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Inequivalent Embeddings of the Koras–Russell Cubic 3-Fold

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1. Introduction

The Koras–Russell cubic 3-fold is the hypersurface X of the complex affine space $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$ defined by the equation

$$P = x + x^2y + z^2 + t^3 = 0.$$

It is well-known that X is an affine contractible smooth 3-fold that is not algebraically isomorphic to an affine 3-space. The main result of this paper is to show that there exists another hypersurface Y of $\mathbb{A}^4$ that is isomorphic to X but such that there exists no automorphism of the ambient 4-space that restricts to an isomorphism between X and Y. In other words, the two hypersurfaces are inequivalent. In order to prove this result, we give a description of the automorphism group of X. It is shown that all algebraic automorphisms of X extend to automorphisms of $\mathbb{A}^4$, which implies that every automorphism of X fixes the point $(0, 0, 0, 0) \in X$.

More specifically, we will study certain properties of the Koras–Russell cubic 3-fold. The point of view comes from the elementary remark that this 3-fold can be interpreted as a 1-parameter family of Danielewski hypersurfaces. A *Danielewski hypersurface* is a subvariety of $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z])$ defined by an equation of the form $x^n y = q(x, z)$, where n is a nonzero natural number and $q(x, z) \in \mathbb{C}[x, z]$ is a polynomial such that $q(0, z)$ is of degree at least 2. Such hypersurfaces have been studied by the authors in [4], [13], and [14]. This interpretation allows us to deduce results similar to the ones for Danielewski hypersurfaces for this 3-fold.

An important question in affine algebraic geometry asks whether every embedding of complex affine k-space $\mathbb{A}^k$ in $\mathbb{A}^n$, where $k < n$, is rectifiable—in other words, is equivalent to an embedding as a linear subspace. The Abyhankar–Moh–Suzuki theorem shows that the answer is "yes" if $n = 2$ [1; 17] and, by a general result proved independantly by Kaliman [8] and Srinivas [16], if $n \geq 2k + 2$ then the answer is also affirmative. However, all other cases remain open.

Here we are interested in the case of embeddings of hypersurfaces. It is easy to find affine varieties of dimension n admitting nonequivalent embeddings into $\mathbb{A}^{n+1}$. For example, the punctured line $\mathbb{A}^1 \setminus \{0\}$ has many nonequivalent embeddings in $\mathbb{A}^2$. For each $n \in \mathbb{N}$, let $P_n = x^n y - 1$. The subvariety defined by the zero

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set of P_n is isomorphic to $\mathbb{A}^1 \setminus \{0\}$; however, the induced embeddings for each n are inequivalent. This can be seen by the fact that the subschemes defined by $P_n - 1 = 0$ are all nonisomorphic. It is more difficult to find examples where all nonzero fibers of the defining polynomial are irreducible.

In Example 6.3 of [10], Kaliman and Zaidenberg gave examples of acyclic surfaces that admit nonequivalent embeddings in 3-space. In these cases, the obstruction for equivalence is essentially topological, since the nonzero fibers of the polynomials that define the two hypersurfaces are nonhomeomorphic. Even if they are contractible, these surfaces are algebraically remote from the affine plane because they have nonnegative logarithmic Kodaira dimension. In contrast, an example is given [6] for Danielewski hypersurfaces where the nonzero fibers are algebraically nonisomorphic but analytically isomorphic, and examples are given in [13] and [14] of Danielewski hypersurfaces where all fibers are algebraically isomorphic but the embeddings are nonequivalent. Danielewski hypersurfaces are rational, hence close to the affine plane from an algebraic point of view, but they have nontrivial singular homology groups. However, the techniques used in the works just cited are purely algebraic and do not depend on the topological properties of these surfaces. As we shall see here, similar ideas can be used to treat the case of a variety diffeomorphic to $\mathbb{R}^6$. For other inequivalent embeddings of hypersurfaces, see, for example, [15].

In this paper we use similar techniques as for the Danielewski hypersurfaces to study the Koras–Russell 3-fold X described previously. For a polynomial $f \in \mathbb{C}[x, y, z, t]$, we denote by $V(f)$ the subscheme of $\mathbb{A}^4$ defined by the zero set of f in $\mathbb{A}^4$; that is, $V(f) = \operatorname{Spec}(\mathbb{C}[x, y, z, t]/(f))$. The hypersurface $X = V(P)$ is smooth and contractible, and it is therefore diffeomorphic to $\mathbb{A}^3$ [2]. However, it was shown by Makar-Limanov that X is not isomorphic to affine 3-space [11]. We show that there is another hypersurface $Y = V(Q)$ that is isomorphic to X but for which there is no algebraic automorphism of a 4-space that restricts to an isomorphism between X and Y. Thus we have at least two inequivalent embeddings of X. For this example, the two hypersurfaces are analytically equivalent by a holomorphic automorphism that preserves the fibers of P and Q; hence, as for certain examples of Danielewski hypersurfaces, there is no topological obstruction to the existence of such an automorphism. In other words, the obstruction to extending automorphisms in this case is purely algebraic. Also, for all $c \in \mathbb{C} \setminus \{1\}$, the fibers $V(P+c)$ and $V(Q+c)$ are isomorphic. It is an open question whether $V(P+1) \cong V(Q+1)$.

The methods for studying the question of equivalent embeddings are similar to those used in the articles already cited for Danielewski hypersurfaces. However, these methods must be adapted in order to consider a higher-dimensional variety. They are based on certain properties of the automorphism group of the varieties. The set of locally nilpotent derivations on a Danielewski hypersurface is explicitly known (see [4]; see also [12]), and the Makar-Limanov invariant is nontrivial when $n \geq 2$. This restricts the possibilities for automorphisms of these surfaces. The Makar-Limanov invariant of the Koras–Russell 3-fold is also nontrivial [11]. In Section 3 we determine the complete automorphism group of X. For this case, new methods are needed because the restrictions given by the Makar-Limanov invariant do not suffice to determine the automorphism group. As a corollary, we find the surprising result that any algebraic automorphism of X fixes the point $(0,0,0,0) \in X$ (see Corollary 4.5). Also, all automorphisms of X extend to automorphisms of $\mathbb{A}^4$.

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2. Inequivalent Embeddings in $\mathbb{A}^4$

We denote by P and Q the following polynomials of $\mathbb{C}^{[4]} = \mathbb{C}[x,y,z,t]$:

$$P = x^2y + z^2 + x + t^3, \qquad Q = x^2y + (1+x)(z^2 + x + t^3).$$

The Koras–Russell cubic 3-fold is the hypersurface $X \subset \mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x,y,z,t])$ defined by the equation $P = 0$, whereas Y denotes the hypersurface defined by the equation $Q = 0$.

We start by giving some definitions to clarify the difference between *inequivalent embeddings* and *inequivalent hypersurfaces.*

DEFINITION 2.1. A regular map $\phi\colon X \to Z$ between two algebraic varieties is a *closed embedding of X in Z* if:

(1) $\phi(X)$ is a closed subvariety of Z; and
(2) $\phi\colon X \to \phi(X)$ is an isomorphism.

DEFINITION 2.2. Two embeddings $\phi_1, \phi_2\colon X \to Z$ are *equivalent* if there exists an automorphism Ψ of Z such that $\phi_2 = \Psi \circ \phi_1$.

