

Plane Sextics with a Type- E_6 Singular Point

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1. Introduction

1.1. The Subject

This paper concludes the series [11; 12], where we give a complete deformation classification and compute the fundamental groups of maximizing irreducible plane sextics with an E -type singular point. (With the common abuse of the language, by the *fundamental group* of a curve $B \subset \mathbb{P}^2$ we mean the group $\pi_1(\mathbb{P}^2 \setminus B)$ of its complement.) Here, we consider sextics $B \subset \mathbb{P}^2$ satisfying the following conditions:

- (*) B has simple (i.e., A – D – E) singularities only,
 B has a distinguished singular point P of type E_6 , and
 B has no singular points of type E_7 or E_8 .

(Singular points of type E_7 and E_8 are excluded in order to reduce the lists. Sextics with a type- E_8 point are considered in [12], and irreducible sextics with a type- E_7 point are considered in [11]. Reducible sextics with a type- E_7 point, as well as the more involved case of a distinguished D -type point, may appear elsewhere.)

Recall that a plane sextic B with simple singularities only is called *maximizing* if the total Milnor number $\mu(B)$ assumes the maximal possible value 19. It is well known that maximizing sextics are defined over algebraic number fields (as they are related to singular $K3$ -surfaces). Furthermore, such sextics are rigid: two maximizing sextics are equisingular deformation equivalent if and only if they are related by a projective transformation.

Another important class is formed by the so-called sextics of *torus type*—in other words, those whose equation can be represented in the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are certain homogeneous polynomials of degree 2 and 3, respectively. (This property turns out to be equisingular deformation invariant.) Each sextic B of torus type can be perturbed to Zariski’s famous six-cuspidal sextic [21], which is obtained when f_2 and f_3 as just described are sufficiently generic. Hence, the group $\pi_1(\mathbb{P}^2 \setminus B)$ factors to the *reduced braid group* $\tilde{\mathbb{B}}_3 := \mathbb{B}_3/(\sigma_1\sigma_2)^3$; in particular, it is never finite. (The existence of two distinct families of irreducible six-cuspidal sextics, those of and not of torus type, was first stated by Del Pezzo and then proved by Segre; see e.g. [19, p. 407]. Zariski [21] later showed that the two families differ by the fundamental groups.)

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A representation of the equation of B in the form $f_2^3 + f_3^2 = 0$ is called a *torus structure*. The points of intersection of the conic $\{f_2 = 0\}$ and cubic $\{f_3 = 0\}$ are always singular for B ; they are called *inner* (with respect to the given torus structure), whereas the other singular points are called *outer*. In the listing that follows (see Tables 1 and 2 in Section 2.6), we indicate sextics of torus type by representing their sets of singularities in the form

$$(\text{inner singularities}) \oplus \text{outer singularities}.$$

An exception is the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$, which is always of torus type and admits four distinct torus structures.

Formally, the deformation classification of plane sextics with simple singularities can be reduced to a purely arithmetical problem (see [6]), and for maximizing sextics this latter problem has been completely solved (see [18; 20]) in the sense that all deformation classes have been enumerated. Unfortunately, this approach—based on the theory of $K3$ -surfaces and the global Torelli theorem—is not constructive, and very little is known about the geometry of the curves. (Sporadic examples using explicit equations are scattered in the literature.) Here we use another approach, suggested in [10] and [11]: a plane curve B with a sufficiently deep (with respect to the degree) singularity is reduced to a trigonal curve \bar{B} in an appropriate Hirzebruch surface. If B is a maximizing sextic (with a triple point), then \bar{B} is a maximal trigonal curve; hence it can be studied using Grothendieck's *dessin d'enfants* of its functional j -invariant. In the end, we obtain an explicit geometric description of \bar{B} and B (rather than equation); among other things, this description is suitable for computing the braid monodromy and hence the fundamental group of the curves.

1.2. Results

The principal results of this paper are Theorem 1.2.1 (the classification) and Theorems 1.2.2 and 1.2.3 (the computation of the fundamental group).

1.2.1. THEOREM. *Up to projective transformation (equivalently, up to equisingular deformation), there are 93 maximizing plane sextics satisfying condition (*) realizing 71 combinatorial sets of singularities; of them, 53 sextics (40 sets of singularities) are irreducible (see Table 1) and 40 sextics (32 sets of singularities) are reducible (see Table 2).*

In Theorem 1.2.1, one set of singularities is common: $\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$ is realized by one reducible and three irreducible sextics. This theorem is proved in Section 2; for more details, see comments to the tables in Section 2.6.

Among the irreducible sextics in Theorem 1.2.1, twelve (eight sets of singularities) are of torus type. Seven of them (four sets of singularities) are “new” in the sense that they have not been extensively studied before.

1.2.2. THEOREM. *Let $B \subset \mathbb{P}^2$ be an irreducible maximizing sextic that satisfies condition (*) and is not of torus type. If the set of singularities of B is $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (nos. 4 and 5 in Table 1), then $\pi_1(\mathbb{P}^2 \setminus B) = \mathrm{SL}(2, \mathbb{F}_5) \rtimes \mathbb{Z}_6$ (see (3.3.3) or (3.5.2) for the presentations). Otherwise, $\pi_1(\mathbb{P}^2 \setminus B) = \mathbb{Z}_6$.*

1.2.3. THEOREM. *Let B be a sextic as in Theorem 1.2.1, and let B' be a proper irreducible perturbation of B that is not of torus type. Then $\pi_1(\mathbb{P}^2 \setminus B') = \mathbb{Z}_6$.*

Theorems 1.2.2 and 1.2.3 are proved in Section 3 and Section 4.4, respectively. The theorems substantiate my conjecture asserting that the fundamental group of an irreducible plane sextic that has simple singularities only and is not of torus type is finite. Recall that the conjecture was originally motivated by certain experimental evidence (which has now been extended) and the fact that the abelianization of the commutant of the fundamental group of an irreducible sextic that is not of torus type is finite (which is a restatement of the proved part of the so-called Oka conjecture; see [5]). At present, the conjecture is essentially settled for sextics with a triple singular point (the case of a D -type point is considered in a forthcoming paper).

In Section 3 we also write down presentations of the fundamental groups of all other sextics as in Theorem 1.2.1, and in Section 4 we consider their perturbations. In particular, we prove the following theorem, computing the groups of all “new” irreducible sextics of torus type.

1.2.4. THEOREM. *The fundamental group of a sextic with the set of singularities $(E_6 \oplus A_{11}) \oplus A_2$, $(E_6 \oplus A_8 \oplus A_2) \oplus A_3$, or $(E_6 \oplus A_8 \oplus A_2) \oplus A_2 \oplus A_1$ (nos. 12, 13, 18, and 40 in Table 1) is isomorphic to the reduced braid group \mathbb{B}_3 . The groups of sextics with the set of singularities $(E_6 \oplus A_5 \oplus 2A_2) \oplus A_4$ (nos. 9 and 41) are not isomorphic to \mathbb{B}_3 ; their presentations are found in Sections 3.3.9 and 3.6(3), respectively.*

1.2.5. PROBLEM. Are the fundamental groups of the two sextics with the set of singularities $(E_6 \oplus A_5 \oplus 2A_2) \oplus A_4$ (nos. 9 and 41 in Table 1) isomorphic to each other? (A similar question still stands for the sextics with the set of singularities $(2E_6 \oplus A_5) \oplus A_2$, nos. 6 and 7 in Table 1; see [4; 17].)

1.2.6. THEOREM. *Let B be a sextic as in Theorem 1.2.1, and let B' be a proper irreducible perturbation of B that is of torus type. Then, with the following few exceptions:*

- a perturbation $E_6 \oplus A_5 \oplus 4A_2 \rightarrow$ (a curve of weight 8) (see [9]) and
- a perturbation that can be further perturbed to a curve B'' with the set of singularities $(6A_2) \oplus 4A_1$ (see [4; 9]),

the group $\pi_1(\mathbb{P}^2 \setminus B')$ is the reduced braid group \mathbb{B}_3 .

(Here, the *weight* of a sextic is understood in the sense of [5] as the total weight of all its singular points, where the weight $w(P)$ of a singular point P is defined via $w(A_{3k-1}) = k$, $w(E_6) = 2$, and $w(P) = 0$ otherwise. The fundamental group of a sextic of weight ≥ 8 is much larger than \mathbb{B}_3 because it has a larger Alexander polynomial. In the second exceptional case in the statement, one has $\pi_1(\mathbb{P}^2 \setminus B') = \mathbb{B}_4/\sigma_1^2\sigma_2\sigma_3^2\sigma_2$; see [4; 9].) Theorem 1.2.6 is proved in Section 4.5.

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2. The Classification

2.1. The Settings

We recall briefly some of the results of [11] concerning the construction and the classification of plane sextics satisfying (*). For details on maximal trigonal curves and their skeletons, see [10] or [11]. We denote by $\Sigma_k, k > 0$, the geometrically ruled rational surface with an exceptional section E of self-intersection $-k$.

2.1.1. PROPOSITION. *There is a natural bijection ϕ , invariant under equisingular deformations, between Zariski open (in each equisingular stratum) subsets of the following two sets:*

- (1) *plane sextics B with a distinguished type- \mathbf{E}_6 singular point P , and*
- (2) *trigonal curves $\bar{B} \subset \Sigma_4$ with a distinguished type- $\tilde{\mathbf{A}}_5$ singular fiber \bar{F} .*

A sextic B is irreducible if and only if so is $\bar{B} = \phi(B)$, and B is maximizing if and only if \bar{B} is maximal and stable—that is, has no singular fibers of types $\tilde{\mathbf{A}}_0^, \tilde{\mathbf{A}}_1^*$, or $\tilde{\mathbf{A}}_2^*$ (these fibers are called unstable).*

Up to fiberwise equisingular deformation (equivalently, up to automorphism of Σ_k), maximal trigonal curves $\bar{B} \subset \Sigma_k$ are classified by their *skeletons* and *type specifications*. The skeleton $\text{Sk} = \text{Sk}_{\bar{B}} \subset S^2$ (which is defined as Grothendieck’s *dessin d’enfants* of the functional j -invariant of \bar{B}) is an embedded connected bipartite graph with all \bullet -vertices of valency ≤ 3 and all \circ -vertices of valency ≤ 2 . The \bullet -vertices of valency ≤ 2 and \circ -vertices of valency 1 are called *singular*; they correspond to the unstable and type- $\tilde{\mathbf{E}}$ singular fibers of \bar{B} . Besides, each n -gonal *region* of Sk (i.e., connected component of the complement $S^2 \setminus \text{Sk}$) contains a single singular fiber of \bar{B} , which is of type $\tilde{\mathbf{A}}_{n-1}$ ($\tilde{\mathbf{A}}_0^*$ if $n = 1$) or $\tilde{\mathbf{D}}_{n+4}$. The type specification is the function choosing, for each singular vertex and each region of Sk , whether the corresponding fiber is of type $\tilde{\mathbf{A}}$ or $\tilde{\mathbf{D}}, \tilde{\mathbf{E}}$.

The skeleton and the type specification of a maximal curve $\bar{B} \subset \Sigma_k$ are subject to the relation

$$\#_{\bullet} + \#_{\circ}(1) + \#_{\bullet}(2) = 2(k - t),$$

where t is the number of triple singular points of the curve, $\#_*(n)$ is the number of $*$ -vertices of valency n , $*$ = \bullet or \circ , and $\#_*$ is the total number of $*$ -vertices. Any pair satisfying this relation gives rise to a unique curve.

