)$ coincides with the familiar definition of $\mu^k(f)$ in the μ^*-sequence of Teissier (apart from $\mu^{N+1}(f)$, which is missing). Therefore, from this and part (iii) of the lemma, we are justified in calling Theorem 4.9 a generalization of the Briançon–Speder–Teissier theorem.

5. Reducing the Number of Invariants

We turn our attention once again to the main idea of the paper: using nontriviality of normal Morse data outside of a stratum to give a converse to Gaffney and Gassler's theorem (stated previously as Theorem 3.5). This time we shall add an extra condition to reduce even further the number of invariants required in Theorem 4.9.

An additional required condition is that, outside the stratum of interest, the family of maps be locally trivial over the family's parameter. At first sight this may seem a strong condition, but it is found in the main examples of interest. In the classic Briançon–Speder–Teissier result, for example, the family has a line of singularities; outside this line, at each point the space is a manifold and for each a small neighborhood is the product of the parameter axis and a neighborhood of the point p in the space above the projection to the axis.

In this section we assume the following. Let $f : (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ be a reduced hypersurface and let $F(x,t) = (\bar{F}(x,t), t)$ be a 1-parameter unfolding such that $\bar{F}(x,0) = f(x)$. Take a representative of $\bar{F}$, also denoted $\bar{F}$, so that $\bar{F} : U \to \mathbb{C}$ is such that $U \subseteq \mathbb{C}^{N+1} \times \mathbb{C}$ is an open contractible set.

Let $X = V(\bar{F})$ and $T = U \cap (\{0\} \times \mathbb{C})$, and let $\pi : \mathbb{C}^{N+1} \times \mathbb{C} \to \mathbb{C}$ denote the natural projection. We can identify T with its image in $\mathbb{C}$ under this map. Let $\pi^{-1}(t) = M_t$. For a stratum S_α of a stratification $\{S_\alpha\}_{\alpha\in\Lambda}$ of $X \backslash T$ we define $S_{\alpha,t} := M_t \cap S_\alpha$. As usual, we assume that strata are connected.

DEFINITION 5.1. We say F has a *product structure over T* if the following statements hold.

(i) The stratification of $X \backslash T$ is Whitney regular with strata of dimension greater than 1.
(ii) The set T is contractible.
(iii) The manifold M_t is transverse to S_α at every point $(x,t) \in S_\alpha \subset X \backslash T$ for all strata S_α in the stratification of $X \backslash T$.
(iv) For all $\alpha \in \Lambda$, $S_{\alpha,0} \neq \emptyset$.

Note that $M_t = \mathbb{C}^{N+1} \times \{t\}$ is a slice such that $\{S_{\alpha,t}\}_{\alpha\in\Lambda}$ with $\{0\}$ is a Whitney stratification of $X_t := M_t \cap X$.

Because of the product structure on X, we can say something about $m(S_\alpha, \mathbf{F}^\bullet)$ for various $\mathbf{F}^\bullet$.

LEMMA 5.2. *Suppose that F has a product structure over T. Then*

$$m(S_{\alpha,0}) = m(S_{\alpha,t}) \quad \textit{and}$$
$$m(S_{\alpha,0}, \phi_{f_0}\mathbb{C}^\bullet_{\mathbb{C}^{N+1}}) = m(S_{\alpha,t}, \phi_{f_t}\mathbb{C}^\bullet_{\mathbb{C}^{N+1}})$$

for all t in the family and for $S_{\alpha,t} \neq \{0\}$.

Proof. Since $S_{\alpha,t} = S_\alpha \cap M_t$ inherits its stratification from S_α, the complex link of $S_{\alpha,t}$ is just the complex link of the stratum S_α. Therefore, $m(S_{\alpha,0}) = m(S_{\alpha,t})$ because strata are connected and $S_{\alpha,0} \neq \emptyset$.

For the second part, note that we have just shown that the characteristic normal Morse data is, in effect, constant along T (recall that T is contractible); hence we must show that, for points $p_t \in S_{\alpha,t}$ and $p_0 \in S_{\alpha,0}$, there exist neighborhoods U_t and U_0 and a stratum-preserving homeomorphism $h\colon U_t \to U_0$ such that

$$h_*(\phi_{f_t}\mathbb{C}^\bullet_{M_t}|_{U_t}) \cong \phi_{f_0}\mathbb{C}^\bullet_{M_0}|_{U_0}$$

in the bounded constructible derived category. Note that $\phi_{f_t}\mathbb{C}^\bullet_{M_t}$ is constructible for all t because $\mathbb{C}^\bullet_{M_t}$ is constructible on M_t and note that M_t can be Whitney stratified, so $V(f_t)$ is a union of strata. Hence the stratification is Thom A_{f_t} (see [5] or [30]) and therefore, by Thom's second isotopy lemma, we have the triviality over strata required for constructibility.

Next, the preceding isomorphism amounts to saying that, at every point, we can find an isomorphism between the vanishing cycles of f_t and f_0 at corresponding points in the homeomorphism of U_t and U_0.

Since F has a product structure over T we have that, at every point outside T, the fibres of F over T are topologically trivial and that this homeomorphism is stratum preserving. By [22] we know that topologically equivalent hypersurface singularities have homotopically equivalent Milnor fibres. Hence, the required result is true. □

Perhaps the most important fact we can deduce from the assumption of a product structure is that the multiplicity of the absolute polar varieties of the strata of fibres is upper semicontinuous.

