

Moduli Spaces of Rank-2 ACM Bundles on Prime Fano Threefolds

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1. Introduction

A vector bundle F on a smooth polarized variety (X, H_X) has no intermediate cohomology if $H^k(X, F \otimes \mathcal{O}_X(tH_X)) = 0$ for all $t \in \mathbb{Z}$ and $0 < k < \dim(X)$. These bundles are also called *arithmetically Cohen–Macaulay (ACM)* because they correspond to maximal Cohen–Macaulay modules over the coordinate ring of X . It is known that an ACM bundle must be a direct sum of line bundles if $X = \mathbb{P}^n$ [39] or a direct sum of line bundles and (twisted) spinor bundles if X is a smooth quadric hypersurface in \mathbb{P}^n [53; 72]. On the other hand, there exists a complete classification of varieties admitting, up to twist, a finite number of isomorphism classes of indecomposable ACM bundles [16; 25]. Only five cases exist besides rational normal curves, projective spaces, and quadrics.

For varieties that are not on this list, the problem of classifying ACM bundles has been taken up only in some special cases. For instance, on general hypersurfaces in \mathbb{P}^n of dimension at least 3, a full classification of ACM bundles of rank 2 is available; see [22; 23; 56; 63; 64]. For dimension 2 and rank 2, a partial classification can be found in [12; 20; 21; 27]. For higher rank, some results are given in [7; 19].

The case of smooth Fano threefolds X with Picard number 1 has also been studied. In this case one has $\text{Pic}(X) \cong \langle H_X \rangle$, with H_X ample, and the canonical divisor class K_X satisfies $K_X = -i_X H_X$, where the *index* i_X satisfies $1 \leq i_X \leq 4$. Recall that $i_X = 4$ implies $X \cong \mathbb{P}^3$ and $i_X = 3$ implies that X is isomorphic to a smooth quadric. Thus, the class of ACM bundles is completely understood in these two cases.

In contrast to this, the cases $i_X = 2, 1$ are highly nontrivial. First of all, there are several deformation classes of these varieties [49; 51; 52]. A different approach to the classification of these varieties was proposed by Mukai [67; 68; 69].

In the second place, it is still unclear how to characterize the invariants of ACM bundles; in fact, the investigation has been thoroughly carried out only in the case of rank 2. For $i_X = 2$, the classification was completed in [5]. For $i_X = 1$, a result of Madonna [57] implies that if a rank-2 ACM bundle F is defined on X , then its

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second Chern class c_2 must take values in $\{1, \dots, g + 3\}$ if $c_1(F) = 1$ or in $\{2, 4\}$ if $c_1(F) = 0$. Partial existence results are given in [6; 56].

Third, the set of ACM rank-2 bundles can have positive dimension. A natural point of view is to study them in terms of the moduli space $M_X(2, c_1, c_2)$ of (Gieseker) semistable rank-2 sheaves F with $c_1(F) = c_1$, $c_2(F) = c_2$, and $c_3(F) = 0$. For $i_X = 2$, such moduli space has been mostly studied when X is a smooth cubic threefold [24; 44; 61]; see [13] for a survey.

If the index i_X equals 1 then the threefold X is said to be *prime*, and one defines the *genus* of X as $g = 1 + H_X^3/2$. The genus satisfies $2 \leq g \leq 12$, $g \neq 11$, and there are 10 deformation classes of prime Fano threefolds. In this case, some of the relevant moduli spaces $M_X(2, 1, c_2)$ are studied in [45] for $g = 3$, in [15; 46; 47] for $g = 7$, in [41; 44] for $g = 8$, in [48] for $g = 9$, and in [6] for $g = 12$.

Arithmetic Cohen–Macaulay bundles of rank 2 also appeared in the framework of determinantal hypersurfaces; indeed, any such bundle provides a pfaffian representation of the equation of the hypersurface [12; 45].

The goal of our paper is to provide the classification of rank-2 ACM bundles F on a smooth prime Fano threefold X —that is, in the case $i_X = 1$. Note that we can assume $c_1(F) \in \{0, 1\}$. Combining our existence theorems (namely, Theorem 3.1 and Theorem 4.1) with the results of Madonna and others mentioned previously, we obtain the following classification.

THEOREM. *Let X be a smooth prime Fano threefold of genus g with $-K_X$ very ample. Then an ACM vector bundle F of rank 2 has the following Chern classes:*

- (i) if $c_1(F) = 1$ then $c_2(F) = 1$ or $\frac{g}{2} + 1 \leq c_2(F) \leq g + 3$;
- (ii) if $c_1(F) = 0$ then $c_2(F) = 2, 4$.

If $c_1(F) = 1$ and $c_2(F) \geq \frac{g}{2} + 2$, we assume, in addition, that X contains a line L with normal bundle $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$. Then there exists an ACM vector bundle F for any case listed above.

Note that the assumption that $-K_X$ is very ample (the threefold X is then called *non-hyperelliptic*) excludes two families of prime Fano threefolds, one with $g = 2$ and the other with $g = 3$. These two cases will be discussed in a forthcoming paper.

The proof is based on deformations of sheaves that are not locally free (hence neither ACM), such as extensions of ideal sheaves. The idea is to work recursively by starting with some well-behaved bundles that have *minimal* c_2 . In order for the induction to work in case (i), the only restriction we need on the threefold X is that it contain a line L with normal bundle $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$ (in this case we will say that X is *ordinary*). This is always verified if $g \geq 9$ unless X is the Mukai–Umemura threefold of genus 12. This condition is verified by a general prime Fano threefold of any genus; see Section 2.3 for more details.

The paper is organized as follows. In the next section we give some preliminary notions. Section 3 is devoted to the proof of the main theorem in the case $c_1(F) = 1$, and Section 4 concerns the case $c_1(F) = 0$. We conclude the paper with Section 5, giving applications to pfaffian representations of quartic threefolds in \mathbb{P}^4 and cubic hypersurfaces of a smooth quadric in \mathbb{P}^5 .

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2. Preliminaries

Given a smooth complex projective n -dimensional polarized variety (X, H_X) and a sheaf F on X , we write $F(t)$ for $F \otimes \mathcal{O}_X(tH_X)$. Given a subscheme Z of X , we write F_Z for $F \otimes \mathcal{O}_Z$ and we denote by $\mathcal{I}_{Z,X}$ the ideal sheaf of Z in X and by $N_{Z,X}$ its normal sheaf. We will frequently drop the second subscript.

Given a pair of sheaves (F, E) on X , we will write $\text{ext}_X^k(F, E)$ for the dimension of the group $\text{Ext}_X^k(F, E)$, and similarly $h^k(X, F) = \dim H^k(X, F)$. The Euler characteristic of a pair of sheaves (F, E) is defined as $\chi(F, E) = \sum_k (-1)^k \text{ext}_X^k(F, E)$, and $\chi(F)$ is defined as $\chi(\mathcal{O}_X, F)$. We denote by $p(F, t)$ the Hilbert polynomial $\chi(F(t))$ of the sheaf F . We write $e_{E,F}$ for the natural evaluation map

$$e_{E,F} : \text{Hom}_X(E, F) \otimes E \rightarrow F.$$

2.1. ACM Sheaves

Let (X, H_X) be an n -dimensional polarized variety and assume H_X very ample, so that $X \subset \mathbb{P}^m$. We denote by I_X the saturated ideal of X in \mathbb{P}^m and by $R(X)$ the coordinate ring of X . Given a sheaf F on X , we define the following $R(X)$ -modules:

$$H_*^k(X, F) = \bigoplus_{t \in \mathbb{Z}} H^k(X, F(t)) \quad \text{for each } k = 0, \dots, n.$$

The variety X is said to be *arithmetically Cohen–Macaulay* (ACM) if $R(X)$ is a Cohen–Macaulay ring. This is equivalent to

$$H_*^1(\mathbb{P}^m, \mathcal{I}_{X, \mathbb{P}^m}) = 0 \quad \text{and} \quad H_*^k(\mathbb{P}^m, \mathcal{O}_X) = 0 \quad \text{for } 0 < k < n.$$

A sheaf F on X is called *locally* Cohen–Macaulay if for any point $x \in X$ we have $\text{depth}(F_x) = \dim(X)$.

DEFINITION 2.1. A sheaf F on an n -dimensional ACM variety X is called *ACM* if F is locally Cohen–Macaulay and has no intermediate cohomology:

$$H_*^k(X, F) = 0 \quad \text{for any } 0 < k < n. \tag{2.1}$$

By [18, Prop. 2.1] there is a one-to-one correspondence, between ACM sheaves on X and graded maximal Cohen–Macaulay modules on $R(X)$, given by $F \mapsto H_*^0(X, F)$. If X is smooth then any ACM sheaf is locally free (see e.g. [1, Lemma 3.2]), so F being ACM is equivalent to condition (2.1).

As already mentioned, on a smooth quadric hypersurface of \mathbb{P}^m with $m \geq 4$ there exist ACM bundles, of rank greater than 1, called *spinor bundles*. We recall here some facts and notation regarding these bundles in the case they have rank 2; for more details see [53; 71]. If $Q_3 \subset \mathbb{P}^4$ is a smooth quadric, then there exists one spinor bundle \mathcal{S} of rank 2 on Q_3 . It is μ -stable (see Section 2.2), globally generated, with first Chern class equal to the hyperplane class H_{Q_3} of Q_3 , and with

$c_2(\mathcal{S}) = [L]$, where L is a line contained in Q_3 . Moreover, we have the following natural exact sequence on Q_3 :

$$0 \rightarrow \mathcal{S}(-1) \rightarrow \mathcal{O}_{Q_3}^4 \xrightarrow{e_{\theta, \mathcal{S}}} \mathcal{S} \rightarrow 0. \tag{2.2}$$

On the other hand, if $Q_4 \subset \mathbb{P}^5$ is a smooth quadric, then there exist two nonisomorphic spinor bundles of rank 2 defined over Q_4 . We denote them by \mathcal{S}_1 and \mathcal{S}_2 . They are both μ -stable (see Section 2.2), globally generated, and satisfy $c_1(\mathcal{S}_i) = H_{Q_4}$ and $c_2(\mathcal{S}_i) = \Lambda_i$, where Λ_1 and Λ_2 are the classes of two disjoint projective planes contained in Q_4 . These planes are parameterized by global sections of the bundles \mathcal{S}_i . These classes generate the cohomology group $H^{2,2}(Q_4)$, and one has the relations $H_{Q_4}^2 = \Lambda_1 + \Lambda_2$ and $\Lambda_i^2 = 1$. Moreover, we have the following natural exact sequences on Q_4 :

$$0 \rightarrow \mathcal{S}_i(-1) \rightarrow \mathcal{O}_{Q_4}^4 \xrightarrow{e_{\theta, \mathcal{S}_{i+1}}} \mathcal{S}_{i+1} \rightarrow 0, \tag{2.3}$$

where we take the indices modulo 2.

2.2. Stability and Moduli Spaces

Let us now recall a few well-known facts about semistable sheaves on projective varieties. We refer to [40] for a more detailed account of these notions.

Let (X, H_X) be a smooth complex projective n -dimensional polarized variety. We recall that a torsion-free coherent sheaf F on X is (Gieseker) *semistable* if, for any coherent subsheaf E with $\text{rk}(E) < \text{rk}(F)$, one has $p(E, t)/\text{rk}(E) \leq p(F, t)/\text{rk}(F)$ for $t \gg 0$. The sheaf F is called *stable* if this inequality is strict for all E as above.

If X has Picard number 1, we can view the first Chern class $c_1(F)$ of a sheaf F on X as an integer. Then the *slope* of a sheaf F of positive rank is defined as $\mu(F) = c_1(F)/\text{rk}(F)$. We say that F is *normalized* if $-1 < \mu(F) \leq 0$. We recall that a torsion-free coherent sheaf F is μ -*semistable* if, for any coherent subsheaf E with $\text{rk}(E) < \text{rk}(F)$, one has $\mu(E) \leq \mu(F)$. The sheaf F is called μ -*stable* if this inequality is strict for all E as above. The two notions are related by the following implications:

$$\mu\text{-stable} \implies \text{stable} \implies \text{semistable} \implies \mu\text{-semistable}.$$

Notice that a rank-2 sheaf F with odd $c_1(F)$ is μ -stable as soon as it is μ -semistable.

Recall that, by Maruyama’s theorem [62], if $\dim(X) = n \geq 2$ and F is a μ -semistable sheaf of rank $r < n$, then the restriction of F to a general hyperplane section of X is still μ -semistable.

Let us introduce some notation concerning moduli spaces. We denote by $M_X(r, c_1, \dots, c_n)$ the moduli space of S -equivalence classes of rank- r torsion-free semistable sheaves on X with Chern classes c_1, \dots, c_n , where c_k lies in $H^{k,k}(X)$. For brevity, sometimes we will write F instead of the class $[F]$. The Chern class c_k will be denoted by an integer as soon as $H^{k,k}(X)$ has dimension 1. We will drop the last values of the classes c_k when they are zero.

We will denote by Spl_X the coarse moduli space of simple sheaves on X . As proved in [2], it is an algebraic space. We denote by $\mathcal{H}_d^s(X)$ the union of components of the Hilbert scheme of closed Z subschemes of X , with Hilbert polynomial $p(\mathcal{O}_Z, t) = dt + 1 - g$, containing integral curves of degree d and arithmetic genus g .

2.3. Prime Fano Threefolds

Let now X be a smooth projective variety of dimension 3. Recall that X is called *Fano* if its anticanonical divisor class $-K_X$ is ample. A Fano threefold is called *non-hyperelliptic* if $-K_X$ is very ample.

We say that X is *prime* if the Picard group is generated by the canonical divisor class K_X . If X is a prime Fano threefold then $\text{Pic}(X) \cong \mathbb{Z} \cong \langle H_X \rangle$, where $H_X = -K_X$ is ample. One defines the *genus* of a prime Fano threefold X as the integer g such that $\deg(X) = H_X^3 = 2g - 2$.

Smooth prime Fano threefolds are classified up to deformation; see, for instance, [52, Chap. IV]. The number of deformation classes is 10. The genus of a smooth non-hyperelliptic prime Fano threefold takes values in $\{3, 4, \dots, 9, 10, 12\}$, and there are only two families (one for $g = 2$, another for $g = 3$) that consist of hyperelliptic threefolds. A hyperelliptic prime Fano threefold of genus 3 is a flat specialization of a quartic hypersurface in \mathbb{P}^4 (see e.g. [58] and the references therein). It is well known that a smooth non-hyperelliptic prime Fano threefold is ACM.

