Weighted C^k Estimates for a Class of Integral Operators on Nonsmooth Domains

DARIUSH EHSANI

1. Introduction

Let X be an n-dimensional complex manifold equipped with a Hermitian metric, and let $D \subset\subset X$ be a strictly pseudoconvex domain with defining function r. Here we do not assume the nonvanishing of the gradient, dr, thus allowing for the possibility of singularities in the boundary, ∂D, of D. We refer to such domains as Henkin–Leiterer domains, as they were first systematically studied by Henkin and Leiterer in [2].

We shall make the additional assumption that r is a Morse function.

Let $\gamma = |\partial r|$. In [1] the author established an integral representation of the following form.

THEOREM 1.1. *There exist integral operators* $\tilde{T}_q : L^2_{(0,q+1)}(D) \to L^2_{(0,q)}(D)$ *with* $0 \le q < n = \dim X$ *such that, for* $f \in L^2_{(0,q)} \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, *one has*

$$\gamma^3 f = \tilde{T}_q \bar{\partial} f + \tilde{T}^*_{q-1} \bar{\partial}^* f + (error\ terms) \quad for\ q \ge 1.$$

Theorem 1.1 is valid under the assumption that we are working with the Levi metric. With local coordinates denoted by $\zeta_1, \ldots, \zeta_n$, we define a Levi metric in a neighborhood of ∂D by

$$ds^2 = \sum_{j,k} \frac{\partial^2 r}{\partial \zeta_j, \partial \bar{\zeta}_k}(\zeta).$$

A Levi metric on X is a Hermitian metric that is a Levi metric in a neighborhood of ∂D. In what follows we will be working with X equipped with a Levi metric.

The author [1] then used properties of the operators in the representation to establish the following estimates.

THEOREM 1.2. *For* $f \in L^2_{0,q}(D) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$ *with* $q \ge 1$,

$$\|\gamma^{3(n+1)} f\|_{L^\infty} \lesssim \|\gamma^2 \bar{\partial} f\|_\infty + \|\gamma^2 \bar{\partial}^* f\|_\infty + \|f\|_2.$$

In this paper we examine the operators in the integral representation, derive more detailed properties of such operators under differentiation, and use the properties to establish C^k estimates. Our main theorem is as follows.

Theorem 1.3.   *Let $f \in L^2_{0,q}(D) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, $q \geq 1$, and $\alpha < 1/4$. Then for $N(k)$ large enough we have*

$$\|\gamma^{N(k)}f\|_{C^{k+\alpha}} \lesssim \|\gamma^{k+2}\bar{\partial}f\|_{C^k} + \|\gamma^{k+2}\bar{\partial}^*f\|_{C^k} + \|f\|_2.$$

We show that we may take any $N(k) > 3(n+6) + 8k$.

Our results are consistent with those obtained by Lieb and Range in the case of smooth strictly pseudoconvex domains [4], where we may take $\gamma = 1$. In [4], an estimate as in Theorem 1.3 with $\gamma = 1$ and $\alpha < 1/2$ was given.

In a separate paper we look to establish C^k estimates for $f \in L^2(D) \cap \mathrm{Dom}(\bar{\partial})$, as the functions used in the construction of the integral kernels in the case $q = 0$ differ from those in the case $q \geq 1$.

One of the difficulties in working on nonsmooth domains is the problem of the choice of frame of vector fields with which to work. In the case of smooth domains a special boundary chart is used in which $\omega^n = \partial r$ is part of an orthonormal frame of $(1,0)$-forms. When ∂r is allowed to vanish, the frame needs to be modified. We get around this difficulty by defining a $(1,0)$-form ω^n by $\partial r = \gamma \omega^n$. In the dual frame of vector fields we are then faced with factors of γ in the expressions of the vector fields with respect to local coordinates, and we deal with these terms by multiplying our vector fields by a factor of γ. This ensures that when vector fields are commuted, there are no error terms that blow up at the singularity.

We organize our paper as follows. In Section 2 we define the types of operators that make up the integral representation established in [1]. Section 3 contains the most essential properties used to obtain our results. In Section 3 we consider the properties of our integral operators under differentiation. Finally, in Section 4 we apply the properties from Section 3 to obtain our C^k estimates.

The author extends thanks to Ingo Lieb, with whom he shared many fruitful discussions over the ideas presented here and from whom he originally had the idea to extend results on smooth domains to Henkin–Leiterer domains.

2. Admissible Operators

Denoting local coordinates by $\zeta_1, \ldots, \zeta_n$, we define a Levi metric in a neighborhood of ∂D by

$$ds^2 = \sum_{j,k} \frac{\partial^2 r}{\partial \zeta_j, \partial \bar{\zeta}_k}(\zeta)\, d\zeta_j\, d\bar{\zeta}_k.$$

A Levi metric on X is a Hermitian metric that is a Levi metric in a neighborhood of ∂D.

We thus equip X with a Levi metric, and we take $\rho(x, y)$ to be a symmetric smooth function on $X \times X$ that coincides with the geodesic distance in a neighborhood of the diagonal Λ and is positive outside of Λ.

For ease of notation, in what follows we will always work with local coordinates ζ and z.

Since D is strictly pseudoconvex and r is a Morse function, we can take $r_\epsilon = r + \epsilon$ for epsilon small enough. Then r_ϵ will be defining functions for smooth, strictly pseudoconvex D_ϵ. For such r_ϵ we have that all derivatives of r_ϵ are independent of ϵ. In particular, $\gamma_\epsilon(\zeta) = \gamma(\zeta)$ and $\rho_\epsilon(\zeta, z) = \rho(\zeta, z)$.

Let F be the Levi polynomial for D_ϵ:

$$F(\zeta, z) = \sum_{j=1}^{n} \frac{\partial r_\epsilon}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 r_\epsilon}{\partial \zeta_j \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k).$$

We note that $F(\zeta, z)$ is independent of ϵ since derivatives of r_ϵ are.

For ϵ small enough we can choose $\delta > 0$ and $\varepsilon > 0$ and a patching function $\varphi(\zeta, z)$, independent of ϵ, on $\mathbb{C}^n \times \mathbb{C}^n$ such that

$$\varphi(\zeta, z) = \begin{cases} 1 & \text{for } \rho^2(\zeta, z) \leq \frac{1}{2}\varepsilon, \\ 0 & \text{for } \rho^2(\zeta, z) \geq \frac{3}{4}\varepsilon. \end{cases}$$

Defining $S_\delta = \{\zeta : |r(\zeta)| < \delta\}$, $D_{-\delta} = \{\zeta : r(\zeta) < \delta\}$, and

$$\phi_\epsilon(\zeta, z) = \varphi(\zeta, z)(F_\epsilon(\zeta, z) - r_\epsilon(\zeta)) + (1 - \varphi(\zeta, z))\rho^2(\zeta, z),$$

we have the following result.

LEMMA 2.1. *On $D_\epsilon \times D_\epsilon \cap S_\delta \times D_{-\delta}$,*

$$|\phi_\epsilon| \gtrsim |\langle \partial r(z), \zeta - z \rangle| + \rho^2(\zeta, z),$$

where the constants in the inequalities are independent of ϵ.

We at times have to be precise and keep track of factors of γ that occur in our integral kernels. We shall write $\mathcal{E}_{j,k}(\zeta, z)$ for those double forms on open sets $U \subset D \times D$ such that $\mathcal{E}_{j,k}$ is smooth on U and satisfies

$$\mathcal{E}_{j,k}(\zeta, z) \lesssim \xi_k(\zeta)|\zeta - z|^j, \tag{1}$$

where ξ_k is a smooth function in D with the property

$$|\gamma^\alpha D_\alpha \xi_k| \lesssim \gamma^k$$

for D_α a differential operator of order α, and such that

$$\Lambda_\zeta \mathcal{E}_{j,k} = \mathcal{E}_{j-1,k} + \mathcal{E}_{j,k-1},$$

where Λ_ζ is a first-order differential operator in ζ.

We shall write $\mathcal{E}_j$ for those double forms on open sets $U \subset X \times X$ such that $\mathcal{E}_j$ is smooth on U, can be extended smoothly to $\bar{D} \times \bar{D}$, and satisfies

$$\mathcal{E}_j(x, y) \lesssim \rho^j(x, y);$$

$\mathcal{E}_{j,k}^*$ will denote forms that can be written as $\mathcal{E}_{j,k}(z, \zeta)$.