PROPOSITION 5.3. *Suppose that F has a product structure over T and that the hypersurface defined by $f_t\colon (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ and given by $f_t(x) = F(x,t)$ is reduced for all $t \in T$. Then, for $\overline{S_{\alpha,t}} \neq \{0\}$, we have that $\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}})$ is upper semicontinuous for $1 \leq k \leq \dim \overline{S_{\alpha,t}}$.*

That is, for sufficiently small t we have

$$\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}}) \leq \operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,0}}) \quad \textit{for } 1 \leq k \leq \dim \overline{S_{\alpha,0}}.$$

Proof. First we need to define the relative ith polar variety, denoted $\Gamma^i(Y, h)$, of a closed complex analytic set $Y \subseteq \mathbb{C}^{N+1} \times \mathbb{C}$ associated to a complex analytic function $h\colon Y \to \mathbb{C}$ such that $h^{-1}(t) \subseteq M_t$. This variety is defined similarly to Definition 4.10, so we shall use the same notation from there. Note that the domain of L^i is still $\mathbb{C}^{N+1}$.

The set $\Gamma^i_{\underline{z}}(Y, h)$ is the closure of the union for all s of the set

$$\Sigma(L^i_{\underline{z}}|_{h^{-1}(s)\backslash(h^{-1}(s)\cap\Sigma Y)})$$

provided that the dimension of this closure is greater than or equal to i. If the dimension condition does not hold, then we define $\Gamma^i_{\underline{z}}(Y, h)$ to be the empty set. As in the remarks following Definition 4.10, by taking sufficiently general projections we can drop the reference to $\underline{z}$ and get the relative ith polar variety of a closed complex analytic set Y associated to a complex analytic function h, denoted $\Gamma^i(Y, h)$.

In [9] this set is denoted $P_j(Y, h)$, where j is its codimension in Y. Similarly, the absolute polar varieties of a set Z are denoted $P_j(Z)$, where j is its codimension in Z. In our setup we shall have $Y = \overline{S_\alpha}$, $h = \pi$, and $Z = \overline{S_{\alpha,t}}$.

Because we shall require the results from [9], we explicitly note the connection between the notation there and here. We have

$$P_j(\overline{S_\alpha}, \pi) = \Gamma^{\dim \overline{S_\alpha} - j}(\overline{S_\alpha}, \pi) = \Gamma^{\dim \overline{S_{\alpha,t}} + 1 - j}(\overline{S_\alpha}, \pi)$$

and

$$P_j(\overline{S_{\alpha,t}}) = \Gamma^{\dim \overline{S_{\alpha,t}} - j}(\overline{S_{\alpha,t}}).$$

In the following we shall assume without further comment that $1 \le k \le \dim \overline{S_{\alpha,t}}$.

By the assumption that F has a product structure, we can apply [9, Lemma 5.3] and see that

$$P_j(\overline{S_\alpha}, \pi) \cap (\mathbb{C}^{N+1} \times \{t\}) = P_j(\overline{S_{\alpha,t}})$$

for all t and for $0 \le j \le \dim \overline{S_{\alpha,t}} - 1$. In our notation this is

$$\Gamma^{k+1}(\overline{S_\alpha}, \pi) \cap (\mathbb{C}^{N+1} \times \{t\}) = \Gamma^k(\overline{S_{\alpha,t}}).$$

Therefore, since the multiplicity of a hyperplane slice of a closed analytic set Z is greater than or equal to the multiplicity of Z [4, Prop. 7], it follows that

$$\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,0}}) \ge \operatorname{mult}_{(0,0)} \Gamma^{k+1}(\overline{S_\alpha}, \pi).$$

(On the left-hand side we take the multiplicity at the origin in $\mathbb{C}^{N+1}$; on the right, we take the multiplicity at the origin in $\mathbb{C}^{N+1} \times \mathbb{C}$.)

Next, Teissier's result [31, Prop. IV.6.1.1] regarding the upper semicontinuity of the multiplicity of relative polar varieties associated to a map gives

$$\operatorname{mult}_{(0,t)} \Gamma^{k+1}(\overline{S_\alpha}, \pi) \le \operatorname{mult}_{(0,0)} \Gamma^{k+1}(\overline{S_\alpha}, \pi)$$

for all t in some neighborhood of 0 in $\mathbb{C}$.

Finally, for all strata S_α of X we have that $T \subseteq \overline{S_\alpha}$. So, essentially by [31, Prop. VI.2.1], there exists a contractible open neighborhood W of $(0, 0)$ in $\mathbb{C}^{N+1} \times \mathbb{C}$ such that $W \cap (T\backslash\{(0, 0)\})$ is a Whitney stratum in the obvious stratification of $X \cap W$ (that is, the one given by the stratification of $(X\backslash T) \cap W$ with the addition of $(T\backslash\{(0, 0)\}) \cap W$ and $\{(0, 0)\}$). Hence, by [9, Thm. 5.6], for all $(0, t) \in T\backslash\{(0, 0)\}$ in this neighborhood we have

$$\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}}) = \operatorname{mult}_{(0,t)} \Gamma^{k+1}(\overline{S_\alpha}, \pi).$$

This provides the result

$$\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}}) \leq \operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,0}})$$

that we seek. □

Now we can state a lemma that will be at the heart of our next theorem.