Any prime Fano threefold X contains lines and conics. The Hilbert scheme $\mathcal{H}_1^0(X)$ of lines contained in X is a projective curve. It is well known that the surface swept out by the lines of a prime Fano threefold X is linearly equivalent to the divisor rH_X for some $r \geq 2$ (see e.g. the table on p. 76 of [52]). Moreover, if $g \geq 4$ then every line meets finitely many other lines (see [51, Thm. 3.4(iii); 58]). If $g = 3$, we know by [33, Sec. 7] that there always exist two disjoint lines in X .

A prime Fano threefold X is said to be *exotic* if the Hilbert scheme $\mathcal{H}_1^0(X)$ has a component that is nonreduced at any point. By [51, Lemma 3.2], this is equivalent to the fact that, for any line $L \subset X$ of this component, the normal bundle N_L splits as $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-2)$. It is well known that a general prime Fano threefold X is not exotic. On the other hand, for $g \geq 9$, the results of [29] and [73] imply that X is nonexotic unless $g = 12$ and X is the Mukai–Umemura threefold (see [70]). We say that a prime Fano threefold X is *ordinary* if the Hilbert scheme $\mathcal{H}_1^0(X)$ has a generically smooth component. This is equivalent to the fact that there exists a line $L \subset X$ whose normal bundle N_L splits as $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$.

If X is a smooth non-hyperelliptic prime Fano threefold, then the Hilbert scheme $\mathcal{H}_2^0(X)$ of conics contained in X is a projective surface, and a general conic C in X has trivial normal bundle [51, Prop. 4.3, Thm. 4.4]. Moreover, the threefold X is covered by conics. In addition, if X is a general prime Fano threefold then $\mathcal{H}_2^0(X)$ is a smooth surface; see [42] for a survey.

A smooth projective surface S is a *K3 surface* if it has trivial canonical bundle and irregularity zero. A general hyperplane section S of a prime Fano threefold X is a K3 surface polarized by the restriction H_S of H_X to S . If X has genus g , then

S has (sectional) genus g and degree $H_S^2 = 2g - 2$. If X is non-hyperelliptic then, by Mořezon's theorem [65], we have $\text{Pic}(S) \cong \mathbb{Z} = \langle H_S \rangle$.

Note that a general hyperplane section of a hyperelliptic prime Fano threefold is still a K3 surface of Picard number 1 if $g = 2$. This is no longer true in the other hyperelliptic case (i.e., for $g = 3$). Indeed, let X be a double cover of a smooth quadric in \mathbb{P}^4 ramified on a general octic surface. Then the general hyperplane section is a K3 surface of Picard number 2.

2.4. Summary of Basic Formulas

From now on, X will be a smooth prime Fano threefold of genus g , polarized by H_X . Let F be a sheaf of (generic) rank r on X with Chern classes c_1, c_2, c_3 . Recall that these classes will be denoted by integers, since $H^{k,k}(X)$ is generated by the class of H_X for $k = 1$, by the class of a line contained in X for $k = 2$, and by the class of a closed point of X for $k = 3$. We will say that F is an r -bundle if it is a vector bundle (i.e. a locally free sheaf) of rank r . The *discriminant* of F is defined as

$$\Delta(F) = 2rc_2 - (r - 1)(2g - 2)c_1^2. \quad (2.4)$$

Bogomolov's inequality (see e.g. [40, Thm. 3.4.1]) states that if F is a μ -semistable sheaf then

$$\Delta(F) \geq 0. \quad (2.5)$$

Applying Hirzebruch–Riemann–Roch to F we get

$$\chi(F) = r + \frac{11 + g}{6}c_1 + \frac{g - 1}{2}c_1^2 - \frac{1}{2}c_2 + \frac{g - 1}{3}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3.$$

We recall by [66] (see also [40, Part II, Chap. 6]) that, given a simple sheaf F of rank r on a K3 surface S of genus g , with Chern classes c_1, c_2 , the dimension at the point $[F]$ of the moduli space Spl_S is

$$\Delta(F) - 2(r^2 - 1), \quad (2.6)$$

where $\Delta(F)$ is still defined by (2.4). If the sheaf F is stable, then this value also equals the dimension at the point $[F]$ of the moduli space $M_S(r, c_1, c_2)$.

Let us focus on vector bundles F of rank 2. Then we have $F \cong F^*(c_1(F))$. Further, the well-known Hartshorne–Serre correspondence relates vector bundles of rank 2 over X with subvarieties Z of X of codimension 2. We refer to [34], [36], and, in particular, [37, Thm. 4.1] (see also [4] for a survey).

PROPOSITION 2.2. *Fix the integers c_1, c_2 . Then we have a one-to-one correspondence between the sets*

- (1) *of equivalence classes of pairs (F, s) , where F is a rank-2 vector bundle on X with $c_i(F) = c_i$ and s is a global section of F , up to multiplication by a nonzero scalar, whose zero locus has codimension 2, and*
- (2) *of locally complete intersection curves $Z \subset X$ of degree c_2 , with $\omega_Z \cong \mathcal{O}_Z(c_1 - 1)$.*

Recall that Z has arithmetic genus $p_a(Z) = 1 - \frac{d(1-c_1)}{2}$. The zero locus of a nonzero global section s of a rank-2 vector bundle F has codimension 2 if F is globally generated and s is general, or if $H^0(X, F(-1)) = 0$.

LEMMA 2.3. *Assume that X is not hyperelliptic and let F be a rank-2 bundle on X . Let s be a global section of F whose zero locus is a curve $D \subset X$. Then we have*

$$H_*^1(X, F) \cong H_*^1(X, \mathcal{I}_{D,X}(c_1(F))). \tag{2.7}$$

In particular, F is ACM if and only if $H_^1(X, \mathcal{I}_{D,X}) = 0$ if and only if $H_*^1(X, \mathcal{I}_{D,\mathbb{P}^m}) = 0$. If D is smooth, then F is ACM if and only if D is projectively normal.*

Proof. The section s gives the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow F \rightarrow \mathcal{I}_D(c_1(F)) \rightarrow 0; \tag{2.8}$$

and, taking cohomology, we obtain the required isomorphism (2.7). By Serre duality it follows that, since F is locally free, the condition $H_*^1(X, F) = 0$ is equivalent to $H_*^2(X, F) = 0$ and thus to F being ACM. Since by (2.7) the module $H_*^1(X, \mathcal{I}_D)$ is isomorphic to $H_*^1(X, F) = 0$ up to the grading, we have $H_*^1(X, \mathcal{I}_D) = 0$ if and only if F is ACM.

Take now $X \subset \mathbb{P}^m$ polarized by H_X (which is very ample by assumption) and consider $D \subset X$. We have the exact sequence

$$0 \rightarrow \mathcal{I}_{X,\mathbb{P}^m} \rightarrow \mathcal{I}_{D,\mathbb{P}^m} \rightarrow \mathcal{I}_{D,X} \rightarrow 0.$$

Recall that X is an ACM variety of dimension 3, so $H_*^k(X, \mathcal{I}_{D,\mathbb{P}^m}) = 0$ for $k = 1, 2$. Therefore, taking cohomology in the preceding sequence, it follows that $H_*^1(X, \mathcal{I}_{D,X}) = 0$ if and only if $H_*^1(X, \mathcal{I}_{D,\mathbb{P}^m}) = 0$.

Finally, if $D \subset \mathbb{P}^m$ is smooth then the condition $H_*^1(X, \mathcal{I}_{D,\mathbb{P}^m}) = 0$ is equivalent to D being projectively normal (see [35, Chap. II, Exer. 5.14]). □

2.5. ACM Bundles of Rank 2

In this section, we recall Madonna’s result in the case of bundles of rank 2 on a smooth prime Fano threefold.

PROPOSITION 2.4 [57]. *Let F be a normalized ACM 2-bundle on X . Then the Chern classes c_1 and c_2 of F satisfy the following restrictions:*

$$\begin{aligned} c_1 = 0 &\implies c_2 \in \{2, 4\}; \\ c_1 = 1 &\implies c_2 \in \{1, \dots, g + 3\}. \end{aligned}$$

REMARK 2.5. Let F and c_1, c_2 be as before, and let t_0 be the smallest integer t such that $H^0(X, F(t)) \neq 0$. In [57] the author computes the following values of t_0 :

- (a) if $(c_1, c_2) = (1, 1)$, then $t_0 = -1$;
- (b) if $(c_1, c_2) = (0, 2)$, then $t_0 = 0$;

- (c) if $(c_1, c_2) = (1, c_2)$ with $2 \leq c_2 \leq g + 2$, then $t_0 = 0$;
- (d) if $(c_1, c_2) = (0, 4)$, then $t_0 = 1$;
- (e) if $(c_1, c_2) = (1, g + 3)$, then $t_0 = 1$.

We observe that F is not semistable in cases (a) and (b) but is strictly μ -semistable in case (b). In the remaining cases, if F exists then it is a μ -stable sheaf.

The existence of F in cases (a) and (b) is well known. It amounts, in view of Proposition 2.2, to the existence of lines and conics contained in X .

The following lemma (cf. [15, Lemma 3.1]) gives a sharp lower bound on the values in Madonna's list. We set

$$m_g = \left\lceil \frac{g+2}{2} \right\rceil. \quad (2.9)$$

LEMMA 2.6. *The moduli space $M_X(2, 1, d)$ is empty for $d < m_g$. In particular, we get the further restriction $c_2 \geq m_g$ in case (c) of Remark 2.5.*

REMARK 2.7. The non-hyperelliptic assumption cannot be dropped. Indeed, if X is a hyperelliptic Fano threefold of genus 3 then the moduli space $M_X(2, 1, 2)$ is not empty. Let $Q \in \mathbb{P}^4$ be a smooth quadric and $\pi : X \rightarrow Q$ a double cover ramified along a general octic surface. Set $F = \pi^*(S)$, where S is the spinor bundle on Q . Then F is a stable vector bundle on X lying in $M_X(2, 1, 2)$. Notice that the restriction F_S to any hyperplane section $S \subset X$ is decomposable (hence strictly semistable).

3. Bundles with Odd First Chern Class

Throughout the paper, we denote by X a smooth non-hyperelliptic prime Fano threefold of genus g . In this section we will prove the existence of the semistable bundles, appearing in Madonna's (restricted) list, whose first Chern class is odd. The main result of this section is the following existence theorem.

THEOREM 3.1. *Let X be a smooth non-hyperelliptic prime Fano threefold of genus g , and let $\frac{g}{2} + 1 \leq d \leq g + 3$. If $d \geq \frac{g}{2} + 2$, we assume, in addition, that X is ordinary. Then there exists an ACM vector bundle F of rank 2 with $c_1(F) = 1$ and $c_2(F) = d$. Moreover, in the range $d \geq \frac{g}{2} + 2$, such a bundle F can be chosen from a generically smooth component of the moduli space $M_X(2, 1, d)$ of dimension $2d - g - 2$.*

We will study first the case of minimal c_2 and then proceed recursively.

3.1. Moduli of ACM 2-bundles with Minimal c_2

In this section we study the moduli space of rank-2 semistable sheaves with odd c_1 (we may assume that c_1 is 1) and minimal c_2 . Namely, given a smooth non-hyperelliptic prime Fano threefold X of genus g , we set $m_g = \left\lceil \frac{g+2}{2} \right\rceil$ according to (2.9) and study $M_X(2, 1, m_g)$. Our goal is to prove the following statement.

THEOREM 3.2. *Let X be a smooth non-hyperelliptic prime Fano threefold of genus g . Then any sheaf F lying in $M_X(2, 1, m_g)$ is locally free and ACM, and it is globally generated if $g \geq 4$.*

Furthermore, there is a line $L \subset X$ such that

$$F \otimes \mathcal{O}_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(1) \tag{3.1}$$

and $M_X(2, 1, m_g)$ can be described as follows:

- (i) *the curve $\mathcal{H}_1^0(X)$ parameterizing lines contained in X if $g = 3$;*
- (ii) *a scheme of length 2 if $g = 4$, smooth if and only if X is contained in a smooth quadric;*
- (iii) *a double cover of the Hesse septic curve if $g = 5$ (see Section 3.1.5);*
- (iv) *a single smooth point if $g = 6, 8, 10, 12$;*
- (v) *a smooth nontetragonal curve of genus 7 if $g = 7$;*
- (vi) *a smooth plane quartic if $g = 9$.*

Moreover, if we assume that X is ordinary if $g = 3$ and that X is contained in a smooth quadric if $g = 4$, then there is a sheaf F in $M_X(2, 1, m_g)$ with

$$\text{Ext}_X^2(F, F) = 0. \tag{3.2}$$

Finally, if X is ordinary, then the line L in (3.1) can be chosen in such a way that $N_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$.

The proof of this theorem is presented in Sections 3.1.1–3.1.5.

3.1.1. Nonemptiness

It is well known that, for any non-hyperelliptic smooth prime Fano threefold X of genus g , the moduli space $M_X(2, 1, m_g)$ is nonempty. To the authors’ knowledge, there is no proof of this fact other than a case-by-case analysis. We refer, for example, to [56] for $g = 3$, to [57] for $g = 4, 5$, to [30] for $g = 6$, to [46; 47; 55] for $g = 7$, to [31; 32; 68] for $g = 8$, to [48] for $g = 9$, to [68] for $g = 10$, and to [54] (see also [26; 74]) for $g = 12$.

Given a sheaf F in $M_X(2, 1, m_g)$, we note that F is locally free and $H^k(X, F) = 0$ for $k \geq 1$ by [15, Prop. 3.5]. The Riemann–Roch theorem implies that

$$h^0(X, F) = g + 3 - m_g,$$

and any section $s \neq 0$ in $H^0(X, F)$ vanishes along a curve C_s ; this gives rise to the exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{s} F \rightarrow \mathcal{I}_{C_s, X}(1) \rightarrow 0, \tag{3.3}$$

where C_s has degree m_g . We immediately have

$$h^0(X, \mathcal{I}_{C_s, X}(1)) = g + 2 - m_g. \tag{3.4}$$

3.1.2. Cases $g \geq 6$

Let F be a sheaf in $M_X(2, 1, m_g)$ (there is such an F by the previous paragraph). From [15, Prop. 3.5] it follows that F is locally free, ACM, and globally generated. Given any line L contained in X , the sheaf F satisfies (3.1); indeed, F has

degree 1 and is globally generated on L . Clearly we can choose L with $N_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$ if X is ordinary.

It only remains to study the structure of $M_X(2, 1, m_g)$. We will do this with the aid of the following two lemmas, which are probably well known to experts but for which we can find no explicit reference in the literature.