DEFINITION 2.3. Two subvarieties X_1 and X_2 of Z are *equivalent* if there exists an automorphism Ψ of Z such that $\Psi(X_1) = X_2$. If X_1 and X_2 are hypersurfaces, then we say they are *equivalent hypersurfaces.*

In this paper, we will show that $V(P)$ and $V(Q)$ are isomorphic as abstract 3-folds but that they are inequivalent as hypersurfaces of $\mathbb{A}^4$ in the sense of Definition 2.3. In other words, no isomorphism between $V(P)$ and $V(Q)$ extends to an automorphism of $\mathbb{A}^4$. We will do this in two steps. First, we will find an isomorphism ϕ between $V(P)$ and $V(Q)$ that does not extend to an automorphism of $\mathbb{A}^4$. This implies that the two embeddings $i_1\colon V(P) \to \mathbb{A}^4$ and $i_2 \circ \phi\colon V(P) \to \mathbb{A}^4$ are inequivalent in the sense of Definition 2.2. (Here, $V(P)$ is the scheme defined as $\operatorname{Spec}(\mathbb{C}[x,y,z,t]/(P))$ and i_1 is the embedding of $V(P)$ in $\mathbb{A}^4$ corresponding to the canonical homomorphism $i_1^*\colon \mathbb{C}[x,y,z,t] \to \mathbb{C}[x,y,z,t]/(P)$; the embedding i_2 is defined similarly by replacing P by Q.)

The second step, discussed in Section 3, will be to study the automorphism group of $V(P)$. We will show in Section 4 that all automorphisms of this hypersurface extend to automorphisms of $\mathbb{A}^4$. Finally, putting these two results together, we will show the stronger result that $V(P)$ and $V(Q)$ are inequivalent hypersurfaces.

In this paper we use the following key result concerning the Makar-Limanov invariant of an irreducible affine variety. Given an irreducible affine variety Z, denote by $\mathbb{C}[Z]$ the ring of regular functions on Z. A *locally nilpotent derivation* of $\mathbb{C}[Z]$ is a $\mathbb{C}$-derivation $\partial\colon \mathbb{C}[Z] \to \mathbb{C}[Z]$ such that, for any $f \in \mathbb{C}[Z]$, there exists an $n \in \mathbb{N}$ such that $\partial^n(f) = 0$. If ∂ is a locally nilpotent derivation of $\mathbb{C}[Z]$ then $\exp(\lambda\partial)$ defines an algebraic action of $(\mathbb{C}, +)$ on Z, and all $(\mathbb{C}, +)$-actions arise in this way. The kernel of a locally nilpotent derivation is the subalgebra of invariants in $\mathbb{C}[Z]$ of the corresponding action. The *Makar-Limanov invariant* $\mathrm{ML}(\mathbb{C}[Z])$ is defined as the intersection of the kernels of all locally nilpotent derivations on $\mathbb{C}[Z]$. If Z is affine space, then the Makar-Limanov invariant is simply $\mathbb{C}$. However, it was shown in [11] that $\mathrm{ML}(\mathbb{C}[X]) = \mathbb{C}[x] \neq \mathbb{C}$. This implies that the Koras–Russell 3-fold X is not isomorphic to the affine 3-space. In Theorem 9.1 and Example 9.1 of [9], the result of Makar-Limanov is generalized; in particular, it is shown that for every $c \in \mathbb{C}$ and every $\lambda \in \mathbb{C}^*$,

$$\mathrm{ML}(\mathbb{C}[x, y, z, t]/(Q - c)) = \mathrm{ML}(\mathbb{C}[x, y, z, t]/(\lambda P - c)) = \mathbb{C}[x].$$

We will use this equality throughout the paper.

Theorem 2.4. *The following statements hold.*

(a) *The endomorphism of* $\mathbb{A}^4$ *defined by* $(x, y, z, t) \mapsto (x, (1 + x)y, z, t)$ *induces an isomorphism* $\phi\colon X \xrightarrow{\sim} Y$.

(b) ϕ *cannot be extended to an algebraic automorphism of* $\mathbb{A}^4$.

Proof. Let Φ and Ψ be the endomorphisms of $\mathbb{A}^4$ defined by

$$\Phi\colon (x, y, z, t) \mapsto (x, (1 + x)y, z, t) \quad \text{and}$$

$$\Psi\colon (x, y, z, t) \mapsto (x, (1 - x)y - x - z^2 - t^3, z, t).$$

Denote by Φ^* the endomorphism of $\mathbb{C}[x, y, z, t]$ corresponding to Φ (Φ^* fixes x, z, and t, and $\Phi^*(y) = (1 + x)y$), and denote by Ψ^* the corresponding endomorphism to Ψ. One checks that $\Phi^*(Q) = (1+x)P$ and $\Psi^*(P) = (1-x)Q$. Thus, Φ induces a well-defined morphism of algebraic varieties $\phi\colon X \to Y$, and Ψ induces a well-defined regular map $\psi\colon Y \to X$.

Since ϕ and ψ are morphisms of schemes over $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, z, t])$, the identities $(\Phi^* \circ \Psi^*)(y) = y - P$ and $(\Psi^* \circ \Phi^*)(y) = y - Q$ guarantee that they are inverse isomorphisms.

Part (b) follows from the remark that $\phi((0, 0, 0, 0)) = (0, 0, 0, 0)$ and from Proposition 2.5(i). □

Proposition 2.5. *The following results hold.*

(i) *Suppose there exists an algebraic automorphism* Ξ *of* $\mathbb{A}^4$ *that restricts to an isomorphism between X and Y. Then* Ξ *does not fix the point* $(0, 0, 0, 0)$; *that is,* $\Xi(0, 0, 0, 0) \neq (0, 0, 0, 0)$.

(ii) *If* Ξ *is an algebraic automorphism of* $\mathbb{A}^4$ *that restricts to an automorphism of X, then* Ξ *fixes the point* $(0, 0, 0, 0)$.

(iii) *If* Ξ *is an algebraic automorphism of* $\mathbb{A}^4$ *that restricts to an automorphism of Y, then* Ξ *fixes the point* $(0, -1, 0, 0)$.

Proof. For part (i), suppose that Ξ is an automorphism of $\mathbb{A}^4$ that extends an isomorphism between X and Y.

Note that, since $Q = x^2y + (1+x)(z^2 + x + t^3)$ is irreducible, it follows that $\Xi^*(Q) = \lambda P$ for some $\lambda \in \mathbb{C}^*$. Thus, Ξ maps $V(P - \lambda^{-1}c)$ isomorphically onto $V(Q - c)$ for every $c \in \mathbb{C}$.

Let us show first that $\Xi^*(x) = \lambda x$. As mentioned previously, for any $c \in \mathbb{C}$, the Makar-Limanov invariant of $(\mathbb{C}[x,y,z,t]/(Q-c))$ and of $(\mathbb{C}[x,y,z,t]/(\lambda P - c))$ is $\mathbb{C}[x]$. This implies that, for every $c \in \mathbb{C}$, Ξ^* restricts to an isomorphism

$$\mathbb{C}[x] = \mathrm{ML}(\mathbb{C}[x,y,z,t]/(Q-c)) \xrightarrow{\sim} \mathrm{ML}(\mathbb{C}[x,y,z,t]/(\lambda P - c)) = \mathbb{C}[x].$$

Combining this with the fact that, for any $c \in \mathbb{C}$, $V(x-a, Q-c)$ and $V(x-a, P-c)$ are isomorphic to $\mathbb{A}^2$ if and only if $a \neq 0$, we find that Ξ^* preserves the ideal (x).