LEMMA 5.4. *Suppose the following.*

(i) *The map F has a product structure over T.*
(ii) *The hypersurface defined by $f_t\colon (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ and given by $f_t(x) = F(x,t)$ is reduced.*
(iii) *The characteristic normal Morse data $m(S_{\alpha,0})$ is nonzero (and hence positive) for all $S_{\alpha,0} \neq \{0\}$.*

Then

$$\mu^i(f_t) \text{ is constant for all } 1 \leq i \leq N - r$$

implies that

$$\lambda^i(f_t) \text{ is constant for all } r + 1 \leq i \leq N - 1,$$

where r is a nonnegative integer.

Proof. From the definition of $\lambda^k(\mathbb{C}^\bullet_{V(f_t)})$ (and as the maps f_t are reduced), at the origin of $\mathbb{C}^{N+1} \times \{0\}$ we see that

$$\lambda^k(\mathbb{C}^\bullet_{V(f_t)}) = \sum_{S_{\alpha,t}} m(S_{\alpha,t}) \operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}}).$$

Since $\Gamma^0(\overline{S_{\alpha,t}}) = \emptyset$ for all $\overline{S_{\alpha,t}} \neq \{0\}$ and since $\Gamma^k(\{0\}) = \emptyset$ for all $k \geq 1$, this reduces to

$$\lambda^k(\mathbb{C}^\bullet_{V(f_t)}) = \sum_{S_{\alpha,t} \neq \{0\}} m(S_{\alpha,t}) \operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}}) \quad \text{for } k \geq 1,$$

$$\lambda^0(\mathbb{C}^\bullet_{V(f_t)}) = m(\{0\}) = \mu^N(f_t).$$

The last equality comes from Lemma 4.16(i).

Because we have a product structure, it follows from Lemma 5.2 that

$$m(S_{\alpha,t}) = m(S_{\alpha,0})$$

for all t in the family.

By Proposition 5.3, $\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}})$ is upper semicontinuous; since $m(S_{\alpha,t}) \neq 0$, we can deduce that, for each $k \geq 1$,

$$\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}}) \text{ constant for all } \overline{S_{\alpha,t}} \neq \{0\} \iff \lambda^k(\mathbb{C}^\bullet_{V(f_t)}) \text{ constant.} \qquad (*)$$

Now consider the sheaf of vanishing cycles for f_t, $\phi_{f_t}\mathbb{C}^\bullet_{M_t}$. Then, for $0 \leq k \leq N$, $\lambda^k(\phi_{f_t}\mathbb{C}^\bullet_{M_t})$ is the Lê number of f_t, $\lambda^k(f_t)$ (see Example 4.13), and we have

$$\lambda^k(\phi_{f_t}\mathbb{C}^\bullet_{M_t}) = \sum_{S_{\alpha,t}} m(S_{\alpha,t}, \phi_{f_t}\mathbb{C}^\bullet_{M_t})\, \mathrm{mult}_0\, \Gamma^k(\overline{S_{\alpha,t}}).$$

Since we have a product structure, it follows from Lemma 5.2 that

$$m(S_{\alpha,t}, \phi_{f_t}\mathbb{C}^\bullet_{M_t})$$

is constant in the family. Therefore, for each $k \geq 1$,

$$\mathrm{mult}_0\, \Gamma^k(\overline{S_{\alpha,t}}) \text{ constant for all } \overline{S_{\alpha,t}} \neq \{0\} \implies \lambda^k(\phi_{f_t}\mathbb{C}^\bullet_{M_t}) \text{ constant.} \qquad (**)$$

Now Lemma 4.16(ii) together with $(*)$ and $(**)$ yields the statement. □

REMARK 5.5. Note that if we have constancy of all the $\mu^i(f_t)$ for $1 \leq i \leq N$, then we control all the Lê numbers except $\lambda^0(f_t)$. In view of the classic Briançon–Speder–Teissier result, this is not surprising. The $\mu^i(f_t)$ are all the numbers in the classical μ^*-sequence except the Milnor number of the original map f_t, which is just $\lambda^0(f_t)$ (see [26] or [27]).

REMARK 5.6. In the proof of Lemma 5.4, a key reason for controlling the higher multiplicities (and hence the relative polar multiplicities) is that the multiplicities of the absolute polar varieties of the strata are kept constant. It is well known that constancy of these (with some other conditions) can be used to control the Whitney conditions; see [9] and [31].

This result is perhaps not surprising when one considers one of the main theorems in [24]. In Théorème 4.1.1 of that paper, the $\mu^i(f)$ (and hence $\lambda^i(\mathbb{C}^\bullet_{V(f)})$) are connected to the terms $\chi_{d_{\alpha_0}+1}(X, X_{\alpha_0})$ and $\chi_{d_{\alpha_0}+2}(X, X_{\alpha_0})$, and the $m(S_\alpha)$ correspond to the $1 - \chi_{d_\alpha+1}(X, X_\alpha)$.

It would be interesting to explore the connection with the work of [24] and make it more explicit.

REMARK 5.7. One of the assumptions of Lemma 5.4 is that the characteristic normal Morse data are positive. This leads to "constancy of the μ^i implies constancy of the λ^i". The same type of proof can be used to show that if the $m(S_{\alpha,t}, \phi_{f_t}\mathbb{C}^\bullet_{\mathbb{C}^{N+1}})$ data are positive, then "constancy of the λ^i implies constancy of the μ^i".