Under the assumptions of this paper (\bar{B} has no unstable fibers or fibers of type $\tilde{\mathbf{E}}_7$ or $\tilde{\mathbf{E}}_8$), all \circ -vertices of Sk are of valency 2 and all its \bullet -vertices are of valency 3 or 1, the latter corresponding to the type- $\tilde{\mathbf{E}}_6$ singular fibers of \bar{B} . Hence the preceding vertex count can be simplified to

$$\#_{\bullet} = 2(k - t). \tag{2.1.2}$$

Furthermore, the \circ -vertices can be ignored, with the convention that a \circ -vertex is to be understood at the middle of each edge connecting two \bullet -vertices.

To summarize, the proof of Theorem 1.2.1 reduces to the enumeration of all pairs (Sk , type specification), where $\text{Sk} \subset S^2$ is a connected graph with all vertices of valency 3 or 1 and with a distinguished hexagonal region.

2.2. The Case of Two Type- E_6 Points

The only maximizing sextic with three type- E_6 singular points (no. 1 in Table 1) is well known; see [4; 17]. Assume that B has two type- E_6 singular points. Then Sk has one monovalent \bullet -vertex and, in view of equation (2.1.2), it can be obtained by attaching the fragment $\bullet\text{---}\bullet$ at the center of an edge of a regular 3-graph Sk' with two or four vertices (see [7]). All possibilities resulting in a skeleton Sk with a hexagonal region are listed in Figure 1, where Sk' is shown in black and the possible position of the insertion is shown in gray. The sextics obtained are nos. 2–8 in Table 1; all curves are irreducible given the existence of a type- \tilde{E}_6 singular fiber.

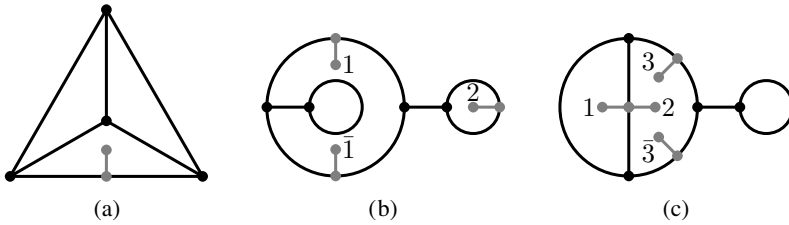


Figure 1 Two type- E_6 points

2.2.1. REMARK. For insertion 2 in Figure 1(c), the skeleton Sk has two hexagonal regions resulting in two sextics (nos. 6 and 7 in Table 1). There are indeed two distinct deformation families of sextics of torus type with the set of singularities $(2E_6 \oplus A_5) \oplus A_2$; see [4; 17] and Remark 2.6.1.

2.3. The Case of a Hexagon with a Loop

For the rest of this section, assume that P is the only type- E_6 singular point of B . Then Sk is a regular 3-graph with a distinguished hexagonal region \bar{H} . Such skeletons can be enumerated using [2]; however, we choose a more constructive descriptive approach.

Combinatorially, there are three possibilities for \bar{H} :

- (i) hexagon with a loop (see Figure 2, left);
- (ii) hexagon with a double loop (see Figure 5); or
- (iii) genuine hexagon (see Section 2.5).

Assume that \bar{H} is a hexagon with a loop (see the shaded area in Figure 2, left). Removing a neighborhood of \bar{H} from Sk and patching vertices u, v in the figure to a single edge, one obtains another regular 3-graph Sk' with at most four vertices

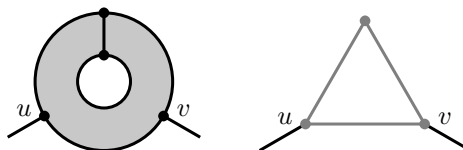


Figure 2 A hexagon with a loop

(see [7]). Conversely, Sk can be obtained from Sk' by inserting a fragment as in Figure 2, left, at the middle of an edge of Sk' . The essentially distinct possibilities for the position of the insertion are shown in Figure 3 for irreducible curves and in Figure 4 for reducible curves (a reducibility criterion is found in [10]). To simplify the drawings, we represent the insertion by a gray triangle (as in Figure 2, right).

The resulting sextics are nos. 9–32 in Table 1 and nos. 1'–19' in Table 2.

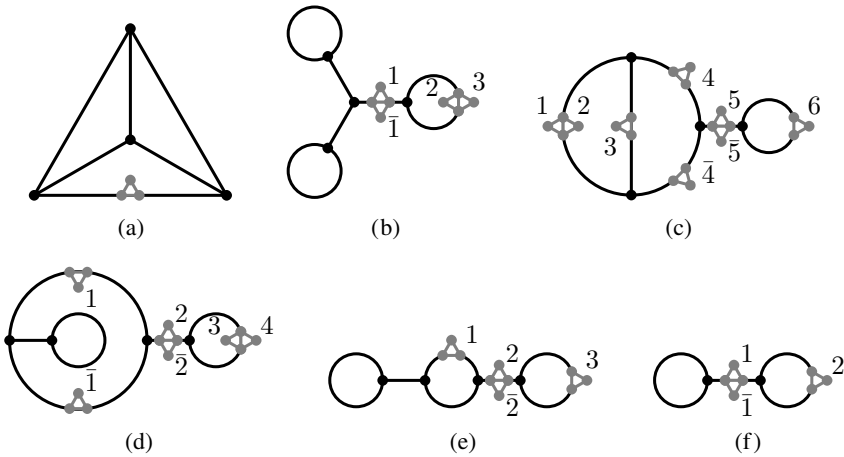


Figure 3 A hexagon with a loop: Irreducible curves

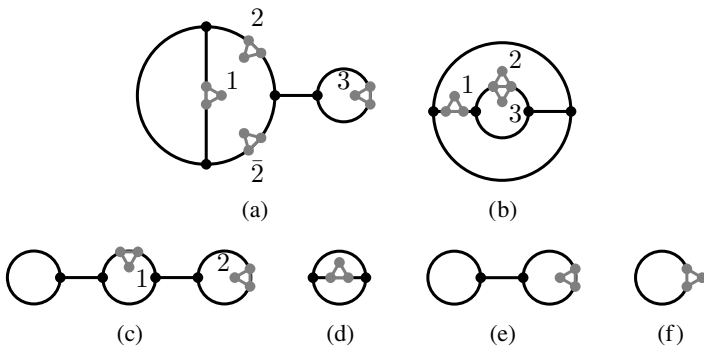


Figure 4 A hexagon with a loop: Reducible curves

2.3.1. **REMARK.** The curves in pairs nos. 27, 28, nos. 31, 32, and nos. 15', 16' in the tables differ by their type specifications: the type- \tilde{D}_5 fiber can be chosen either inside one of the “free” loops of Sk' shown in the figure or inside the inner loop of the hexagon. We assume that the latter possibility corresponds to curves nos. 28, 32 in Table 1 and no. 16' in Table 2.

2.4. *The Case of a Hexagon with a Double Loop*

Now assume that the distinguished hexagon \tilde{H} looks like the outer region in Figure 5, left. Each of the remaining fragments A, B of Sk has an odd number of

vertices, and the total number of remaining vertices is at most four. Hence, we can assume that A has one vertex and B has at most three vertices. Then A is a single loop and the graph can be redrawn as shown in Figure 5, right, where \bar{H} is represented by the shaded area. In other words, Sk can be obtained from a regular 3-graph Sk' with two or four vertices and with a loop (see [7]) by replacing a loop with the fragment shown in Figure 5, right. The five possibilities are listed in Figure 6; the resulting sextics are nos. 33–38 in Table 1. (Using [10], one can easily show that the existence of a fragment as in Figure 5, right implies that the curve is irreducible.)

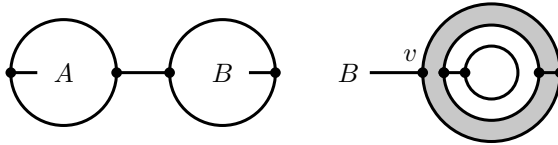


Figure 5 A hexagon with a double loop

2.4.1. REMARK. The skeleton in Figure 6(f) has a symmetry interchanging its two monogons and two pentagons. For this reason, unlike the case described in Remark 2.3.1, items nos. 37 and 38 are realized by one deformation family each.

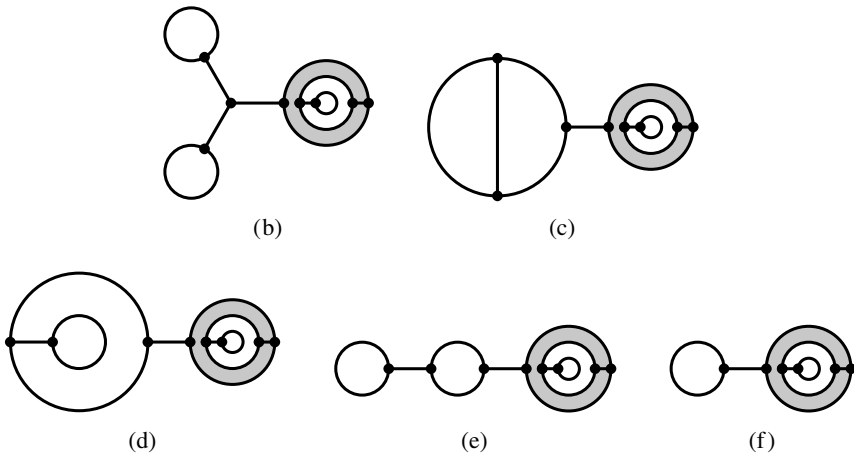


Figure 6 A hexagon with a double loop: The five skeletons

2.5. The Case of a Genuine Hexagon

Finally, assume that \bar{H} is a genuine hexagon (i.e., all six vertices in the boundary $\partial\bar{H}$ are pairwise distinct). In other words, $\partial\bar{H}$ is the equator in S^2 , and Sk is obtained from $\partial\bar{H}$ by completing it to a regular 3-graph by inserting at most two vertices and connecting edges into one of the two hemispheres. All possibilities are listed in Figure 7 for irreducible curves and in Figure 8 for reducible curves; the resulting sextics are nos. 39–42 in Table 1 and nos. 20'–38' in Table 2.