For $N \geq 0$, we let R_N denote an N-fold product, or a sum of such products, of first derivatives of $r(z)$, with the notation $R_0 = 1$. Here

$$P_\epsilon(\zeta, z) = \rho^2(\zeta, z) + \frac{r_\epsilon(\zeta)}{\gamma(\zeta)} \frac{r_\epsilon(z)}{\gamma(z)}.$$

DEFINITION 2.2. A double differential form $\mathcal{A}^\epsilon(\zeta, z)$ on $\bar{D}_\epsilon \times \bar{D}_\epsilon$ is an *admissible* kernel if it has the following properties.

(i) $\mathcal{A}^\epsilon$ is smooth on $\bar{D}_\epsilon \times \bar{D}_\epsilon - \Lambda_\epsilon$.

(ii) For each point $(\zeta_0, \zeta_0) \in \Lambda_\epsilon$ there is a neighborhood $U \times U$ of (ζ_0, ζ_0) on which $\mathcal{A}^\epsilon$ or $\bar{\mathcal{A}}^\epsilon$ has the representation

$$R_N R_M^* \mathcal{E}_{j,\alpha} \mathcal{E}_{k,\beta}^* P_\epsilon^{-t_0} \phi_\epsilon^{t_1} \bar{\phi}_\epsilon^{t_2} \phi_\epsilon^{*t_3} \bar{\phi}_\epsilon^{*t_4} r_\epsilon^l r_\epsilon^{*m} \tag{2}$$

with $N, M, \alpha, \beta, j, k, t_0, \ldots, m$ integers and $j, k, t_0, l, m \geq 0$, $-t = t_1 + \cdots + t_4 \leq 0$, $N, M \geq 0$, and with $N + \alpha \geq 0$ and $M + \beta \geq 0$.

This representation is of *smooth type s* for

$$s = 2n + j + \min\{2, t - l - m\} - 2(t_0 + t - l - m).$$

We define the *type* of $\mathcal{A}^\epsilon(\zeta, z)$ to be

$$\tau = s - \max\{0, 2 - N - M - \alpha - \beta\}.$$

We say that $\mathcal{A}^\epsilon$ has *smooth type $\geq s$* if at each point (ζ_0, ζ_0) there is a representation (2) of smooth type $\geq s$; $\mathcal{A}^\epsilon$ has *type $\geq \tau$* if at each point (ζ_0, ζ_0) there is a representation (2) of type $\geq \tau$. We shall also refer to the *double type* of an operator (τ, s) if the operator is of type τ and of smooth type s.

The definition of smooth type just given is taken from [5]. In this paper, $(r_\epsilon(x))^* = r_\epsilon(y)$, where the asterisk has a similar meaning for other functions of one variable.

Let $\mathcal{A}_j^\epsilon$ be kernels of type j. We denote by $\mathcal{A}_j$ the pointwise limit as $\epsilon \to 0$ of $\mathcal{A}_j^\epsilon$ and define the double type of $\mathcal{A}_j$ to be the double type of the $\mathcal{A}_j^\epsilon$ of which it is a limit. We also denote by A_j^ϵ those operators with kernels of the form $\mathcal{A}_j^\epsilon$; A_j will denote the operators with kernels $\mathcal{A}_j$. We use the notation $\mathcal{A}_{(j,k)}^\epsilon$ (resp. $\mathcal{A}_{(j,k)}$) to denote kernels of double type (j, k).

We let $\mathcal{E}_{j-2n}^i(\zeta, z)$ be a kernel of the form

$$\mathcal{E}_{j-2n}^i(\zeta, z) = \frac{\mathcal{E}_{m,0}(\zeta, z)}{\rho^{2k}(\zeta, z)}, \quad j \geq 1,$$

where $m - 2k \geq j - 2n$. We denote by E_{j-2n} the corresponding isotropic operator.

From [1], we have our next theorem.

THEOREM 2.3. *For $f \in L_{(0,q)}^2(D) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, there exist integral operators T_q, S_q, and P_q such that*

$$\gamma(z)^3 f(z) = \gamma^* T_q \bar{\partial}(\gamma^2 f) + \gamma^* S_q \bar{\partial}^*(\gamma^2 f) + \gamma^* P_q(\gamma^2 f).$$

The operators T_q, S_q, and P_q have the form

$$T_q = E_{1-2n} + A_1,$$

$$S_q = E_{1-2n} + A_1,$$

$$P_q = \frac{1}{\gamma} A_{(-1,1)}^\epsilon + \frac{1}{\gamma^*} A_{(-1,1)}^\epsilon.$$

3. Estimates

We begin with estimates on the kernels of a certain type. In [1] we proved the following statement.

PROPOSITION 3.1. *Let A_j be an operator of type $j > 0$. Then*

$$A_j : L^p(D) \to L^s(D), \qquad \frac{1}{s} > \frac{1}{p} - \frac{j}{2n+2}.$$

We describe what we shall call tangential derivatives on the Henkin–Leiterer domain D. A nonvanishing vector field T in $\mathbb{R}^{2n}$ will be called *tangential* if $Tr = 0$ on $r = 0$. Near a boundary point, we choose a coordinate patch on which we have an orthonormal frame $\omega^1, \ldots, \omega^n$ of $(1,0)$-forms with $\partial r = \gamma \omega^n$. Let $L_1, \ldots, L_n$ denote the dual frame. Then $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}$, and $Y = L_n - \bar{L}_n$ are tangential vector fields and $N = L_n + \bar{L}_n$ is a normal vector field. We say that a given vector field X is a *smooth* tangential vector field if it is a tangential field and if, near each boundary point, X is a combination of such vector fields $L_1, \ldots, L_{n-1}$, $\bar{L}_1, \ldots, \bar{L}_{n-1}$, Y, and rN with coefficients in $C^\infty(\bar{D})$. We make the important remark here that, in the coordinate patch of a critical point, the smooth tangential vector fields are not smooth combinations of derivatives with respect to the coordinate system described in Lemma 3.9. In fact, they are combinations of derivatives with respect to the coordinates of Lemma 3.9 with coefficients only in $C^0(\bar{D})$ owing to factors of γ that occur in the denominators of such coefficients. In general, a kth-order derivative of such coefficients is in $\mathcal{E}_{0,-k}$. Thus, when integrating by parts, special attention has to be paid to these nonsmooth terms.

DEFINITION 3.2. We say an operator with kernel, $\mathcal{A}$, is of *commutator* type j if $\mathcal{A}$ is of type j and if, in the representation of $\mathcal{A}$ in (2), we have $t_1 t_3 \geq 0$, $t_2 t_4 \geq 0$, and $(t_1 + t_3)(t_2 + t_4) \leq 0$.

DEFINITION 3.3. Let W be a smooth tangential vector field on $\bar{D}$. We call W *allowable* if, for all $\zeta \in \partial D$,

$$W^\zeta \in T_\zeta^{1,0}(\partial D) \oplus T_\zeta^{0,1}(\partial D).$$

The following theorem is obtained by a modification of Theorem 2.20 in [4] (see also [3]). The new details, which arise because here we do not assume $|\partial r| \neq 0$, require careful consideration and so we shall work out the calculations explicitly.

THEOREM 3.4. *Let A_1 be an admissible operator of commutator type ≥ 1 and X a smooth tangential vector field. Then*

$$\gamma^* X^z A_1 = -A_1 \tilde{X}^\zeta \gamma + A_1^{(0)} + \sum_{\nu=1}^{l} A_1^{(\nu)} W_\nu^\zeta \gamma,$$

where $\tilde{X}$ is the adjoint of X, the W_ν are allowable vector fields, and the $A_j^{(\nu)}$ are admissible operators of commutator type $\geq j$.

Proof. We use a partition of unity and suppose that X has arbitrarily small support on a coordinate patch near a boundary point in which we have an orthonormal frame $\omega^1, \ldots, \omega^n$ of $(1,0)$-forms with $\partial r = \gamma \omega^n$, as described previously, with $L_1, \ldots, L_n$ constituting the dual frame. We have $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}$, and $Y = L_n - \bar{L}_n$ as tangential vector fields and $N = L_n + \bar{L}_n$ as a normal vector field.

We have the following decomposition of the tangential vector field X:

$$X = \sum_{j=0}^{n-1} a_j L_j + \sum_{j=0}^{n-1} b_j \bar{L}_j + aY + brN,$$

where the a_j, b_j, a, and b are smooth with compact support. We shall prove the theorem for each term in the decomposition.