This may be of interest because there are cases where the $m(S_{\alpha,t}, \phi_{f_t}\mathbb{C}^\bullet_{\mathbb{C}^{N+1}})$ are positive—for example, the classical Briançon–Speder–Teissier result. Since there are few other obvious examples, we have chosen not to state precisely this version of the lemma. It would, however, be interesting to find more examples.

We state another useful lemma for relating invariants in families.

LEMMA 5.8. *Suppose that* $\lambda^i(f_t)$ *is constant for all* $1 \leq i \leq N-1$. *Then*

$$\tilde{\chi}^{N+1}(f_t) \text{ is constant} \iff \lambda^0(f_t) \text{ is constant.}$$

Proof. By, for example, [27, p. 73], the reduced Euler characteristic of the Milnor fibre of f_t (which is equal to $\tilde{\chi}^{N+1}(f_t)$) is equal to the alternating sum of the Lê numbers $\lambda^i(f_t)$. From this the lemma follows. □

The main theorem is that we can give a converse to Gaffney and Gassler's Theorem 6.3 (stated here as Theorem 3.5) with fewer invariants (cf. [16, Lemma 3.1]). We return now to the setup for f and F from the start of this section.

THEOREM 5.9. *Suppose the following.*

- *The map F has a product structure over T.*
- *The hypersurface defined by $f_t\colon (\mathbb{C}^{N+1},0) \to (\mathbb{C},0)$ and given by $f_t(x) = \bar{F}(x,t)$ is reduced.*
- *The characteristic normal Morse data $m(S_{\alpha,0})$ is nonzero (and hence positive) for all $S_{\alpha,0} \neq \{0\}$.*

Then, the following statements are equivalent.

(i) $(\mu^1(f_t),\dots,\mu^N(f_t),\lambda^0(f_t))$ *is constant in the family.*
(ii) $(\mu^1(f_t),\dots,\mu^N(f_t),\tilde{\chi}^{N+1}(f_t))$ *is constant in the family.*
(iii) $(m_1(f_t),\dots,m_N(f_t),\lambda^0(f_t),\dots,\lambda^{N-1}(f_t))$ *is constant in the family.*
(iv) $(m_1(f_t),\dots,m_N(f_t),\tilde{\chi}^2(f_t),\dots,\tilde{\chi}^{N+1}(f_t))$ *is constant in the family.*
(v) *The stratum T is Whitney equisingular over all the strata S_α.*

Proof. (i) $\Rightarrow$ (iii): We have $\mu^i(f_t) = m_{N-i+1}(f_t)$ for all $1 \le i \le N-1$ by Lemma 4.16. The implication then follows from Lemma 5.4 with $r = 0$.

(iii) $\Rightarrow$ (i): This is obvious.

(iii) $\Leftrightarrow$ (iv): This is shown on page 726 of [12].

(ii) $\Rightarrow$ (iii): From Lemma 5.4 we know that $\mu^i(f_t)$ constant for all $1 \le i \le N$ implies $\lambda^k(f_t)$ constant for $1 \le k \le N-1$. From Lemma 5.8 we then deduce that $\lambda^0(f_t)$ is constant also.

(iv) $\Rightarrow$ (ii): This is obvious given $\mu^i(f_t) = m_{N-i+1}(f_t)$.

(iv) $\Leftrightarrow$ (v): This is Theorem 4.9. □

REMARK 5.10. If $m(S_\alpha) = 0$ for the stratum S_α, then one can see from the proof of Lemma 5.4 that if—instead of assuming $m(S_\alpha) \neq 0$—we assume $\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}})$ is constant in the family for all k, then the conclusion of the theorem still holds.

The statement of such a theorem would obviously be ugly, so we have chosen to omit it. However, it is an obvious generalization that may be of some interest in certain cases.

REMARK 5.11. Since $m_i(f_t) = \mu^{N-i+1}(f_t)$, there exist several other obvious equivalences that could have been stated in Theorem 5.9.

REMARK 5.12. It should be noted that (iii) $\Leftrightarrow$ (iv) holds in more generality; see [12, p. 726].

REMARK 5.13. In light of Remark 5.7, if we replace the condition that $m(S_{\alpha,t}) > 0$ with $m(S_{\alpha,t},\phi_{f_t}\mathbb{C}^\bullet_{\mathbb{C}^{N+1}}) > 0$, then we can produce the additional statement that $\chi^*(f_t)$ constant is equivalent to T being a Whitney stratum. This allows us yet another way to deduce the classical Briançon–Speder–Teissier result and again demonstrates that we should find more examples where the $m(S_{\alpha,t},\phi_{f_t}\mathbb{C}^\bullet_{\mathbb{C}^{N+1}}) > 0$ condition holds.

6. Applications

We now apply Theorem 5.9 to re-prove some old results and improve others as well as to give some new results. In particular, we will consider what happens for equisingularity of families of finitely $\mathcal{A}$-determined multi-germ maps $f_t\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$.

The Classic Briançon–Speder–Teissier Result

The first application is to show that Theorem 5.9 gives the classic Briançon–Speder–Teissier result. The demonstration of this is included because we hope that the proof will shed light on the application of the theorem to families of finitely $\mathcal{A}$-determined maps.