LEMMA 3.3. *Assume $g \geq 6$, and let F and F' be sheaves in $M_X(2, 1, m_g)$. Then we have $\text{Ext}_X^2(F, F') = 0$. In particular, the space $M_X(2, 1, m_g)$ is smooth. If g is even then this implies that $M_X(2, 1, m_g)$ consists of a single smooth point.*

Proof. We have said that F' is globally generated and so we write the natural exact sequence

$$0 \rightarrow K \rightarrow H^0(X, F') \otimes \mathcal{O}_X \xrightarrow{e_{\mathcal{O}, F'}} F' \rightarrow 0, \quad (3.5)$$

where the sheaf K is locally free; then we have

$$\text{rk}(K) = g - m_g + 1, \quad c_1(K) = -1.$$

Note that K is a stable bundle by Hoppe's criterion (see e.g. [3, Thm. 12; 38, Lemma 2.6]). Indeed, note that $H^0(X, K) = 0$ and that we have $-1 < \mu(\wedge^p K) < 0$ for $0 < p < \text{rk}(K)$. By the inclusion

$$\wedge^p K \hookrightarrow \wedge^{p-1} K \otimes H^0(X, F)$$

we obtain, recursively, $H^0(X, \wedge^p K) = 0$ for all $p \geq 0$.

Now, since F is stable and ACM, we have $H^k(X, F^*) = 0$ for all k . Thus, tensoring (3.5) by F^* , we obtain

$$\text{ext}_X^2(F, F') = h^2(X, F^* \otimes F') = h^3(X, F^* \otimes K) = h^0(X, K^* \otimes F^*) = 0,$$

where the last equality holds by stability. Indeed, $c_1(K^* \otimes F^*) = m_g - g + 1 < 0$ for $g \geq 6$.

Note that, when g is even, we have $\chi(F, F') = 1$. This gives $\text{Hom}_X(F, F') \neq 0$. But a nonzero morphism $F \rightarrow F'$ must be an isomorphism. This concludes the proof. \square

LEMMA 3.4. *Assume $g \geq 6$ and g odd. Then the space $M_X(2, 1, m_g)$ is fine and isomorphic to a smooth irreducible curve.*

Proof. Let F be a sheaf in $M_X(2, 1, m_g)$. By Lemma 3.3, the moduli space is smooth and, since $\chi(F, F) = 0$, we have $\text{ext}_X^1(F, F) = \text{hom}_X(F, F) = 1$. Thus $M_X(2, 1, m_g)$ is a nonsingular curve.

It is well known (from classical results due to Narasimhan, Ramanan, and Grothendieck) that the obstruction to the existence of a universal sheaf on $X \times M_X(2, 1, m_g)$ corresponds to an element of the Brauer group of $M_X(2, 1, m_g)$. But this group vanishes as soon as the variety $M_X(2, 1, m_g)$ is a nonsingular curve [17; 28]. Hence we have a universal vector bundle on $X \times M_X(2, 1, m_g)$. We consider a component M of $M_X(2, 1, m_g)$ and let \mathcal{E} be the restriction of the universal sheaf to $X \times M$. We let p and q be the projections of $X \times M$ to X and M , respectively.

To prove the irreducibility of $M_X(2, 1, m_g)$, we denote by \mathcal{E}_y the restriction of \mathcal{E} to $X \times \{y\}$. We have $\mathcal{E}_y \cong \mathcal{E}_z$ if and only if $y = z$ for $y, z \in M$. Moreover, for any sheaf F in $M_X(2, 1, m_g)$, we have

$$\text{Ext}_X^k(\mathcal{E}_y, F) = 0 \quad \text{for } k = 2, 3 \text{ and for all } k \text{ if } F \not\cong \mathcal{E}_y,$$

where the vanishing for $k = 2$ follows from Lemma 3.3. Hence we have

$$\begin{aligned} \mathbf{R}^k q_* (\mathcal{E}^* \otimes p^*(F)) &= 0 \quad \text{for } k \neq 1; \\ \mathbf{R}^1 q_* (\mathcal{E}^* \otimes p^*(F)) &\cong \mathcal{O}_y \quad \text{for } F \cong \mathcal{E}_y. \end{aligned}$$

In particular, for any sheaf F in $M_X(2, 1, m_g)$, the sheaf $\mathbf{R}^1 q_* (\mathcal{E}^* \otimes p^*(F))$ has rank 0 and we have $\chi(\mathbf{R}^1 q_* (\mathcal{E}^* \otimes p^*(F))) = 1$, since this value can be computed by the Grothendieck–Riemann–Roch formula. Thus there must be a point $y \in M$ such that $\text{Ext}_X^1(\mathcal{E}_y, F) \neq 0$; hence $\text{Hom}_X(\mathcal{E}_y, F) \neq 0$ and so $F \cong \mathcal{E}_y$. This implies that F belongs to M . □

Lemma 3.3 thus proves (3.2) as well as part (iv) of Theorem 3.2. The irreducibility statement of Lemma 3.4 proves that $M_X(2, 1, m_g)$ is a curve of the desired type by [46] for $g = 7$ (in this case irreducibility was already known) and by [48] for $g = 9$. Theorem 3.2 is thus proved for $g \geq 6$, and it remains to establish it for $g = 3, 4, 5$; this we do in Sections 3.1.3–3.1.5.

3.1.3. Case $g = 3$

A smooth non-hyperelliptic prime Fano threefold of genus 3 is a smooth quartic threefold in \mathbb{P}^4 . To prove Theorem 3.2, we need the following proposition.

PROPOSITION 3.5. *Let $X \subset \mathbb{P}^4$ be a smooth quartic threefold. Then any element F in $M_X(2, 1, 3)$ is an ACM bundle and fits into an exact sequence of the form*

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow H^0(X, F) \otimes \mathcal{O}_X \xrightarrow{e_{6,F}} F \rightarrow \mathcal{O}_L(-2) \rightarrow 0, \tag{3.6}$$

where L is a line contained in X .

The map $F \mapsto L$ gives an isomorphism of $M_X(2, 1, 3)$ to $\mathcal{H}_1^0(X)$.

Proof. We consider a sheaf F in $M_X(2, 1, 3)$ and a cubic curve C_s associated to a nonzero global section s by (3.3). By (3.4), the curve C_s spans a projective plane $\Lambda \subset \mathbb{P}^4$ that intersects X along $D = C_s \cup L$, where L is a line. Then we have the exact sequence

$$0 \rightarrow \mathcal{I}_{D,X}(1) \rightarrow \mathcal{I}_{C_s,X}(1) \rightarrow \mathcal{I}_{C_s,D}(1) \rightarrow 0.$$

Note that $\mathcal{I}_{C_s,D}$ is a torsion-free sheaf of rank 1 supported on L and hence of the form $\mathcal{O}_L(t)$. By calculating Chern classes one can easily show that $t = -3$, so the preceding exact sequence now reads

$$0 \rightarrow \mathcal{I}_{D,X}(1) \rightarrow \mathcal{I}_{C_s,X}(1) \rightarrow \mathcal{O}_L(-2) \rightarrow 0.$$

Since D is cut by two hyperplanes, we also have a surjective map $\mathcal{O}_X^2 \rightarrow \mathcal{I}_{D,X}(1)$ whose kernel is $\mathcal{O}_X(-1)$. It is easy to patch these exact sequences together with (3.3) to obtain a long exact sequence of the form

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{O}_X^3 \rightarrow F \rightarrow \mathcal{O}_L(-2) \rightarrow 0, \quad (3.7)$$

which amounts to (3.6) because $\mathcal{O}_L(-2)$ has no nonzero global sections.

A straightforward Hilbert polynomial computation—together with the remark that all sheaves in $M_X(2, 1, 3)$ are stable—show that we can apply [40, Cor. 4.6.6] to get a universal sheaf \mathcal{E} on $X \times M_X(2, 1, 3)$. We denote by p and q the projections from $X \times M_X(2, 1, 3)$ to X and $M_X(2, 1, 3)$, respectively. We can thus globalize the exact sequence (3.7) over $X \times M_X(2, 1, 3)$ and write the middle arrow as the fiber over a point of $M_X(2, 1, 3)$ of the natural map:

$$q^*(q_*(\mathcal{E})) \rightarrow \mathcal{E}.$$

Taking the support of the cokernel sheaf of this map yields a family of lines contained in X and parameterized by $M_X(2, 1, 3)$; hence, by the universal property of the Hilbert scheme, this family is induced by a morphism $\alpha: M_X(2, 1, 3) \rightarrow \mathcal{H}_1^0(X)$.

We observe that, dualizing and twisting (3.7), we easily obtain an exact sequence of the form

$$0 \rightarrow F \rightarrow \mathcal{O}_X^3(1) \rightarrow \mathcal{I}_{L,X}(2) \rightarrow 0. \quad (3.8)$$

Let now L be any line contained in X . Since L is cut by three hyperplanes, we have a projection $\mathcal{O}_X^3(1) \rightarrow \mathcal{I}_{L,X}(2)$. It is easy to see that the kernel of this projection is a stable bundle lying in $M_X(2, 1, 3)$. In order to globalize (3.8), we denote by \mathcal{I} the universal ideal sheaf on $X \times \mathcal{H}_1^0(X)$ and by f and g the projections from $X \times \mathcal{H}_1^0(X)$ to X and $\mathcal{H}_1^0(X)$, respectively. Thus we have the surjective map

$$f^*(\mathcal{O}_X(1)) \otimes g^*(g_*(\mathcal{I} \otimes f^*(\mathcal{O}_X(1)))) \rightarrow \mathcal{I} \otimes f^*(\mathcal{O}_X(2)).$$

We therefore have a family of sheaves in $M_X(2, 1, 3)$ parameterized by $\mathcal{H}_1^0(X)$ and hence a classifying map $\beta: \mathcal{H}_1^0(X) \rightarrow M_X(2, 1, 3)$. Since α and β are mutually inverse, the schemes $M_X(2, 1, 3)$ and $\mathcal{H}_1^0(X)$ are isomorphic. \square

Let us note that the foregoing analysis implies Theorem 3.2 for X . We know that F is locally free and ACM. This last fact can be seen directly as follows. The curve C_s is a complete intersection in \mathbb{P}^4 . Thus we have $H_*^1(\mathbb{P}^4, \mathcal{I}_{C_s, \mathbb{P}^4}) = 0$, so F is ACM by Lemma 2.3. Condition (3.2) holds for F as soon as F corresponds to a smooth point of $\mathcal{H}_1^0(X)$, and such points exist as soon as X is ordinary. Finally, let $L' \subset X$ be a line that does not meet L (it exists for any X by [33, Sec. 7]). Restricting (3.7) to L' , we see that the splitting required for (3.1) holds on L' .

3.1.4. Case $g = 4$

A smooth prime Fano threefold X of genus 4 must be the complete intersection of a quadric Q and a cubic in \mathbb{P}^5 . Almost all the results we need for the next proposition follow from [57, Sec. 3.2].

PROPOSITION 3.6. *Let X be a smooth prime Fano threefold of genus 4, and let $Q \subset \mathbb{P}^5$ be the unique quadric containing X .*

- (i) If the quadric Q is smooth, then $M_X(2, 1, 3)$ consists of two smooth points given by two globally generated stable ACM bundles F_1 and F_2 . Moreover, we have the natural exact sequence

$$0 \rightarrow F_i(-1) \rightarrow \mathcal{O}_X^4 \xrightarrow{e_{\sigma, F_{i+1}}} F_{i+1} \rightarrow 0, \tag{3.9}$$

where we take the indices modulo 2.

- (ii) If the quadric Q is singular, then $M_X(2, 1, 3)$ consists of a length-2 scheme supported at a point that corresponds to a globally generated stable ACM bundle F with

$$\text{hom}_X(F, F) = \text{ext}_X^2(F, F) = 1, \quad \text{ext}_X^1(F, F) = 2, \tag{3.10}$$

and we have the natural exact sequence

$$0 \rightarrow F(-1) \rightarrow \mathcal{O}_X^4 \xrightarrow{e_{\sigma, F}} F \rightarrow 0. \tag{3.11}$$

Proof. Given a sheaf F in $M_X(2, 1, 3)$, we consider a cubic curve C_s arising as the zero locus of a global section of F . By (3.4), the curve C_s is contained in three independent hyperplanes. Therefore, C_s spans a projective plane Λ that must be contained in Q by degree reasons. The curve C_s is thus a complete intersection in \mathbb{P}^5 , so that $H_*^1(\mathbb{P}^5, \mathcal{I}_{C_s, \mathbb{P}^5}) = 0$ and F is ACM by Lemma 2.3. This gives a direct argument besides [15, Prop. 3.5] to prove that F is ACM.

Assume now that Q is nonsingular. Then one considers the bundles F_1 and F_2 obtained by restricting to X the two nonisomorphic spinor bundles S_1 and S_2 on Q . Note that F_1 and F_2 are not isomorphic, as $\text{Ext}_X^k(F_i, F_i) = 0$ for $k \geq 1$. One can check this by computing the vanishing of $H^k(Q, S_i \otimes S_i^*(-3))$ for all k , which in turn follows from Bott’s theorem. It is easy to deduce that the F_i are stable and hence provide two smooth points of $M_X(2, 1, 3)$.

Note that Λ arises as the zero locus of a section of S_i for some i such that F is the restriction of S_i to X . We have thus proved that $M_X(2, 1, 3)$ consists of two smooth points. Finally, restricting the exact sequence (2.3) to X , we obtain (3.9). This finishes the proof in case (i).

We consider now the case when Q is singular—namely, Q is a cone with vertex v over a smooth quadric Q' contained in $\mathbb{P}^4 \subset \mathbb{P}^5$. Here we have one spinor bundle S on Q' lifting to a rank-2 sheaf \tilde{F} on Q that is locally free away from v . It is easy to check that, by restricting \tilde{F} to X , we get a stable bundle in $M_X(2, 1, 3)$.

A plane $\Lambda \subset Q$ must be the span of v and a line L contained in Q' . Recall that L arises as the zero locus of a global section of S . This easily implies that Λ is the zero locus of a global section of \tilde{F} , so that F is the restriction of \tilde{F} . Therefore $M_X(2, 1, 3)$ is supported at a single point $[F]$. By specialization from the case (i), it follows that $M_X(2, 1, 3)$ is a scheme structure of length 2 over $[F]$.