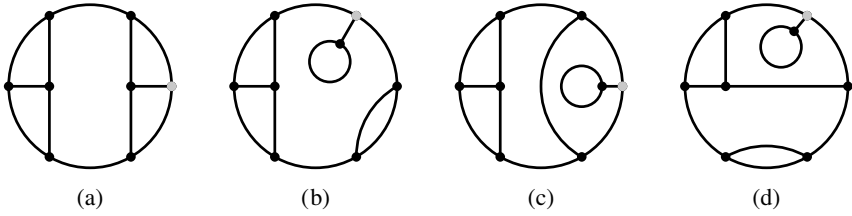


Figure 7 A genuine hexagon: Irreducible curves

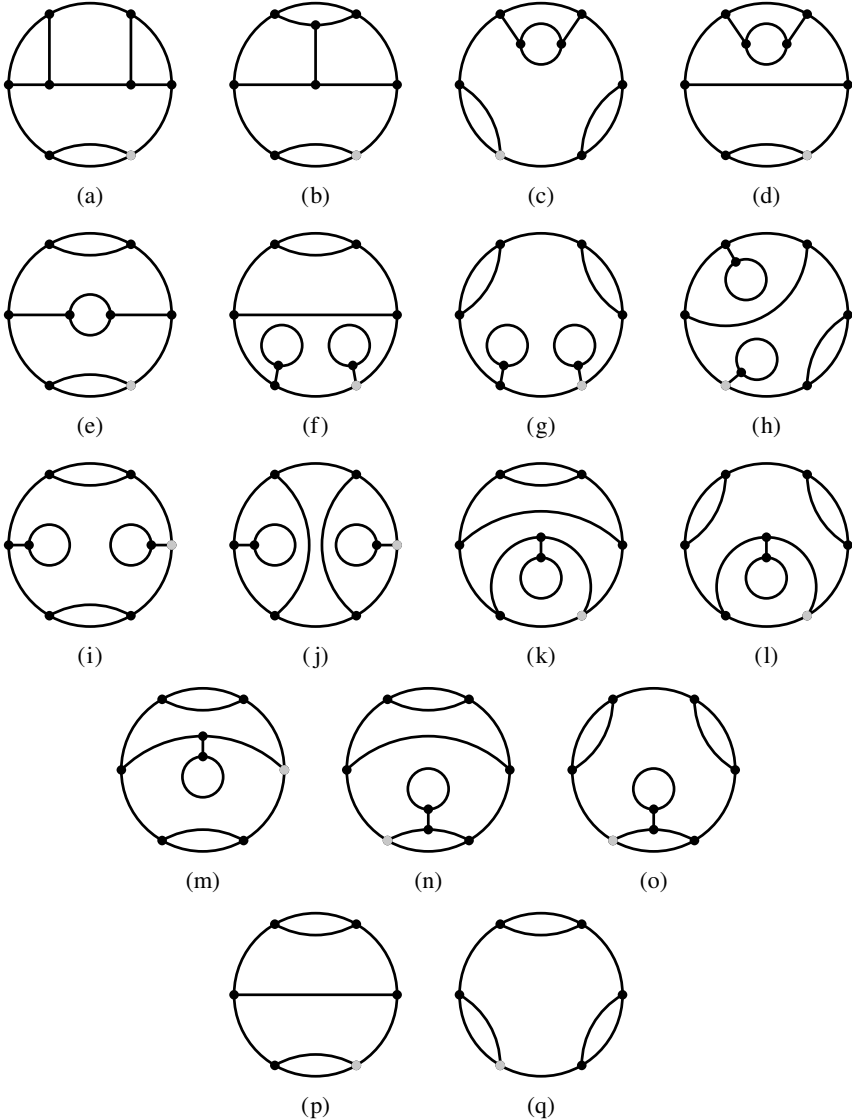


Figure 8 A genuine hexagon: Reducible curves

2.6. The List

The results of the classification are collected in Table 1 for irreducible curves and in Table 2 for reducible curves, where we list the combinatorial types of singularities and references to the corresponding figures. (For sextics of torus type, the inner singularities are also indicated; the exception is no. 39 in Table 1, which admits four distinct torus structures.) Equal superscripts precede combinatorial types shared by several items in the tables. (The set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$ prefixed with ¹⁰ is common for both tables.) The Count column lists the numbers (n_r, n_c) of real curves and pairs of complex conjugate curves. The last two columns refer to the computation of the fundamental group and indicate the parameters used (explained in the relevant sections). In addition, the curves with nonabelian fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus B)$ are prefixed with one of the following symbols.

- *: The group π_1 is not abelian.
- ? : The group π_1 is not known to be abelian.
- **: For curves of torus type, $\pi_1 \neq \overline{\mathbb{B}}_3$.

2.6.1. **REMARK.** For pairs nos. 4, 5 and nos. 6, 7, one can ask if the two curves remain nonequivalent if a permutation of the two type- \mathbf{E}_6 points is allowed. For nos. 6 and 7, there still are two distinct equisingular deformation families (see [4; 17]); for nos. 4 and 5, the two curves become equivalent (see [18]). (Alternatively, if nos. 4 and 5 were not equivalent, then each of the curves would have a symmetry interchanging its two type- \mathbf{E}_6 points. Because the curve is maximizing, the symmetry would necessarily be stable—contradicting [7].)

2.6.2. **REMARK.** The sets of singularities nos. 3 and 8 with $(n_r, n_c) = (0, 1)$ can be realized by real curves; see [18]. However, with respect to this real structure, the two type- \mathbf{E}_6 points must be complex conjugate.

3. The Computation

3.1. Preliminaries and Notation

To compute the fundamental groups, we use Zariski–van Kampen’s method [14], applying it to the ruling of Σ_4 . The principal steps of the computation are outlined here; for more details, see [11; 12].

Fix a maximizing sextic B satisfying $(*)$ and let \bar{B} be the maximal trigonal curve given by Proposition 2.1.1. For the fiber at infinity F_∞ we take the distinguished type- $\bar{\mathbf{A}}_5$ fiber \bar{F} of \bar{B} , and for the reference fiber F we take the fiber over an appropriate vertex v of the skeleton Sk of \bar{B} in the boundary $\partial\bar{H}$ of the hexagonal region \bar{H} containing \bar{F} . Choose a marking at v (see [10]) so that the edges e_2 and e_3 at v belong to the boundary $\partial\bar{H}$, and let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a canonical basis in F defined by this marking; see [10] or Figure 9. (The precise choice of the vertex v and the marking is explained below on a case-by-case basis.) Denote $\rho = \alpha_1\alpha_2\alpha_3$.

Table 1 Maximal Sets of Singularities with a Type- E_6 Point
Represented by Irreducible Sextics

#	Set of singularities	Figure	Count	π_1	Parameters
**1	$(3E_6) \oplus A_1$		(1, 0)	see [4]	
**2	$(2E_6 \oplus 2A_2) \oplus A_3$	1(a)	(1, 0)	see [4]	
3	$2E_6 \oplus A_7$	1(b)-1	(0, 1)	3.3	$(-, -, 1, -)$
*4	$^1 2E_6 \oplus A_4 \oplus A_3$	1(b)-2	(1, 0)	3.5	$l = 4$
*5	$^1 2E_6 \oplus A_4 \oplus A_3$	1(c)-1	(1, 0)	3.3	$(4, 5, -, -)$
**6	$^2 (2E_6 \oplus A_5) \oplus A_2$	1(c)-2	(1, 0)	see [4]	
**7	$^2 (2E_6 \oplus A_5) \oplus A_2$	1(c)-2	(1, 0)	see [4]	
8	$2E_6 \oplus A_6 \oplus A_1$	1(c)-3	(0, 1)	3.2	
**9	$^3 (E_6 \oplus A_5 \oplus 2A_2) \oplus A_4$	3(a)	(1, 0)	3.3.9	$(6, 5, 3, -)$
10	$E_6 \oplus A_{13}$	3(b)-1	(0, 1)	3.3	$(-, -, 1, -)$
11	$^4 E_6 \oplus A_{10} \oplus A_3$	3(b)-2	(1, 0)	3.3.4	
*12	$(E_6 \oplus A_{11}) \oplus A_2$	3(b)-3	(1, 0)	3.3.7	
*13	$(E_6 \oplus A_8 \oplus A_2) \oplus A_3$	3(c)-1	(1, 0)	3.3.10	$(9, 4, 3, -)$
14	$^5 E_6 \oplus A_7 \oplus A_4 \oplus A_2$	3(c)-2	(1, 0)	3.3	$(5, 8, 3, -)$
15	$^6 E_6 \oplus A_5 \oplus 2A_4$	3(c)-3	(1, 0)	3.3	$(5, 5, 6, -)$
16	$^7 E_6 \oplus A_8 \oplus A_4 \oplus A_1$	3(c)-4	(0, 1)	3.3	$(-, -, -, 1)$
17	$^8 E_6 \oplus A_{10} \oplus A_2 \oplus A_1$	3(c)-5	(0, 1)	3.3	$(-, -, 1, -)$
*18	$^9 (E_6 \oplus A_8 \oplus A_2) \oplus A_2 \oplus A_1$	3(c)-6	(1, 0)	3.3.10	$(9, 3, -, 3)$
19	$E_6 \oplus A_7 \oplus A_6$	3(d)-1	(0, 1)	3.3	$(-, -, -, 1)$
20	$^{10} E_6 \oplus A_9 \oplus A_4$	3(d)-2	(0, 1)	3.3	$(-, -, 1, -)$
21	$E_6 \oplus A_6 \oplus A_4 \oplus A_3$	3(d)-3	(1, 0)	3.3	$(4, 7, -, -)$
22	$^5 E_6 \oplus A_7 \oplus A_4 \oplus A_2$	3(d)-4	(1, 0)	3.3	$(3, 8, -, 5)$
23	$^4 E_6 \oplus A_{10} \oplus A_3$	3(e)-1	(1, 0)	3.3	$(-, -, -, 1)$
24	$E_6 \oplus A_{12} \oplus A_1$	3(e)-2	(0, 1)	3.3	$(-, -, 1, -)$
25	$^8 E_6 \oplus A_{10} \oplus A_2 \oplus A_1$	3(e)-3	(1, 0)	3.3	$(11, 3, -, 2)$
26	$E_6 \oplus D_{13}$	3(f)-1	(1, 0)	3.3	$(-, -, 1, -)$
27	$^{11} E_6 \oplus D_5 \oplus A_8$	3(f)-1	(0, 1)	3.3	$(-, -, 1, -)$
28	$^{11} E_6 \oplus D_5 \oplus A_8$	3(f)-1	(1, 0)	3.3.6	
29	$E_6 \oplus D_{11} \oplus A_2$	3(f)-2	(1, 0)	3.3	$(-, -, -, 1)$
30	$E_6 \oplus D_7 \oplus A_6$	3(f)-2	(1, 0)	3.3	$(-, -, -, 1)$
31	$^{12} E_6 \oplus D_5 \oplus A_6 \oplus A_2$	3(f)-2	(1, 0)	3.3	$(7, 3, -, -)$
32	$^{12} E_6 \oplus D_5 \oplus A_6 \oplus A_2$	3(f)-2	(1, 0)	3.3.5	
33	$^{10} E_6 \oplus A_9 \oplus A_4$	6(b)	(1, 0)	3.5	$l = 10$
34	$^{13} E_6 \oplus A_6 \oplus A_4 \oplus A_2 \oplus A_1$	6(c)	(1, 0)	3.5	$l = 7$
35	$^6 E_6 \oplus A_5 \oplus 2A_4$	6(d)	(1, 0)	3.5	$l = 6$
36	$^7 E_6 \oplus A_8 \oplus A_4 \oplus A_1$	6(e)	(1, 0)	3.5	$l = 9$
37	$E_6 \oplus D_9 \oplus A_4$	6(f)	(1, 0)	3.5	
38	$E_6 \oplus D_5 \oplus 2A_4$	6(f)	(1, 0)	3.5	$l = 5$
**39	$E_6 \oplus A_5 \oplus 4A_2$	7(a)	(1, 0)	3.6(1)	$(3, 3, 6, 3, 3, -)$
*40	$^9 (E_6 \oplus A_8 \oplus A_2) \oplus A_2 \oplus A_1$	7(b)	(0, 1)	3.6(2)	$(9, -, 3, 3, -, 2)$
**41	$^3 (E_6 \oplus A_5 \oplus 2A_2) \oplus A_4$	7(c)	(1, 0)	3.6(3)	$(6, -, 5, 3, 3, -)$
42	$^{13} E_6 \oplus A_6 \oplus A_4 \oplus A_2 \oplus A_1$	7(d)	(0, 1)	3.6	$(7, -, 3, 5, 2, -)$

Table 2 Maximal Sets of Singularities with a Type- E_6 Point Represented by Reducible Sextics