Let $\Xi^*(x) = \mu x$ with $\mu \in \mathbb{C}^*$. Then $\Xi^*(Q - x) = \lambda P - \mu x$. One checks easily that the variety $Z = \mathrm{Spec}(\mathbb{C}[x,y,z,t]/(Q-x))$ is singular along the line $L_z = \{x = z = t = 0\}$ in $\mathbb{A}^4$. On the other hand, it follows from the Jacobian criterion that the variety $\Xi^{-1}(Z) = V(\lambda P - \mu x)$ is singular if and only if $\lambda = \mu$. This implies that $\lambda = \mu$.

In other words, we have shown that $\Xi^*(Q - x) = \lambda(P - x)$. Thus, Ξ restricts to an isomorphism between $W_P = V(P - x)$ and $W_Q = V(Q - x)$.

For parts (ii) and (iii), a similar argument shows that any automorphism of $\mathbb{A}^4$ that extends an automorphism of X will preserve the subvariety W_P, and any automorphism of $\mathbb{A}^4$ that extends an automorphism of Y will preserve the subvariety W_Q.

Now we look more carefully at the two subvarieties W_P and W_Q. They are both singular along the line $\{x = z = t = 0\}$. We look now at the tangent cone of each singular point of W_P and W_Q. Let $p_0 = (0, y_0, 0, 0, 0)$.

We deduce from the identity

$$P - x = y_0x^2 + z^2 + x^2(y - y_0) + t^3$$

that the tangent cone $\mathrm{TC}_{p_0}(W_P)$ of W_P at p_0 is isomorphic to $\mathrm{Spec}(\mathbb{C}[x,y,z,t]/(z^2 + y_0x^2))$. In particular, $\mathrm{TC}_{p_0}W_P$ consists of two distinct hyperplanes for $y_0 \neq 0$ and of the double hyperplane $\{z = 0\}$ if $y_0 = 0$. On the other hand, we deduce from the identity

$$Q - x = (y_0 + 1)x^2 + z^2 + x^2(y - y_0) + t^3 + xz^2 + xt^3$$

that the tangent cone of W_Q at a point $p_0 = (0, y_0, 0, 0)$ is isomorphic to $\mathrm{Spec}(\mathbb{C}[x,y,z,t]/(z^2 + (y_0+1)x^2))$. Thus the tangent cone $\mathrm{TC}_{p_0}(W_Q)$ consists of two distinct hyperplanes for $y_0 \neq -1$ and of the double hyperplane $\{z = 0\}$ if $y_0 = -1$.

For part (i), since $\Xi(W_P) = W_Q$, we have that $\Xi(0,0,0,0) = (0,-1,0,0)$. For part (ii), any automorphism of W_P fixes the point $(0,0,0,0)$. Finally, for part (iii), any automorphism of W_Q fixes the point $(0,-1,0,0)$. This completes the proof. □

Remark. In Proposition 2.5, we show that any automorphism of $V(P - x)$ must fix the origin because the tangent cone of the origin is distinct from all the other tangent cones of this variety. However, the argument cannot be used to study all automorphisms of $V(P)$. In fact, $V(P)$ is smooth, so the tangent cones of all points are isomorphic. Later on it will be shown that all automorphisms of $X = V(P)$ extend to automorphisms of $\mathbb{A}^4$. Thus, using part (ii) of Proposition 2.5, one finds that any automorphism of X fixes the origin (see Corollary 4.5).

COROLLARY 2.6. *The Koras–Russell cubic 3-fold admits at least two nonequivalent embeddings in* $\mathbb{A}^4$.

Proof. Consider the inclusions $i_1 : X \hookrightarrow \mathbb{A}^4$ and $i_2 : Y \hookrightarrow \mathbb{A}^4$. By Theorem 2.4, embeddings i_1 and $i_2 \circ \phi$ are inequivalent. □

3. The Automorphism Group of X

We will now determine the structure of the automorphism group $\operatorname{Aut}(X)$. We start with some notation. If S is a ring and R is a subring, then $\operatorname{Aut}_R(S)$ denotes the group of ring automorphisms of S that fix R. Denote by $\mathbb{C}[X] = \mathbb{C}^{[4]}/(P)$ the ring of regular functions on X. The group $\operatorname{Aut}(X)$ is isomorphic to the group $\operatorname{Aut}(\mathbb{C}[X]) = \operatorname{Aut}_{\mathbb{C}}(\mathbb{C}[X])$.

NOTATION 3.1. Denote by $I = (x^2, z^2 + t^3 + x) \subset \mathbb{C}[x, z, t]$ the ideal generated by x^2 and $z^2 + t^3 + x$. Let $\mathcal{A}$ be the subgroup of $\operatorname{Aut}(\mathbb{C}[x, z, t])$ of automorphisms that preserve the ideals (x) and I. Let $\mathcal{A}_1$ be the subgroup of $\mathcal{A}$ of automorphisms $\varphi \in \mathcal{A}$ such that φ fixes x, and let $\varphi \equiv \mathrm{id}$ modulo (x). Finally, let $\mathcal{A}_2$ be the subgroup of $\operatorname{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z, t])$ that is equivalent to the identity modulo x^2:

$$\mathcal{A} = \{\varphi \in \operatorname{Aut}(\mathbb{C}[x, z, t]) \mid \varphi((x)) = (x),\ \varphi(I) = I\};$$

$$\mathcal{A}_1 = \{\varphi \in \operatorname{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z, t]) \mid \varphi(I) = I,\ \varphi \equiv \mathrm{id} \bmod (x)\};$$

$$\mathcal{A}_2 = \{\varphi \in \operatorname{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z, t]) \mid \varphi \equiv \mathrm{id} \bmod (x^2)\}.$$

It is clear that $\mathcal{A}_2$ is a normal subgroup of $\mathcal{A}_1$ and that $\mathcal{A}_1$ is a normal subgroup of $\mathcal{A}$.

The ring $\mathbb{C}[X]$ can be viewed as the subalgebra of $\mathbb{C}[x, x^{-1}, z, t]$ that is generated by x, z, t, and $(z^2 + t^3 + x)/x^2$. In particular, it contains $\mathbb{C}[x, z, t]$ as a subring.

The following proposition can be deduced from the results of Makar-Limanov concerning the set of locally nilpotent derivations on X. See [11] and [5].

PROPOSITION 3.2. *The automorphism group* $\operatorname{Aut}(X) \cong \operatorname{Aut}(\mathbb{C}[X])$ *is isomorphic to the group* $\mathcal{A}$. *The isomorphism of* $\operatorname{Aut}(\mathbb{C}[X])$ *to* $\mathcal{A}$ *is induced by restriction of any automorphism of* $\mathbb{C}[X]$ *to the subalgebra* $\mathbb{C}[x, z, t]$.

Proof. In [11] it was shown that the Makar-Limanov invariant of $\mathbb{C}[X]$ is $\mathbb{C}[x]$. In fact, more was proven. It was shown that, for any locally nilpotent derivation ∂ of $\mathbb{C}[X]$, we have $\ker \partial^2 \subset \mathbb{C}[x, z, t]$. Now note that there exist locally nilpotent derivations ∂_1 and ∂_2 on $\mathbb{C}[X]$ such that $\partial_1(z) = \partial_1(x) = 0$ and $\partial_2(t) = \partial_1(x) = 0$. In particular, this implies that the union of kernels of all locally nilpotent derivations of $\mathbb{C}[X]$ generates the ring $\mathbb{C}[x, z, t]$. (This ring is also known as the Derksen invariant.) Thus, any automorphism of $\mathbb{C}[X]$ restricts to an automorphism of

$\mathbb{C}[x,z,t]$. Also, any automorphism $\varphi \in \mathbb{C}[X]$ stabilizes the ideal (x). Indeed, since the Makar-Limanov invariant $\mathbb{C}[x] \subset \mathbb{C}[X]$ is stable, there exist a $\lambda \in \mathbb{C}^*$ and a $b \in \mathbb{C}$ such that $\varphi(x) = \lambda x + b$. Also, the zero set of $\lambda x + b$ in X is singular if and only if $b = 0$. Thus, since the zero set of x in X is singular, we have that $b = 0$ and thus the ideal (x) is preserved.