Case 1: $X = a_j L_j$ or $b_j \bar{L}_j$, $j \leq n-1$, or aY. We write

$$\gamma^* X^z \mathcal{A}_1 = -\gamma X^\zeta \mathcal{A}_1 + (\gamma X^\zeta + \gamma^* X^z) \mathcal{A}_1.$$

Then an integration by parts gives

$$\gamma^* X^z \mathcal{A}_1 f = -\mathcal{A}_1(\tilde{X}^\zeta \gamma f) + (f, (\gamma X^\zeta + \gamma^* X^z) \mathcal{A}_1).$$

We now use the following relations:

$$(\gamma X^\zeta + \gamma^* X^z) \mathcal{E}_{j,\alpha} = \mathcal{E}_{j,\alpha},$$
$$(\gamma X^\zeta + \gamma^* X^z) \mathcal{E}_{j,\beta}^* = \mathcal{E}_{j,\beta}^*,$$
$$(\gamma X^\zeta + \gamma^* X^z) P = \mathcal{E}_{2,0} + \frac{rr^*}{\gamma\gamma^*} \mathcal{E}_{0,0} \qquad (3)$$
$$= \mathcal{E}_{0,0} P + \mathcal{E}_{2,0},$$
$$(\gamma X^\zeta + \gamma^* X^z) \phi = \mathcal{E}_{1,1} + \mathcal{E}_{2,0}.$$

Any type-1 kernel

$$\mathcal{A}_1(\zeta, z) = R_N R_M^* \mathcal{E}_{j,\alpha} \mathcal{E}_{k,\beta}^* P^{-t_0} \phi^{t_1} \bar{\phi}^{t_2} \phi^{*t_3} \bar{\phi}^{*t_4} r^l r^{*m} \qquad (4)$$

can be decomposed into terms

$$\mathcal{A}_1 = \mathcal{A}_1' + \mathcal{A}_2,$$

where $\mathcal{A}_1'$ is of *pure* type—meaning that it has a representation as in (4) but with $t_3 = t_4 = 0$ and $t_1 t_2 \leq 0$ [4]. From the relations (3) we have

$$(\gamma X^\zeta + \gamma^* X^z) \mathcal{A}_2 = \gamma \mathcal{A}_1 + \mathcal{A}_2.$$

In calculating $(\gamma X^\zeta + \gamma^* X^z) \mathcal{A}_1'$, we find the term that is not immediately seen to be of type $\mathcal{A}_1$ is the one resulting from the operator $\gamma X^\zeta + \gamma^* X^z$ falling on ϕ^{t_1}, in which case we obtain the term of double type $(0,0)$,

$$B := R_N R_M^* \mathcal{E}_{j+1,\alpha+1} \mathcal{E}_{k,\beta}^* P^{-t_0} \phi^{t_1-1} \bar{\phi}^{t_2} r^l r^{*m},$$

where $N + \alpha \geq 2$, plus a term that is $\mathcal{A}_1$. We follow [3] in reducing to the case where B can be written as a sum of terms B_σ such that B_σ or $\bar{B}_\sigma$ is of the form

$$\gamma^2 \phi^\sigma (\phi + \bar{\phi})^{\tau_1 + \tau_2 - \sigma} R_N R_M^* \mathcal{E}_{j+1,\alpha-1} \mathcal{E}_{k,\beta}^* P^{-t_0} r^l r^{*m},$$

where $\tau_1 + \tau_2 \leq -3$ and either $\tau_1 \leq \sigma \leq \tau_1 + \tau_2$ or $\tau_2 \leq \sigma \leq \tau_1 + \tau_2$.

We fix a point z and choose local coordinates ζ such that

$$d\zeta_j(z) = \omega_j(z).$$

Working in a neighborhood of a singularity in the boundary (where we can use a coordinate system as in (13) to follow), we see that $\partial/\partial\zeta_n$ is a combination of derivatives with coefficients of the form $\xi_0(z)$ and that L_n is a combination of derivatives with coefficients of the form $\xi_0(\zeta)$, where ξ_0 is defined in (1). We have that $\Lambda_n - \partial/\partial z_n$ is a sum of terms of the form

$$\mathcal{E}_{1,-1}\Lambda^\epsilon,$$

where Λ is a first-order differential operator.

Using these special coordinates, we note that

$$Y\phi = \gamma + \mathcal{E}_{1,0} + \mathcal{E}_{2,-1},$$
$$Y\bar\phi = -\gamma + \mathcal{E}_{1,0} + \mathcal{E}_{2,-1},$$
$$YP = \mathcal{E}_{1,0} + \frac{\mathcal{E}_{0,0}}{\gamma}(P + \mathcal{E}_{2,0})$$

and write

$$
\begin{aligned}
B_\sigma &= \gamma^2\phi^\sigma(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_N R_M^* \mathcal{E}_{j+1,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m}\\
&= \gamma Y(\phi^{\sigma+1}(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_N R_M^* \mathcal{E}_{j+1,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m})\\
&\quad + \gamma\phi^\sigma(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_N R_M^* \mathcal{E}_{j+2,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m}\\
&\quad + \gamma\phi^\sigma(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_N R_M^* \mathcal{E}_{j+3,\alpha-2}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m}\\
&\quad + \gamma\phi^{\sigma+1}(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma-1}R_N R_M^* \mathcal{E}_{j+2,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m}\\
&\quad + \gamma\phi^{\sigma+1}(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma-1}R_N R_M^* \mathcal{E}_{j+3,\alpha-2}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m}\\
&\quad + \gamma\phi^{\sigma+1}(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_{N-1} R_M^* \mathcal{E}_{j+1,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m}\\
&\quad + \gamma\phi^{\sigma+1}(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_N R_M^* \mathcal{E}_{j,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m}\\
&\quad + \gamma\phi^{\sigma+1}(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_N R_M^* \mathcal{E}_{j+2,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0-1}r^l r^{*m}\\
&\quad + \phi^{\sigma+1}(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_N R_M^* \mathcal{E}_{j+1,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0}r^l r^{*m}\\
&\quad + \phi^{\sigma+1}(\phi + \bar\phi)^{\tau_1+\tau_2-\sigma}R_N R_M^* \mathcal{E}_{j+3,\alpha-1}\mathcal{E}_{k,\beta}^* P^{-t_0-1}r^l r^{*m}.
\end{aligned}
$$

Thus

$$B_\sigma = \gamma Y\mathcal{A}_{(1,2)} + \mathcal{A}_1'.$$

By the strict pseudoconvexity of D there exist allowable vector fields W_1, W_2, W_3 and a function φ, smooth on the interior of D, that satisfies

$$\Phi^k\varphi = \mathcal{E}_{0,1-k},$$

where Φ is a first-order differential operator, such that Y can be written as

$$Y = \varphi[W_1, W_2] + W_3.$$

Thus

$$\gamma Y \mathcal{A}_{(2,1)} = \gamma \varphi [W_1, W_2] \mathcal{A}_{(1,2)} + \gamma W_3 \mathcal{A}_{(1,2)}$$
$$= \gamma [W_1, W_2] \varphi \mathcal{A}_{(1,2)} + \mathcal{A}_1'$$

with $\mathcal{A}_1'$ of commutator type ≥ 1.

An integration by parts gives

$$(f, \gamma [W_1, W_2](\varphi \mathcal{A}_2)) = (\tilde{W}_1 \gamma f, W_2(\varphi \mathcal{A}_{(1,2)})) - (\tilde{W}_2 \gamma f, W_1(\varphi \mathcal{A}_{(1,2)})).$$

Here $\tilde{W}_1$ and $\tilde{W}_2$ are allowable vector fields and both $W_2(\varphi \mathcal{A}_{(1,2)})$ and $W_1(\varphi \mathcal{A}_{(1,2)})$ are of the form $\mathcal{A}_1'$, where $\mathcal{A}_1'$ is of commutator type. This proves the theorem for Case 1.

Case 2: $X = \mathcal{E}_0 r N$. We use

$$(r \gamma N^{\zeta} + r^* \gamma^* N^z) \mathcal{E}_{j,\alpha} = \mathcal{E}_{j,\alpha},$$

$$(r \gamma N^{\zeta} + r^* \gamma^* N^z) P = \mathcal{E}_{2,0} + \frac{r}{\gamma} \frac{r^*}{\gamma^*} \mathcal{E}_{0,0} \tag{5}$$

$$= \mathcal{E}_{2,0} + P \mathcal{E}_{0,0},$$

$$(r \gamma N^{\zeta} + r^* \gamma^* N^z) \phi = r \mathcal{E}_{0,0} + r^* \mathcal{E}_{0,0}.$$

Thus

$$\gamma^* X A_1 f = (\mathcal{E}_0 r^* f, \gamma^* N^z \mathcal{A}_1)$$
$$= (-\mathcal{E}_0 r f, \gamma N^{\zeta} \mathcal{A}_1) + (f, \mathcal{E}_0 (r \gamma N^{\zeta} + r^* \gamma^* N^z) \mathcal{A}_1)$$
$$= (-\tilde{N}^{\zeta} (\mathcal{E}_0 r \gamma f), \mathcal{A}_1) + (f, \mathcal{E}_0 (r \gamma N^{\zeta} + r^* \gamma^* N^z) \mathcal{A}_1).$$

We have

$$\tilde{N}^{\zeta} (\mathcal{E}_0 r \gamma f) = \mathcal{E}_{0,0} f + \mathcal{E}_0 r \tilde{N}^{\zeta} \gamma f,$$

and $\mathcal{E}_0 r \tilde{N}^{\zeta}$ is an allowable vector field. The relations in (5) show that

$$(r \gamma N^{\zeta} + r^* \gamma^* N^z) \mathcal{A}_1$$

is of commutator type ≥ 1. Case 2 therefore follows. $\qquad\square$

We will use a criterion for Hölder continuity given by Schmalz.