THEOREM 6.1. *Let $F\colon (\mathbb{C}^{N+1} \times \mathbb{C}, 0) \to (\mathbb{C}, 0 \times 0)$ be family of maps $f_t(x) = F(x,t)$ such that each f_t defines a reduced isolated singularity at the origin. Then, the singular set of F is Whitney over the nonsingular set if and only if $\mu^*(f_t)$ is constant in the family. Here $\mu^i(f_t)$ is the classic Milnor number of f_t restricted to a generic i-plane in $\mathbb{C}^{N+1}$.*

Proof. Since the Milnor number $\mu^{N+1}(f_t) = \mu(f_t)$ is constant, the set $X = \{x \in f_t^{-1}(0) \text{ for some } t\}$ has singular set equal to $T = \{0\} \times \mathbb{C} \subset \mathbb{C}^{N+1} \times \mathbb{C}$. Thus we can partition X into the manifolds by $\{X \backslash T, T\}$. We have a product structure along T because $X \backslash T$ is a manifold and $X \cap (\mathbb{C}^{N+1} \times \{t\}) = f_t^{-1}(0)$ has a stratification that is obviously Whitney. The normal Morse data of $X \backslash T$ is equal to 1 by Example 4.5.

The $\mu^1(f_t), \dots, \mu^N(f_t)$ of Theorem 5.4 are the usual Milnor numbers (by Example 4.18). The reduced Euler characteristic $\tilde{\chi}^{N+1}(f_t)$ is $(-1)^N \mu^{N+1}(f_t)$ because the Milnor fibre of f_t is a wedge of spheres, the number of which is $\mu^{N+1}(f_t)$. Hence by Theorem 5.4, where (ii) $\Leftrightarrow$ (v), we deduce the result. $\square$

Families of Finitely $\mathcal{A}$-Determined Map Germs

So far the emphasis has been on hypersurfaces. We shall now generalize to a wider class of maps. Suppose that we have a complex analytic multi-germ $f\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, y)$, where $\underline{x} = \{x_1, \dots, x_s\}$ is a finite set of points in $\mathbb{C}^n$. Such a map germ is *stable* at y if all small perturbations of f are $\mathcal{A}$-equivalent to f—in other words, if there exist local diffeomorphisms of source and target between the perturbation and f. See [8] or [32] for detailed definitions. We remark that unfoldings of stable maps are stable.

Let $J(f)$ be the Jacobian of f and let $\Sigma(f) = \{x \in \mathbb{C}^n \mid \operatorname{rank} df < p\}$. This is the *critical set of f*. Define the *discriminant of f*, denoted $\Delta(f)$, to be the image germ of $\Sigma(f)$ under f. Note that for $n < p$ this is just the image of f.

We say that $f\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, y)$ is *finitely $\mathcal{A}$-determined at y* if there exists a neighborhood $U \subseteq \mathbb{C}^p$ of y such that, for all $z \in U \backslash \{y\}$, the germ

$$f'\colon (\mathbb{C}^n, f^{-1}(z) \cap \Sigma(f)) \to (\mathbb{C}^p, z)$$

is stable. That is, f has an *isolated instability at* y. This definition is analogous to isolated singularity in the case of spaces.

If we have a finitely $\mathcal{A}$-determined multi-germ $f\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$, where $n \geq p - 1$, then the discriminant of f is a hypersurface (see [8, p. 446]). We apply Theorem 5.9 in this context; that is, we show when the discriminant of family of maps is Whitney equisingular. Later we will define equisingularity for maps rather than just for complex analytic sets.

In order to apply Theorem 5.9 we stratify the discriminant by stable type. Good references for proofs of the following are [8, Sec. 2.5] and [9, Sec. 6].

Let $G\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$ be a stable map. There exist open sets $U \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^p$ such that $G^{-1}(W) = U$ and $G\colon U \to W$ is a representative of G. We can partition $\Delta(G)$ by stable type. That is, y_1 and y_2 in $\mathbb{C}^p$ have the same stable type if $G_1\colon (\mathbb{C}^n, G_1^{-1}(y_1) \cap \Sigma(G_1)) \to (\mathbb{C}^p, y_1)$ and $G_2\colon (\mathbb{C}^n, G_2^{-1}(y_2) \cap \Sigma(G_2)) \to (\mathbb{C}^p, y_2)$ are $\mathcal{A}$-equivalent. These sets are complex analytic manifolds.

We can take strata in $U \subset \mathbb{C}^n$ by taking the partition

$$G^{-1}(S) \cap \Sigma(G), \quad G^{-1}(S) \backslash \Sigma(G)), \quad \text{and} \quad U \backslash G^{-1}(\Delta(G)),$$

where S is a stratum in the discriminant.

Definition 6.2 [9]. A finitely $\mathcal{A}$-determined multi-germ f has *discrete stable type* if there exists a versal unfolding of f in which only a finite number of stable types appear.

We shall consider two main classes of discrete stable type maps: corank-1 maps and those in Mather's nice dimensions. Recall that a map is called *corank 1* at a point x if its differential is at most one less than maximal at that point. We say that the map is corank 1 if it is corank 1 at all points. The precise conditions for a map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ to be in the nice dimensions are given in [28]. In particular, maps with $p \leq 7$ are in the nice dimensions.