#	Set of singularities	Figure	Count	π_1	Parameters
**1'	$^{14}(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$	4(a)-1	(1, 0)	3.4(1)	(6, 4, 6, -)
2'	$^{15}\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$	4(a)-2	(0, 1)	3.4	(-, -, 2, -)
*3'	$^{16}\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	4(a)-3	(1, 0)	3.4(2)	(4, 8, -, 3)
4'	$^{17}\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$	4(b)-1	(1, 0)	3.4	(-, -, 2, -)
5'	$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$	4(b)-2	(1, 0)	3.4	(7, 4, 4, -)
*6'	$^{18}\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	4(b)-3	(1, 0)	3.4(3)	(5, 6, 4, -)
7'	$^{10}\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$	4(c)-1	(1, 0)	3.4	(-, -, -, 1)
8'	$^{19}\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	4(c)-2	(1, 0)	3.4	(4, 10, -, 2)
9'	$\mathbf{E}_6 \oplus \mathbf{D}_9 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	4(d)	(1, 0)	3.4	(-, -, 2, -)
10'	$\mathbf{E}_6 \oplus \mathbf{D}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	4(d)	(1, 0)	3.4	(-, -, 2, -)
11'	$\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	4(d)	(1, 0)	3.4	(5, 4, -, -)
12'	$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	4(d)	(1, 0)	3.4.3	(-, -, 2, -)
13'	$\mathbf{E}_6 \oplus \mathbf{D}_{10} \oplus \mathbf{A}_3$	4(e)	(1, 0)	3.4	(-, -, -, 1)
14'	$\mathbf{E}_6 \oplus \mathbf{D}_8 \oplus \mathbf{A}_5$	4(e)	(1, 0)	3.4	(-, -, -, 1)
*15'	$^{20}\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$	4(e)	(1, 0)	3.4(4)	(4, 6, -, -)
*16'	$^{20}\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$	4(e)	(1, 0)	3.4.3	(4, 6, -, 1)
17'	$\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{D}_6$	4(f)	(1, 0)	3.4.2	
18'	$\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_1$	4(f)	(1, 0)	3.4.3	(-, 2, -, -)
19'	$\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_2$	4(f)	(1, 0)	3.4.3	(3, -, -, -)
**20'	$(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	8(a)	(1, 0)	3.7(1)	(2, 6, 3, 4, 3, -)
?21'	$\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	8(b)	(1, 0)	3.7(2)	(2, 5, 4, 3, 4, -)
*22'	$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_1$	8(c)	(1, 0)	3.10(1)	(2, 8, 2, -, 4, -)
*23'	$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$	8(d)	(1, 0)	3.10(2)	(2, 4, 6, 4, -, -)
**24'	$(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus 3\mathbf{A}_1$	8(e)	(1, 0)	3.10(3)	(2, 6, 6, 2, -, -)
25'	$^{19}\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	8(f)	(1, 0)	3.8	(10, -, 4, 2, -, -)
26'	$^{21}\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus 2\mathbf{A}_1$	8(g)	(1, 0)	3.8	(12, -, 2, -, 2, -)
27'	$^{15}\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$	8(h)	(0, 1)	3.8	(8, -, 2, -, 6, -)
**28'	$^{21}(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus 2\mathbf{A}_1$	8(i)	(1, 0)	3.8	(12, -, 2, -, -, 2)
**29'	$^{14}(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$	8(j)	(1, 0)	3.8	(6, -, 4, 6, -, -)
*30'	$^{18}\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	8(k)	(1, 0)	3.7(3)	(6, 5, 4, 2, -, -)
31'	$^{17}\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$	8(l)	(1, 0)	3.7	(6, 7, 2, -, 2, -)
32'	$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_1$	8(m)	(1, 0)	3.7	(8, 5, 2, -, -, 2)
*33'	$^{16}\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	8(n)	(1, 0)	3.7(4)	(8, 3, -, 4, 2, -)
*34'	$\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$	8(o)	(1, 0)	3.7(5)	(10, 3, -, 2, -, 2)
35'	$\mathbf{E}_6 \oplus \mathbf{D}_8 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_1$	8(p)	(1, 0)	3.9	(-, 2, -, 4, 2, -)
*36'	$\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$	8(p)	(1, 0)	3.10(4)	(2, 4, 4, -, -, -)
*37'	$\mathbf{E}_6 \oplus \mathbf{D}_{10} \oplus 3\mathbf{A}_1$	8(q)	(1, 0)	3.10(5)	(2, -, 2, -, 2, -)
38'	$\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$	8(q)	(1, 0)	3.9	(-, 6, 2, -, 2, -)

According to [11], the generators $\alpha_1, \alpha_2, \alpha_3$ are subject to the so-called *relation at infinity*:

$$\rho^4 = (\alpha_2 \alpha_3)^3. \tag{3.1.1}$$

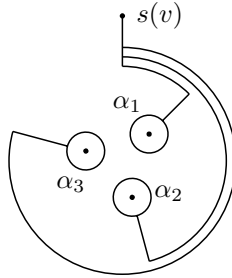


Figure 9 A canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$

Let F_1, \dots, F_r be the singular fibers of \bar{B} other than \bar{F} . Dragging F about F_j and keeping the base point in an appropriate section, one obtains an automorphism $m_j \in \mathbb{B}_3 \subset \text{Aut}(\alpha_1, \alpha_2, \alpha_3)$, called the *braid monodromy* about F_j . In this notation, the group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus B)$ has a presentation of the form

$$\pi_1 = \langle \alpha_1, \alpha_2, \alpha_3 \mid m_j = \text{id}, j = 1, \dots, r, \text{ and (3.1.1)} \rangle, \tag{3.1.2}$$

where each *braid relation* $m_j = \text{id}$, $j = 1, \dots, r$, should be understood as the triple of relations $m_j(\alpha_i) = \alpha_i$, $i = 1, 2, 3$. Furthermore, in the presence of the relation at infinity, (any) one of the braid relations $m_j = \text{id}$, $j = 1, \dots, r$, can be omitted. The braid monodromies m_j are computed using [10]; all necessary details are explained in this section.

Throughout the paper, all finite groups/quotients are analyzed using GAP [13]. Most infinite groups are handled by means of the following obvious lemma, which we state here for future reference.

3.1.3. LEMMA. *Let G be a group, and let $a \in G$ be a central element whose projection to the abelianization $G/[G, G]$ has infinite order. Then the projection $G \rightarrow G/a$ restricts to an isomorphism $[G, G] = [G/a, G/a]$.*

Given two elements α, β of a group and a nonnegative integer m , we introduce the notation

$$\{\alpha, \beta\}_m = \begin{cases} (\alpha\beta)^k(\beta\alpha)^{-k} & \text{if } m = 2k \text{ is even,} \\ ((\alpha\beta)^k\alpha)((\beta\alpha)^k\beta)^{-1} & \text{if } m = 2k + 1 \text{ is odd.} \end{cases}$$

The relation $\{\alpha, \beta\}_m = 1$ is equivalent to $\sigma^m = \text{id}$, where σ is the generator of the braid group \mathbb{B}_2 acting on the free group $\langle \alpha, \beta \rangle$. Hence,

$$\{\alpha, \beta\}_m = \{\alpha, \beta\}_n = 1 \text{ is equivalent to } \{\alpha, \beta\}_{\text{g.c.d.}(m,n)} = 1. \tag{3.1.4}$$

For the small values of m , the relation $\{\alpha, \beta\}_m = 1$ takes the following form:

- $m = 0$, tautology;
- $m = 1$, the identification $\alpha = \beta$;
- $m = 2$, the commutativity relation $[\alpha, \beta] = 1$;
- $m = 3$, the braid relation $\alpha\beta\alpha = \beta\alpha\beta$.

3.2. Two Type- E_6 Singular Points

It suffices to consider the set of singularities $2E_6 \oplus A_6 \oplus A_1$ (no. 8 in Table 1) only. The fundamental groups of all sextics of torus type with two type- E_6 singular points are found in [4], and the remaining curves that are not of torus type are covered in Sections 3.3 and 3.5.

Thus, consider the skeleton Sk given by Figure 1(c)-3 (i.e., insertion 3 in Figure 1(c)). Let u be the monovalent \bullet -vertex of Sk , and let v be the trivalent \bullet -vertex adjacent to u . Mark v so that $[u, v]$ is the edge e_2 at v . Then, in addition to relation (3.1.1), the group π_1 has the relations

$$\{\alpha_2\alpha_3\alpha_2^{-1}, \alpha_1\}_7 = \{\alpha_1\alpha_2\alpha_3\alpha_2^{-1}\alpha_1^{-1}, \alpha_2\}_2 = 1, \quad \rho\alpha_2\rho^{-1} = \alpha_2\alpha_3\alpha_2^{-1},$$

obtained from the heptagon, the bigon, and the monovalent \bullet -vertex u of Sk , respectively. The resulting group is abelian.

3.3. Hexagon with a Loop: Irreducible Curves

Assume that \bar{H} is a hexagon with a loop (see Section 2.3 and Figure 10), and take for v the vertex shown in the figure.

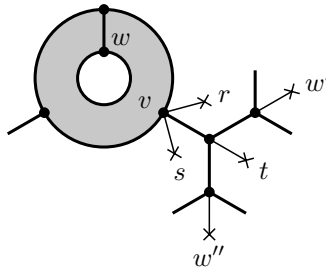


Figure 10 A hexagon with a loop: The regions

The inner loop of \bar{H} (the monogonal region containing w in its boundary) gives the relation

$$\alpha_1 = \alpha_3^{-1}\alpha_2\alpha_3. \tag{3.3.1}$$

Extend Sk to a dessin (see [10]), and consider the \times -vertices r, s, t , and w' shown in Figure 10. (We do not assert that all these vertices are distinct.) Assume that they are at the centers of l -, m -, n -, and k -gonal regions of Sk , respectively. Then the braid relations about the singular fibers of \bar{B} over these vertices are

$$\begin{aligned} r: & \{\alpha_1, \alpha_2\}_l = 1, \\ s: & \{\alpha_1, \alpha_2\alpha_3\alpha_2^{-1}\}_m = 1, \\ t: & \{\alpha_2, \rho\alpha_3\rho^{-1}\}_n = 1, \\ w': & \{\alpha_2^{-1}\alpha_1\alpha_2, \rho\alpha_3\rho^{-1}\}_k = 1 \end{aligned} \tag{3.3.2}$$

if we assume that the fibers are of type $\tilde{\mathbf{A}}$. For a $\tilde{\mathbf{D}}$ -type fiber, we omit the corresponding relation in (3.3.2) and indicate this omission by a “-” in the parameter list. (Sometimes, we also omit a relation just because it is not necessary to prove that π_1 is abelian.) Using the values of (l, m, n, k) shown in Table 1, one concludes that the groups of most sextics that are not of torus type are abelian. The few exceptional cases are treated separately in what follows.

The same arguments apply to the sets of singularities $2\mathbf{E}_6 \oplus \mathbf{A}_7$ (no. 3 in Table 1) and $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (no. 5 in Table 1) because the corresponding skeletons can be represented as shown in Figure 11(a)-1, $\bar{1}$ and Figure 11(a)-2, respectively. The former fundamental group is abelian. The latter is the order-720 group given by

$$\pi_1 = \langle \alpha_1, \alpha_2, \alpha_3 \mid (3.1.1), (3.3.1), (3.3.2) \rangle \tag{3.3.3}$$

with $(l, m, n, k) = (4, 5, -, -)$; it splits into the semidirect product $\mathrm{SL}(2, \mathbb{F}_5) \rtimes \mathbb{Z}_6$. (Recall that the braid relation about the remaining type- $\tilde{\mathbf{E}}_6$ singular fiber can be ignored.) An alternative presentation of this group is given by (3.5.2).