Lemma 3.5 [6, Lemma 4.1]. *Let $D \subseteq \mathbb{R}^m$ ($m \geq 1$) be an open set, and let $B(D)$ denote the space of bounded functions on D. Suppose r is a C^2 function on $\mathbb{R}^m$, $m \geq 1$, such that $D := \{r < 0\} \subseteq \mathbb{R}^m$. Then there exists a constant $C < \infty$ such that the following statement holds: If a function $u \in B(D)$ satisfies for some $0 < \alpha \leq 1/2$ and for all $z, w \in D$ the estimate*

$$|u(z) - u(w)| \leq |z - w|^{\alpha} + \max_{y=z, w} \frac{|\nabla r(y)||z - w|^{1/2+\alpha}}{|r(y)|^{1/2}},$$

then

$$|u(z) - u(w)| \leq C|z - w|^\alpha$$

for all $z, w \in D$.

We will also refer to another lemma of Schmalz [6, Lemma 3.2] that provides a useful coordinate system in which to prove estimates.

LEMMA 3.6. *Define* x_j *by* $\zeta_j = x_j + ix_{j+n}$ *for* $1 \leq j \leq n$. *Let* $E_\delta(z) := \{\zeta \in D : |\zeta - z| < \delta\gamma(z)\}$ *for* $\delta > 0$. *Then there exist a constant* c *and numbers* $l, m \in \{1, \ldots, 2n\}$ *such that, for all* $z \in D$,

$$\{-r(\zeta), \operatorname{Im}\phi(\cdot, z), x_1, \ldots_{\hat{l}, \hat{m}} \ldots, x_{2n}\},$$

where x_l *and* x_m *are omitted, forms a coordinate system in* $E_c(z)$. *We have the estimate*

$$dV \lesssim \frac{1}{\gamma(z)^2}|dr(\zeta) \wedge d\operatorname{Im}\phi(\cdot, z) \wedge dx_1 \wedge \cdots_{\hat{l}, \hat{m}} \cdots \wedge dx_{2n}| \text{ on } E_c(z),$$

where dV *is the Euclidean volume form on* $\mathbb{R}^{2n}$.

We next define the function spaces with which we will be working.

DEFINITION 3.7. Let $0 \leq \beta$ and $0 \leq \delta$. We define

$$\|f\|_{L^{\infty,\beta,\delta}(D)} = \sup_{\zeta \in D}|f(\zeta)||\gamma^\beta(\zeta)||r(\zeta)|^\delta.$$

DEFINITION 3.8. For $0 < \alpha < 1$ we set

$$\Lambda_\alpha(D) = \left\{f \in L^\infty(D) : \|f\|_{\Lambda_\alpha} := \|f\|_{L^\infty} + \sup \frac{|f(\zeta) - f(z)|}{|\zeta - z|^\alpha} < \infty\right\}.$$

We also define the spaces $\Lambda_{\alpha,\beta}$ by

$$\Lambda_{\alpha,\beta} := \{f : \|f\|_{\Lambda_{\alpha,\beta}} = \|\gamma^\beta f\|_{\Lambda_\alpha} < \infty\}.$$

From [1], we have the following lemma.

LEMMA 3.9.
$$\frac{r_\epsilon}{\gamma} \in C^1(D_\epsilon)$$

with C^1 *estimates independent of* ϵ.

For our C^k estimates later, we will need the following properties.

THEOREM 3.10. *Let* T *be a smooth first-order tangential differential operator on* D. *For* A *an operator of type 1, we have*:
  (i) $A: L^{\infty,2+\epsilon,0}(D) \to \Lambda_{\alpha,2-\epsilon'}(D)$ *with* $0 < \epsilon, \epsilon'$ *and* $\alpha + \epsilon + \epsilon' < 1/4$;
  (ii) $\gamma^*TA: L^{\infty,2+\epsilon,0}(D) \to L^{\infty,\epsilon',\delta}(D)$ *with* $1/2 < \delta < 1$ *and* $\epsilon < \epsilon' < 1$;
  (iii) $A: L^{\infty,\epsilon,\delta}(D) \to L^{\infty,\epsilon',0}(D)$ *with* $\epsilon < \epsilon'$ *and* $\delta < 1/2 + (\epsilon' - \epsilon)/2$.

Proof. (i) We will prove part (i) of the theorem in the cases that $\mathcal{A}$, the kernel of A, is of double type $(1, 1)$ satisfying the inequality

$$|\mathcal{A}| \lesssim \frac{\gamma(\zeta)^2}{P^{n-1/2-\mu}|\phi|^{\mu+1}}, \quad \mu \geq 1,$$

and $\mathcal{A}$ is of double type $(1, 2)$ satisfying

$$|\mathcal{A}| \lesssim \frac{\gamma(\zeta)}{P^{n-1-\mu}|\phi|^{\mu+1}}, \quad \mu \geq 1.$$

All other cases are handled by the same methods.

Case A: $\mathcal{A}$, the kernel of A, is of double type $(1, 1)$. We estimate

$$\int_D \frac{1}{\gamma^\epsilon(\zeta)} \left| \frac{\gamma(z)^{2-\epsilon'}}{(\phi(\zeta,z))^{\mu+1}P(\zeta,z)^{n-1/2-\mu}} - \frac{\gamma(w)^{2-\epsilon'}}{(\phi(\zeta,w))^{\mu+1}P(\zeta,w)^{n-1/2-\mu}} \right| dV(\zeta).$$
(6)

Then the integral in (6) is bounded by

$$\int_D \frac{1}{\gamma^\epsilon(\zeta)} \left| \frac{\gamma(z)^{2-\epsilon'}(\phi(\zeta,w))^{\mu+1} - \gamma(w)^{2-\epsilon'}(\phi(\zeta,z))^{\mu+1}}{(\phi(\zeta,w))^{\mu+1}(\phi(\zeta,z))^{\mu+1}P(\zeta,z)^{n-1/2-\mu}} \right| dV(\zeta)$$

$$+ \int_D \frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \left| \frac{P(\zeta,z)^{n-1/2-\mu} - P(\zeta,w)^{n-1/2-\mu}}{(\phi(\zeta,w))^{\mu+1}P(\zeta,z)^{n-1/2-\mu}P(\zeta,w)^{n-1/2-\mu}} \right| dV(\zeta)$$

$$= I + II.$$

In I we use

$$(\phi(\zeta,w))^{\mu+1} - (\phi(\zeta,z))^{\mu+1} = \sum_{l=0}^{\mu} (\phi(\zeta,w))^{\mu-l}(\phi(\zeta,z))^l(\phi(\zeta,w) - \phi(\zeta,z))$$

and

$$\phi(\zeta,w) - \phi(\zeta,z) = O(\gamma(\zeta) + |\zeta - z|)|z - w|.$$

Therefore,

$$I \lesssim \sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{(\gamma(\zeta) + |\zeta - z|)|z - w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-1-2\mu}} dV(\zeta)$$

$$+ \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{|\gamma(z)^{2-\epsilon'} - \gamma(w)^{2-\epsilon'}|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu}} dV(\zeta)$$

$$\lesssim \sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{3-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z - w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-1-2\mu}} dV(\zeta)$$

$$+ \sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z - w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-2-2\mu}} dV(\zeta)$$

$$+ \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{|\gamma(z)^{2-\epsilon'} - \gamma(w)^{2-\epsilon'}|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu}} dV(\zeta)$$

$$= I_a + I_b + I_c.$$

For the integral I_a we break the region of integration into two parts, $\{|\zeta - w| \leq |\zeta - z|\}$ and $\{|\zeta - z| \leq |\zeta - w|\}$. By symmetry, we need only consider the region $\{|\zeta - z| \leq |\zeta - w|\}$.