Theorem 6.3. *Suppose that* $G\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$ *is a stable map and that one of the following holds*:

(i) *G is in the nice dimensions*;
(ii) *G is corank 1 and $n < p$.*

Then, the stratification of G by stable type is Whitney regular and any Whitney stratification of G is a refinement of this; *that is, this stratification is canonical.*

Proof. See [9, Lemma 7.2] or [8, Sec. 2.5]. □

We shall use this theorem without comment.

Definition 6.4. A stable type is called *0-stable* if the stratification by stable type has a 0-dimensional stratum.

Examples 6.5. The Whitney cross-cap $(x, y) \mapsto (x, y^2, xy)$ is 0-stable. The multi-germ from $(\mathbb{C}^2, \{x_1, x_2, x_3\})$ to $(\mathbb{C}^3, 0)$ that gives an ordinary triple point is 0-stable.

By counting the "0-stables" that appeared in a stable perturbation of a map with an isolated instability, Mond [29] produced interesting and useful invariants of finitely $\mathcal{A}$-determined maps $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$.

Now suppose that we have a multi-germ $f\colon (\mathbb{C}^n,\underline{x}) \to (\mathbb{C}^p,0)$ with an isolated instability at $0 \in \mathbb{C}^p$ and that $F(x,t) = (f_t(x),t)$ is a 1-parameter unfolding of f such that $f_0 = f$ and $f_t(x) = 0$ for all $x \in \underline{x}$.

Suppose that we have a representative $F\colon U \to W$ of F with $F^{-1}(W) = U$. The *parameter axes* in source and target are, respectively,

$$S := (\{\underline{x}\} \times \mathbb{C}) \cap U \subset \mathbb{C}^n \times \mathbb{C} \quad \text{and}$$
$$T := (\{0\} \times \mathbb{C}) \cap W \subset \mathbb{C}^p \times \mathbb{C}.$$

We would like stratifications of F so that these are strata. First we can aim to stratify the discriminant of F so that T is a stratum. If $n \geq p-1$, then the discriminant is a hypersurface and so we can apply Theorem 5.9. A harder problem is to stratify the map itself so that both S and T are strata. Here our strategy is to use a stratification of the discriminant and pull it back to one on the source. We could then apply Thom's second isotopy lemma to show that the family is topologically trivial.

DEFINITION 6.6. We say that the 0-stables are *constant in the family* f_t if there does not exist a curve $X(t)$ in $\Delta(F)$ whose closure contains $0 \in \mathbb{C}^p$ and such that f_t has a 0-stable at $X(t)$.

DEFINITION 6.7. The *locus of instability* of F is the set of points $(y,t) \in (\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$ such that the map $F\colon (\mathbb{C}^n \times \mathbb{C}, F^{-1}(y,t) \cap \Sigma(F)) \to (\mathbb{C}^p \times \mathbb{C}, (y,t))$ is not stable.

We can now define the types of unfoldings required for applying Theorem 5.9 to discriminants. This was defined (for mono-germs) by Gaffney in [9].

DEFINITION 6.8. Suppose that $f\colon (\mathbb{C}^n,\underline{x}) \to (\mathbb{C}^p,0)$ is finitely $\mathcal{A}$-determined (i.e., has an isolated instability) and has discrete stable type. Suppose that F is a 1-parameter unfolding with a representative $F\colon U \to W$ such that $F|\Sigma(F) \cap U \to W$ is proper and finite-to-one and $F^{-1}(0) \cap \Sigma(F) \cap U = \{(x_1,0),\ldots,(x_s,0)\}$.

We call F an *excellent unfolding* if all of the following statements hold.

(i) $F^{-1}(W) = U$.
(ii) $F(U \cap \Sigma(F) \backslash S) = W \backslash T$.
(iii) The locus of instability is T.
(iv) The 0-stables are constant in the family.
(v) If $n = p$, then the degree of the map f_t is constant in the family.

REMARKS 6.9. (1) In [9], Gaffney calls unfoldings *good* when all the conditions except the 0-stable one hold.

(2) These conditions can often be checked by analyzing invariants of the members of the family (see e.g. Prop. 6.6 or Thm. 8.7 of [9]). See [18] for the case of corank-1 maps with $n < p$.

(3) Representatives $F: U \to W$ such that $F|\Sigma(F) \cap U \to W$ is proper and finite-to-one and $F^{-1}(0) \cap \Sigma(F) \cap U = \{(x_1, 0), \ldots, (x_s, 0)\}$ can always be found; see [8, p. 31].

Main Theorem on Families of Discriminants of Map Germs

We now come to the main theorem in the case that our family of hypersurfaces arises as the discriminant of the unfolding of a finitely $\mathcal{A}$-determined map germ.

We use the following notation. If $f: (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$, $n \geq p-1$, is a finitely $\mathcal{A}$-determined multi-germ, then the discriminant is a hypersurface. If F is a 1-parameter unfolding of f of the form $F(x,t) = (\bar{F}(x,t), t)$, then we shall define f_t to be the family $f_t(x) = \bar{F}(x,t)$ and define $g_t: (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ to be the family of functions defining the discriminants of f_t. We can choose g_0 reduced so that g_t will be reduced for all t in some neighborhood of 0.