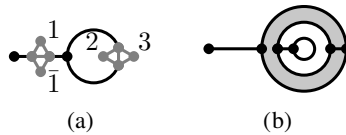


Figure 11

3.3.4. THE SET OF SINGULARITIES $\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$ (no. 11 in Table 1). In addition to (3.1.1), (3.3.1), and (3.3.2) with $(l, m, n, k) = (4, 11, -, -)$, one also has the relation

$$(\alpha_1 \alpha_2 \alpha_3 \alpha_2^{-1}) \alpha_1 (\alpha_1 \alpha_2 \alpha_3 \alpha_2^{-1})^{-1} = (\alpha_2^{-1} \alpha_1 \alpha_2) \alpha_3 (\alpha_2^{-1} \alpha_1 \alpha_2)^{-1}$$

from the lower left loop in Figure 3(b). The resulting group is abelian.

Alternatively, choosing a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ in the fiber over the upper left \bullet -vertex in Figure 3(b), one obtains the relations $\alpha_2 = \alpha_3$ (from the upper left loop) and $\alpha_2^{-1} \alpha_1 \alpha_2 = \rho^{-1} \alpha_2 \rho$ (from the lower left loop). In view of the former, the latter simplifies to the braid relation $\alpha_1 \alpha_2 \alpha_1 = \alpha_2 \alpha_1 \alpha_2$; that is, $\{\alpha_1, \alpha_2\}_3 = 1$. On the other hand, the 11-gonal outer region of the skeleton gives the relation $\{\alpha_1, \alpha_2\}_{11} = 1$. Hence $\{\alpha_1, \alpha_2\}_1 = 1$ (see (3.1.4)) and the group is cyclic.

3.3.5. THE SET OF SINGULARITIES $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$ (no. 32 in Table 1). In this case we assume that the type- $\tilde{\mathbf{D}}_5$ singular fiber is chosen inside the inner loop of the insertion; see Remark 2.3.1. Hence the group has no relation (3.3.1). However, relation (3.1.1) and relations (3.3.2) with $(l, m, n, k) = (7, 3, -, 1)$ suffice to show that the group is abelian.

3.3.6. THE SET OF SINGULARITIES $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_8$ (no. 28 in Table 1). As before, the type- $\tilde{\mathbf{D}}_5$ fiber is chosen inside the inner loop of the insertion; see Remark 2.3.1.

Hence the group has no relation (3.3.1). Still, it has relations (3.1.1) and (3.3.2) with $(l, m, n, k) = (9, -, 1, -)$ and, in addition, the relation

$$\alpha_2\alpha_3\alpha_2^{-1} = (\alpha_1\alpha_2\alpha_3)\alpha_2(\alpha_1\alpha_2\alpha_3)^{-1}$$

resulting from the left loop in Figure 3(f). These relations suffice to show that the group is abelian.

3.3.7. THE SET OF SINGULARITIES $(E_6 \oplus A_{11}) \oplus A_2$ (no. 12 in Table 1). As explained in Section 3.3.4, the group π_1 is a quotient of the braid group \mathbb{B}_3 (for the braid relations, only the two left loops in Figure 3(b) are used). Therefore, $\pi_1 = \bar{\mathbb{B}}_3$ by the following simple lemma (see e.g. [8, Lemma 3.6.1]).

3.3.8. LEMMA. *Let B be an irreducible plane sextic of torus type. Then any epimorphism $\mathbb{B}_3 \rightarrow \pi_1(\mathbb{P}^2 \setminus B)$ factors through an isomorphism $\bar{\mathbb{B}}_3 = \pi_1(\mathbb{P}^2 \setminus B)$.*

3.3.9. THE SET OF SINGULARITIES $(E_6 \oplus A_5 \oplus 2A_2) \oplus A_4$ (no. 9 in Table 1). The group π_1 has presentation (3.3.3) with $(l, m, n, k) = (6, 5, 3, -)$. Using GAP [13], one can see that the quotient π_1/α_1^5 is a perfect group of order 7680 whereas $\bar{\mathbb{B}}_3/\sigma_1^5 = A_5$ has order 60. Hence the natural epimorphism $\pi_1 \rightarrow \bar{\mathbb{B}}_3$ is proper.

3.3.10. OTHER CURVES OF TORUS TYPE. The remaining sextics of torus type appearing in this section are nos. 13 and 18 in Table 1. The group π_1 has presentation (3.3.3) with the values of the parameters (l, m, n, k) given in the table. Both groups factor to $\bar{\mathbb{B}}_3$. Entering the presentations into GAP [13] and then simplifying them via

```
P := PresentationNormalClosure(g, Subgroup(g, [g.1/g.2]));
SimplifyPresentation(P);
```

one finds that the commutant $[\pi_1, \pi_1]$ is a free group on two generators. Since all groups involved are residually finite and hence Hopfian, it follows that both epimorphisms $\pi_1 \rightarrow \bar{\mathbb{B}}_3$ are isomorphisms. (This approach was suggested to me by E. Artal Bartolo.)

3.4. Hexagon with a Loop: Reducible Curves

Choose a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ as in Section 3.3. Then (3.1.1) and (3.3.1) imply that $\pi_1/[\pi_1, \pi_1] = \mathbb{Z}$ and that the projection $\pi_1 \rightarrow \mathbb{Z}$ is given by $\alpha_1, \alpha_2 \mapsto 1$ and $\alpha_3 \mapsto -5$. In particular, it follows that the sextic splits into an irreducible quintic and a line.

In addition to (3.1.1) and (3.3.1), consider the relations

$$\begin{aligned} r: & \{\alpha_1, \alpha_2\}_l = 1, \\ s: & \{\alpha_1, \alpha_2\alpha_3\alpha_2^{-1}\}_m = 1, \\ t: & \{\alpha_2, \rho\alpha_3\rho^{-1}\}_n = 1, \\ w'': & \{\alpha_2^{-1}\alpha_1\alpha_2, \rho^{-1}\alpha_2\rho\}_k = 1 \end{aligned} \tag{3.4.1}$$

arising from the \times -vertices r, s, t, w'' in Figure 10 (assuming that the fibers over these vertices are of type $\tilde{\mathbf{A}}$; if a fiber is of type $\tilde{\mathbf{D}}$, the corresponding relation is omitted).

In order to analyze the group using GAP [13], observe that relation (3.1.1) implies $[\alpha_1, (\alpha_2\alpha_3)^3] = 1$; then, in view of (3.3.1), the element $(\alpha_2\alpha_3)^3$ commutes with α_3 and hence with α_2 . Thus $(\alpha_2\alpha_3)^3 \in \pi_1$ is a central element, and its projection to the abelianization of π_1 is the element -12 of infinite order. Because of Lemma 3.1.3, it suffices to study the commutant of the quotient $\pi_1/(\alpha_2\alpha_3)^3$. The abelianization of the latter quotient is \mathbb{Z}_{12} .

The sets of parameters (l, m, n, k) used in the calculation are listed in Table 2. The curves with the following sets of singularities have nonabelian groups.

- (1) $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$ (no. 1'): the curve is of torus type, so $\text{ord}[\pi_1, \pi_1] = \infty$. Note that $\pi_1/\alpha_1^2 = \text{GL}(2, \mathbb{F}_3)$ has order 48; in particular, π_1 is not isomorphic to any braid group \mathbb{B}_n .
- (2) $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 3'): one has $\pi_1 = \text{SL}(2, \mathbb{F}_7) \times \mathbb{Z}$.
- (3) $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 6'): one has $\pi_1 = \text{SL}(2, \mathbb{F}_5) \rtimes \mathbb{Z}$.
- (4) $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$ (no. 15'): one has $\pi_1 = ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}$.

(In items (2), (3), and (4), the centralizer of $[\pi_1, \pi_1]$ projects to a subgroup of index 1, 2, and 4, respectively, in the abelianization. In the former case, it follows that the product is direct.) The group in item (2) was first computed in [1].

3.4.2. THE SET OF SINGULARITIES $\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{D}_6$ (no. 17' in Table 2). The curve has no $\tilde{\mathbf{A}}$ -type singular fibers outside the insertion, and we replace (3.4.1) with the relations

$$(\rho\alpha_1\alpha_2\alpha_1)\alpha_2(\rho\alpha_1\alpha_2\alpha_1)^{-1} = \alpha_1, \quad (\rho\alpha_1\alpha_2)\alpha_1(\rho\alpha_1\alpha_2)^{-1} = \alpha_2$$

resulting from the type- $\tilde{\mathbf{D}}_7$ singular fiber over r . The resulting group is abelian.

3.4.3. A $\tilde{\mathbf{D}}$ -TYPE FIBER INSIDE THE INSERTION. If the singular fiber of \bar{B} in the loop inside the insertion is of type $\tilde{\mathbf{D}}_5$, then relation (3.3.1) should be replaced with

$$\rho\beta_2\beta_3\beta_2^{-1}\rho^{-1} = \beta_2, \quad \rho\beta_2\rho^{-1} = \beta_3, \tag{3.4.4}$$

where $\{\beta_1, \beta_2, \beta_3\}$ is an appropriate canonical basis over the \bullet -vertex w in Figure 10. Using [10], one has $\beta_1 = \alpha_1\alpha_3\alpha_1^{-1}$, $\beta_2 = \alpha_1$, and $\beta_3 = \alpha_3^{-1}\alpha_2\alpha_3$ (in particular, $\beta_1\beta_2\beta_3 = \rho$). From (3.4.4) it follows that $\delta := \rho^2\beta_2\beta_3$ is a central element of π_1 ; since the projection of δ to the abelianization of π_1 is the element -4 of infinite order, one can use Lemma 3.1.3 and study the commutant of the quotient π_1/δ .

The sets of parameters (l, m, n, k) used in the calculation are listed in Table 2. The only nonabelian group in this series is the one corresponding to the set of singularities $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$ (no. 16'); it can be represented as a semidirect product $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}$ and is isomorphic to the one described in item (4).

3.5. Hexagon with a Double Loop

Assume that \bar{H} is a hexagon with a double loop (see Section 2.4), and choose for v the vertex shown in Figure 5, right. We can assume that the singular fibers inside

the insertion are all of type $\tilde{\mathbf{A}}$; see Remark 2.4.1. Then, the braid relations resulting from the inner pentagon and monogon are

$$\{\alpha_1, \alpha_3^{-1}\alpha_2\alpha_3\}_5 = 1, \quad \alpha_1\alpha_3^{-1}\alpha_2\alpha_3\alpha_1^{-1} = \alpha_3^{-1}\alpha_2^{-1}\alpha_3\alpha_2\alpha_3. \tag{3.5.1}$$

In addition, for all curves except no. 37 in Table 1 there is a relation $\{\alpha_1, \alpha_2\}_l = 1$, where $l = 10, 7, 6, 9$, or 5 (in the order of appearance in Table 1). For no. 37, one has the commutativity relation $[\alpha_3, \alpha_1\alpha_2] = 1$ from the $\tilde{\mathbf{D}}_9$ -type fiber. In all cases, we can use GAP [13] to conclude that the group is abelian.

The same arguments apply to the set of singularities $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (no. 4 in Table 1) because the corresponding skeleton can be represented as shown in Figure 11(b) (so that one has $l = 4$). The resulting group has order 720, and its presentation is

$$\pi_1 = \langle \alpha_1, \alpha_2, \alpha_3 \mid (3.1.1), (3.5.1), \{\alpha_1, \alpha_2\}_4 = 1 \rangle. \tag{3.5.2}$$

This group is isomorphic to (3.3.3); see Remark 2.6.1.