To complete the argument, we use a general result from [10]. The variety X is the affine modification of $\mathbb{C}[x,z,t]$ along $-x^2$ with center $I = (x^2, x+z^2+t^3)$. In other words, $\mathbb{C}[X] = \mathbb{C}[x,z,t,(x+z^2+t^3/x^2)]$. Since the sequence $x^2, x+z^2+t^3$ is regular, one can easily verify that the intersection of the principal ideal (x^2) in $\mathbb{C}[X]$ and $\mathbb{C}[x,z,t]$ is exactly I. Therefore, any automorphism of $\mathbb{C}[X]$ preserves I. Proposition 2.1 of [10] implies that any automorphism of $\mathbb{C}[x,z,t]$ that is in $\mathcal{A}$ extends to a unique automorphism of $\mathbb{C}[X]$. □

REMARK 3.3. Proposition 3.2 has the following geometric interpretation. The inclusion $\mathbb{C}[x,z,t] \subset \mathbb{C}[X]$ induces a dominant morphism $\sigma: X \to \mathbb{A}^3$. Any automorphism of X is the lifting by σ of a unique automorphism of $\mathbb{A}^3$. More precisely, if $\tilde{\varphi}$ is an automorphism of X then there is a unique automorphism φ of $\mathbb{A}^3$ such that $\varphi \circ \sigma = \sigma \circ \tilde{\varphi}$. Also, an automorphism φ of $\mathbb{A}^3$ has a lifting as an automorphism of X if and only if φ preserves the ideals (x) and I.

We will now discuss the structure of the group $\mathcal{A}$. Note that it contains a subgroup isomorphic to $\mathbb{C}^*$ (corresponding to a $\mathbb{C}^*$-action on X) given by the $\mathbb{C}^*$-action where x has weight 6, z has weight 3, and t has weight 2.

PROPOSITION 3.4. $\mathcal{A} = \mathcal{A}_1 \rtimes \mathbb{C}^*$.

Proof. It is clear that $\mathcal{A}_1 \rtimes \mathbb{C}^*$ is a subgroup of $\mathcal{A}$. We will now show that $\mathcal{A}_1$ and $\mathbb{C}^*$ generate $\mathcal{A}$. First note that if $\varphi \in \mathcal{A}$ then, since φ preserves the ideal (x), it induces an automorphism $\bar{\varphi}$ of $\mathbb{C}[x,z,t]/(x) \cong \mathbb{C}[z,t]$. Also, since I is preserved, the ideal (z^2+t^3) is preserved by $\bar{\varphi}$. By composing with an automorphism in $\mathbb{C}^*$, we can assume that $\bar{\varphi}(z^2+t^3) = z^2+t^3$. In particular, for all $c \in \mathbb{C}$, $\bar{\varphi}$ induces an automorphism of $V(z^2+t^3+c)$. If $c \neq 0$, then this variety is a smooth elliptic curve E with one point p removed. The group of automorphisms of this affine curve is the group of automorphisms of E that fix the point p. This group is of order 6, and it is generated by the automorphism that fixes t and sends z to $-z$ and by the automorphism that fixes z and sends t to $e^{i2\pi/3}t$ (see e.g. [7]). Hence there are only six automorphisms of $\mathbb{C}[z,t]$ that fix the polynomial z^2+t^3, and they are all in the image of $\mathbb{C}^*$. We can therefore suppose that $\bar{\varphi}(z) = z$ and $\bar{\varphi}(t) = t$. This means exactly that $\varphi \in \mathcal{A}_1$. □

Now we are left with the problem of understanding the group $\mathcal{A}_1$. For this part, we will consider a more general situation. First note that the group $\mathcal{A}_1$ is exactly the group of automorphisms φ of $\mathbb{C}[x,z,t]$ that fix x and such that $\varphi \equiv \mathrm{id} \bmod (x)$ and $\varphi(z^2+t^3) \in (x^2, z^2+t^3)$.

NOTATION 3.5. Let $r \in \mathbb{C}[z,t]$ be a polynomial. Denote by $\mathcal{A}_1(r)$ the group

$$\mathcal{A}_1(r) = \{\varphi \in \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z,t]) \mid \varphi \equiv \mathrm{id} \bmod (x),\ \varphi(r) \in (x^2, r)\}.$$

Thus we have that $\mathcal{A}_1 = \mathcal{A}_1(z^2+t^3)$ and that, for any r, $\mathcal{A}_2$ is a normal subgroup of $\mathcal{A}_1(r)$.

We use the following standard notation for partial derivatives. If $h \in \mathbb{C}[z,t]$, $h_z = \partial h/\partial z$, $h_t = \partial h/\partial t$, and $h, f \in \mathbb{C}[z,t]$, then the Poisson bracket of h and f is given by $\{h, f\} = h_z f_t - h_t f_z$.

Proposition 3.6. *Let $r \in \mathbb{C}[z,t]$ be a polynomial with no multiple irreducible factor and such that the zero set $V(r) \in \mathbb{A}^2$ is connected. Then for every automorphism there exist a $\varphi \in \mathcal{A}_1(r)$ and a polynomial $\alpha \in \mathbb{C}[z,t]$ such that $\varphi(z) \equiv z + x(r\alpha)_t \bmod (x^2)$ and $\varphi(t) \equiv t - x(r\alpha)_z \bmod (x^2)$.*

Moreover, $\theta\colon \mathcal{A}_1(r) \to \mathbb{C}[z,t]$, $\varphi \mapsto \alpha$, is a surjective group homomorphism whose kernel is $\mathcal{A}_2$. In particular, the quotient group $\mathcal{A}_1(r)/\mathcal{A}_2$ is isomorphic to the additive group $(\mathbb{C}[z,t],+)$.

Proof. For any $\varphi \in \mathcal{A}_1(r)$, we have that $\varphi(z) \equiv z + xf \bmod (x^2)$ and $\varphi(t) \equiv t + xg \bmod (x^2)$, where $f, g \in \mathbb{C}[z,t]$. By hypothesis, φ is an automorphism; therefore, its Jacobian equals 1. This implies in particular that $f_z + g_t = 0$. In other words, there exists an $h \in \mathbb{C}[z,t]$ such that $h_t = f$ and $h_z = -g$.

Now consider $\varphi(r) \equiv r + x(\{r,h\}) \bmod (x^2)$. Since $\varphi(r) \in (r, x^2)$, the Poisson bracket $\{r,h\}$ is in the ideal (r). This implies that there exists a constant $c \in \mathbb{C}$ such that $h - c \in (r)$. To see this, note that $\mathrm{d}h \wedge \mathrm{d}r$ is identically zero along the zero set $V(r)$ of r. Thus h is locally constant as a function on $V(r)$ in a neighborhood of every smooth point of $V(r)$. Since r has no multiple irreducible factor, the set of smooth points is dense. Since $V(r)$ is connected, h is constant along $V(r)$.