We first consider the region E_c, where c is chosen as in Lemma 3.5. Without loss of generality we can choose c sufficiently small so that $\gamma(z) \lesssim \gamma(\zeta)$ holds in $E_c(z)$. We thus estimate

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \leq |\zeta - w|}} \gamma(z)^{3-\epsilon'-\epsilon} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-1-2\mu}} \, dV(\zeta). \quad (7)$$

We use $\gamma(z) \lesssim \gamma(w) + |z - w|$ and

$$|z - w|^\beta \lesssim |\zeta - z|^\beta + |\zeta - w|^\beta$$
$$\lesssim |\zeta - w|^\beta \quad (8)$$

for $\beta > 0$ to bound the integral in (7) by a constant times

$$|z - w|^{1/2+\alpha} \int_{\substack{D \cap E_c \\ |\zeta - z| \leq |\zeta - w|}} \frac{\gamma(z)^2 \gamma(w) |\zeta - w|^{1/2-\alpha}}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-1-2\mu+\epsilon+\epsilon'}} \, dV(\zeta)$$

$$+ |z - w|^\alpha \int_{\substack{D \cap E_c \\ |\zeta - z| \leq |\zeta - w|}} \frac{\gamma(z)^2 |\zeta - w|^{2-\alpha}}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-1-2\mu+\epsilon+\epsilon'}} \, dV(\zeta). \quad (9)$$

We use a coordinate system $s_1, s_2, t_1, \ldots, t_{2n-2}$ as given by Lemma 3.6 with $s_1 = -r(\zeta)$ and $s_2 = \operatorname{Im}\phi$, and we use the estimate from that lemma on the volume element

$$dV(\zeta) \lesssim \frac{t^{2n-3}}{\gamma(z)^2} |ds_1 \, ds_2 \, dt|, \quad (10)$$

where $t = \sqrt{t_1^2 + \cdots + t_{2n-2}^2}$; the second line follows from $\gamma(\zeta) \lesssim \gamma(z)$ on $E_c(z)$.

We have the estimates

$$\phi(\zeta,z) \gtrsim s_1 + |s_2| + t^2,$$
$$\phi(\zeta,w) \gtrsim -r(w) + s_1 + t^2.$$

After redefining s_2 to be positive, we bound the first integral of (9) by

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

$$\int_V \frac{|\zeta - w|^{1/2-\alpha}}{(s_1 + s_2 + t^2)^{\mu+1-l}(s_1 + |\zeta - w|^2)^{l+1/2} t^{2n-1-2\mu+\epsilon+\epsilon'}} t^{2n-3} \, ds_1 \, ds_2 \, dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{t^{2\mu-2-\epsilon-\epsilon'}}{(s_1 + s_2 + t^2)^{\mu+1-l}(s_1 + t^2)^{l+1/4+\alpha/2}} \, ds_1 \, ds_2 \, dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{1}{s_1^{7/8}(s_1 + s_2) t^{3/4+\alpha+\epsilon+\epsilon'}} \, ds_1 \, ds_2 \, dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{1}{s_1^{15/16} s_2^{15/16} t^{3/4+\alpha+\epsilon+\epsilon'}} \, ds_1 \, ds_2 \, dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w), \quad (11)$$

where V is a bounded subset of $\mathbb{R}^3$.

The second integral of (9) can be bounded by a constant times

$$|z - w|^\alpha \int_V \frac{|\zeta - w|^{2-\alpha}}{(s_1 + s_2 + t^2)^{\mu+1-l}(s_1 + |\zeta - w|^2)^{l+1}t^{2n-1-2\mu+\epsilon+\epsilon'}} t^{2n-3}\, ds_1\, ds_2\, dt$$

$$\lesssim |z - w|^\alpha \int_V \frac{t^{2\mu-2-\epsilon-\epsilon'}}{(s_1 + s_2 + t^2)^{\mu+1-l}(s_1 + t^2)^{l+\alpha/2}}\, ds_1\, ds_2\, dt$$

$$\lesssim |z - w|^\alpha,$$

where again V is a bounded subset of $\mathbb{R}^3$. The last line follows by the estimates in (11).

In estimating the integrals of I_a over the region $D \setminus E_c$, we write

$$\int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{|z - w|}{|\phi(\zeta, z)|^{\mu+1-l}|\phi(\zeta, w)|^{l+1}|\zeta - z|^{2n-4-2\mu+\epsilon'}}\, dV(\zeta)$$

$$\lesssim |z - w|^\alpha$$

$$\int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{|\zeta - w|^{1-\alpha}}{|\phi(\zeta, z)|^{\mu+1-l}|\phi(\zeta, w)|^{l+1}|\zeta - z|^{2n-4-2\mu+\epsilon'}}\, dV(\zeta)$$

$$\lesssim |z - w|^\alpha$$

$$\int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\phi(\zeta, z)|^{\mu+1-l}|\phi(\zeta, w)|^{l+1/2+\alpha/2}|\zeta - z|^{2n-4-2\mu+\epsilon'}}\, dV(\zeta)$$

$$\lesssim |z - w|^\alpha \int_{D \setminus E_c} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta - z|^{2n-1+\alpha+\epsilon'}}\, dV(\zeta). \tag{12}$$

We denote the critical points of r by $p_1, \ldots, p_k$ and take ε small enough so that, in each

$$U_{2\varepsilon}(p_j) = \{\zeta : D \cap |\zeta - p_j| < 2\varepsilon\},$$

for $j = 1, \ldots, k$ there are (by the Morse lemma) coordinates $u_{j_1}, \ldots, u_{j_m}, v_{j_{m+1}}, \ldots, v_{j_{2n}}$ such that

$$-r(\zeta) = u_{j_1}^2 + \cdots + u_{j_m}^2 - v_{j_{m+1}}^2 - \cdots - v_{j_{2n}}^2 \tag{13}$$

with $u_{j_\alpha}(p_j) = v_{j_\beta}(p_j) = 0$ for all $1 \le \alpha \le m$ and $m + 1 \le \beta \le 2n$. Let $U_\varepsilon = \bigcup_{j=1}^k U_\varepsilon(p_j)$. We break the problem of estimating (12) into subcases depending on whether $z \in U_\varepsilon$.

Subcase a: $z \in U_\varepsilon(p_j)$. Define $w_1, \ldots, w_{2n}$ by

$$w_\alpha = \begin{cases} u_{j_\alpha} & \text{for } 1 \le \alpha \le m, \\ v_{j_\alpha} & \text{for } m + 1 \le \alpha \le 2n. \end{cases} \tag{14}$$

Let $x_1, \ldots, x_{2n}$ be defined by $\zeta_\alpha = x_\alpha + ix_{n+\alpha}$. By the Morse lemma, the Jacobian of the transformation from coordinates $x_1, \ldots, x_{2n}$ to $w_1, \ldots, w_{2n}$ is bounded from below and above; thus we have

$$|\zeta - z| \simeq |w(\zeta) - w(z)|$$

for $\zeta, z \in U_{2\varepsilon}(p_j)$.

From (13) we have $\gamma(z) \gtrsim |w(z)|$ and thus

$$|w(\zeta) - w(z)| \simeq |\zeta - z|$$
$$\gtrsim \gamma(z)$$
$$\gtrsim |w(z)|$$
$$\geq |w(\zeta)| - |w(\zeta) - w(z)|,$$

so we obtain

$$|w(\zeta)| \lesssim |w(\zeta) - w(z)|$$
$$\simeq |\zeta - z|.$$

Using $|w(\zeta)| \lesssim \gamma(\zeta)$, we use the preceding coordinates to estimate

$$|z - w|^\alpha \int_{U_\varepsilon \setminus E_c} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta - z|^{2n-1+\alpha+\epsilon'}} \, dV(\zeta)$$

$$\lesssim |z - w|^\alpha \int_V \frac{u^{m-1} v^{2n-m-1}}{(u+v)^{2n-1+\alpha+\epsilon'+\epsilon}} \, du \, dv$$

$$\lesssim |z - w|^\alpha,$$

where we use $u = \sqrt{u_{j_1}^2 + \cdots + u_{j_m}^2}$ and $v = \sqrt{v_{j_{m+1}}^2 + \cdots + v_{j_{2n}}^2}$ and where V is a bounded set.

In integrating over the region $D \setminus U_\varepsilon$, we have

$$|z - w|^\alpha \int_{(D \setminus U_\varepsilon) \setminus E_c} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta - z|^{2n-1+\alpha+\epsilon'}} \, dV(\zeta)$$

$$\lesssim |z - w|^\alpha \int_{(D \setminus U_\varepsilon) \setminus E_c} \frac{1}{\gamma^\epsilon(\zeta)} \, dV(\zeta) \lesssim |z - w|^\alpha,$$

which follows by using the coordinates $w_1, \ldots, w_{2n}$.