3.6. Genuine Hexagon: Irreducible Curves

Consider one of the four skeletons shown in Figure 7, and take for v any vertex in $\partial\tilde{H}$. Let $v_0 = v, v_1, \dots, v_5$ be the vertices in $\partial\tilde{H}$ numbered starting from v in the *clockwise* direction (this is the *counterclockwise* direction in the figures, which represent the complementary hexagon $S^2 \setminus \tilde{H}$). Mark each vertex similar to $v_0 = v$ and denote by R_i the region of Sk whose boundary contains the edges e_1 and e_2 at $v_i, i = 0, \dots, 5$. Let n_i be the number of vertices in $\partial R_i, i = 0, \dots, 5$; in other words, assume that R_i is an n_i -gon. Then, in addition to the common relation at infinity (3.1.1), the group π_1 has the relations

$$\{\sigma_2^i(\alpha_2), \alpha_1\}_{n_i} = 1, \quad i = 0, \dots, 5, \tag{3.6.1}$$

resulting from the singular fibers in R_i . If R_i and \tilde{H} are all the regions but at most one of Sk (which is always the case for irreducible curves; see Figure 7), then (3.1.1) and (3.6.1) form a complete set of relations for π_1 . Furthermore, in the sequence R_0, \dots, R_5 , some of the regions coincide. For each region, it suffices to consider only one instance in the sequence and ignore the other relations by letting the corresponding parameters n_i equal 0; these relations would follow from the others.

The values of the parameters (n_0, \dots, n_5) used in the calculation are listed in Table 1, and the initial vertex $v = v_0$ is shown in Figure 7 in gray. For the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 42 in Table 1), the resulting group is abelian. The other three curves are of torus type, so their groups factor to $\tilde{\mathbb{B}}_3$. Here is some additional information on these groups.

- (1) $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$ (no. 39): the curve is a sextic of torus type of weight 8 in the sense of [5]. Hence π_1 is much larger than $\tilde{\mathbb{B}}_3$; its Alexander module is a direct sum of *two* copies of $\mathbb{Z}[t]/(t^2 - t + 1)$. An alternative presentation of this group is found in [9].
- (2) $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 40): much as in Section 3.3.10, one can use GAP [13] to show that the natural epimorphism $\pi_1 \rightarrow \tilde{\mathbb{B}}_3$ is an isomorphism.

- (3) $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$ (no. 41): the quotient $\tilde{\pi}_1/\alpha_1^5$ is a perfect group of order 7680, whereas $\tilde{\mathbb{B}}_3/\sigma_1^5 = \mathbb{A}_5$. Hence the natural epimorphism $\pi_1 \rightarrow \tilde{\mathbb{B}}_3$ is proper. (The values of the parameters actually used are $(n_0, \dots, n_5) = (3, 3, 5, 6, 6, 5)$.) I do not know whether this group is isomorphic to the one considered in Section 3.3.9.

3.6.2. REMARK. In items (2) and (3), if the reference fiber is chosen as shown in Figure 7 then the group has also relation $(\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2) = \alpha_3$; see (3.8.1). Although formally this extra relation follows from the others, it simplifies the analysis of the group.

3.7. Genuine Hexagon: Reducible Curves

The approach of Section 3.6 applies to reducible curves \bar{B} as well (see Figure 8)—provided that the skeleton of \bar{B} has at most one region or $\bar{\mathbf{D}}$ -type singular fiber strictly inside the complementary hexagon $S^2 \setminus \bar{H}$ (i.e., to all skeletons except those shown in Figures 8(f)–(j)). (In the case of a $\bar{\mathbf{D}}$ -type fiber, shown in Figures 8(p) and (q), the region R_i containing this fiber should be ignored in (3.6.1).)

The first two curves (Figures 8(a) and (b)) do not seem to have any extra features that would facilitate the study of their groups. (Each of these curves splits into an irreducible quintic and a line; hence $\pi_1/[\pi_1, \pi_1] = \mathbb{Z}$.) The values of the parameters (n_0, \dots, n_5) are listed in Table 2 (assuming that $v = v_0$ is the vertex shown in the figures in gray), and the resulting groups are as follows.

- (1) $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 20’): the curve is of torus type, so π_1 factors to the braid group \mathbb{B}_3 ; in particular, $[\pi_1, \pi_1]$ is infinite.
- (2) $\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 21’): the commutant $[\pi_1, \pi_1]$ is perfect (one can compute the Alexander module $A = 0$ as in Section 3.10); it appears to be infinite, but at present I do not even know whether it is nontrivial.

In the rest of this section we consider the skeletons with one monogonal region inside $\mathbb{P}^1 \setminus H$ (i.e., those shown in Figures 8(k)–(o)). Choose for the initial vertex $v = v_0$ the one shown in the figures in gray. Then the monogonal region gives an extra relation

$$(\alpha_2\alpha_1\alpha_2)^{-1}\alpha_1(\alpha_2\alpha_1\alpha_2) = \alpha_3. \tag{3.7.1}$$

(Strictly speaking, this relation follows from the others, but its presence simplifies the calculations. In particular, since each sextic B in question is known to be reducible, it follows that it splits into an irreducible quintic and a line, where the projection $\pi_1 \rightarrow \pi_1/[\pi_1, \pi_1] = \mathbb{Z}$ is given by $\alpha_1, \alpha_3 \mapsto 1$ and $\alpha_2 \mapsto -5$.) On the other hand, relation (3.6.1) implies that $\delta := (\alpha_1\alpha_2)^{n_0}$ commutes with α_1 and α_2 ; hence δ is a central element and, using Lemma 3.1.3, one can study the commutant of the quotient π_1/δ .

The parameters (n_0, \dots, n_5) are listed in Table 2. The following three sets of singularities result in nonabelian fundamental groups.

- (3) $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 30’): one has $\pi_1 = \mathrm{SL}(2, \mathbb{F}_5) \times \mathbb{Z}$.
- (4) $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 33’): one has $\pi_1 = \mathrm{SL}(2, \mathbb{F}_7) \times \mathbb{Z}$.

- (5) $E_6 \oplus A_9 \oplus A_2 \oplus 2A_1$ (no. 34'): the commutant of π_1 is a perfect group; it appears infinite, but I do not know a proof. The commutants of π_1/α_1^2 and π_1/α_1^3 have orders 60 and 51840, respectively.

(In items (3) and (4), the centralizer of $[\pi_1, \pi_1]$ projects to a subgroup of index 2 and 1, respectively, in $\pi_1/[\pi_1, \pi_1]$. The groups are isomorphic to those of items (3) and (2), respectively, in Section 3.4.) The group in item (4) was first computed in [1], where it was also shown that sextics nos. 3' and 33' in Table 2 are Galois conjugate over $\mathbb{Q}(\sqrt{2})$.

3.8. Genuine Hexagon: Two Monogons inside $S^2 \setminus \tilde{H}$

In this section, we consider a skeleton with two monogonal regions strictly inside $S^2 \setminus \tilde{H}$ (i.e., one of those shown in Figures 8(f)–(j)). Take for $v = v_0$ the vertex shown in the figures in gray. Then, in addition to (3.1.1) and (3.6.1), the group has an extra relation

$$(\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2) = \alpha_3 \tag{3.8.1}$$

resulting from the monogon closest to v . In particular, it follows that the curve splits into an irreducible quartic and an irreducible conic. Furthermore, because $\delta := (\alpha_1\alpha_2)^{n_0}$ commutes with α_1 and α_2 , it is a central element and one can use Lemma 3.1.3 to study the commutant of the quotient π_1/δ .

The parameters (n_0, \dots, n_5) are listed in Table 2. For the first three curves, the groups are abelian. (As a consequence, the curve defined by the skeleton in Figure 8(g), no. 26' in Table 2, is not of torus type.) The other two curves are of torus type; an alternative way to construct these curves and to compute their fundamental groups is found in [8]. (To prove that these curves are of torus type, one can argue that the existence of such curves is shown in [8] and nos. 28, 29 are the only candidates left.)

3.9. Genuine Hexagon with a \tilde{D} -Type Fiber

Consider one of the two skeletons shown in Figure 8(p) or (q), and choose the initial vertex $v = v_0$ so that R_0 is the region containing the \tilde{D} -type fiber F of \tilde{B} . Then, as before, the defining relations for π_1 are (3.1.1) and (3.6.1), with the contribution of R_0 ignored in the latter.

However, we do make use of the region R_0 in order to find central elements in π_1 . Let $n = n_0$. Then F is of type \tilde{D}_{n+4} , and the braid relations about F are

$$\alpha_3^{-1}\alpha_i\alpha_3 = \sigma_1^{n+2}(\alpha_i), \quad i = 1, 2.$$

As a consequence,

$$[\alpha_3, \alpha_1\alpha_2] = 1; \quad \text{hence} \quad [\alpha_3, \rho] = [\alpha_1\alpha_2, \rho] = 1, \tag{3.9.1}$$

and $\delta := \alpha_3(\alpha_1\alpha_2)^{1+n/2}$ is a central element of π_1 . (Note that n is even in all cases.) Since $[\alpha_2\alpha_3, \rho^4] = 1$ (see (3.1.1)), by (3.9.1) one has $[\alpha_2, \rho^4] = 1$ and then $[\alpha_1, \rho^4] = 1$. Thus, $\rho^4 = (\alpha_2\alpha_3)^3$ is also a central element of π_1 , and Lemma 3.1.3 applied twice implies that the commutant $[\pi_1, \pi_1]$ of π_1 is isomorphic to that of the quotient $\pi_1/\langle \delta, \rho^4 \rangle$. (It is worth mentioning that $n \geq 2$ and so the images of δ and ρ in the abelianization of π_1 are linearly independent.)

The values of the parameters (n_0, \dots, n_5) are listed in Table 2. (Recall that the initial vertex v_0 is determined by the position of the $\tilde{\mathbf{D}}$ -type fiber, which depends on the curve.) We can use GAP [13] to conclude that, for the sets of singularities

$$\mathbf{E}_6 \oplus \mathbf{D}_8 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_1 \quad \text{and} \quad \mathbf{E}_6 \oplus \mathbf{D}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$$

(nos. 35' and 38' in Table 2), the group π_1 is abelian whereas for the other two curves (nos. 36' and 37') it has infinite commutant. For a more precise statement see Section 3.10, where we compute the Alexander modules of these and a few other groups.

3.9.2. REMARK. The fact that the fundamental groups of the reducible sextics with the sets of singularities $\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$ and $\mathbf{E}_6 \oplus \mathbf{D}_{10} \oplus 3\mathbf{A}_1$ have infinite commutants can also be explained as follows. Each curve splits into an irreducible quartic B_4 with a type- \mathbf{E}_6 singular point and a pair of lines. One of the lines B_1 either is double tangent to B_4 or has a single point of 4-fold intersection with B_4 . Hence, even after patching back in the other line (which corresponds to letting one of the canonical generators of π_1 equal 1), one obtains a curve with large fundamental group (which is, respectively, \mathbb{B}_3 or $\mathbb{T}_{3,4} = \langle \alpha, \beta \mid \alpha^3 = \beta^4 \rangle$; see [3]).

3.10. Other Sextics with Two Linear Components

With the exception of the two curves mentioned in the previous section, all maximizing sextics splitting into an irreducible quartic (with a type- \mathbf{E}_6 singular point) and two lines have fundamental groups with infinite commutants. In order to prove this statement and make it more precise, we compute the so-called Alexander modules of the groups (see [15]).

3.10.1. DEFINITION. Let G be a group, and let $G' = [G, G]$ be its commutant. The *Alexander module* of G is the abelian group $G'/[G', G']$ regarded as a $\mathbb{Z}[G/G']$ -module via the conjugation action $(a, x) \mapsto a^{-1}xa$ with $a \in G/G'$ and $x \in G'/[G', G']$.

Abbreviate $\pi'_1 = [\pi_1, \pi_1]$ and denote the Alexander module of π_1 by A .