Further, an exact sequence of the form (2.2) takes place on Q' . Lifting this sequence to Q and restricting to X , we obtain the exact sequence (3.11). It is now easy to obtain (3.10) by applying the functor $\text{Hom}_X(F, \cdot)$ to (3.11), noting that $\chi(F, F) = 0$, and using Serre duality. □

All the statements of Theorem 3.2 are now proved for X , except the splitting (3.1). But since F and F_i are globally generated, (3.1) holds for any line $L \subset X$.

3.1.5. *Case $g = 5$*

Let us first recall some basic facts concerning prime Fano threefolds of genus 5, for which we refer to [11, Sec. 1.5]. The threefold X is defined as the complete intersection of a net Π of quadrics in \mathbb{P}^6 ; that is, for each point y of the projective plane Π we have a quadric $Q_y \subset \mathbb{P}^6$. This defines a quadric fibration $f: \mathcal{X} \rightarrow \Pi$, where \mathcal{X} is the set of pairs of points $(x, y) \in \mathbb{P}^6 \times \Pi$ and where the point x lies in Q_y and f is the projection onto the second factor. The plane Π contains the Hesse septic curve \mathcal{H} of singular quadrics, and \mathcal{H} is smooth away from finitely many ordinary double points. Each quadric Q_y in \mathcal{H} has rank at least 5 (for X is smooth) and admits one or two rulings according to whether $\text{rk}(Q_y)$ equals 5 or 6. The curve parameterizing these rulings is denoted by $\tilde{\mathcal{H}}$. It admits an involution τ whose only fixed points lie over the singularities of \mathcal{H} , and we have $\mathcal{H} \cong \tilde{\mathcal{H}}/\tau$. This defines $\tilde{\mathcal{H}}$ as a double cover of \mathcal{H} , and we say that $\tilde{\mathcal{H}}$ is *associated* to X . Further, we consider the set of projective spaces $\mathbb{P}^3 \subset Q_y$ belonging to the same ruling of Q_y . This defines a \mathbb{P}^3 -bundle $G(f) \rightarrow \tilde{\mathcal{H}}$, and we denote by $\mathbb{P}_{\tilde{y}}^3$ the fiber over $\tilde{y} \in \tilde{\mathcal{H}}$.

PROPOSITION 3.7. *Let X be a smooth prime Fano threefold of genus 5, and let $\tilde{\mathcal{H}}$ be associated to X . Then the space $M_X(2, 1, 4)$ is isomorphic to $\tilde{\mathcal{H}}$, and any element $F \in M_X(2, 1, 4)$ is globally generated and ACM.*

Moreover, there is an involution ρ on $M_X(2, 1, 4)$ that associates to F the sheaf F^ρ fitting into

$$0 \rightarrow F^\rho(-1) \rightarrow \mathcal{O}_X^4 \xrightarrow{e_{\mathcal{O}, F}} F \rightarrow 0, \tag{3.12}$$

and ρ corresponds to τ under the isomorphism $M_X(2, 1, 4) \cong \tilde{\mathcal{H}}$.

Proof. We will identify the fiber $\mathbb{P}_{\tilde{y}}^3$ of the \mathbb{P}^3 -bundle $G(f) \rightarrow \tilde{\mathcal{H}}$ with the projectivized space of sections of a rank-2 sheaf on Q_y . We distinguish the two cases $\text{rk}(Q_y) = 5, 6$ for $y \in \mathcal{H}$.

If the quadric Q_y has rank 6, then it is a cone whose vertex is a point v over a smooth quadric Q'_y in a \mathbb{P}^5 contained in \mathbb{P}^6 . The set of projective three-spaces Λ contained in Q_y is thus parameterized by the set of planes in Q'_y . These planes are in bijection with the elements of $\mathbb{P}(H^0(Q'_y, S_i))$ for $i = 1, 2$, where S_1, S_2 are the spinor bundles on Q'_y . Each of the bundles S_i extends to a sheaf \tilde{F}_i on Q_y that is locally free away from v , and we easily compute $h^0(Q_y, \tilde{F}_i) = h^0(Q'_y, S_i) = 4$. We note, incidentally, that there is a natural exact sequence (we take the indices modulo 2)

$$0 \rightarrow \tilde{F}_i(-1) \rightarrow \mathcal{O}_{Q_y}^4 \xrightarrow{e_{\mathcal{O}, \tilde{F}_{i+1}}} \tilde{F}_{i+1} \rightarrow 0 \tag{3.13}$$

that lifts to Q_y the exact sequence (2.3).

Summing up, a subspace $\Lambda = \mathbb{P}^3$ contained in Q_y corresponds to an element of $\mathbb{P}(H^0(Q_y, \tilde{F}))$ with $\tilde{F} = \tilde{F}_1$ or $\tilde{F} = \tilde{F}_2$. So $\mathbb{P}_{\tilde{y}}^3$ is canonically identified with $\mathbb{P}(H^0(Q_y, \tilde{F}))$. Note also that, given $\Lambda \subset Q_y$, we have

$$0 \rightarrow \mathcal{O}_{Q_y} \rightarrow \tilde{F} \rightarrow \mathcal{I}_{\Lambda, Q_y}(1) \rightarrow 0. \tag{3.14}$$

If $\text{rk}(Q_y) = 5$, then Q_y is a cone with vertex on a line $L \subset \mathbb{P}^6$ over a smooth quadric $Q''_y \subset \mathbb{P}^4$. In this case the subspaces $\Lambda = \mathbb{P}^3$ of Q_y are given by lines in Q''_y , and any of these lines can be described as the zero locus of a section of S , the spinor bundle on Q''_y . One can lift S to a rank-2 sheaf \tilde{F} on Q_y that is locally free away from L . We still have $h^0(Q_y, \tilde{F}) = 4$ and (3.14) still holds, so $\mathbb{P}^3_{\tilde{y}}$ is again identified with $\mathbb{P}(H^0(Q_y, \tilde{F}))$.

In order to prove our statement, we show that the \mathbb{P}^3 -bundle $G(f)$ is isomorphic over the base curve to the \mathbb{P}^3 -bundle on $M_X(2, 1, 4)$ consisting of pairs $([s], F)$, where F lies in $M_X(2, 1, 4)$ and $[s]$ lies in $\mathbb{P}(H^0(X, F))$. Recall that s gives rise to the curve C_s , which has degree 4. By (3.4), the curve C_s is contained in three independent hyperplanes, so C_s spans a \mathbb{P}^3 that must be contained in a singular quadric Q_y . Note that this quadric is unique, for otherwise X would contain a quadric surface, contradicting $\text{Pic}(X) \cong \langle H_X \rangle$. Note also that the curve C_s is a complete intersection in \mathbb{P}^6 , so $H^1(\mathbb{P}^6, \mathcal{I}_{C_s, \mathbb{P}^6}) = 0$ and hence F is ACM by Lemma 2.3. This confirms [15, Prop. 3.5].

Given this setup, we associate to $[s] \in \mathbb{P}(H^0(X, F))$ the element $[\tilde{s}]$ of $\mathbb{P}(H^0(Q_y, \tilde{F}))$ that corresponds to the space spanned by C_s . It is easy to see that $H^0(Q_y, \tilde{F})$ is naturally isomorphic to $H^0(X, F)$ and that, restricting (3.14) from Q_y to X , we obtain (3.3). Then $\mathbb{P}^3_{\tilde{y}}$ is identified with $\mathbb{P}(H^0(X, F))$ and we can associate \tilde{y} to F .

Note that this construction is reversible: to a point \tilde{y} of $\tilde{\mathcal{H}}$ we associate the bundle F on X such that, for any element Λ in $\mathbb{P}^3_{\tilde{y}}$, the intersection $\Lambda \cap X$ is a curve of degree 4 obtained as zero locus of a section of F . This proves that $M_X(2, 1, 4)$ is isomorphic to $\tilde{\mathcal{H}}$. The vanishing (3.2) holds for F as soon as F corresponds to a smooth point of $\tilde{\mathcal{H}}$. We remark that any such F is globally generated because it is the restriction to X of \tilde{F} , which is globally generated (this is clear, for instance, by (3.14)). We have thus constructed an isomorphism $M_X(2, 1, 4) \cong \tilde{\mathcal{H}}$.

It remains to check the statement regarding the involutions on $M_X(2, 1, 4)$ and $\tilde{\mathcal{H}}$. We have to check that ρ is well-defined and (under the isomorphism $M_X(2, 1, 4) \cong \tilde{\mathcal{H}}$ constructed previously) agrees with τ , which by definition interchanges the rulings of Q_y as soon as $\text{rk}(Q_y) = 6$. Recall that any sheaf F in $M_X(2, 1, 4)$ is the restriction to X of a sheaf \tilde{F} on Q_y , say of \tilde{F}_1 , that corresponds to one ruling of Q_y . We thus have (3.13) (with $i = 0$), and restricting to X yields an exact sequence of the form (3.12) for some sheaf F^ρ lying in $M_X(2, 1, 4)$. Note that F^ρ is then the restriction to X of the sheaf \tilde{F}_2 on Q_y . Since \tilde{F}_2 corresponds to the second ruling of Q_y , we have proved that ρ agrees with τ . \square

Proposition 3.7 proves Theorem 3.2 for X , once we check (3.1). But this splitting holds for any line $L \subset X$, since any $F \in M_X(2, 1, 4)$ is globally generated.

We have now finished the proof of Theorem 3.2.

3.2. Moduli of ACM 2-bundles with Intermediate c_2

This section is devoted to the proof of Theorem 3.1 in the cases $m_g + 1 \leq d \leq g + 2$. This will prove in particular the existence of case (c) of Madonna's list;

see Remark 2.5. We will need a series of lemmas to prove recursively the existence of ACM bundles of rank 2. The following lemma is proved in [15, Thm. 3.9], once we take care of the special case of Fano threefolds of genus 4 contained in a singular quadric. Note that this is the only case for which no sheaf F in $M_X(2, 1, m_g)$ satisfies (3.15).

LEMMA 3.8. *Let X be ordinary and let L be a general line in X . Then, for any integer $d \geq m_g + 1$, there exists a rank-2, stable, locally free sheaf F with $c_1(F) = 1$ and $c_2(F) = d$ that satisfies*

$$\mathrm{Ext}_X^2(F, F) = 0, \quad (3.15)$$

$$H^1(X, F(-1)) = 0, \quad (3.16)$$

$$F \otimes \mathcal{O}_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(1), \quad (3.17)$$

where L is a line with $N_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$.

Proof. All statements are proved in [15, Thm. 3.9] by induction on $d \geq m_g + 1$ (except when $g = 4$ and X is contained in a singular quadric). The induction step proceeds as follows. Given a stable 2-bundle F_{d-1} with $c_1(F_{d-1}) = 1$ and $c_2(F_{d-1}) = d - 1$ that satisfies (3.17) for a given line $L \subset X$ (with $N_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$), we have the unique exact sequence

$$0 \rightarrow \mathcal{S}_d \rightarrow F_{d-1} \xrightarrow{\sigma} \mathcal{O}_L \rightarrow 0, \quad (3.18)$$

where σ is the natural surjection and $\mathcal{S}_d = \ker(\sigma)$ is a nonreflexive sheaf in $M_X(2, 1, d)$. We have proved in [15, Thm. 3.9] that if (3.15) and (3.16) hold for F_{d-1} , then we get a vector bundle of $M_X(2, 1, d)$ satisfying (3.15), (3.16), and (3.17) by flatly deforming \mathcal{S}_d .

Assume thus $g = 4$ and that X is contained in a singular quadric. Given a line L contained in X such that $N_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$ and the vector bundle $F_3 \in M_X(2, 1, 3)$ (see Proposition 3.6), we set $\mathcal{S}_4 = \ker(F_3 \rightarrow \mathcal{O}_L)$ (where the map is nonzero). We obtain an exact sequence of the form (3.18) with $d = 4$.

The sheaf \mathcal{S}_4 sits in $M_X(2, 1, 4)$, and we want to prove that $\mathrm{Ext}_X^2(\mathcal{S}_4, \mathcal{S}_4) = 0$. Applying $\mathrm{Hom}_X(\mathcal{S}_4, \cdot)$ to (3.18) yields

$$\mathrm{Ext}_X^1(\mathcal{S}_4, \mathcal{O}_L) \rightarrow \mathrm{Ext}_X^2(\mathcal{S}_4, \mathcal{S}_4) \rightarrow \mathrm{Ext}_X^2(\mathcal{S}_4, F_3).$$

It is easy to check that the first term vanishes by applying $\mathrm{Hom}_X(\cdot, \mathcal{O}_L)$ to (3.18) and then using [15, Rem. 2.1] and the fact that (3.17) holds for F_3 . To prove the vanishing of the last term, we apply $\mathrm{Hom}_X(\mathcal{S}_4, \cdot)$ to (3.11) and note that

$$\mathrm{ext}_X^3(\mathcal{S}_4, F_3(-1)) = \mathrm{hom}(F_3, \mathcal{S}_4) = 0$$

by Serre duality and stability.

In this setup, the sheaf \mathcal{S}_4 admits a smooth neighborhood in $M_X(2, 1, m_g + 1)$, which has dimension 2 in view of an easy Riemann–Roch computation. On the

other hand, the sheaves fitting in (3.18) fill in a curve in $M_X(2, 1, m_g + 1)$ by [15, Lemma 3.8]. Therefore, the remaining part of the argument of [15, Thm. 3.9] goes through. \square

DEFINITION 3.9. Let X be ordinary. Let $M(m_g)$ be a component of $M_X(2, 1, m_g)$ containing a stable locally free sheaf F that satisfies the three conditions (3.15), (3.16), and (3.17). (When $g = 4$ and X is contained in a singular quadric, we just set $M(3) = \{F\}$ with F given by Proposition 3.6.) This component exists by Theorem 3.2, and it coincides with $M_X(2, 1, m_g)$ for $g \geq 6$. For each $d \geq m_g + 1$, we recursively define $N(d)$ as the set of nonreflexive sheaves \mathcal{S}_d fitting as kernel in an exact sequence of the form (3.18), with $F_{d-1} \in M(d - 1)$ (and F_{d-1} satisfying (3.15), (3.16), and (3.17)) and $M(d)$ as the component of the moduli scheme $M_X(2, 1, d)$ containing $N(d)$. We have

$$\dim(M(d)) = 2d - g - 2.$$

LEMMA 3.10. *Let X be ordinary. For each $m_g \leq d \leq g + 2$, the general element F_d of $M(d)$ satisfies*

$$\begin{aligned} h^0(X, F_d) &= g + 3 - d, \\ H^k(X, F_d) &= 0 \text{ for } k \geq 1. \end{aligned}$$

Proof. The proof works by induction on d . The first step of the induction corresponds to $d = m_g$ and follows from Theorem 3.2. Note that $H^3(X, F_d) = 0$ for all d by Serre duality and stability, and by Riemann–Roch we have $\chi(F_d) = g + 3 - d$.