Subcase b: $z \notin U_\varepsilon$. We have $|\zeta - z| \gtrsim \gamma(z)$, but $\gamma(z)$ is bounded from below since $z \notin U_\varepsilon$. We therefore have to estimate

$$\int_D \frac{1}{\gamma^\epsilon(\zeta)} \, dV(\zeta),$$

which is easily done by working with the coordinates $w_1, \ldots, w_{2n}$.

The region in which $|\zeta - w| \leq |\zeta - z|$ is handled in the same manner, and thus we are finished bounding I_a.

We now estimate I_b, and again we consider only the region $|\zeta - z| \leq |\zeta - w|$. We first estimate the integrals of I_b over the region $E_c(z)$, where c is chosen as in Lemma 3.6 and sufficiently small so that $|\zeta - z| \lesssim \gamma(\zeta)$. As we chose coordinates for the integrals in I_a, we choose a coordinate system in which $s_1 = -r(\zeta)$ and $s_2 = \text{Im}\,\phi$, and we use the estimate on the volume element given by (10). We thus write

$$\int_{\substack{D \cap E_c \\ |\zeta-z| \le |\zeta-w|}} \gamma(z)^2 \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-2-2\mu+\epsilon+\epsilon'}} \, dV(\zeta)$$

$$\lesssim |z-w|^\alpha$$

$$\int_{\substack{D \cap E_c \\ |\zeta-z| \le |\zeta-w|}} \gamma(z)^2 \frac{1}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1/2+\alpha/2}|\zeta-z|^{2n-2-2\mu+\epsilon+\epsilon'}} \, dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_V \frac{t^{2n-3}}{(s_1+s_2+t^2)^{\mu+1-l}(s_1+t^2)^{l+1/2+\alpha/2}t^{2n-2-2\mu+\epsilon+\epsilon'}} \, ds_1 \, ds_2 \, dt$$

$$\lesssim |z-w|^\alpha \int_0^M \int_0^N \frac{t^{2\mu-1-\epsilon-\epsilon'}}{(s_1+t^2)^{\mu-l}(s_1+t^2)^{l+1/2+\alpha/2}} \, ds_1 \, dt$$

$$\lesssim |z-w|^\alpha \int_0^M \int_0^N \frac{1}{s_1^{7/8}t^{1/4+\alpha+\epsilon+\epsilon'}} \, ds \, dt$$

$$\lesssim |z-w|^\alpha, \tag{15}$$

where we have redefined the coordinate s_2 to be positive, V is a bounded subset of $\mathbb{R}^3$, and $M, N > 0$ are constants. The integrals of I_b over the region $D \setminus E_c$ are estimated by (12).

For the integral I_c we use

$$|\gamma(w)^{2-\epsilon'} - \gamma(z)^{2-\epsilon'}| \lesssim |z-w|(\gamma(w)^{1-\epsilon'} + \gamma(z)^{1-\epsilon'})$$

and estimate

$$\int_{\substack{D \\ |\zeta-z| \le |\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{|z-w|(\gamma(w)^{1-\epsilon'} + \gamma(z)^{1-\epsilon'})}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-1-2\mu}} \, dV(\zeta). \tag{16}$$

Let us first consider the case $\gamma(w) \le \gamma(z)$ and integrate (16) over the region E_c. We use a coordinate system $s, t_1, \ldots, t_{2n-1}$ with $s = -r$ and the estimate

$$dV(\zeta) \lesssim \frac{t^{2n-2}}{\gamma(z)} \, ds \, dt$$

for $t = \sqrt{t_1^2 + \cdots + t_{2n-1}^2}$. We thus bound (16) by

$$\int_{\substack{D \cap E_c \\ |\zeta-z| \le |\zeta-w|}} \frac{|z-w|\gamma(z)^{1-\epsilon'}}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-1-2\mu+\epsilon}} \, dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_{\substack{D \cap E_c \\ |\zeta-z| \le |\zeta-w|}} \frac{\gamma(z)}{|\phi(\zeta,w)|^{\mu+1/2+\alpha/2}|\zeta-z|^{2n-1-2\mu+\epsilon+\epsilon'}} \, dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_V \frac{t^{2n-2}}{(s+t^2)^{\mu+1/2+\alpha/2}t^{2n-1-2\mu+\epsilon+\epsilon'}} \, ds \, dt$$

$$\lesssim |z-w|^\alpha \int_V \frac{1}{s^{3/4}t^{1/2+\epsilon+\epsilon'+\alpha/2}} \, ds \, dt$$

$$\lesssim |z-w|^\alpha, \tag{17}$$

where V is here a bounded region of $\mathbb{R}^2$.

Over the complement of E_c, (16) is bounded by

$$|z-w|^\alpha \int_{\substack{D \\ |\zeta-z| \le |\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\phi(\zeta,w)|^{\mu+1/2+\alpha/2}|\zeta-z|^{2n-2-2\mu+\epsilon'}} \, dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta-z|^{2n-1+\epsilon'+\alpha}} \, dV(\zeta)$$

$$\lesssim |z-w|^\alpha,$$

which follows from the estimates of (12).

For the case $\gamma(z) \le \gamma(w)$ we estimate (16) over the region E_c, using coordinates as before, by

$$\int_{\substack{D \cap E_c \\ |\zeta-z| \le |\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{|z-w|\gamma(w)^{1-\epsilon'}}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-1-2\mu}} \, dV(\zeta)$$

$$\lesssim \frac{|z-w|^{1/2+\alpha/2}}{|r(w)|^{1/2}} \gamma(w)$$

$$\int_{\substack{D \cap E_c \\ |\zeta-z| \le |\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta-z|^{2n-1-2\mu+\epsilon'}} \, dV(\zeta)$$

$$\lesssim \frac{|z-w|^{1/2+\alpha/2}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{t^{2n-2}}{(s+t^2)^{\mu+1/4+\alpha/2} t^{2n-2\mu+\epsilon+\epsilon'}} \, ds \, dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha/2}}{|r(w)|^{1/2}} \gamma(w), \tag{18}$$

where the last line follows as before. Over the complement of E_c, we use $\gamma(w) \lesssim |\zeta - w|$ to bound (16) by

$$|z-w|^\alpha \int_{\substack{D \\ |\zeta-z| \le |\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\phi(\zeta,w)|^{\mu+\epsilon'/2+\alpha/2}|\zeta-z|^{2n-1-2\mu}} \, dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta-z|^{2n-1+\epsilon'+\alpha}} \, dV(\zeta)$$

$$\lesssim |z-w|^\alpha.$$

We are now done with integral I.

For II we again break the integral into regions $|\zeta - z| \le |\zeta - w|$ and $|\zeta - w| \le |\zeta - z|$, again considering only the region $|\zeta - z| \le |\zeta - w|$ since the other case is handled similarly.

We write

$$(P(\zeta,z)^{1/2})^{2n-1-2\mu} - (P(\zeta,w)^{1/2})^{2n-1-2\mu}$$

$$= \sum_{l=0}^{2n-2\mu-2} (P(\zeta,z)^{1/2})^{2n-2-2\mu-l} (P(\zeta,w)^{1/2})^l (P(\zeta,z)^{1/2} - P(\zeta,w)^{1/2})$$

and use

$$|P(\zeta,z)^{1/2} - P(\zeta,w)^{1/2}| = \frac{|P(\zeta,z) - P(\zeta,w)|}{P(\zeta,z)^{1/2} + P(\zeta,w)^{1/2}}$$

$$\lesssim \frac{|\zeta - z| + \frac{|r(\zeta)|}{\gamma(\zeta)}}{|\zeta - z|}|z - w|$$

$$\lesssim \frac{|\zeta - w| + \frac{|r(w)|}{\gamma(w)}}{|\zeta - z|}|z - w|,$$

which follows from Lemma 3.9.