The sextics in question are represented by the skeletons shown in Figures 8(c)–(e), (p), and (q), and the defining relations for π_1 are (3.1.1) and (3.6.1). (As usual, if a $\tilde{\mathbf{D}}$ -type fiber is present then the corresponding relation in (3.6.1) should be ignored.) The abelianization $\pi_1/\pi'_1 = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the images s, t of α_1, α_2 , respectively, and the group ring $\mathbb{Z}[\pi_1/\pi'_1]$ can be identified with the ring $\Lambda := \mathbb{Z}[s, s^{-1}, t, t^{-1}]$ of Laurent polynomials in s, t .

Each skeleton in question has a bigonal region not containing a $\tilde{\mathbf{D}}$ -type fiber, and we choose the initial vertex v_0 so that this region is R_0 . (See the gray vertex in the figures; note that, for Figures 8(p) and (q), this choice of v_0 differs from that used in Section 3.9.) Then $[\alpha_1, \alpha_2] = 1$ and, using the Reidemeister–Schreier method (see e.g. [16]), one can see that A is generated over Λ by a single element $a := \alpha_1^4 \alpha_2 \alpha_3$. The relation at infinity (3.1.1) transforms into $(s^{-4} + s^{-8})a = (s^{-3} + s^{-6} + s^{-9})a$, or

$$Q_\infty(s)a = 0, \quad \text{where } Q_\infty(s) := (s^2 - s + 1)(s^4 - s^2 + 1). \quad (3.10.2)$$

Alternatively, (3.10.2) can be rewritten in the form $(s - 1)s(s^4 + s + 1)a = -a$, which means that the multiplication by $s - 1$ is invertible in A . For this reason, we cancel the factor $s - 1$ in all other relations.

For an integer $m \geq 0$, denote $P_m(x) = (x^m - 1)/(x - 1)$. (In particular, $P_0 \equiv 0$ and $P_1 \equiv 1$.) Observe that, for curves with two linear components, all integers n_i in (3.6.1) are even; see Table 2. A relation $\{\sigma_2^j(\alpha_2), \alpha_1\}_{2r} = 1$ results in the following relation for A :

$$\begin{aligned} (t - 1)P_r(st)P_j(s^4)a &= 0 && \text{if } i = 2j \text{ is even, or} \\ P_r(s^3t)[(1 - s^4t)P_j(s^4) + s^{4j}t]a &= 0 && \text{if } i = 2j - 1 \text{ is odd.} \end{aligned}$$

(Recall that we cancel all factors $s - 1$.) Using the values of (n_0, \dots, n_5) listed in Table 2, one arrives at the following Alexander modules.

- (1) $E_6 \oplus A_7 \oplus A_3 \oplus 3A_1$ (no. 22'): $A = \mathbb{Z}[s]/Q_\infty(s)$ and $ta = a$.
- (2) $E_6 \oplus A_5 \oplus 2A_3 \oplus 2A_1$ (no. 23'): $A = \mathbb{Z}[s]/Q_\infty(s)$ and $ta = -s^{-3}a$.
- (3) $(E_6 \oplus 2A_5) \oplus 3A_1$ (no. 24'): $A = \mathbb{Z}[s]/(s^2 - s + 1)$ and $ta = (1 + s^{-4})a$.
- (4) $E_6 \oplus D_6 \oplus 2A_3 \oplus A_1$ (no. 36'): $A = \mathbb{Z}[s]/(s^2 - s + 1)$ and $ta = -s^{-3}a$.
- (5) $E_6 \oplus D_{10} \oplus 3A_1$ (no. 37'): $A = \mathbb{Z}[s]/Q_\infty(s)$ and $ta = a$.

(Note that $Q_\infty|(s^{12} - 1)$; hence s is invertible in $\mathbb{Z}[s]/Q_\infty$ and one need not consider Laurent polynomials explicitly.)

3.10.3. REMARK. In items (1) and (5), one has $n_2 = 2$; hence $(t - 1)a = 0$. In items (2) and (4), one has $n_1 = 4$; hence $(1 + s^3t)a = 0$. In each case, t is a Laurent polynomial in s and one can represent A as a quotient of the Λ -module $\mathbb{Z}[s]/Q_\infty$ (with an appropriate action of t); then, in most cases, all extra relations follow from the relation at infinity (3.10.2).

Denote $Q(s) = s^2 - s + 1$ and $R(s) = s^4 - s^2 + 1$, so that $Q_\infty(s) = Q(s)R(s)$. In item (4), in addition to (3.10.2) one has the relation $Q(s)S_1(s)a = 0$, where $S_1(s) := (s + 1)^2$. Since $R(s) - (s - 1)^2S_1(s) = s^2$ is an invertible element, the relation ideal (in $\mathbb{Z}[s]$) is generated by a single element $s^2 - s + 1$.

In item (3), the additional relations are

$$P_3(s^3t)a = (t - 1)P_3(st)a = (1 - s^4t + s^4)a = 0.$$

From the last relation, one obtains $ta = (1 + s^{-4})a$. Hence, again t is a polynomial in s and, substituting $t = 1 + s^{-4}$ into the first two relations, one has

$$(s^2 - s + 1)S_1(s)a = (s^2 - s + 1)S_2(s)a = 0,$$

where

$$S_1(s) := s^6 + s^5 + 2s^2 + 2s + 1, \quad S_2(s) := s^6 + 2s^5 + 2s^4 + s + 1.$$

One can easily check that

$$\begin{aligned} s(s^3 - s - 2)(s^4 + s^3 + s^2 - s + 1)R(s) - s^2(s^4 + s^3 + s^2 - s + 1)S_1(s) \\ + (s^5 + s^4 + s^3 - s^2 + s + 1)S_2(s) = 1. \end{aligned}$$

Hence again the relation ideal is generated by a single element $s^2 - s + 1$.

3.10.4. REMARK. In items (1) and (2), the fact that the groups are infinite can be explained similarly to Remark 3.9.2. Each curve splits into an irreducible quartic B_4 and a pair of lines, one of which is either double tangent to B_4 or has a single point of 4-fold intersection with B_4 .

4. Perturbations

We fix a maximizing plane sextic B satisfying condition (*) in the Introduction and consider a perturbation B' of B . Throughout this section, \bar{B} stands for the maximal trigonal curve corresponding to B via Proposition 2.1.1.

4.1. Perturbations of the Type- \mathbf{E}_6 Point P

Let U be a Milnor ball about the distinguished type- \mathbf{E}_6 singular point P of B . The group $\pi_1(U \setminus B)$ is generated by three elements $\beta_1, \beta_2, \beta_3$ subject to the relations

$$\beta_3 = (\beta_1\beta_2\beta_3)\beta_2(\beta_1\beta_2\beta_3)^{-1}, \quad \beta_2 = (\beta_1\beta_2\beta_3)\beta_1(\beta_1\beta_2\beta_3)^{-1}. \quad (4.1.1)$$

According to [11], the inclusion homomorphism $\pi_1(U \setminus B) \rightarrow \pi_1(\mathbb{P}^2 \setminus B)$ is given by

$$\beta_1 \mapsto \rho\alpha_1\rho^{-1}, \quad \beta_2 \mapsto \alpha_1, \quad \beta_3 \mapsto \rho^{-1}\alpha_1\rho,$$

where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis for $\pi_1(\mathbb{P}^2 \setminus B)$ as in Section 3.1. Note that a type- \mathbf{E}_6 singularity has an order-3 automorphism inducing the automorphism

$$\beta_1 \mapsto (\beta_1\beta_2)\beta_3(\beta_1\beta_2)^{-1}, \quad \beta_2 \mapsto \beta_1, \quad \beta_3 \mapsto \beta_2$$

of the local fundamental group. However, on the image in $\pi_1(\mathbb{P}^2 \setminus B)$, this latter automorphism is inner—namely, it is induced by the conjugation $g \mapsto \rho g \rho^{-1}$. (For proof, one needs to use the commutativity relation $[\alpha_1, (\alpha_2\alpha_3)^3] = 1$, which follows from (3.1.1).) Hence, the local automorphisms at P can be ignored when studying the extra relations in $\pi_1(\mathbb{P}^2 \setminus B')$ resulting from a perturbation $B \rightarrow B'$.

The fundamental groups $\pi_1(U \setminus B')$ of small perturbations $B \rightarrow B'$ are found in [4]; they are as follows.

- (1) $\mathbf{E}_6 \rightarrow 2\mathbf{A}_2 \oplus \mathbf{A}_1$: one has $\pi_1(U \setminus B) = \mathbb{B}_4$, where the additional relations are $[\beta_1, \beta_3] = \{\beta_1, \beta_2\}_3 = \{\beta_2, \beta_3\}_3 = 1$.
- (2) $\mathbf{E}_6 \rightarrow 2\mathbf{A}_2$ (as a further perturbation of (1)) and $\mathbf{E}_6 \rightarrow \mathbf{A}_5$: one has $\pi_1(U \setminus B) = \mathbb{B}_3$, where the additional relations are $\beta_1 = \beta_3$ and $\{\beta_1, \beta_2\}_3 = 1$.
- (3) All others: one has $\pi_1(U \setminus B) = \mathbb{Z}$, and the relations are $\beta_1 = \beta_2 = \beta_3$.

Note that, in the presence of (4.1.1), the additional relations in items (1) and (2) follow from the first relation; namely, $[\beta_1, \beta_3] = 1$ and $\beta_1 = \beta_3$, respectively. The first two perturbations are of *torus type* (i.e., they preserve torus structures of B); the other perturbations destroy any torus structure with respect to which P is an inner singularity.

Combining items (1)–(3) with the inclusion homomorphism, we obtain the following extra relations for the perturbed group $\pi_1(\mathbb{P}^2 \setminus B')$.

- (1) $E_6 \rightarrow 2A_2 \oplus A_1$: the extra relation is $[\alpha_1, \rho^{-2}\alpha_1\rho^2] = 1$.
- (2) $E_6 \rightarrow 2A_2$ and $E_6 \rightarrow A_5$: the extra relation is $[\alpha_1, \rho^2] = 1$.
- (3) All others: the extra relation is $[\alpha_1, \rho] = 1$.

As a consequence, if P is perturbed as in (3), then one has $[\alpha_1, \alpha_2\alpha_3] = 1$ and (3.1.1) becomes

$$\alpha_1^4\alpha_2\alpha_3 = 1. \tag{4.1.2}$$

In particular, π_1 is generated by α_1 and α_2 (or by α_1 and α_3) in this case.

4.1.3. LEMMA. *Assume that the distinguished hexagon \bar{H} of Sk is genuine, that one of the regions R_i (see Section 3.6) is a bigon, and that the point P is perturbed as in item (3). Then the group $\pi_1(\mathbb{P}^2 \setminus B')$ is abelian.*

Proof. Taking R_i for R_0 , one concludes that $[\alpha_1, \alpha_2] = 1$. Together with (4.1.2), this relation implies that the group is abelian. □

4.1.4. REMARK. Note that almost all skeletons with a genuine hexagon have a bigonal region; the exceptions are Figures 7(a) and (c) and Figure 8(j).

4.2. Perturbations of A -Type Points

Extend the skeleton Sk of B to a dessin by inserting a \circ -vertex at the middle of each edge, inserting a \times -vertex v_R at the center of each region R , and connecting v_R to the vertices in the boundary ∂R by appropriate edges (see [10]). Let Q be a type- A_p singular point of B ; it is located in a type- \tilde{A}_p singular fiber F of B at the center of a certain $(p + 1)$ -gonal region R of Sk . If Q is perturbed then F splits into singular fibers of types \tilde{A}_{s_i} (\tilde{A}_0^* if $s_i = 0$), $i = 1, \dots, r$, with $\sum(s_i + 1) = p + 1$. Geometrically, the $2(p + 1)$ -valent \times -vertex v_R splits into several \times -vertices of valencies $2(s_i + 1)$.