Assume now that the statement holds for F_{d-1} with $d \leq g + 2$, and let us prove it for a general element \mathcal{S}_d of $N_X(d)$. By semicontinuity the claim will follow for the general element $F_d \in M(d)$. So let F_{d-1} be a locally free sheaf in $M(d - 1)$. By induction we know that $h^0(X, F_{d-1}) = g + 3 - d + 1 \geq 2$. A nonzero global section s of F_{d-1} gives the exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{s} F_{d-1} \rightarrow \mathcal{I}_C(1) \rightarrow 0, \tag{3.19}$$

where C is a curve of degree $d - 1$ and arithmetic genus 1. We want to show that we can choose a line $L \subset X$ and a section s such that C does not meet L , and this will prove that

$$\mathcal{O}_L \otimes \mathcal{I}_C(1) \cong \mathcal{O}_L(1). \tag{3.20}$$

To do this, we note that $h^0(X, \mathcal{I}_C(1)) = g + 3 - d \geq 1$, so C is contained in some hyperplane section surface S given by a global section t of $\mathcal{I}_C(1)$. Let L be a general line such that $F_{d-1} \otimes \mathcal{O}_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(1)$ and L meets S at a single point x . We may assume the latter condition because there exists a line in X not contained in S (indeed, the lines contained in X sweep a divisor of degree > 1 ; see Section 2.3). Then we write down the exact commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \mathcal{O}_X & \xlongequal{\quad} & \mathcal{O}_X & \\
 & & & \downarrow t & & \downarrow t & \\
 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s} & F_{d-1} & \xrightarrow{s^\top} & \mathcal{I}_C(1) \longrightarrow 0 \\
 & & \parallel & & \downarrow t^\top & & \downarrow & \\
 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s} & \mathcal{I}_D(1) & \longrightarrow & \mathcal{O}_S(H_S - C) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
 \end{array}$$

which in turn yields the exact sequence

$$0 \rightarrow \mathcal{O}_X^2 \xrightarrow{\binom{s}{t}} F_{d-1} \rightarrow \mathcal{O}_S(H_S - C) \rightarrow 0;$$

dualizing, we obtain

$$0 \rightarrow F_{d-1}^* \xrightarrow{\binom{s^\top}{t^\top}} \mathcal{O}_X^2 \rightarrow \mathcal{O}_S(C) \rightarrow 0. \tag{3.21}$$

Thus the curve C moves in a pencil without base points in the surface S , and each member C' of this pencil corresponds to a global section s' of F_{d-1} that vanishes on C' . Therefore we can choose s such that C does not contain x .

Now let σ be the natural surjection $F_{d-1} \rightarrow \mathcal{O}_L$ and $\mathcal{S}_d = \ker(\sigma)$. We thus have the exact sequence (3.18). Taking cohomology, from induction hypotheses we obtain $H^2(X, \mathcal{S}_d) = 0$.

By tensoring (3.19) by \mathcal{O}_L , in view of (3.20) we see that the composition $\sigma \circ s$ must be nonzero (in fact, it is surjective). Thus the section s does not lift to \mathcal{S}_d and so $h^0(X, \mathcal{S}_d) \leq h^0(X, F_{d-1}) - 1$. Therefore,

$$h^0(\mathcal{S}_d) \geq \chi(\mathcal{S}_d) = \chi(F_{d-1}) - 1 = h^0(F_{d-1}) - 1 \geq h^0(\mathcal{S}_d)$$

and our claim follows. □

Using Lemma 2.3, it is straightforward to deduce the following corollary from Lemma 3.10.

COROLLARY 3.11. *Let X be ordinary. For $d \leq g + 2$, let D be the zero locus of a nonzero global section of a general element F of $M(d)$. Then we have*

$$h^0(X, \mathcal{I}_D(1)) = g + 2 - d.$$

We are now in position to prove Theorem 3.1 in the cases when $c_2 \leq g + 2$.

Proof of Theorem 3.1 for $d \leq g + 2$. We work by induction on $d \geq m_g$. By Theorem 3.2, the statement holds for $d = m_g$.

Assume now that $m_g < d \leq g + 2$. By Lemma 3.8 we can consider a general sheaf F in $M(d)$. Recall that F is obtained as a general deformation of a sheaf \mathcal{S}_d fitting into an exact sequence of the form (3.18), where F_{d-1} is a vector bundle in $M(d - 1)$. By induction we assume that F_{d-1} is ACM. It remains to prove that F , too, is ACM.

Since $d - 1 \leq g + 2$, we can choose (as in the proof of Lemma 3.10) a line $L \subset X$, a projection $\sigma: F_{d-1} \rightarrow \mathcal{O}_L$, and a global section $s \in H^0(X, F_{d-1})$ such that $\sigma \circ s$ is surjective. We can assume $\mathcal{S}_d = \ker(\sigma)$. Let C be the zero locus of s . Then we have the following exact diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{I}_L & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_L \longrightarrow 0 \\
 & & \downarrow & & \downarrow s & & \parallel \\
 0 & \longrightarrow & \mathcal{S}_d & \longrightarrow & F_{d-1} & \xrightarrow{\sigma} & \mathcal{O}_L \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{I}_C(1) & \equiv & \mathcal{I}_C(1) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array} \tag{3.22}$$

Since L is projectively normal, the leftmost column implies that $H_*^1(X, \mathcal{S}_d) \subset H_*^1(X, \mathcal{I}_C(1))$. By Lemma 2.3 we have $H_*^1(X, \mathcal{I}_C(1)) \cong H_*^1(X, F_{d-1})$, and this module vanishes by the induction hypothesis. So we obtain $H_*^1(X, \mathcal{S}_d) = 0$; hence, by semicontinuity, the module $H_*^1(X, F)$ is zero as well. Then by Serre duality the vector bundle F is ACM. \square

The following lemma will be needed later on.

LEMMA 3.12. *Let D be the zero locus of a global section of a sheaf F lying in $M_X(2, 1, d)$, satisfying (3.15) and (3.16), and such that $H^1(X, \mathcal{I}_D(1)) = 0$. Then $\text{Ext}_X^2(\mathcal{I}_D, \mathcal{I}_D) = 0$ and we obtain the exact sequence*

$$0 \rightarrow H^0(X, \mathcal{I}_D(1)) \rightarrow \text{Ext}_X^1(\mathcal{I}_D, \mathcal{I}_D) \rightarrow \text{Ext}_X^1(F, F) \rightarrow 0, \tag{3.23}$$

so $\text{ext}_X^1(\mathcal{I}_D, \mathcal{I}_D) = d$.

Proof. We apply the functor $\text{Hom}_X(F, \cdot)$ to the exact sequence (2.8). It is easy to check that $\text{Ext}_X^k(F, \mathcal{O}_X) = 0$ for any k and thus we obtain, for each k , an isomorphism

$$\text{Ext}_X^k(F, \mathcal{I}_D(1)) \cong \text{Ext}_X^k(F, F).$$

Hence, by applying $\text{Hom}_X(\cdot, \mathcal{I}_D(1))$ to (2.8) we get the vanishing $\text{Ext}_X^2(\mathcal{I}_D, \mathcal{I}_D) = 0$ and, since F is a stable (and thus simple) sheaf, we obtain the exact sequence (3.23). The value of $\text{ext}_X^1(\mathcal{I}_D, \mathcal{I}_D)$ can now be computed by Riemann–Roch. \square

3.3. Moduli of ACM 2-bundles with Maximal c_2

In order to complete the proof of Theorem 3.1 we have to consider the case $d = g + 3$. This will give the existence of case (e) of Madonna's list. We need the following lemma.

LEMMA 3.13. *Let F be a rank-2 stable bundle on X with $c_1(F) = 1$. Then F is ACM if*

$$H^k(X, F) = H^k(X, F(-1)) = 0 \quad \text{for any } k.$$

Proof. We would like to prove that $H^1(X, F(t))$ vanishes for any integer t . This holds for $t < 0$ in view of [15, Prop. 3.7], so we need only show it for $t \geq 0$.

Let S be a general hyperplane section of X . Taking cohomology of the restriction exact sequence

$$0 \rightarrow F(-1+t) \rightarrow F(t) \rightarrow F_S(t) \rightarrow 0, \quad (3.24)$$

we obtain that $H^k(S, F_S) = 0$ for any k . By Serre duality, since $F^* \cong F(-1)$ we also have $H^k(S, F_S(-1)) = 0$ for any k . It follows that $H^k(C, F_C) = 0$ for any k , where C is the general sectional curve of X . Now, since $H^0(C, F_C(t)) = 0$ for any $t \leq 0$, from the restriction exact sequence

$$0 \rightarrow F_S(-1+t) \rightarrow F_S(t) \rightarrow F_C(t) \rightarrow 0$$

we deduce that $H^1(S, F_S(t)) = 0$ for any $t \leq 0$. By Serre duality this also implies that $H^1(S, F_S(t)) = 0$ for any $t \geq 0$. Now from (3.24) we obtain $H^1(X, F(t)) = 0$ for any $t \geq 0$, and we have proved $H^1_*(X, F) = 0$. By Serre duality we immediately obtain the vanishing $H^2_*(X, F) = 0$, and we are done. \square

Proof of Theorem 3.1 for $d = g + 3$. By Lemma 3.8, there exists a sheaf F_{g+3} in $M(g+3)$ obtained as a general deformation of a sheaf \mathcal{S}_{g+3} fitting into the exact sequence

$$0 \rightarrow \mathcal{S}_{g+3} \rightarrow F_{g+2} \rightarrow \mathcal{O}_L \rightarrow 0, \quad (3.25)$$

where $F_{g+2} \in M(g+2)$ and L is a line contained in X and representing a smooth point of $\mathcal{H}_1^0(X)$. We already know that Theorem 3.1 holds for $c_2 = g + 2$; hence we can assume that F_{g+2} is ACM, so $h^0(X, F_{g+2}) = 1$. It remains to prove that F_{g+3} is ACM, too.

We can assume that F_{g+3} satisfies condition (3.16) because \mathcal{S}_{g+3} does, so by [15, Prop. 3.7] we have $H^k(X, F_{g+3}(-1)) = 0$ for all k . Taking cohomology of (3.25) yields $H^2(X, \mathcal{S}_{g+3}) = 0$, so by semicontinuity we can assume $H^2(X, F_{g+3}) = 0$. On the other hand, $H^3(X, F_{g+3}) = 0$ by Serre duality and stability.

Note that $h^0(X, F_{g+3}) \leq 1$ by semicontinuity and (3.25). If $h^0(X, F_{g+3}) = 0$ then, by the Riemann–Roch formula, we also have $H^1(X, F_{g+3}) = 0$. Hence we can apply Lemma 3.13 and so conclude that F_{g+3} is ACM.

In order to complete the proof, we can assume that there is an open dense neighborhood $\Omega \subset M_X(2, 1, g+3)$ of the point representing \mathcal{S}_{g+3} such that all elements F_{g+3} (including \mathcal{S}_{g+3}) satisfy $h^0(X, F_{g+3}) = 1$ and then show that this leads to a

contradiction. By Lemma 3.8, we can assume $\text{Ext}_X^2(F_{g+3}, F_{g+3}) = 0$, which by Riemann–Roch implies that

$$\dim(\Omega) = \text{ext}_X^1(F_{g+3}, F_{g+3}) = g + 4. \tag{3.26}$$

For any F_{g+3} in Ω we consider the curve D , which is the zero locus of the (unique up to scalar) nonzero global section of F_{g+3} . This gives a map

$$\beta: \Omega \rightarrow \mathcal{H}_{g+3}^1(X).$$

We observe that the sheaf F_{g+3} can be recovered from D in view of Proposition 2.2, so β is injective. We will prove that $\mathcal{H}_{g+3}^1(X)$ is smooth and locally of dimension $g + 3$ around the point representing D , which contradicts β being injective because $\dim(\Omega) = g + 4$. In order to do this, we will prove

$$\text{Ext}_X^2(\mathcal{I}_D, \mathcal{I}_D) = 0 \quad \text{and} \quad \text{ext}_X^1(\mathcal{I}_D, \mathcal{I}_D) = g + 3, \tag{3.27}$$

where the second equality follows from the first vanishing by Riemann–Roch.

Consider now a nonzero global section s of the (nonreflexive) sheaf \mathcal{S}_{g+3} . We will say that a curve $B \subset X$ is the *zero locus* of s if we have an exact sequence:

$$0 \rightarrow \mathcal{O}_X(-1) \xrightarrow{s} \mathcal{S}_{g+3}(-1) \rightarrow \mathcal{I}_B \rightarrow 0.$$

Note that the section s induces a (nonzero) global section of F_{g+2} whose zero locus is a curve $C \subset X$. The exact sequence

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow F_{g+2}(-1) \rightarrow \mathcal{I}_C \rightarrow 0 \tag{3.28}$$

induces, in view of (3.25) twisted by $\mathcal{O}_X(-1)$, the exact sequences

$$0 \rightarrow \mathcal{I}_{C \cup L} \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_L(-1) \rightarrow 0 \quad \text{and} \tag{3.29}$$

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{S}_{g+3}(-1) \rightarrow \mathcal{I}_{C \cup L} \rightarrow 0, \tag{3.30}$$

so $C \cup L$ is the zero locus of s .

Note that our neighborhood Ω gives a flat family of curves in X ; that is, at the point corresponding to a sheaf F we associate the zero locus of its (unique up to scalar) nonzero global section. The central fiber of this family (the one corresponding to the sheaf \mathcal{S}_{g+3}) is $C \cup L$ and the general fiber is D , so D is a deformation of $C \cup L$. Then it will suffice to prove (3.27) on $C \cup L$. The rest of the proof is devoted to this task.