Theorem 6.10. *Suppose that $f: (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$, $n \geq p-1$, is a finitely $\mathcal{A}$-determined multi-germ of discrete stable type and that F is a 1-parameter unfolding of f. Assume that the following conditions hold.*

(i) *The unfolding is excellent.*
(ii) *The characteristic normal Morse data is nonzero for strata that appear in the stratification by stable types of the discriminant of F.*

Then the discriminant of F is Whitney equisingular along the parameter axis T if and only if the sequence $(\mu^1(g_t), \ldots, \mu^{p-1}(g_t), \tilde{\chi}^p(g_t))$ is constant in the family.

Proof. Because F is excellent, there are no 1-dimensional strata other than those contained in the parameter axis. Furthermore, and again since F is excellent, we know from Propositions 6.3, 6.4, and 6.5 of [9] that (a) the stratification of $F|U \backslash F^{-1}(T) \to W \backslash T$ by stable types is a Whitney stratification and (b) the induced stratification of $f_t: U \cap (\mathbb{C}^n \times \{t\}) \to W \cap (\mathbb{C}^p \times \{t\})$ is Whitney and has a product structure over T at the origin.

Thus, by Theorem 5.9 applied to the family g_t, we get the conclusion. □

Remarks 6.11. (1) Obviously, by Theorem 5.9, other equivalent statements are possible—for example, involving the Lê numbers of g_t. The invariants chosen here are the easiest to define and are clearly topological in nature.

(2) Note that, in analogy with the Briançon–Speder–Teissier result, we seem to have the smallest number of invariants possible without making any further assumptions.

We can now prove a result similar to [13, Thm. 6.6].

Corollary 6.12 (cf. [13]). *Suppose that $f: (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$ is a finitely $\mathcal{A}$-determined multi-germ and that F is an excellent 1-parameter unfolding of f. Suppose also that f is in Mather's nice dimensions with $n \geq p$. Then the discriminant of F is Whitney equisingular along the parameter axis T if and only if the sequence $(\mu^1(g_t), \ldots, \mu^{p-1}(g_t), \tilde{\chi}^p(g_t))$ is constant in the family.*

Proof. Since f is in the nice dimensions, it is of discrete stable type. The stable types appearing in any unfolding will obviously be in the nice dimensions, so the complex links of the stable types are noncontractible by Example 4.7. Thus the characteristic normal data is nonzero and so condition (ii) of Theorem 6.10 is satisfied. □

We can now state a theorem for the case of images with $p = n+1$.

COROLLARY 6.13. *Suppose that* $f\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^{n+1}, 0)$ *is a corank-1, finitely* $\mathcal{A}$*-determined multi-germ and that* F *is an excellent 1-parameter unfolding of* f*. Then the image of* F *is Whitney equisingular along the parameter axis* T *if and only if the sequence* $(\mu^1(g_t), \dots, \mu^n(g_t), \tilde{\chi}^{n+1}(g_t))$ *is constant in the family.*

Proof. The map f is of discrete stable type because f_0 is corank 1. Furthermore, the stable types appearing in any unfolding will also be corank 1. The complex links of the stable types are nontrivial by Example 4.6, so condition (ii) of Theorem 6.10 is satisfied. □

REMARK 6.14. See [18] for conditions on members of the family to show that F is an excellent unfolding.

Main Theorem on Families of Map Germs

We can stratify the map F so that the parameter axes S and T are strata. Gaffney initiated this study of equisingularity of finitely $\mathcal{A}$-determined maps (rather than just hypersurfaces) in [9]. His statements were for mono-germs, but the extension to multi-germs is fairly straightforward.

DEFINITION 6.15. Let $F : (\mathbb{C}^n \times \mathbb{C}, \underline{x} \times 0) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$ be a family of maps $F(x,t) = (f_t(x), t)$ such that each $f_t\colon (\mathbb{C}^n, \underline{x}) \to (\mathbb{C}^p, 0)$ has an isolated instability at the origin.

We say that F is *Whitney equisingular* (*along the parameter axes*) if there is a representative $F\colon U \to W$ such that $U \subseteq \mathbb{C}^n \times \mathbb{C}$ and $W \subseteq \mathbb{C}^p \times \mathbb{C}$ can be Whitney stratified so that:

(i) F satisfies Thom's A_F condition; and
(ii) the parameter axes $S = \{\underline{x}\} \times \mathbb{C} \subseteq \mathbb{C}^n \times \mathbb{C}$ and $T = \{0\} \times \mathbb{C} \subseteq \mathbb{C}^p \times \mathbb{C}$ are strata.

REMARK 6.16. By Thom's second isotopy lemma, if a family is Whitney equisingular then it is topologically trivial.

For mono-germs, we can improve on the main theorem in [16].

THEOREM 6.17 (cf. [16, Thm. 3.3]). *Suppose that* $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ *is a corank-1, finitely* $\mathcal{A}$*-determined mono-germ and that* F *is an excellent 1-parameter unfolding of* f*. Then* F *is Whitney equisingular along the parameter axes* S *and* T *if and only if the sequence* $(\mu^1(g_t), \dots, \mu^{p-1}(g_t), \tilde{\chi}^p(g_t))$ *is constant in the family.*

Proof. Clearly, if F is Whitney equisingular along S and T, then the image is Whitney equisingular along T. Hence, by Corollary 6.13, the sequence is constant in the family.

For the converse, the same corollary implies that if the sequence is constant then the image is Whitney equisingular along T. The main theorem of [11] implies that the source may also be Whitney stratified so that S is a stratum. Since Gaffney proves this only for mono-germs, we must also be restricted to mono-germs.