We thus estimate

$$\int_{\substack{D \\ |\zeta-z| \le |\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \left| \frac{P(\zeta,z)^{n-1/2-\mu} - P(\zeta,w)^{n-1/2-\mu}}{(\phi(\zeta,w))^{\mu+1}P(\zeta,z)^{n-1/2-\mu}P(\zeta,w)^{n-1/2-\mu}} \right| dV(\zeta)$$

$$\lesssim \sum_{l=0}^{2n-2\mu-2}$$

$$\int_{\substack{D \\ |\zeta-z| \le |\zeta-w|}} \frac{\frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(w)}|z - w|\left(|\zeta - z| + \frac{|r(w)|}{\gamma(w)}\right)}{|\phi(\zeta,w)|^{\mu+1}(P(\zeta,z)^{1/2})^{l+1}(P(\zeta,w)^{1/2})^{2n-1-2\mu-l}|\zeta - z|} dV(\zeta)$$

$$\lesssim \int_{\substack{D \\ |\zeta-z| \le |\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z - w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2\mu}} dV(\zeta)$$

$$+ \int_{\substack{D \\ |\zeta-z| \le |\zeta-w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|r(w)||z - w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n+1-2\mu}} dV(\zeta)$$

$$= II_a + II_b.$$

For II_a, we break the integral into the regions $E_c(z)$ and its complement. We first consider

$$\int_{\substack{D \setminus E_c \\ |\zeta-z| \le |\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z - w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2\mu}} dV(\zeta)$$

$$\lesssim |z - w|^\alpha \int_{\substack{D \setminus E_c \\ |\zeta-z| \le |\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\phi(\zeta,w)|^{\mu-1/2+\alpha/2+\epsilon'/2}|\zeta - z|^{2n-2\mu}} dV(\zeta)$$

$$\lesssim |z - w|^\alpha \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta - z|^{2n-1+\alpha+\epsilon'}} dV(\zeta)$$

$$\lesssim |z - w|^\alpha, \tag{19}$$

where we use $\gamma(w) \lesssim |\zeta - w|$ and the estimates for (12).

We then bound the integral II_a over the region $E_c(z)$ by considering the different cases $\gamma(w) \le \gamma(z)$ and $\gamma(z) \le \gamma(w)$. In the case $\gamma(w) \le \gamma(z)$, we use a coordinate system, $s, t_1, \ldots, t_{2n-1}$ in which $s = -r(\zeta)$; then, using the estimate

$$dV(\zeta) \lesssim \frac{t^{2n-2}}{\gamma(z)} ds\, dt, \tag{20}$$

we have

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z - w|}{|\phi(\zeta, w)|^{\mu+1}|\zeta - z|^{2n-2\mu}} \, dV(\zeta)$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(z)^{1-\epsilon'} \frac{1}{|\phi(\zeta, w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon}} \, dV(\zeta)$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{t^{2\mu-2-\epsilon-\epsilon'}}{(s + t^2)^{\mu+1/4+\alpha/2}} \, ds \, dt$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{1}{s^{7/8} t^{3/4+\alpha+\epsilon+\epsilon'}} \, ds \, dt$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w). \tag{21}$$

In the case $\gamma(z) \le \gamma(w)$ we estimate, as before,

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z - w|}{|\phi(\zeta, w)|^{\mu+1}|\zeta - z|^{2n-2\mu}} \, dV(\zeta)$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(w) \frac{1}{|\phi(\zeta, w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} \, dV(\zeta)$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{(\gamma(z) + |\zeta - w|)}{|\phi(\zeta, w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} \, dV(\zeta).$$

The integral involving $\gamma(z)$ is estimated exactly as before. We thus have to deal with

$$\frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{|\zeta - w|}{|\phi(\zeta, w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} \, dV(\zeta),$$

which we estimate using the coordinates $s, t_1, \ldots, t_{2n-1}$ as

$$\frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{|\zeta - w|}{|\phi(\zeta, w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} \, dV(\zeta)$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{t^{2n-2}}{(s + t^2)^{\mu-1/4+\alpha/2}(s + t)^{2n-2\mu+1+\epsilon+\epsilon'}} \, ds \, dt$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{1}{s^{3/4+\alpha/2+\epsilon+\epsilon'+\delta} t^{1-\delta}} \, ds \, dt$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w),$$

where $0 < \delta < 1/4 - (\alpha/2 + \epsilon + \epsilon')$.

For II_b we first estimate

$$\int_{\substack{D\setminus E_c \\ |\zeta-z|\leq|\zeta-w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|r(\zeta)||z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n+1-2\mu}} dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta-z|\leq|\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{|\zeta-w|^{2-\alpha-\epsilon'}}{|\phi(\zeta,w)|^\mu|\zeta-z|^{2n+1-2\mu}} dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta-z|^{2n-1+\alpha+\epsilon'}} dV(\zeta)$$

$$\lesssim |z-w|^\alpha,$$

where c is chosen as in Lemma 3.6 and we use $\gamma(w) \lesssim |\zeta-w|$ on $D\setminus E_c(z)$.

We now finish the estimates for II_b. We have

$$\int_{\substack{D\cap E_c \\ |\zeta-z|\leq|\zeta-w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|r(\zeta)||z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n+1-2\mu}} dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_{\substack{D\cap E_c \\ |\zeta-z|\leq|\zeta-w|}} \gamma(w)^{1-\epsilon'} \frac{1}{|\phi(\zeta,w)|^{\mu-1/2+\alpha/2}|\zeta-z|^{2n+1-2\mu+\epsilon}} dV(\zeta). \tag{22}$$

We again consider the different cases $\gamma(w) \leq \gamma(z)$ and $\gamma(z) \leq \gamma(w)$ separately. With $\gamma(w) \leq \gamma(z)$, we use coordinates $s, t_1, \ldots, t_{2n-1}$ as before with the volume estimate (20) to estimate (22) as

$$|z-w|^\alpha \int_V \frac{t^{2n-2}}{(s+t^2)^{\mu-1/2+\alpha/2}(s+t)^{2n+1-2\mu+\epsilon+\epsilon'}} ds\, dt$$

$$\lesssim |z-w|^\alpha \int_V \frac{1}{s^{1/2+\alpha/2+\epsilon+\epsilon'+\delta}t^{1-\delta}} ds\, dt$$

$$\lesssim |z-w|^\alpha,$$

where $0 < \delta < 1/2-(\alpha/2+\epsilon+\epsilon')$ and V again denotes a bounded subset of $\mathbb{R}^2$.

In the case $\gamma(z) \leq \gamma(w)$, we write $\gamma(w) \lesssim \gamma(z)+|\zeta-w|$ and estimate (22) as

$$|z-w|^\alpha \int_{\substack{D\cap E_c \\ |\zeta-z|\leq|\zeta-w|}} \frac{\gamma(z)+|\zeta-w|}{|\phi(\zeta,w)|^{\mu-1/2+\alpha/2}|\zeta-z|^{2n+1-2\mu+\epsilon+\epsilon'}} dV(\zeta).$$

The integral involving $\gamma(z)$ is handled exactly as before, so we estimate

$$|z-w|^\alpha \int_{\substack{D\cap E_c \\ |\zeta-z|\leq|\zeta-w|}} \frac{|\zeta-w|}{|\phi(\zeta,w)|^{\mu-1/2+\alpha/2}|\zeta-z|^{2n+1-2\mu+\epsilon+\epsilon'}} dV(\zeta)$$

$$\lesssim |z-w|^\alpha \int_{\substack{D\cap E_c \\ |\zeta-z|\leq|\zeta-w|}} \frac{1}{|\phi(\zeta,w)|^{\mu-1+\alpha/2}|\zeta-z|^{2n+1-2\mu+\epsilon+\epsilon'}} dV(\zeta).$$

The case of $\mu = 1$ is trivial, so we assume $\mu \geq 2$ and use the coordinates $s, t_1, \ldots, t_{2n-1}$ to estimate

$$|z - w|^\alpha \int_V \frac{t^{2n-2}}{(s + t^2)^{\mu-1+\alpha/2}(s + t)^{2n+2-2\mu+\epsilon+\epsilon'}} \, ds \, dt$$

$$\lesssim |z - w|^\alpha \int_V \frac{1}{s^{3/4+\alpha/2+\epsilon+\epsilon'}t^{1/2}} \, ds \, dt$$

$$\lesssim |z - w|^\alpha.$$

Case B: A is of double type $(1, 2)$. Following the previous arguments, we see that we need to estimate

$$\int_D \frac{1}{\gamma(\zeta)^{1+\epsilon}} \left| \frac{\gamma(z)^{2-\epsilon'}(\phi(\zeta,w))^{\mu+1} - \gamma(w)^{2-\epsilon'}(\phi(\zeta,z))^{\mu+1}}{(\phi(\zeta,w))^{\mu+1}(\phi(\zeta,z))^{\mu+1}P(\zeta,z)^{n-1-\mu}} \right| dV(\zeta)$$

$$+ \int_D \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \left| \frac{P(\zeta,z)^{n-1-\mu} - P(\zeta,w)^{n-1-\mu}}{(\phi(\zeta,w))^{\mu+1}P(\zeta,z)^{n-1-\mu}P(\zeta,w)^{n-1-\mu}} \right| dV(\zeta)$$

$$= III + IV.$$

Following the calculations for integral I in Case A, we estimate III by the integrals

$$\sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{2-\epsilon'}}{\gamma(\zeta)^\epsilon} \frac{|z - w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta)$$

$$+ \sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-3-2\mu}} \, dV(\zeta)$$

$$+ \int_D \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{|\gamma(z)^{2-\epsilon'} - \gamma(w)^{2-\epsilon'}|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta)$$

$$= III_a + III_b + III_c.$$

Estimates for the integral III_a are given by I_b in Case A.