Applying the functor $\text{Hom}_X(\cdot, \mathcal{O}_L(-1))$ to (3.28), we obtain

$$\begin{aligned} 0 \rightarrow \text{Hom}_X(F_{g+2}, \mathcal{O}_L) &\rightarrow \text{Hom}_X(\mathcal{O}_X, \mathcal{O}_L) \\ &\rightarrow \text{Ext}_X^1(\mathcal{I}_C, \mathcal{O}_L(-1)) \rightarrow \text{Ext}_X^1(F_{g+2}, \mathcal{O}_L). \end{aligned}$$

Indeed, we have

$$\text{Hom}_X(\mathcal{I}_C, \mathcal{O}_L(-1)) \cong H^0(X, \mathcal{H}om_X(\mathcal{I}_C, \mathcal{O}_X) \otimes \mathcal{O}_L(-1)) = 0. \tag{3.31}$$

We also have $\text{hom}_X(\mathcal{O}_X, \mathcal{O}_L) = 1$, and (3.17) for F_{g+2} implies that

$$\text{hom}_X(F_{g+2}, \mathcal{O}_L) = 1 \quad \text{and} \quad \text{Ext}_X^1(F_{g+2}, \mathcal{O}_L) = 0.$$

Then we deduce the vanishing

$$\mathrm{Ext}_X^1(\mathcal{I}_C, \mathcal{O}_L(-1)) = 0. \quad (3.32)$$

Let us now apply $\mathrm{Hom}_X(\mathcal{I}_{C \cup L}, \cdot)$ to (3.29). We obtain

$$\mathrm{Ext}_X^1(\mathcal{I}_{C \cup L}, \mathcal{O}_L(-1)) \rightarrow \mathrm{Ext}_X^2(\mathcal{I}_{C \cup L}, \mathcal{I}_{C \cup L}) \rightarrow \mathrm{Ext}_X^2(\mathcal{I}_{C \cup L}, \mathcal{I}_C).$$

We want to show that the middle term in this sequence is zero by showing that the remaining terms vanish. Applying $\mathrm{Hom}_X(\cdot, \mathcal{O}_L(-1))$ to (3.29), we get

$$\mathrm{Ext}_X^1(\mathcal{I}_C, \mathcal{O}_L(-1)) \rightarrow \mathrm{Ext}_X^2(\mathcal{I}_{C \cup L}, \mathcal{O}_L(-1)) \rightarrow \mathrm{Ext}_X^2(\mathcal{O}_L, \mathcal{O}_L).$$

The leftmost term vanishes by (3.32), and the rightmost term vanishes by [15, Rem. 2.1]. It follows that $\mathrm{Ext}_X^2(\mathcal{I}_{C \cup L}, \mathcal{O}_L(-1)) = 0$. Now, we apply $\mathrm{Hom}_X(\cdot, \mathcal{I}_C)$ to (3.29). We get

$$\mathrm{Ext}_X^2(\mathcal{I}_C, \mathcal{I}_C) \rightarrow \mathrm{Ext}_X^2(\mathcal{I}_{C \cup L}, \mathcal{I}_C) \rightarrow \mathrm{Ext}_X^3(\mathcal{O}_L(-1), \mathcal{I}_C).$$

Note that $\mathrm{Ext}_X^3(\mathcal{O}_L(-1), \mathcal{I}_C) \cong \mathrm{Hom}_X(\mathcal{I}_C, \mathcal{O}_L(-2))^* = 0$, where the vanishing follows from (3.31). On the other hand, by Lemma 3.12 we can assume $\mathrm{Ext}_X^2(\mathcal{I}_C, \mathcal{I}_C) = 0$ and hence $\mathrm{Ext}_X^2(\mathcal{I}_{C \cup L}, \mathcal{I}_C) = 0$.

Summing up, we conclude that $\mathrm{Ext}_X^2(\mathcal{I}_{C \cup L}, \mathcal{I}_{C \cup L}) = 0$ and, by applying Riemann–Roch, we obtain $\mathrm{ext}_X^1(\mathcal{I}_{C \cup L}, \mathcal{I}_{C \cup L}) = g + 3$. By semicontinuity, we obtain the same vanishing for the curve D as well. We have thus shown (3.27), and this finishes the proof. \square

4. Bundles with Even First Chern Class

We let again X be any smooth non-hyperelliptic prime Fano threefold. In this section, we study semistable sheaves F with Chern classes $c_1(F) = 0$, $c_2(F) = 4$, and $c_3(F) = 0$ on X , and we prove the existence of case (d) of Madonna’s list. The main result of this part is the following.

THEOREM 4.1. *Let X be a smooth non-hyperelliptic prime Fano threefold. Then there exists a rank-2, ACM, stable locally free sheaf F with $c_1(F) = 0$ and $c_2(F) = 4$. The bundle F lies in a generically smooth component of dimension 5 of the space $\mathcal{M}_X(2, 0, 4)$.*

We start with a review of some facts concerning conics contained in X .

4.1. Conics and Rank-2 Bundles with $c_2 = 2$

Here we study rank-2 sheaves on X with $c_1 = 0$ and $c_2 = 2$ as well as their relation to the Hilbert scheme $\mathcal{H}_2^0(X)$ of conics contained in X . We rely on well-known properties of the Hilbert scheme $\mathcal{H}_2^0(X)$ (see Section 2.3).

LEMMA 4.2. *Any Cohen–Macaulay curve $C \subset X$ of degree 2 has $p_a(C) \leq 0$. Moreover, if C is nonreduced then it must be a Gorenstein double structure on a line L defined by the exact sequence*

$$0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{I}_L \rightarrow \mathcal{O}_L(t) \rightarrow 0, \tag{4.1}$$

where $t \geq -1$ and we have $p_a(C) = -1 - t$ and $\omega_C \cong \mathcal{O}_C(-2 - t)$.

Proof. If C is reduced, clearly it must be a conic (then $p_a(C) = 0$) or the union of two skew lines (then $p_a(C) = -1$). So assume that C is nonreduced and hence is a double structure on a line L . By [59, Lemma 2] we have the exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{I}_C/\mathcal{I}_L^2 \rightarrow \mathcal{I}_L/\mathcal{I}_L^2 \rightarrow \mathcal{O}_L(t) \rightarrow 0, \\ 0 \rightarrow \mathcal{O}_L(t) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_L \rightarrow 0, \end{aligned} \tag{4.2}$$

and C is a Gorenstein structure given by Ferrand’s doubling (see [9; 60]). Recall that $\mathcal{I}_L/\mathcal{I}_L^2 \cong N_L^*$. By [51, Lemma 3.2] we have either $N_L^* \cong \mathcal{O}_L \oplus \mathcal{O}_L(1)$ or $N_L^* \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(2)$. It follows that $t \geq -1$ and we obtain (4.1). We compute that $c_3(\mathcal{I}_L) = -1$ and $c_3(\mathcal{O}_L(t)) = 1 + 2t$; hence $c_3(\mathcal{I}_C) = -2 - 2t$, so $p_a(C) = -1 - t$.

Dualizing (4.1), we can use the fundamental local isomorphism to obtain the exact sequence

$$0 \rightarrow \mathcal{O}_L(-2) \rightarrow \omega_C \rightarrow \mathcal{O}_L(-2 - t) \rightarrow 0, \tag{4.3}$$

which by functoriality is (4.2) twisted by $\mathcal{O}_X(-2 - t)$. This concludes the proof. \square

COROLLARY 4.3. *All conics contained in X are reduced if and only if $\mathcal{H}_0^1(X)$ is smooth. This occurs if X is general.*

Proof. By [52, Prop. 4.2.2], the Hilbert scheme $\mathcal{H}_0^1(X)$ is smooth if and only if we have $N_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$ for any line L in X . By Lemma 4.2 this is equivalent to the fact that any conic contained in X is reduced. Recall that if X is general then, by [52, Thm. 4.2.7], $\mathcal{H}_0^1(X)$ is smooth. \square

Given a conic D , in view of Proposition 2.2 (and by Lemma 4.2) there is a μ -semistable vector bundle F^D , with $c_1(F^D) = 0$ and $c_1(F^D) = 2$, that fits into

$$0 \rightarrow \mathcal{O}_X \xrightarrow{\varphi} F^D \rightarrow \mathcal{I}_D \rightarrow 0. \tag{4.4}$$

One can easily prove the vanishing $\text{Ext}_X^2(F^D, F^D) = 0$, since the normal bundle to D is trivial for generic D .

LEMMA 4.4. *Let F be a locally free sheaf on X , $C \subset X$ a conic with normal bundle $N_C \cong \mathcal{O}_C^2$, and x a point of C . Assume that $F \otimes \mathcal{O}_C \cong \mathcal{O}_C^2$ and that $\text{Ext}_X^2(F, F) = 0$. Let \mathcal{F} be a sheaf fitting into an exact sequence of the form*

$$0 \rightarrow \mathcal{F} \rightarrow F \rightarrow \mathcal{O}_C \rightarrow 0. \tag{4.5}$$

Then we have $H^0(C, \mathcal{F}(-x)) = 0$ and $\text{Ext}_X^2(\mathcal{F}, \mathcal{F}) = 0$.

Proof. To prove the vanishing of $H^0(C, \mathcal{F}(-x))$, we tensor (4.5) by \mathcal{O}_C and obtain the following exact sequence of sheaves on C :

$$0 \rightarrow \text{Tor}_1^X(\mathcal{O}_C, \mathcal{O}_C) \rightarrow \mathcal{F} \otimes \mathcal{O}_C \rightarrow F \otimes \mathcal{O}_C \rightarrow \mathcal{O}_C \rightarrow 0. \tag{4.6}$$

Recall that $\mathcal{F}or_1^X(\mathcal{O}_C, \mathcal{O}_C)$ is isomorphic to $N_C^* \cong \mathcal{O}_C^2$. Now, twisting (4.6) by $\mathcal{O}_C(-x)$ and taking global sections, we easily get $H^0(C, \mathcal{F}(-x)) = 0$. Next let us prove the vanishing of $\text{Ext}_X^2(\mathcal{F}, \mathcal{F})$. Applying the functor $\text{Hom}_X(\cdot, \mathcal{F})$ to (4.5), we obtain the exact sequence

$$\text{Ext}_X^2(F, \mathcal{F}) \rightarrow \text{Ext}_X^2(\mathcal{F}, \mathcal{F}) \rightarrow \text{Ext}_X^3(\mathcal{O}_C, \mathcal{F}).$$

We will prove that both the first and the last term of this sequence vanish. Consider the first term, and apply $\text{Hom}_X(F, \cdot)$ to (4.5). We get the exact sequence

$$\text{Ext}_X^1(F, \mathcal{O}_C) \rightarrow \text{Ext}_X^2(F, \mathcal{F}) \rightarrow \text{Ext}_X^2(F, F).$$

By assumption we have $\text{Ext}_X^2(F, F) = 0$ and $\text{Ext}_X^1(F, \mathcal{O}_C) \cong H^1(C, F) = 0$. We obtain $\text{Ext}_X^2(F, \mathcal{F}) = 0$. To show the vanishing of the group $\text{Ext}_X^3(\mathcal{O}_C, \mathcal{F})$, we apply Serre duality and obtain

$$\text{Ext}_X^3(\mathcal{O}_C, \mathcal{F})^* \cong \text{Hom}_X(\mathcal{F}, \mathcal{O}_C(-1)) \cong H^0(X, \mathcal{H}om_X(\mathcal{F}, \mathcal{O}_C(-1))).$$

To show that this group is zero, apply the functor $\mathcal{H}om_X(\cdot, \mathcal{O}_C)$ to the sequence (4.5) to get

$$0 \rightarrow \mathcal{O}_C \rightarrow F^* \otimes \mathcal{O}_C \rightarrow \mathcal{H}om_X(\mathcal{F}, \mathcal{O}_C) \rightarrow N_C \rightarrow 0,$$

which implies $\mathcal{H}om_X(\mathcal{F}, \mathcal{O}_C(-1)) \cong \mathcal{O}_C(-1)^3$, and this sheaf has no nonzero global sections. \square

LEMMA 4.5. *Let C and D be smooth disjoint conics contained in X with trivial normal bundle. Then a sheaf \mathcal{F} fitting into a nontrivial extension of the form*

$$0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{F} \rightarrow \mathcal{I}_D \rightarrow 0 \tag{4.7}$$

is simple.

Proof. In order to prove the simplicity, apply $\text{Hom}_X(\mathcal{F}, \cdot)$ to (4.7) and get

$$\text{Hom}_X(\mathcal{F}, \mathcal{I}_C) \rightarrow \text{Hom}_X(\mathcal{F}, \mathcal{F}) \rightarrow \text{Hom}_X(\mathcal{F}, \mathcal{I}_D).$$

The first term vanishes; indeed, applying $\text{Hom}_X(\cdot, \mathcal{I}_C)$ to (4.7) and since $C \cap D = \emptyset$ we get

$$0 \rightarrow \text{Hom}_X(\mathcal{F}, \mathcal{I}_C) \rightarrow \text{Hom}_X(\mathcal{I}_C, \mathcal{I}_C) \xrightarrow{\delta} \text{Ext}_X^1(\mathcal{I}_D, \mathcal{I}_C).$$

Clearly the map $\delta: \mathbb{C} \rightarrow \mathbb{C}$ is nonzero, so $\text{Hom}_X(\mathcal{F}, \mathcal{I}_C) = 0$. On the other hand, applying $\text{Hom}_X(\cdot, \mathcal{I}_D)$ to (4.7) yields

$$\text{Hom}_X(\mathcal{F}, \mathcal{I}_D) \cong \text{Hom}_X(\mathcal{I}_D, \mathcal{I}_D) \cong \mathbb{C},$$

from which we deduce $\text{hom}_X(\mathcal{F}, \mathcal{F}) = 1$ —that is, the sheaf \mathcal{F} is simple. \square

4.2. ACM Bundles of Rank 2 with $c_1 = 0$ and $c_2 = 4$

This section is devoted to the proof of Theorem 4.1. The idea is to produce the required ACM bundle of rank 2 as a deformation of a simple sheaf obtained as extension of the ideal sheaves of two sufficiently general conics.

Step 1. *Choose two smooth disjoint conics C and D in X with trivial normal bundle.* It is well known that there are two smooth conics C and D in X with trivial normal bundle (see Section 2.3). Let us check that we can assume that C and D are disjoint. Let S be a hyperplane section surface containing C . A general conic D intersects S at two points. Since X is covered by conics, moving D in $\mathcal{H}_2^0(X)$, these two points sweep out S . Thus, a general conic D meets C at most at a single point. This gives a rational map $\varphi: \mathcal{H}_2^0(X) \rightarrow C$. Note that, for any point $x \in C$, we have $H^0(C, N_C(-x)) = 0$. So there are only finitely many conics contained in X through x , for this space parameterizes the deformations of C that pass through x . Thus the general fiber of φ is finite, which is a contradiction.