We may assume that the constant $c = 0$. Thus $h = r\alpha$ with $\alpha \in \mathbb{C}[z,t]$, and we have $\varphi(z) \equiv z + x(r\alpha)_t \bmod (x^2)$ and $\varphi(t) \equiv t - x(r\alpha)_z \bmod (x^2)$.

It is easy to check that $\theta\colon \mathcal{A}_1(r) \to \mathbb{C}[z,t]$, $\varphi \mapsto \alpha$, is a group homomorphism whose kernel is $\mathcal{A}_2$. We now prove that it is surjective. For any $\alpha \in \mathbb{C}[z,t]$ define an automorphism $\bar{\varphi} \in \mathrm{Aut}_R\, R[z,t]$, where $R = \mathbb{C}[x]/(x^2)$, that is given by $\bar{\varphi}(z) = z + x(r\alpha)_t$ and $\bar{\varphi}(t) = t - x(r\alpha)_z$. Note that the inverse of $\bar{\varphi}$ is given by $\bar{\varphi}^{-1}(z) = z - x(r\alpha)_t$ and $\bar{\varphi}^{-1}(t) = t + x(r\alpha)_z$. Also, $\bar{\varphi}$ is indeed an automorphism, and its Jacobian is equal to 1. By a result of van den Essen, Maubach and Vénéreau [18], there exists an automorphism φ of $\mathbb{C}[x][z,t]$ that projects to $\bar{\varphi}$. By construction, $\varphi \in \mathcal{A}_1(r)$ and $\theta(\varphi) = \alpha$. □

4. Extensions of Automorphisms

In this section, we will continue with the more general setting in order to prove Lemma 4.2. We then will apply the lemma to the hypersurface X.

Notation 4.1. Let $r \in \mathbb{C}[z,t]$ be a polynomial with no multiple irreducible factor and whose zero set is connected, and let F be any polynomial in $\mathbb{C}[x,z,t]$. We define $P_{r,F} = x^2y + r + xF \in \mathbb{C}[x,y,z,t]$, and we let $X_{r,F} = V(P_{r,F})$. Thus, for example, $X = X_{z^2+t^3,1}$ and $Y = X_{z^2+t^3,(1+z^2+t^3+x)}$.

As in the proof of Proposition 3.2, for any $\varphi \in \mathcal{A}_1(r)$ we can construct an endomorphism Φ of $\mathbb{C}[x,y,z,t] = \mathbb{C}^{[4]}$ that induces a unique automorphism $\tilde{\varphi}$ of $\mathbb{C}[X_{r,F}]$, as follows. First, Φ is an extension of φ where we determine $\Phi(y)$. Now suppose that $\theta(\varphi) = \alpha$. Let $\beta = \{r, \alpha\}$. Then it is easily checked that $\varphi(r + xF) \equiv (1 + x\beta)(r + xF) \bmod (x^2)$. Hence there exists a unique $G \in \mathbb{C}[x, z, t]$ such that $\Phi(P_{r,F}) = (1 + x\beta)P_{r,F}$ if we put $\Phi(y) = (1 + x\beta)y + G$. We will denote by $\tilde{\varphi}$ the induced automorphism on $\mathbb{C}[X_{r,F}]$. In this way, $\mathcal{A}_1(r)$ can be considered as a subgroup of $\mathrm{Aut}(\mathbb{C}[X_{r,F}])$. We will now show that any such automorphism of $X_{r,F}$ lifts to an automorphism of $\mathbb{A}^4$. This is clear for the case that $\beta = 0$, since in this case Φ is an automorphism of $\mathbb{C}^{[4]}$. In particular, any automorphism of $\mathcal{A}_2$ induces an automorphism of $X_{r,F}$ that extends. However, even if $\beta \neq 0$, we will show that by adding an appropriate multiple of $P_{r,F}$ we can lift $\tilde{\varphi}$ to an automorphism of $\mathbb{C}[x, y, z, t]$.

LEMMA 4.2. *Let $\varphi \in \mathcal{A}_1(r)$. Then $\tilde{\varphi}$, the corresponding automorphism of $\mathbb{C}[X_{r,F}]$, lifts to an automorphism of $\mathbb{C}^{[4]}$.*

Proof. Let $\varphi \in \mathcal{A}_1(r)$, and suppose that $\theta(\varphi) = \alpha$. Similarly to the method used in [13], we create a family of endomorphisms of $\mathbb{A}^4$ each of which restricts to an automorphism of a fiber of $P_{r,F}$. Consider c as a variable, and denote by R_c the ring $R_c = \mathbb{C}[x, c]/(x^2)$. Consider now the automorphism $\bar{\phi} \in \mathrm{Aut}_{R_c}(R_c[z, t])$ given by $\bar{\phi}(z) = \varphi(z) + xc\alpha_t$ and $\bar{\phi}(t) = \varphi(t) - xc\alpha_z$. One checks easily that the Jacobian of $\bar{\phi}$ is 1 and therefore, by [18], there exists an automorphism $\phi \in \mathrm{Aut}_{\mathbb{C}[c,x]}(\mathbb{C}[x, c][z, t])$ that restricts to $\bar{\phi}$.

For each $c \in \mathbb{C}$, denote by $\varphi_c \in \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z, t])$ the automorphism defined by $\varphi_c(z) = \phi(z)$ and $\varphi_c(t) = \phi(t)$, where $\phi(z)$ and $\phi(t)$ are viewed as polynomials with coefficients in $\mathbb{C}[c]$. Note that $\varphi_c \in \mathcal{A}_1(r + c)$ and that the expression for φ_c depends polynomially on c.

For each $c \in \mathbb{C}$, we now construct, much as before, an automorphism $\tilde{\varphi}_c$ on $\mathbb{C}[X_{r+c,F}]$. Note that the expression for $\tilde{\varphi}_c$ depends polynomially on c. By making a formal substitution of c by $-P_{r,F}$, we construct an automorphism $\Psi = \tilde{\varphi}_{(-P_{r,F})}$ of $\mathbb{C}[x, y, z, t]$ that preserves the ideal $(P_{r,F})$ (see [13, Lemma 3.4]). Note that Ψ is a lift of the automorphism $\tilde{\varphi}_0 \in \mathrm{Aut}(\mathbb{C}[X_{r,F}])$. Also, φ_0 and φ are equivalent modulo (x^2). More precisely, $\varphi_0^{-1} \circ \varphi$ is an element of $\mathcal{A}_2$. By the comment preceding the lemma, $\varphi_0^{-1} \circ \varphi$ induces an automorphism $\widetilde{\varphi_0^{-1} \circ \varphi}$ that lifts to an automorphism of $\mathbb{C}^{[4]}$. By the unicity of the extension of an element of $\mathcal{A}_1(r)$ to an automorphism of $\mathbb{C}[X_{r,F}]$, we have that $\widetilde{\varphi_0^{-1} \circ \varphi} = \tilde{\varphi}_0^{-1} \circ \tilde{\varphi}$. Since $\tilde{\varphi}_0$ also lifts to an automorphism of $\mathbb{C}^{[4]}$, the same is true for $\tilde{\varphi}$. □

THEOREM 4.3. *Every automorphism of $X = V(P)$ extends to an automorphism of $\mathbb{A}^4$.*

Proof. The automorphism group of X is isomorphic to $\mathcal{A} = \mathcal{A}_1 \rtimes \mathbb{C}^*$. The automorphisms in $\mathbb{C}^*$ extend and, by Lemma 4.2, the automorphisms in $\mathcal{A}_1$ extend. Therefore, all automorphisms extend to an automorphism of $\mathbb{A}^4$. □

EXAMPLE 4.4. Consider the automorphism φ of $\mathbb{C}[x, z, t]$ given by $\varphi(x) = x$, $\varphi(z) = z + 3xt^5$, and $\varphi(t) = t + 2x(z + 3xt^5)^3$. It is indeed an automorphism, because it is the composition of two triangular automorphisms.