The Thom A_F condition follows automatically because if $h\colon Y \to Z$ is a finite complex analytic map—with Y and Z Whitney stratified so that strata map to strata by local diffeomorphisms—then h satisfies the Thom A_f condition because the kernels in the definition of Thom A_F are all $\{0\}$. The map F is finite and so the submersions formed by taking restrictions to strata are, in fact, local diffeomorphisms. Hence, F is Whitney equisingular. □

7. Final Remarks

Remark 7.1. In [9], in particular Propositions 8.4, 8.5, and 8.6, there are a number of formulas relating various polar multiplicities and other invariants. The polar multiplicities appear with coefficient equal to ± 1 in all these propositions. This same behavior can be seen in the work of Jorge Pérez [19; 20] and Saia [21].

That these coefficients are equal to ± 1 appear to be a reflection of the fact that, in the case of $\mathbb{C}^n$ to $\mathbb{C}^p$ with $n < p$ and corank 1, the stable types appearing have characteristic normal data equal to 1 by Example 4.6.

More specifically, the alternating sum of the multiplicities of the characteristic polar cycle of the constant sheaf $\mathbb{C}^\bullet_{V(g_t)}$ in Example 4.12 can often be written as some other well-known invariant—for example, a Milnor number. Since

$$\lambda^k(\mathbb{C}^\bullet_{V(g_t)}) = \sum_{S_{\alpha,t}} m(S_{\alpha,t}) \operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}})$$

and since $m(S_{\alpha,t}) = 1$ (in the notation of Lemma 5.4), we get an alternating sum of the polar multiplicities $\operatorname{mult}_0 \Gamma^k(\overline{S_{\alpha,t}})$.

Alternatively, it is well known (see e.g. [23]) that the alternating sum of polar multiplicities is equal to the Euler obstruction. Hence corank-2 maps (and, in particular, their characteristic normal data) will need to be studied to determine the precise explanation.

Remark 7.2. As remarked before, the analogy with the Briançon–Speder–Teissier theorem shows that the number of invariants in Theorem 5.9 cannot be reduced any further without extra conditions being imposed. More than this, it seems likely that one needs the nontriviality of the complex links in the theorem. If one has a contractible complex link of some stratum, then it is probable that one can create examples where the $(\mu^*, \tilde{\chi}^{N+1})$ sequence is constant yet the parameter axis does not satisfy the Whitney conditions. Such an example therefore needs to be found.

It is difficult to express succinctly the reasons behind this strong probability in the current space. The interested reader is directed to Section 4 of Part III (in particular, Prop. 4.10) in [27].

REMARK 7.3. Although it is satisfying to reduce the number of required invariants to p, it is unsatisfactory that they are not defined consistently. In $(\mu^*(g_t), \tilde{\chi}^p(g_t))$ we have a mix of higher multiplicities and an Euler characteristic. On the other hand, many other theorems give an even less consistent mix of polar multiplicities, Milnor numbers, Lê numbers, and so on.

However, for equisingularity of maps it is possible to define still another sequence of invariants using the *disentanglement* of a map (see [17] and [7, Sec. 4] for a discussion). This sequence is denoted by $\mu_I^i(f_t)$, $1 \leq i \leq p$, since it depends on the map f_t and not on the function g_t defining the discriminant. It is possible to show that $\mu^i(g_t) = \mu_I^i(f_t)$ for $1 \leq i \leq p-1$. In low-dimensional examples it can be shown that $\tilde{\chi}^{p+1}(g_t)$ and $\mu_I^p(f_t)$ are connected though not equal; see [17]. Thus Whitney equisingularity of these maps is controlled by the p invariants $\mu_I^*(f_t)$. It would be interesting to prove in general that constancy of $(\mu^*(g_t), \tilde{\chi}^p(g_t))$ is equivalent to constancy of $\mu_I^*(f_t)$.

Another reason for studying this is that $\mu_I^p(f_t)$ is involved in the control over an unfolding being excellent (see [17; 18]).

REMARK 7.4. Also of interest is to find when equisingularity of the discriminant implies equisingularity of the map. For a family of corank-1 maps $f_t\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$, Gaffney [11] shows that if the image is Whitney equisingular then the family is Whitney equisingular. It would be good to know how generally this claim is valid. It seems unlikely, particularly for $n < p$, that an unfolding should have a source that is not Whitney stratified with S a stratum such that the image is Whitney stratified with T a stratum. In other words: one would expect an image to be more complicated than its source, and the map should not "repair" faults with stratifications.

Furthermore, Gaffney has pointed out to me that, in the case of maps, one cares about the topological triviality and so need not be restricted to Whitney stratifications. One could use the c-regular stratifications of Bekka [1] for source and target because this would imply topological triviality of the family by Thom's second isotopy lemma (since the lemma holds for these types of stratifications). Alternatively, one could attempt to find conditions under which Whitney equisingularity of the discriminant implies c-regularity of the source; since Whitney stratification is stronger than c-stratification, this would again imply topological triviality.

REMARK 7.5. An alternative viewpoint to the methods described here is given in [3]. The authors of that paper are concerned with families of mappings from $\mathbb{C}^2$ to $\mathbb{C}^3$ and introduce the concept of equisingularity at the normalization. Since their method gives a lot of information about families of maps, it would be interesting to generalize to higher dimensions and combine their method with the one of this paper.

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