Step 2. *Given the conics C and D , define a simple sheaf \mathcal{F} with*

$$c_1(\mathcal{F}) = 0, \quad c_2(\mathcal{F}) = 4, \quad c_3(\mathcal{F}) = 0.$$

Given the conic D , we have the bundle F^D fitting in (4.4). Tensoring by \mathcal{O}_C this exact sequence, we obtain $F_C^D \cong \mathcal{O}_C^2$. Then we have $\text{hom}_X(F^D, \mathcal{O}_D) = 2$, and for any nonzero morphism $F_C^D \rightarrow \mathcal{O}_C$ we denote by σ the surjective composition $\sigma: F^D \rightarrow F_C^D \rightarrow \mathcal{O}_C$. We can choose σ such that the composition $\sigma \circ \varphi: \mathcal{O}_X \rightarrow \mathcal{O}_C$ is nonzero—that is, such that

$$\ker(\sigma) \not\supset \text{Im}(\varphi) \otimes \mathcal{O}_C. \tag{4.8}$$

We denote by \mathcal{F} the kernel of σ . We have the exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow F^D \xrightarrow{\sigma} \mathcal{O}_C \rightarrow 0 \tag{4.9}$$

and, patching this exact sequence together with (4.4), we see that \mathcal{F} fits into (4.7). It is easy to compute the Chern classes of \mathcal{F} and to prove that \mathcal{F} is stable. By (4.7) we get $H^k(X, \mathcal{F}) = 0$ for all k . More than that, since a smooth conic is projectively normal, again by (4.7) we obtain

$$H_*^1(X, \mathcal{F}) = 0. \tag{4.10}$$

Note that the sheaf \mathcal{F} is strictly semistable and also simple by Lemma 4.5. This concludes Step 2.

Step 3. *Flatly deform the sheaf \mathcal{F} to a simple sheaf G that does not fit into an exact sequence of the form (4.9).* Note that Lemma 4.4 gives $\text{Ext}_X^2(\mathcal{F}, \mathcal{F}) = 0$. Hence, by [8] we know that there exists a universal deformation of the simple sheaf \mathcal{F} . Since semistability is an open property, by a result of Maruyama, we may assume that the deformation of \mathcal{F} is semistable. In other words, we can deform \mathcal{F} in the open subset Σ of Spl_X given by simple semistable sheaves of rank 2 and Chern classes $c_1 = 0, c_2 = 4$. By Riemann–Roch and by the simplicity of \mathcal{F} we get that $\text{ext}^1(\mathcal{F}, \mathcal{F}) = 5$. This implies that Σ is locally of dimension 5 around the point $[\mathcal{F}]$.

Now we want to prove that the set of sheaves in Σ fitting into an exact sequence of the form (4.9) forms a subset of codimension 1 in Σ .

We have proved that a sheaf \mathcal{F} fitting into an exact sequence of the form (4.9), for some disjoint conics $C, D \subset X$, fits also into (4.7). Therefore, we need only

prove that the set of sheaves fitting into (4.7) is a closed subset of dimension 4 of Σ . Since C and D belong to the surface $\mathcal{H}_2^0(X)$, it is enough to prove that there is in fact a unique (up to isomorphism) nontrivial such extension; that is, $\text{ext}_X^1(\mathcal{I}_D, \mathcal{I}_C) = 1$.

Note that $\text{Hom}_X(\mathcal{I}_D, \mathcal{I}_C) = \text{Ext}_X^3(\mathcal{I}_D, \mathcal{I}_C) = 0$; hence, by Riemann–Roch it suffices to prove $\text{Ext}_X^2(\mathcal{I}_D, \mathcal{I}_C) = 0$. After applying $\text{Hom}_X(\cdot, \mathcal{I}_C)$ to the exact sequence defining D in X , we need only prove the vanishing of $\text{Ext}_X^3(\mathcal{O}_D, \mathcal{I}_C)$. But this group is dual to

$$\text{Hom}_X(\mathcal{I}_C, \mathcal{O}_D(-1)) \cong \text{H}^0(X, \mathcal{H}om_X(\mathcal{I}_C, \mathcal{O}_D(-1))) \cong \text{H}^0(X, \mathcal{O}_C(-1)) = 0.$$

So we have proved that $\text{ext}_X^1(\mathcal{I}_D, \mathcal{I}_C) = 1$ and that the set of sheaves in Σ fitting into an exact sequence of the form (4.9) has codimension 1 in Σ . Then we can choose a deformation G of \mathcal{F} in Σ that does not fit into (4.9), thereby concluding Step 3.

Setting $E = G^{**}$, we write the double dual sequence

$$0 \rightarrow G \rightarrow E \rightarrow T \rightarrow 0. \tag{4.11}$$

Step 4. *Compute the Chern classes of T , and prove*

$$c_1(T) = 0, \quad -c_2(T) \in \{1, 2\}.$$

By semicontinuity, we may assume $\text{hom}_X(G, G) = 1$ and $\text{H}^1(X, G(-1)) = 0$. We may also assume that, for any given line L contained in X , we have the vanishing $\text{Ext}_X^1(\mathcal{O}_L(t), G) = 0$ for all $t \in \mathbb{Z}$. Indeed, applying $\text{Hom}_X(\mathcal{O}_L(t), \cdot)$ to (4.9) yields

$$\text{Hom}_X(\mathcal{O}_L(t), \mathcal{O}_C) \rightarrow \text{Ext}_X^1(\mathcal{O}_L(t), \mathcal{F}) \rightarrow \text{Ext}_X^1(\mathcal{O}_L(t), F^D);$$

observe that the leftmost term vanishes as soon as L is not contained in C (but C is irreducible) and that the rightmost term vanishes for F^D locally free. Clearly E is a semistable sheaf, so $\text{H}^1(X, G(-1)) = 0$ implies $\text{H}^0(X, T(-1)) = 0$; hence T must be a pure sheaf supported on a Cohen–Macaulay curve $B \subset X$. Summing up, we have $c_1(T) = 0$ and $c_2(T) < 0$.

Let us show $c_2(T) \geq -2$. We have already proved that $\text{H}^0(X, T(-1)) = 0$, and this implies that $\chi(T(t)) = -\text{h}^1(X, T(t))$ for any negative integer t . Recall that, by [37, Rem. 2.5.1], the reflexive sheaf E satisfies $\text{H}^1(X, E(t)) = 0$ for all $t \ll 0$. Thus, tensoring by $\mathcal{O}_X(t)$ the exact sequence (4.11) and taking cohomology, we obtain $\text{h}^1(X, T(t)) \leq \text{h}^2(X, G(t))$ for all $t \ll 0$. Further, for any integer t we can easily compute the following Chern classes:

$$c_1(T(t)) = 0, \quad c_2(T(t)) = c_2(T) = 4 - c_2(E), \quad c_3(T(t)) = c_3(E) - 2tc_2(T);$$

hence, by the Riemann–Roch formula we have

$$\chi(T(t)) = -tc_2(T) + \frac{1}{2}(c_3(E) - c_2(T)).$$

Since G is a general deformation of the sheaf \mathcal{F} , we also have, by semicontinuity, $\text{h}^2(X, G(t)) \leq \text{h}^2(X, \mathcal{F}(t))$. On the other hand, by (4.9) we have $\text{h}^2(X, \mathcal{F}(t)) = \text{h}^1(X, \mathcal{O}_C(t)) = -\chi(\mathcal{O}_C(t)) = -2t - 1$. In sum, for all $t \ll 0$ we have the following inequality:

$$-tc_2(T) + \frac{1}{2}(c_3(E) - c_2(T)) \geq 2t + 1,$$

which implies $c_2(T) \geq -2$.

Step 5. *Prove that B must be a smooth conic, and deduce that*

$$T \cong \mathcal{O}_B.$$

Assume the contrary, and note that either $T \cong \mathcal{O}_{L_1}(a_1)$ for some line $L_1 \subset X$ and some $a_1 \in \mathbb{Z}$ if $c_2(T) = -1$; if $c_2(T) = -2$ then, in view of Lemma 4.2, there must be a second line $L_2 \subset X$ (possibly coincident with L_1) and $a_2 \in \mathbb{Z}$ such that T fits into

$$0 \rightarrow \mathcal{O}_{L_1}(a_1) \rightarrow T \rightarrow \mathcal{O}_{L_2}(a_2) \rightarrow 0. \tag{4.12}$$

But we have seen that $\text{Ext}_X^1(\mathcal{O}_L(t), \mathcal{F}) = 0$ for all $t \in \mathbb{Z}$ and for any line $L \subset X$. By semicontinuity we get $\text{Ext}_X^1(\mathcal{O}_L(t), G) = 0$ for all $t \in \mathbb{Z}$ and any line $L \subset X$. In particular, $\text{Ext}_X^1(\mathcal{O}_{L_i}(a_i), G) = 0$ for $i = 1, 2$, so $\text{Ext}_X^1(T, G) = 0$ and (4.11) should split—which is absurd. Therefore, T must be of the form $\mathcal{O}_B(ax)$ for some integer a and for some point x of a smooth conic $B \subset X$. By [37, Prop. 2.6] we have $c_3(E) = c_3(T) = 2a \geq 0$, and $H^0(X, T(-1)) = 0$ implies $a - 2 < 0$.

We are left with the cases $a = 0$ and $a = 1$, and we want to exclude the latter. We do this by proving that $\text{Ext}_X^1(\mathcal{O}_B(x), G)$ is zero for any conic $B \subset X$. This fact can be checked by semicontinuity if we apply $\text{Hom}_X(\mathcal{O}_B(x), \cdot)$ to (4.9), obtaining

$$\text{Hom}_X(\mathcal{O}_B(x), \mathcal{O}_C) \rightarrow \text{Ext}_X^1(\mathcal{O}_B(x), \mathcal{F}) \rightarrow \text{Ext}_X^1(\mathcal{O}_B(x), F^D).$$

The rightmost term in this sequence vanishes because F^D is locally free. The leftmost term is isomorphic to $H^0(X, \mathcal{H}om(\mathcal{O}_B(x), \mathcal{O}_C))$, and for $B \neq C$ the sheaf $\mathcal{H}om(\mathcal{O}_B(x), \mathcal{O}_C)$ is zero. On the other hand, if $B = C$ then $\text{Hom}_X(\mathcal{O}_C(x), \mathcal{O}_C) \cong H^0(C, \mathcal{O}_C(-x)) = 0$.

We have thus obtained $T \cong \mathcal{O}_B$. But then G would fit into an exact sequence of the form (4.9), a contradiction. Summing up, we have proved that T must be zero, so G is isomorphic to E and thus locally free. Since $H^0(X, G) = 0$, the sheaf G must be stable. By (4.10) and semicontinuity we can assume $H_*^1(X, G(t)) = 0$, so by Serre duality we get that G is ACM. This concludes the proof of Theorem 4.1.

5. Applications

We devote this final section to some applications of the existence results for ACM bundles of rank 2 to pfaffian hypersurfaces of projective spaces and quadrics.

5.1. Pfaffian Cubics in a 4-dimensional Quadric

Here we show that the equation of a cubic hypersurface in a smooth quadric $Q \subset \mathbb{P}^5$ can be written as the pfaffian of a skew-symmetric 6×6 matrix of linear forms on the coordinate ring $R(Q)$.

THEOREM 5.1. *Let X be a smooth prime Fano threefold of genus 4 contained in a nonsingular quadric hypersurface $Q \subset \mathbb{P}^5$. Then the equation f of X in the*

coordinate ring of Q is the pfaffian of a skew-symmetric matrix M representing a map

$$\psi: \mathcal{O}_Q(-1)^6 \rightarrow \mathcal{O}_Q^6. \quad (5.1)$$

Recall that we denote by \mathcal{S}_1 and \mathcal{S}_2 the two nonisomorphic spinor bundles on Q (see Section 2.1). We denote by ι the inclusion of X in Q and by H_X the hyperplane class of $X \subset \mathbb{P}^5$.

LEMMA 5.2. *Let X be as before, and let F_i be the restriction of \mathcal{S}_i to X . Let C be a conic contained in X . Then we have*

$$H^k(X, F_i \otimes \mathcal{I}_{C,X}) = 0$$

for all $k = 0, \dots, 3$.

Proof. The inclusions $C \subset X \subset Q$ induce an exact sequence:

$$0 \rightarrow \mathcal{O}_Q(-3) \rightarrow \mathcal{I}_{C,Q} \rightarrow \mathcal{I}_{C,X} \rightarrow 0.$$

Recall that the bundles \mathcal{S}_i are ACM, and by stability we have

$$H^0(Q, \mathcal{S}_i(-1)) = 0. \quad (5.2)$$

Hence, after twisting the previous sequence by \mathcal{S}_i , we need only show $H^k(Q, \mathcal{S}_i \otimes \mathcal{I}_{C,Q}) = 0$. Now, the conic C is the intersection of the quadric Q and of three hyperplanes of \mathbb{P}^5 . Thus we have an exact sequence:

$$0 \rightarrow \mathcal{O}_Q(-3) \rightarrow \mathcal{O}_Q(-2)^3 \rightarrow \mathcal{O}_Q(-1)^3 \rightarrow \mathcal{I}_{C,Q} \rightarrow 0.$$

Twisting this exact sequence by \mathcal{S}_i and taking cohomology, we obtain the result by using (5.2) and the fact that the bundles \mathcal{S}_i are ACM. \square

LEMMA 5.3. *Let X be as before, and let E be a stable locally free sheaf in $M_X(2, 0, 4)$. Then we have*

$$\text{Ext}_Q^1(\iota_*(E(1)), \mathcal{S}_i(a)) = 0 \quad \text{for } a \leq -3. \quad (5.3)$$

If the sheaf E is general in the component of $M_X(2, 0, 4)$ provided by Theorem 4.1, then we also have

$$\text{Ext}_Q^1(\iota_*(E(1)), \mathcal{S}_i(-2)) = 0. \quad (5.4)$$

Proof. We first prove (5.3). Since $\mathcal{H}om_Q(\iota_*(E(1)), \mathcal{S}_i(a)) = 0$, the local-to-global spectral sequence provides the isomorphism

$$\text{Ext}_Q^1(\iota_*(E(1)), \mathcal{S}_i(a)) \cong H^0(Q, \mathcal{E}xt_Q^1(\iota_*(E(1)), \mathcal{O}_Q) \otimes \mathcal{S}_i(a)).$$

By Grothendieck duality, we have

$$\mathcal{E}xt_Q^1(\iota_*(E(1)), \mathcal{O}_Q) \cong \iota_*(E^*(-1)) \otimes \mathcal{O}_Q(3) \cong \iota_*(E(2)),$$

where the second isomorphism holds because E is locally free. Hence we are reduced to showing that

$$H^0(Q, \iota_*(E(2)) \otimes \mathcal{S}_i(a)) \cong H^0(X, E \otimes F_i(2+a)) = 0 \tag{5.5}$$

for $a \leq -3$. Note that the sheaf $E \otimes F_i(2+a)$ is semistable of slope $5/2+a$ and thus has no nonzero global sections if $a \leq -3$. Hence (5.3) is proved.