Also we can check that φ is an element of $\mathcal{A}_1$. It is obvious that $\varphi(x) = x$ and $\varphi \equiv \mathrm{id} \bmod (x)$; we will now show that $\varphi(z^2 + t^3 + x)$ is in the ideal $(x^2, z^2 + t^3 + x)$. Indeed, we find that there exists an element $G \in \mathbb{C}[x, z, t]$ such that $\varphi(z^2 + t^3 + x) = (z^2 + t^3 + x) + x(6zt^5 + 6t^2z^3) + x^2G$. This yields $\varphi(z^2 + t^3 + x) = (z^2 + t^3 + x)(1 + 6xzt^2) + x^2(G - 6zt^2)$.

Thus, φ is an element of $\mathcal{A}_1$. To find the corresponding automorphism of $\mathbb{C}[X]$, we extend φ to the automorphism $\tilde{\varphi}$ of $\mathbb{C}[X]$ where $\tilde{\varphi}(y) = (1 + 6xzt^2)y - (G - 6zt^2)$. In order to lift this automorphism to an automorphism of $\mathbb{C}^{[4]}$, we apply the same procedure as before.

We have that $\alpha = (t^3 - z^2)/2$. We define, for each $c \in \mathbb{C}$, an automorphism $\varphi_c \in \mathcal{A}_1(z^2 + t^3 + c)$ as follows. We have $\varphi_c(z) = z + 3xt^2(t^3 + c/2)$ and $\varphi_c(t) = t + 2x\varphi_c(z)(\varphi_c(z)^2 + c/2)$. More precisely, we have that $\varphi(z) \equiv z + x((z^2 + t^3 + c)\alpha)_t \bmod (x^2)$ and $\varphi(t) \equiv t - x((z^2 + t^3 + c)\alpha)_z \bmod (x^2)$.

Now we can define, for each $c \in \mathbb{C}$, an automorphism $\tilde{\varphi}_c$ of $\mathbb{C}[X_{z^2+t^3+c,1}]$ if we put $\tilde{\varphi}_c(y) = (1 + 6xzt^2)y + G_c$ for a suitable polynomial $G_c \in \mathbb{C}[x, z, t, c]$. Finally, to find the automorphism of $\mathbb{C}[x, y, z, t]$ that is a lift of $\tilde{\varphi}$, we make a formal substitution of c by $-P$.

Corollary 4.5. *The origin $o = (0,0,0,0) \in X$ is fixed by all automorphisms of X.*

Proof. By Proposition 2.5(ii), any automorphism of X that extends to $\mathbb{A}^4$ fixes the origin. By Theorem 4.3, all automorphisms of X extend to $\mathbb{A}^4$. □

Remark. This corollary was first proven in collaboration with G. Freudenburg but using a different method.

5. Inequivalent Hypersurfaces

Consider now the two hypersurfaces $X = V(P)$ and $Y = V(Q)$. (As before, $P = x^2y + z^2 + x + t^3$ and $Q = x^2y + (1 + x)(z^2 + x + t^3)$). We know from Theorem 2.4 that, as abstract varieties, X and Y are isomorphic. We now show the following result.

Theorem 5.1. *X and Y are inequivalent as hypersurfaces of $\mathbb{C}^4$.*

Proof. Suppose there were an automorphism Ψ of $\mathbb{A}^4$ such that $\Psi(X) = Y$. Then $\Psi(o) \neq o$ by Proposition 2.5(i). Consider now the isomorphism ϕ defined in Theorem 2.4 between X and Y. Then $(\Psi^{-1})|_Y \circ \phi$ is an automorphism of X that does not preserve o. This contradicts Corollary 4.5. □

It should be noted that, as another consequence of the description of the automorphism group of X, we can show that all automorphisms of $\mathbb{C}[Y]$ that fix the variable x also extend to automorphisms of $\mathbb{A}^4$.

This is the case because $\mathrm{Aut}(Y) \cong \mathrm{Aut}(X) \cong \mathcal{A} = \mathcal{A}_1 \rtimes \mathbb{C}^*$. The subgroup of automorphisms that fix x (for X or for Y) corresponds via this isomorphism to the subgroup $\mathcal{A}_1 \rtimes C_6 \subset \mathcal{A}_1 \rtimes \mathbb{C}^*$, where C_6 is the subgroup of the sixth roots of unity in $\mathbb{C}^*$. The automorphisms corresponding to elements of $\mathcal{A}_1$ extend to Y

by Lemma 4.2, and the automorphisms corresponding to elements in C_6 extend to linear automorphisms on $\mathbb{A}^4$.

However, the action of $\mathbb{C}^*$ on Y does not extend to an action on $\mathbb{A}^4$. More precisely, for any $\lambda \in \mathbb{C}$ such that $\lambda^6 \neq 1$, the action of $\lambda \in \mathbb{C}^*$ on X, when conjugated by ϕ to give an automorphism of Y, does not extend to an automorphism of $\mathbb{A}^4$. To see this, note that the action of λ does not fix the line $x = z = t = 0$. The only fixed point on this line is the origin. However, by Proposition 2.5(iii), the point $(0, -1, 0, 0)$ must be fixed.

6. Remarks and Some Open Questions

6.1. Locally Nilpotent Derivations on $\mathbb{C}[X]$

We can give a complete description of the locally nilpotent derivations of $\mathbb{C}[X]$. We denote by $\mathrm{LND}(\mathbb{C}[X])$ the set of locally nilpotent derivations of $\mathbb{C}[X]$. Denote by $\mathrm{LND}_x(\mathbb{C}[x][z,t])$ the $\mathbb{C}[x]$-module of locally nilpotent derivations of $\mathbb{C}[x,z,t]$ having x in the kernel. If ∂ is a locally nilpotent derivation on $\mathbb{C}[X]$, it restricts to a unique locally nilpotent derivation of $\mathrm{LND}_x(\mathbb{C}[x][z,t])$.

Proposition 6.1. $\mathrm{LND}(\mathbb{C}[X]) = x^2(\mathrm{LND}_x(\mathbb{C}[x][z,t]))$.