Assume that the braid relation in $\pi_1(\mathbb{P}^2 \setminus B)$ resulting from the region R just described is of the form $\{\delta_1, \delta_2\}_{p+1} = 1$, where δ_1, δ_2 are certain words in $\alpha_1, \alpha_2, \alpha_3$ (cf. (3.3.2), (3.4.1), (3.6.1)). In other words, δ_1 and δ_2 are the first two elements of a canonical basis over a vertex $u \in \partial R$ defined by a marking at u (see [10]) with respect to which e_1 and e_2 belong to ∂R .

4.2.1. LEMMA. *In the notation just introduced, the relation for the new group $\pi_1(\mathbb{P}^2 \setminus B')$ resulting from R is $\{\delta_1, \delta_2\}_s = 1$, where $s = \text{g.c.d.}(s_i + 1), i = 1, \dots, r$.*

Proof. The statement follows from the description of the braid monodromy found in [10] and from (3.1.4). □

4.2.2. COROLLARY. *If Q is perturbed then the relation $\{\delta_1, \delta_2\}_{p+1} = 1$ changes to the relation $\{\delta_1, \delta_2\}_s = 1$, where $s < p + 1$ is a divisor of $(p + 1)$. In particular, one has $s = 1$ if $(p + 1)$ is a prime.*

4.2.3. COROLLARY. *If Q is a point of intersection of two components of B (and hence p is odd) and if the perturbation is to be irreducible in a Milnor ball about Q , then s in Corollary 4.2.2 must be an odd divisor of $p + 1$.*

4.2.4. COROLLARY. *If B is of torus type and Q is an inner singularity (hence $p + 1 = 0 \pmod{3}$) and if the torus structure is to be destroyed, then s in Corollary 4.2.2 must not be divisible by 3.*

4.3. Perturbations of \mathbf{D} -Type Points

Let Q be a type- \mathbf{D}_p , $p \geq 5$, singular point of B ; it is located in a type- $\tilde{\mathbf{D}}_p$ singular fiber F of \tilde{B} at the center of a $(p - 4)$ -gonal region R of Sk .

4.3.1. LEMMA. *If a perturbation $B \rightarrow B'$ is irreducible in a Milnor ball U_Q about Q , then the group $\pi_1(\mathbb{P}^2 \setminus B')$ is abelian.*

Proof. Under the assumptions, the group $\pi_1(U_Q \setminus B')$ is abelian (see [12]). On the other hand, the inclusion homomorphism $\pi_1(U_Q \setminus B') \rightarrow \pi_1(\mathbb{P}^2 \setminus B')$ is onto, since Q is a triple point of a trigonal curve. \square

Let δ_1 and δ_2 be the first two elements of a canonical basis over a vertex $u \in \partial R$ defined by a marking at u (see [10]) with respect to which e_1 and e_2 belong to ∂R . (In other words, if R contained an $\tilde{\mathbf{A}}$ -type fiber then the resulting braid relation would be $\{\delta_1, \delta_2\}_{p-4} = 1$; cf. (3.3.2), (3.4.1), (3.6.1).)

4.3.2. LEMMA. *Assume that a proper perturbation $B \rightarrow B'$ is still reducible in a Milnor ball U_Q about Q . Then, with notation as before, the new group $\pi_1(\mathbb{P}^2 \setminus B)$ has an extra relation $\{\delta_1, \delta_2\}_s = 1$ for some integer s , $1 \leq s \leq p - 2$. If $B \cap U_Q$ has three components (p is even) but $B' \cap U_Q$ has two components, then s is odd.*

Proof. The statement follows from the computation of the fundamental group $\pi_1(U_Q \setminus B')$ found in [12]. \square

4.4. Proof of Theorem 1.2.3

We skip sextics of torus type with two or more type- \mathbf{E}_6 singular points (nos. 1, 2, 6, and 7 in Table 1; they are considered in detail in [4]) as well as the maximizing sextic of weight 8 (no. 39 in Table 1; considered in [9]).

All other perturbations with nonabelian group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus B)$ can be handled on a case-by-case basis, using GAP [13] and modifying the presentation for π_1 found in Section 3 according to Section 4.1, Corollaries 4.2.2–4.2.4, and Lemmas 4.3.1 and 4.3.2. (Recall that, by [9, Prop. 5.1.1], all singular points of B can be perturbed arbitrarily and independently.) A few details are given in Sections 4.4.1–4.4.5. In some cases, the same approach shows that any proper perturbation, including reducible ones, that is not of torus type has abelian fundamental group.

- no. 1': All perturbations not of torus type are abelian.
- no. 3': All proper perturbations are abelian (see Section 4.4.3).
- nos. 6', 15', 16': All proper perturbations are abelian.
- no. 20': All perturbations not of torus type are abelian (see Section 4.4.4).
- no. 24': All perturbations not of torus type are abelian (see Section 4.4.5).
- nos. 28', 29': All perturbations not of torus type are abelian.

- nos. 30', 33': All proper perturbations are abelian.
- no. 34': All proper perturbations, except $\mathbf{E}_6 \rightarrow 2\mathbf{A}_2 \oplus \mathbf{A}_1$, are abelian; for the (reducible) nonabelian one, the perturbation epimorphism appears to be an isomorphism.

4.4.1. THE SET OF SINGULARITIES $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (nos. 4 and 5). The two curves are equivalent (see Remark 2.6.1), and it is easier to use no. 5 and its model given in Figure 11(a)-2 (see Section 3.3).

4.4.2. THE SETS OF SINGULARITIES $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_2$, $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$, AND $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (nos. 12, 13, 18, and 40). One has $\pi_1 = \overline{\mathbb{B}}_3$, and the statement on the perturbations follows from [8, Lemma 3.12].

4.4.3. THE SET OF SINGULARITIES $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 3'). The \mathbf{A}_1 -type point is not involved in the computation of the original group π_1 (see Section 3.4). Using Corollary 4.2.2 and computing the braid monodromy as explained in [10], one can easily see that the additional relation resulting from perturbing this point is $(\alpha_2^{-1}\alpha_1\alpha_2)\alpha_3(\alpha_2^{-1}\alpha_1\alpha_2)^{-1} = \rho^{-1}\alpha_2\rho$.

4.4.4. THE SET OF SINGULARITIES $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 20'). The perturbations of the distinguished type- \mathbf{E}_6 point P are covered by Lemma 4.1.3. If a type- \mathbf{A}_5 point Q is perturbed then one can number the regions R_i (see Section 3.6) so that R_0 is a bigon and Q is over R_1 . According to Corollary 4.2.4, the new group π'_1 has relations $[\alpha_1, \alpha_2] = [\alpha_1, \alpha_3] = 1$; then (3.1.1) turns into (4.1.2) and π'_1 is generated by two commuting elements, α_1 and α_2 .

Suppose that a type- \mathbf{A}_2 point Q is perturbed. Because the skeleton is symmetric, one can assume that Q is over the left triangle in Figure 8(a). Taking this triangle for R_0 (see Section 3.6), one obtains the relations $\alpha_1 = \alpha_2$ (Corollary 4.2.2) and $[\alpha_2, \alpha_3\alpha_2\alpha_3^{-1}] = 1$ (from the bigon R_2); hence also $[\alpha_2, \alpha_3^{-1}\alpha_2\alpha_3] = 1$. On the other hand, (3.1.1) implies $[\alpha_2, (\alpha_2\alpha_3)^3] = 1$ (in view of $\alpha_1 = \alpha_2$). Combining these relations, one arrives at $[\alpha_2, \alpha_3^3] = 1$; that is, α_3^3 is a central element. Now, using Lemma 3.1.3 and GAP [13], one concludes that the perturbed group is abelian.

4.4.5. THE SET OF SINGULARITIES $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus 3\mathbf{A}_1$ (no. 24'). Perturbation of the type- \mathbf{E}_6 or type- \mathbf{A}_5 points are handled similarly to their treatment in Section 4.4.4.

4.5. Proof of Theorem 1.2.6

As in Section 4.4, we ignore the sets of singularities with two or more type- \mathbf{E}_6 points (see [4]) and the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$ of weight 8 (see [9]). These sets of singularities give rise to the exceptions mentioned in the statement.

Consider a dessin with one of the two fragments shown in Figure 12. We assume that a region of the skeleton may contain several \times -vertices of valencies $2(s_i + 1)$, $i = 1, \dots, r$; the parameter s shown in the figures stands for the greatest common divisor $\text{g.c.d.}(s_i + 1)$ (cf. Lemma 4.2.1).

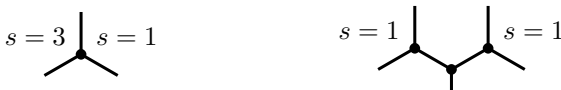


Figure 12 Two special fragments

4.5.1. LEMMA. *Let B' be an irreducible sextic of torus type and with a type- \mathbf{E}_6 singular point, and assume that the dessin of the corresponding trigonal curve \bar{B}' has one of the two fragments shown in Figure 12. Then $\pi_1(\mathbb{P}^2 \setminus B') = \bar{\mathbb{B}}_3$.*

Proof. The statement follows immediately from Lemma 4.2.1. In the first case, in an appropriate canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ over the \bullet -vertex shown in the figure, one has $\alpha_1 = \alpha_2$ and $\{\alpha_2, \alpha_3\}_3 = 1$. The second case (the right fragment in the figure) is essentially considered in Section 3.3.4. In both cases, one obtains an epimorphism $\mathbb{B}_3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus B')$ and then Lemma 3.3.8 applies. \square

Considering sextics of torus type one by one and perturbing their \mathbf{A} -type points, while taking into account Corollaries 4.2.2 and 4.2.3, one finds only three perturbations not covered by Lemma 4.5.1:

- $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4 \rightarrow (\mathbf{E}_6 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_4$ (no. 9);
- $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1 \rightarrow (\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 18);
- $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1 \rightarrow (\mathbf{E}_6 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 20').

Using equivalence of dessins (see [10]), one can easily see that the curves obtained are deformation equivalent to perturbations of the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$ (no. 39 in Table 1). Their fundamental groups are found in [9]. Alternatively, the second case (as well as any perturbation of nos. 12, 13, 18, or 40) is covered by [8, Cor. 3.10], and in the first case one can show that $\pi_1 = \bar{\mathbb{B}}_3$ similarly to the demonstration in Section 3.3.10. In the last case, the group is different: $\pi_1 = \mathbb{B}_4/\sigma_1^2\sigma_2\sigma_3^2\sigma_2$ (see [9]), which agrees with the statement.

It remains to consider perturbations of the distinguished \mathbf{E}_6 -type point P . The only “new” (i.e., not considered in [4] or [9]) set of singularities realized by irreducible curves of torus type with $\pi_1 \neq \bar{\mathbb{B}}_3$ is $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$ (nos. 9 and 41 in the table), and the “worst” perturbation is the one in item (1) of Section 4.1, resulting in the extra relation $[\alpha_1, \rho^{-2}\alpha_1\rho^2] = 1$. Adding this relation to the presentations of π_1 (see Sections 3.3.9 and 3.6(3), respectively) and using GAP [13] as in Section 3.3.10, one finds that the resulting groups are isomorphic to $\bar{\mathbb{B}}_3$. (In order to make the approach work for no. 41, we must also add the relation $(\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2) = \alpha_3$, which is known to hold in the group; see Remark 3.6.2.) \square

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