In order to prove (5.4), we let E be general in the component provided by Theorem 4.1 and then show that (5.5) holds for $a = -2$. In particular, we assume that E is a deformation of a simple sheaf \mathcal{F} given as the middle term of an extension of the form (4.7) of two ideal sheaves $\mathcal{I}_D, \mathcal{I}_C$ of two conics C, D contained in X . By semicontinuity, it will thus suffice to show that

$$H^0(X, \mathcal{F} \otimes F_i) = 0$$

for $i = 1, 2$. In turn, since \mathcal{F} is an extension of ideal sheaves of conics, it will be enough to prove $H^0(X, \mathcal{I}_C \otimes F_i) = 0$ for C a conic contained in X . But we have shown this in Lemma 5.2. □

Proof of Theorem 5.1. Let E be a stable ACM bundle of rank 2 in the component of $M_X(2, 0, 4)$ provided by Theorem 4.1. By Lemma 5.3 we may assume that E satisfies the cohomology vanishing conditions (5.3) and (5.4).

We consider a sheafified minimal graded free resolution of $\iota_*(E(1))$. In particular, we have a bundle \mathcal{P} on Q of the form

$$\mathcal{P} = \bigoplus_{i=1}^s \mathcal{O}_Q(b_i) \quad \text{with } b_1 \geq \dots \geq b_s \tag{5.6}$$

equipped with a projection $\pi : \mathcal{P} \rightarrow \iota_*(E(1))$ such that π induces a surjective map:

$$H_*^0(Q, \mathcal{P}) \rightarrow H_*^0(Q, \iota_*(E(1))). \tag{5.7}$$

Let \mathcal{K} be the kernel of π . It is clear that, since (5.7) is surjective, we have

$$H_*^1(Q, \mathcal{K}) = 0.$$

Moreover, since E is ACM on X and \mathcal{P} is ACM on Q , it easily follows that the sheaf \mathcal{K} is ACM on Q . By a well-known theorem of Knörrer [53], this implies that \mathcal{K} splits as a direct sum:

$$\mathcal{K} \cong \bigoplus_{j=1}^t \mathcal{S}_{i_j}(c_j) \oplus \bigoplus_{h=1}^u \mathcal{O}_Q(a_h) \quad \text{with } a_1 \geq \dots \geq a_u, \tag{5.8}$$

where $i_j \in \{1, 2\}$. Note that, since $H^0(X, E) = 0$, we have $b_i \leq 0$ for all i . Therefore we also have $a_i \leq -1$ (by the minimality of the resolution) and $c_j \leq -1$ (since $\text{Hom}_Q(\mathcal{S}_i(c), \mathcal{O}_Q) = 0$ for $c \geq 0$ by (5.2)).

Let us now use Lemma 5.3. The vanishing results (5.3) and (5.4) imply that no $c_j \leq -2$ occurs in the expression (5.8). For in that case, it would easily follow that \mathcal{P} contains $\mathcal{S}_{i_j}(c_j)$ as a direct summand, which is not the case. The remaining possibility is excluded by the following claim.

CLAIM 5.4. *We have $t = 0$ and $s = 6$. Moreover, for all $i = 1, \dots, 6$, we have $b_i = 0$ and $a_i = -1$.*

Once this claim is proved, the proof of Theorem 5.1 will be finished. Indeed, the sheaf \mathcal{K} is a direct sum of line bundles. Therefore, the argument of [12, Thm. B] applies to our setup, and the matrix M representing the morphism $\psi: \mathcal{K} \rightarrow \mathcal{P}$ can be chosen skew-symmetric with pfaffian equal to the equation defining $X \subset Q$. \square

Proof of Claim 5.4. We have the exact sequence

$$0 \rightarrow \mathcal{K} \xrightarrow{\psi} \mathcal{P} \rightarrow \iota_*(E(1)) \rightarrow 0.$$

Recall that, in view of the foregoing analysis, c_j can only be -1 whereas $a_i \leq -1$ and $b_i \leq 0$ for all i . One easily computes $h^0(X, E(1)) = 6$, so $b_i = 0$ for $i = 1, \dots, 6$ and $b_i \leq -1$ for $i \geq 7$. Then, recalling that $c_j = -1$ for all j and noting that $S_{i_j}(-1)$ must be mapped by the injective map $\psi: \mathcal{K} \rightarrow \mathcal{P}$ to \mathcal{O}_Q^6 , we deduce that $t \leq 3$. The two equations $\text{rk}(\iota_*(E(1))) = 0$ and $c_1(\iota_*(E(1))) = 6H_Q$ imply (respectively)

$$2t + u = s \quad \text{and}$$

$$\sum_{j=1}^s b_j - \sum_{i=1}^{s-2t} a_i + t = 6. \quad (5.9)$$

To prove our statement, we adapt an argument of Bohnhorst and Spindler [14]. Namely, we write $\psi = (\psi'_{r,1}, \dots, \psi'_{r,t}, \psi''_{r,1}, \dots, \psi''_{r,u})_{1 \leq r \leq s}$ and note that $\psi'_{r,j} = 0$ for any $r \geq 7$ and $1 \leq j \leq t$. Now, for each $\ell \leq s - 2t = u$, we let r_ℓ be the maximum integer r such that

$$(\psi'_{r,1}, \dots, \psi'_{r,t}, \psi''_{r,1}, \dots, \psi''_{r,\ell}) \neq 0.$$

Since the map ψ is injective, this easily implies that

$$2t + \ell \leq r_\ell.$$

Hence there must be $j \leq t$ such that $\psi'_{r_\ell, j} \neq 0$ or $j \leq \ell$ such that $\psi''_{r_\ell, j} \neq 0$. In the first case we have $r_\ell \leq 6$, so $2t + \ell \leq 6$ and hence $b_{2t+\ell} = 0$. In the second case, by the minimality of the resolution map ψ we get $b_{r_\ell} - a_j \geq 1$. We deduce, for each $\ell \leq s - 2t$, the inequality $b_{2t+\ell} - a_\ell \geq b_{r_\ell} - a_j \geq 1$. In both cases we have

$$b_{2t+\ell} - a_\ell \geq 1. \quad (5.10)$$

From equation (5.9) we get

$$t + \sum_{\ell=1}^{s-2t} (b_{2t+\ell} - a_\ell) \leq 6,$$

and by (5.10) we obtain

$$s - t = t + s - 2t \leq t + \sum_{\ell=1}^{s-2t} (b_{2t+\ell} - a_\ell) = 6.$$

One can now easily compute $c_2(\iota_*(E(1))) = 21(\Lambda_1 + \Lambda_2)$. This implies that t must be even, for otherwise we would have $c_2(\mathcal{K}) = \alpha_1\Lambda_1 + \alpha_2\Lambda_2$ with $\alpha_1 \neq \alpha_2$,

which easily leads to a contradiction. Since $t \leq 3$, $s \geq 6$, and $b_j \leq -1$ for $j \geq 7$, we are left with the following cases:

$$t = 0, s = 6 \implies \underline{b} = (0, 0, 0, 0, 0, 0), \underline{-a} = (1, 1, 1, 1, 1, 1); \tag{5.11}$$

$$t = 2, s = 6 \text{ and } \underline{b} = (0, 0, 0, 0, 0, 0), \underline{-a} = (2, 2); \tag{5.12}$$

$$t = 2, s = 6 \text{ and } \underline{b} = (0, 0, 0, 0, 0, 0), \underline{-a} = (1, 3); \tag{5.13}$$

$$t = 2, s = 7 \text{ and } \underline{b} = (0, 0, 0, 0, 0, 0, b_7), \underline{-a} = (1, 1, 2 - b_7); \tag{5.14}$$

$$t = 2, s = 7 \text{ and } \underline{b} = (0, 0, 0, 0, 0, 0, b_7), \underline{-a} = (1, 2, 1 - b_7); \tag{5.15}$$

$$t = 2, s = 8 \implies \underline{b} = (0, 0, 0, 0, 0, 0, b_7, b_8), \\ \underline{-a} = (1, 1, 1 - b_7, 1 - b_8). \tag{5.16}$$

Here \underline{a} and \underline{b} denote the vectors of \mathbb{Z}^{s-2t} and \mathbb{Z}^s representing (respectively) the sequence of the a_i and b_j , and we have $-1 \geq b_7 \geq b_8$. It is now an easy exercise to check that, in all the displayed cases except (5.1), the difference of the Hilbert polynomials of \mathcal{P} and \mathcal{K} does not equal $p(\iota_*(E(1)), t) = 2t^3 + 9t^2 + 13t + 6$. This is a contradiction, which leaves the desired case as the only possibility. \square

5.2. Pfaffian Quartic Threefolds in \mathbb{P}^4

A theorem of Iliev and Markushevich [45] asserts that a general quartic threefold X in \mathbb{P}^4 is a *linear pfaffian*—namely, its equation f is the pfaffian of an 8×8 skew-symmetric matrix of linear forms on \mathbb{P}^4 . Similarly, a result of Madonna [56] says that X is a *quadratic pfaffian*; that is, f can be written as the pfaffian of a 4×4 skew-symmetric matrix of quadratic forms. Their proofs are carried out with the aid of the computer algebra package Macaulay2.

Here we prove a result in the same spirit as an application of our existence theorems. We show that any ordinary quartic threefold in \mathbb{P}^4 is a linear pfaffian, and that any smooth quartic threefold in \mathbb{P}^4 is a quadratic pfaffian, with at most two more rows and columns of linear forms.

THEOREM 5.5. *Let X be a smooth quartic threefold in \mathbb{P}^4 defined by an equation f .*

- (i) *There is a skew-symmetric matrix M representing a map of one of the two forms*

$$\mathcal{O}_{\mathbb{P}^4}(-2)^4 \rightarrow \mathcal{O}_{\mathbb{P}^4}^4 \text{ or } \mathcal{O}_{\mathbb{P}^4}(-2)^4 \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^2 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^4}^4 \tag{5.17}$$

such that $\text{Pf}(M) = f$.

- (ii) *If X is ordinary, then there is a skew-symmetric matrix N representing a map*

$$\mathcal{O}_{\mathbb{P}^4}(-1)^8 \rightarrow \mathcal{O}_{\mathbb{P}^4}^8$$

such that $\text{Pf}(N) = f$.

Proof. We work as in Theorem 5.1. Let E be an ACM bundle on X , and consider the sheafified minimal graded free resolution of $\iota_*(E(1))$. We have a bundle \mathcal{P} on \mathbb{P}^4 of the form (5.6) and a projection $\pi : \mathcal{P} \rightarrow \iota_*(E(1))$ such that π is surjective

on global sections for each twist. The kernel \mathcal{H} of this projection is ACM on \mathbb{P}^4 , so we have

$$\mathcal{H} \cong \bigoplus_{h=1}^s \mathcal{O}_{\mathbb{P}^4}(a_h) \quad \text{with } a_1 \geq \dots \geq a_s.$$

The matrix representing the map $\mathcal{H} \rightarrow \mathcal{P}$ can be chosen skew-symmetric by [12, Thm. B], and its pfaffian is f . In particular, the integer s must be even. Assuming $H^0(X, E) = 0$, we have $b_i \leq 0$ for all i and so, by the minimality of the resolution, $a_i \leq -1$. We further have

$$\sum_{\ell=1}^s (b_\ell - a_\ell) = 8, \tag{5.18}$$

and, by the argument of Bohnhorst and Spindler, we can assume

$$b_\ell - a_\ell \geq 1 \quad \text{for all } 1 \leq \ell \leq s. \tag{5.19}$$

Now, to prove (ii) we choose as E a general bundle with $c_1(E) = 1$ and $c_2(E) = 6$ given by Theorem 3.1. It is straightforward to compute $H^0(X, E) = 0$ and $h^0(X, E(1)) = 8$. Therefore we have $s = 8$ and $b_i = 0$ for all i . Thus, by (5.18) and (5.19), we have $a_i = -1$ for all i and we are done.

Let us now show (i). This time we pick a general bundle E with $c_1(E) = 0$ and $c_2(E) = 4$ provided by Theorem 4.1. One computes $H^0(X, E) = 0$ and $h^0(X, E(1)) = 4$, so that $b_i = 0$ for $i = 1, 2, 3, 4$ and $b_i \leq -1$ for $i \geq 5$ and $s \in \{4, 6, 8\}$. The Hilbert polynomial $p(\iota_*(E(1)), t)$ reads

$$\frac{4}{3}t^3 + 6t^2 + \frac{26}{3}t + 4. \tag{5.20}$$

To finish the proof, we divide it into different cases according to the value of s . We want to show that E can be chosen so that $s = 4$ with $a_i = -2$ for all i or $s = 6$ and $a_i = b_i = -1$ for $i = 5, 6$.

Case 1: $s = 8$. In this case, in view of (5.18) and (5.19) we must have $b_i - a_i = 1$ for all i ; hence $a_i = -1$ for $i = 1, 2, 3, 4$ (and recall that $b_i \leq -1$). Looking at the quadratic term of the Hilbert polynomial, one sees that (5.20) forces $b_i = -1$ for all $i = 5, 6, 7, 8$. Therefore the pfaffian of the matrix N is the square of the determinant of a 4×4 matrix of linear forms, which is impossible because f is not a square.

Case 2: $s = 6$. In this case $b_6 \leq b_5 \leq -1$, and we have the possibilities

$$\begin{aligned} -\underline{a} &= (1, 1, 2, 2, 1 - b_5, 1 - b_6), \\ -\underline{a} &= (1, 1, 1, 2, 2 - b_5, 1 - b_6), \\ -\underline{a} &= (1, 1, 1, 2, 1 - b_5, 2 - b_6), \\ -\underline{a} &= (1, 1, 1, 1, 2 - b_5, 2 - b_6). \end{aligned}$$

Looking again at the quadratic term of the Hilbert polynomial, it is easy to see that the only case left by (5.20) is the first one, with $b_7 = b_8 = -1$. This gives rise to the second alternative in (5.17).

Case 3: $s = 4$. If $s = 4$, then there are finitely many choices for the a_i according to (5.18) and (5.19). These are:

$$-\underline{a} \in \{(1, 1, 1, 5), (1, 1, 2, 4), (1, 1, 3, 3), (1, 2, 2, 3), (2, 2, 2, 2)\}.$$

A straightforward computation shows that only in the last case does the Hilbert polynomial agree with (5.20). Since that case corresponds to the first alternative in (5.17), this finishes the proof. \square

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