Proof. If $\partial = x^2\partial_0$ is an element of $x^2(\mathrm{LND}_x(\mathbb{C}[x][z,t])$, then one can extend it to a locally nilpotent derivation on $\mathbb{C}[X]$ by putting $\partial(y) = -\partial_0(z^2+t^3)$. For the converse, if ∂ is a locally nilpotent derivation on $\mathbb{C}[X]$, then $\partial(z^2+t^3) = 2z\partial(z) + 3t^3\partial(t) \in (x^2)$. Consider the derivation $\bar{\partial}$ of $\mathbb{C}[z,t]$ defined by $\bar{\partial}(f) \equiv \partial(f) \bmod (x)$. Then $\bar{\partial}$ induces an action of $(\mathbb{C},+)$ on $\mathbb{A}^2$ that stabilizes the cuspidal curve $z^2+t^3 = 0$. This implies that the action is trivial and therefore that $\bar{\partial} = 0$. In other words, there exists an element $\partial_1 \in \mathrm{LND}_x(\mathbb{C}[x][z,t])$ satisfying $\partial = x\partial_1$. Now, since $\partial(z^2+t^3) \in (x^2)$, we have that $\partial_1(z^2+t^3)$ belongs to (x); and the same argument proves that there exists a ∂_0 such that $\partial_1 = x\partial_0$. □

6.2. Nonzero Fibers of P and Q

Proposition 6.2. *The following statements hold.*

(a) *For every $c \in \mathbb{C}$, $V(Q-c)$ is isomorphic to the hypersurface $V(F_c) = Z_c$ of $\mathbb{A}^4$ defined by the equation*
$$F_c = x^2y + z^2 + (1+c)x + t^3 - c = 0.$$
(b) *For every $c \in \mathbb{C}\setminus\{-1,0\}$ and every $c' \in \mathbb{C}\setminus\{0\}$, Z_c and $V(P-c')$ are isomorphic as abstract affine varieties.*

Proof. Recall that, by definition,
$$\begin{aligned} V(Q-c) &\simeq \mathrm{Spec}(\mathbb{C}[x,y,z,t]/(Q-c)) \\ &= \mathrm{Spec}(\mathbb{C}[x,y,z,t]/(x^2y + (1+x)(z^2+t^3+x) - c)). \end{aligned}$$

We claim that the endomorphisms
$$\Phi_c\colon (x,y,z,t) \mapsto (x, (1-x)y - z^2 - x - t^3, z, t) \quad \text{and}$$
$$\Psi_c\colon (x,y,z,t) \mapsto (x, (1+x)y + c, z, t)$$

of $\mathbb{A}^4$ restrict respectively to isomorphisms $\phi_c\colon V(Q-c) \xrightarrow{\sim} Z_c$ and $\psi_c\colon Z_c \xrightarrow{\sim} V(Q-c)$, which are inverse to each other. Indeed, one checks that $\Phi_c^*(F_c) = (1-x)(Q-c)$ whereas $\Psi_c^*(Q-c) = (1+x)F_c$, so that Φ_c and Ψ_c induce morphisms $\phi_c\colon V(Q-c) \to Z_c$ and $\psi_c\colon Z_c \to V(Q-c)$, respectively. Since ϕ_c and ψ_c are morphisms of schemes over $\mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[x,z,t])$, the identities $(\Phi_c^* \circ \Psi_c^*)(y) = y - (Q-c)$ and $(\Psi_c^* \circ \Phi_c^*)(y) = y - F_c$ guarantee that they are inverse isomorphisms. This proves part (a).

Now note that $V(P-c) \simeq V(P-1)$ for every $c \in \mathbb{C} \setminus \{0\}$. Indeed, if $c \in \mathbb{C}^*$, consider the automorphism of $\mathbb{A}^4$ defined by $(x,y,z,t) = (a^6x, a^{-6}y, a^3z, a^2t)$ for a suitable constant $a \in \mathbb{C}$ such that $a^{-6} = c$. This automorphism maps $V(P-c)$ isomorphically onto $V(P-1)$.

Finally, part (b) follows because, for every $c \in \mathbb{C} \setminus \{-1\}$, the automorphism of $\mathbb{A}^4$ defined by $(x,y,z,t) \mapsto ((1+c)x, (1+c)^{-2}y, z, t)$ maps $V(P-c)$ isomorphically onto Z_c. □

Together with the previous discussion, this result motivates the following question.

QUESTION 1. *Are the subvarieties of* $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x,y,z,t])$ *defined by the equations*

$$x^2y + z^2 + x + t^3 + 1 = 0 \quad \textit{and} \quad x^2y + z^2 + t^3 + 1 = 0$$

isomorphic?

REMARK 6.3. This question has an affirmative answer in holomorphic category. Indeed, one can easily check that the analytic automorphism of $\mathbb{A}^4$ defined by

$$(x,y,z,t) \mapsto \left(x, y + 1 - \frac{e^x - 1 - x}{x^2}(z^2 + t^3), e^{x/2}z, e^{x/3}t\right)$$

induces an isomorphism between the hypersurfaces $V(Q+1)$ and $V(P+1)$.

6.3. *Holomorphic and Stable Equivalence*

Recall that two closed algebraic smooth subvarieties X and Y of $\mathbb{A}^4$ that are isomorphic as abstract algebraic varieties are called *holomorphically equivalent* if there exists a biholomorphism of $\mathbb{A}^4$ restricting to a biholomorphism between X and Y considered as complex manifolds. Similarly, we say that X and Y are *stably equivalent* if there exist an $n \in \mathbb{N}$ and an algebraic automorphism of $\mathbb{A}^{4+n}$ restricting to an isomorphism between $X \times \mathbb{A}^n$ and $Y \times \mathbb{A}^n$. Here we show the following.

PROPOSITION 6.4. *The subvarieties X and Y of* $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x,y,z,t])$ *defined by the equations*

$$P = x^2y + z^2 + x + t^3 = 0 \quad \textit{and} \quad Q = x^2y + (1+x)(z^2 + x + t^3) = 0$$

are holomorphically equivalent and stably equivalent.

Proof. By virtue of Theorem 2.4, X and Y are isomorphic as abstract algebraic varieties. Holomorphic equivalence follows from the observation that the map $\zeta\colon \mathbb{A}^4 \to \mathbb{A}^4$ defined by

$$\zeta(x,y,z,t) = (x, e^x y + x^{-2}(e^x - 1 - x)(z^2 + x + t^3), z, t)$$

is a biholomorphism of $\mathbb{A}^4$ that maps X isomorphically onto Y. Moreover, we remark that ζ can be viewed as a holomorphic extension of the isomorphism $\phi\colon X \to Y$ defined in Theorem 2.4. Indeed, we have

$$\zeta(x,y,z,t) = (x, (1+x)y + x^{-2}(e^x - 1 - x)P, z, t).$$

For stable algebraic equivalence, we consider the $\mathbb{C}[x]$-endomorphism Ψ^* of $\mathbb{C}[x,y,z,t,w]$ defined by:

$$\begin{cases} y \mapsto y + 1 + x^{-2}\big(((1+\frac{1}{2}x)z + x^2 w)^2 - (1+x)z^2\big) \\ \qquad + x^{-2}\big(((1+\frac{1}{3}x)t + x^2 w)^3 - (1+x)t^3\big), \\ z \mapsto (1+\frac{1}{2}x)z + x^2 w, \\ t \mapsto (1+\frac{1}{3}x)t + x^2 w, \\ w \mapsto -\frac{3}{4}z + \frac{2}{9}t + (1-\frac{5}{6}x)w. \end{cases}$$

By definition, Ψ^* restricts to a linear $\mathbb{C}[x]$-endomorphism $\tilde{\Psi}^*$ of $\mathbb{C}[x,z,t,w]$, which is an automorphism because the Jacobian is invertible:

$$\begin{pmatrix} (1+\frac{1}{2}x) & 0 & x^2 \\ 0 & (1+\frac{1}{3}x) & x^2 \\ -\frac{3}{4} & \frac{2}{9} & (1-\frac{5}{6}x) \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}[x]).$$

Since $\Psi^*(y)$ depends triangularly on the variables x, z, t, and w, this implies in turn that Ψ^* is a $\mathbb{C}[x]$-automorphism of $\mathbb{C}[x,y,z,t,w]$. Now one checks easily that $\Psi^* P = Q$, which completes the proof. $\square$

REMARK 6.5. In the foregoing proof we have shown that the isomorphism ϕ: $X \to Y$—which cannot be extended to an algebraic automorphism of $\mathbb{A}^4$ (Theorem 2.4)—can be extended to a holomorphic automorphism of $\mathbb{A}^4$. In particular, there is no topological obstruction to extending the isomorphism ϕ to an automorphism.

7. Locally Nilpotent Derivations on the Cylinder over the Koras–Russell 3-Fold

Recently, the first author proved that the Makar-Limanov invariant of the cylinder over the Koras–Russell 3-fold is trivial [3]. The idea of the proof is as follows. Let $X^0 = X \setminus V(t)$. Now consider the polynomial $P_1 = xy + z^2 + t^3 + x$, and let $X_1 = V(P_1)$ and $X_1^0 = X_1 \setminus V(t)$. It is shown that the cylinders $X^0 \times \mathbb{C}$ and $X_1^0 \times \mathbb{C}$ are isomorphic. Then one uses the triviality of the Makar-Limanov invariant of X_1^0 to show that the Makar-Limanov invariant of $X^0 \times \mathbb{C}$ is trivial, and this implies the result.

We denote by $A = \mathbb{C}[X]$ the coordinate ring of X, by $B = \mathbb{C}[X_1]$ the coordinate ring of X_1, by A_t the coordinate ring of X^0, and by B_t the coordinate ring of X_1^0. In this section we will construct an explicit isomorphism and then obtain a locally nilpotent derivation ∂ on the coordinate ring $A[w]$ of $X \times \mathbb{A}^1$ such that $\partial(x) \neq 0$.

Proposition 7.1. *The algebraic endomorphism Φ of $\mathbb{C}[x,y,z,t^{\pm 1},w]$ defined by*

$$\begin{cases} \Phi(x) = x, \\ \Phi(y) = xy - xw^2 - 2zw, \\ \Phi(z) = z + xw, \\ \Phi(t) = t, \\ \Phi(w) = 2w + yz + 3xyw - 3zw^2 - xw^3 \end{cases}$$

induces an isomorphism

$$\begin{aligned} \phi\colon B_t[w] &= \mathbb{C}[x,y,z,t^{\pm 1},w]/(xy + z^2 + x + t^3) \\ &\xrightarrow{\sim} A_t[w] = \mathbb{C}[x,y,z,t^{\pm 1},w]/(x^2y + z^2 + x + t^3) \end{aligned}$$

whose inverse isomorphism $\phi^{-1}\colon A_t[w] \to B_t[w]$ is induced by the endomorphism of $\mathbb{C}[x,y,z,t^{\pm 1},w]$ defined by

$$\begin{cases} \Psi(x) = x, \\ \Psi(y) = -\frac{1}{t^3}(y + y^2 + wz) - \frac{1}{4t^6}(yz - xw)^2, \\ \Psi(z) = z - \frac{1}{2t^3}x(yz - xw), \\ \Psi(t) = t, \\ \Psi(w) = \frac{1}{2t^3}(yz - xw). \end{cases}$$

Proof. Recall that P denotes the polynomial $P = x^2y + z^2 + x + t^3$. Let S be the polynomial defined by $S = xy + z^2 + x + t^3$.

The endomorphisms Φ and Ψ induce well-defined algebraic morphisms ϕ: $B_t[w] \to A_t[w]$ and $\psi\colon A_t[w] \to B_t[w]$. Indeed, one checks that $\Phi(S) = P$ and $\Psi(P) = (1 - xyt^{-3})S$.

Since ϕ and ψ are $\mathbb{C}[x,t^{\pm 1}]$-morphisms, the following equalities prove that ϕ and ψ are inverse isomorphisms:

$$\Psi \circ \Phi(z) = z, \qquad \Psi \circ \Phi(y) = y - \frac{y}{t^3}S,$$

$$\Psi \circ \Phi(w) = w + \frac{xyw - y^2z - t^3w}{t^6}S;$$

$$\Phi \circ \Psi(w) = w - \frac{w}{t^3}P, \qquad \Phi \circ \Psi(z) = z + \frac{xw}{t^3}P,$$

$$\Phi \circ \Psi(y) = y - \frac{y - w^2}{t^3}P - \frac{w^2}{t^6}P^2. \qquad \square$$

Proposition 7.2. *Let Δ be the locally nilpotent derivation on $\mathbb{C}[x,y,z,t^{\pm 1},w]$ defined by*

$$\Delta = t^6\left(-2z\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial z}\right).$$

Then, the derivation ∂ of $\mathbb{C}[x,y,z,t^{\pm 1},w]$ defined by

$$\partial = (\Phi \circ \Delta \circ \Psi)(x)\frac{\partial}{\partial x} + (\Phi \circ \Delta \circ \Psi)(y)\frac{\partial}{\partial y} + (\Phi \circ \Delta \circ \Psi)(z)\frac{\partial}{\partial z} + (\Phi \circ \Delta \circ \Psi)(w)\frac{\partial}{\partial w}$$

induces a locally nilpotent derivation on

$$A[w] = \mathbb{C}[x, y, z, t, w]/(x^2y + z^2 + x + t^3)$$

that does not contain the variable x *in its kernel.*

Proof. Note first that $\partial(x)$, $\partial(y)$, $\partial(z)$, and $\partial(w)$ are all elements of $\mathbb{C}[x, y, z, t, w]$ and so ∂ restricts to a well-defined derivation on $\mathbb{C}[x, y, z, t, w]$. (For example, one checks that $\partial(x) = -2t^6(z + xw)$.)

Second, since $\Delta(xy+z^2+x+t^3) = 0$, it follows that Δ induces a locally nilpotent derivation on $B_t[w] = \mathbb{C}[x, y, z, t^{\pm 1}, w]/(xy + z^2 + x + t^3)$. Therefore, in light of Proposition 7.1, one can conclude that ∂ induces a locally nilpotent derivation $\tilde{\partial}$ on $A_t[w] = \mathbb{C}[x, y, z, t^{\pm 1}, w]/(x^2y + z^2 + x + t^3)$.

The inclusion $\mathbb{C}[x, y, z, t, w] \subset \mathbb{C}[x, y, z, t^{\pm 1}, w]$ induces an inclusion of $A[w]$ in $A_t[w]$. More precisely, let $\pi : \mathbb{C}[x, y, z, t^{\pm 1}, w] \to A_t[w]$ be the canonical projection. Since P is prime in $\mathbb{C}[x, y, z, t, w]$, one can identify $A[w]$ with the image $\pi(\mathbb{C}[x, y, z, t, w])$ of the subalgebra $\mathbb{C}[x, y, z, t, w]$. We have that $\pi \circ \tilde{\partial} = \partial \circ \pi$, and therefore $\tilde{\partial}$ restricts to a locally nilpotent deriviation on $A[w] = \pi(\mathbb{C}[x, y, z, t, w])$. □

COROLLARY 7.3. *The Makar-Limanov invariant of* $A[w]$ *is trivial.*

Proof. Let ∂_1 and ∂_2 be the locally nilpotent derivations on $A[w]$ defined by

$$\partial_1 = 2z\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial z} \quad \text{and} \quad \partial_2 = 3t^2\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial t}.$$

We have $\mathrm{Ker}(\partial_1) \cap \mathrm{Ker}(\partial_2) = \mathbb{C}[x]$. Thus, $\mathrm{ML}(A[w]) \subset \mathbb{C}[x]$. Now Proposition 7.2 allows us to conclude that $\mathrm{ML}(A[w]) = \mathbb{C}$. □

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