For the integrals of III_b, we consider separately the regions $E_c(z)$ and its complement. We again consider only the case $|\zeta - z| \leq |\zeta - w|$.

In the region $D \cap E_c(z)$, we use a coordinate system in which $s = -r(\zeta)$ is a coordinate, and we use the estimate on the volume element in $E_c(z)$ given by (20). We can also assume that c is sufficiently small to guarantee that $|\zeta - z| \lesssim \gamma(\zeta)$ in E_c.

The integrals

$$\int_{\substack{D \cap E_c \\ |\zeta-z|\leq|\zeta-w|}} \frac{\gamma(z)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-3-2\mu}} \, dV(\zeta)$$

can thus be bounded by

$$\frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z)$$

$$\int_V \frac{|\zeta-w|^{1/2-\alpha}}{(s+|\zeta-z|^2)^{\mu+1/2-l}(s+|\zeta-w|^2)^{l+1}|\zeta-z|^{2n-2-2\mu+\epsilon+\epsilon'}}t^{2n-2}\,ds\,dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z)\int_V \frac{t^{2n-2}}{(s+|\zeta-z|^2)^{\mu+5/4+\alpha/2}|\zeta-z|^{2n-2-2\mu+\epsilon+\epsilon'}}\,ds\,dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z)\int_V \frac{t^{2\mu-\epsilon-\epsilon'}}{(s+t^2)^{\mu+5/4+\alpha/2}}\,ds\,dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z)\int_V \frac{1}{s^{7/8}t^{3/4+\alpha+\epsilon+\epsilon'}}\,ds\,dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z),$$

where V is a bounded subset of $\mathbb{R}^2$.

We now estimate

$$\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}}\frac{\gamma(z)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}}\frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-3-2\mu}}\,dV(\zeta)$$

$$\lesssim |z-w|^\alpha\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}}\frac{1}{\gamma(\zeta)^{1+\epsilon}}\frac{1}{|\phi(\zeta,w)|^{l+1/2+\alpha/2}|\zeta-z|^{2n-3-2l+\epsilon'}}\,dV(\zeta).$$

We use coordinates $u_{j_1},\ldots,u_{j_m},v_{j_{m+1}},\ldots,v_{j_{2n}}$ as in (13) and the neighborhoods $U_{2\varepsilon}(p_j)$ defined previously. We break the problem into subcases depending on whether $z\in U_\varepsilon$.

Subcase a: $z\in U_\varepsilon(p_j)$. As we did before, define $w_1,\ldots,w_{2n}$ by

$$w_\alpha = \begin{cases} u_{j_\alpha} & \text{for } 1\le\alpha\le m,\\ v_{j_\alpha} & \text{for } m+1\le\alpha\le 2n,\end{cases}$$

and let $x_1,\ldots,x_{2n}$ be defined by $\zeta_\alpha = x_\alpha + ix_{n+\alpha}$. Recall that we have $|w(\zeta)|\lesssim |\zeta-z|$ and $|w(\zeta)|\lesssim\gamma(\zeta)$. Thus we estimate, using the coordinates just defined,

$$|z-w|^\alpha\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}}\frac{1}{\gamma(\zeta)^{1+\epsilon}}\frac{1}{|\phi(\zeta,w)|^{l+1/2+\alpha/2}|\zeta-z|^{2n-3-2l+\epsilon'}}\,dV(\zeta)$$

$$\lesssim |z-w|^\alpha\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}}\frac{1}{\gamma(\zeta)^{1+\epsilon}}\frac{1}{|\zeta-z|^{2n-2+\alpha+\epsilon'}}\,dV(\zeta)$$

$$\lesssim |z-w|^\alpha\int_V \frac{u^{m-1}v^{2n-m-1}}{(u+v)^{2n-1+\alpha+\epsilon+\epsilon'}}\,du\,dv$$

$$\lesssim |z-w|^\alpha\int_V \frac{1}{u^{1/2}v^{1/2+\alpha+\epsilon+\epsilon'}}\,du\,dv$$

$$\lesssim |z-w|^\alpha, \tag{23}$$

where we use $u = \sqrt{u_{j_1}^2 + \cdots + u_{j_m}^2}$ and $v = \sqrt{v_{j_{m+1}}^2 + \cdots + v_{j_{2n}}^2}$ and where V is a bounded set.

Subcase b: $z \notin U_\varepsilon$. We have $|\zeta - z| \gtrsim \gamma(z)$, but $\gamma(z)$ is bounded from below since $z \notin U_\varepsilon$. We therefore have to estimate

$$\int_D \frac{1}{\gamma(\zeta)^{1+\epsilon}} \, dV(\zeta),$$

which is easily done by working with the coordinates $w_1, \ldots, w_{2n}$.

We now estimate integral III_c. We use

$$|\gamma(z)^{2-\epsilon'} - \gamma(w)^{2-\epsilon'}| \lesssim |z - w|(\gamma(z)^{1-\epsilon'} + \gamma(w)^{1-\epsilon'})$$

to write

$$III_c \lesssim \int_{\substack{D \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)^{1-\epsilon'} + \gamma(w)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta).$$

We first assume $\gamma(w) \le \gamma(z)$. Then we estimate

$$\int_{\substack{D \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta) \qquad (24)$$

by breaking the integral into the regions E_c and $D \setminus E_c$. In E_c, again assuming c is sufficiently small so that $|\zeta - z| \lesssim \gamma(\zeta)$, we see that (24) is bounded by

$$\int_{\substack{D \\ |\zeta-z|\le|\zeta-w|}} \gamma(z)^{1-\epsilon'} \frac{|z - w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu+\epsilon}} \, dV(\zeta),$$

which we showed to be bounded by $|z - w|^\alpha$ in (17). In the region $D \setminus E_c$, we estimate

$$\int_{\substack{D \setminus E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta)$$

$$\lesssim |z - w|^\alpha \int_{\substack{D \setminus E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\phi(\zeta,w)|^{\mu+1/2+\alpha/2}|\zeta - z|^{2n-3-2\mu+\epsilon'}} \, dV(\zeta)$$

$$\lesssim |z - w|^\alpha,$$

where the last line follows from (23).

We therefore now consider the case $\gamma(z) \le \gamma(w)$ so that

$$III_c \lesssim \int_{\substack{D \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta).$$

In the region E_c, we estimate

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta)$$

$$\lesssim \int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(w) \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu+\epsilon+\epsilon'}} \, dV(\zeta)$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-1-2\mu+\epsilon+\epsilon'}} \, dV(\zeta).$$

Using the coordinate system $s = -r(\zeta)$, $t_1 \ldots, t_{2n-2}$ with volume estimate (20), as before we can estimate

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{t^{2n-2}}{(s+t^2)^{\mu+1/4+\alpha/2}t^{2n-2\mu+\epsilon+\epsilon'}} \, ds \, dt \lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

by (21).

In the region $D \setminus E_c$, we use $\gamma(w) \lesssim |\zeta - w|$ to estimate

$$\int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta)$$

$$\lesssim \int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{|\zeta - w|^{1-\epsilon'}|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta)$$

$$\lesssim |z-w|^{\alpha} \int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\phi(\zeta,w)|^{\mu+\epsilon'/2+\alpha/2}|\zeta - z|^{2n-2-2\mu}} \, dV(\zeta)$$

$$\lesssim |z-w|^{\alpha} \int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\zeta - z|^{2n-2+\epsilon'+\alpha}} \, dV(\zeta)$$

$$\lesssim |z-w|^{\alpha} \int_V \frac{u^{m-1}v^{2n-1-m}}{(u+v)^{2n-1+\epsilon+\epsilon'+\alpha}} \, du \, dv$$

$$\lesssim |z-w|^{\alpha}, \tag{25}$$

where the coordinates u and v are defined as in (23) and where the last line follows from (23). We have finished estimating integral *III* and now turn to *IV*.

As in Case A for integral *II*, we estimate *IV* by the integrals

$$\int_{\substack{D \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu}} \, dV(\zeta)$$

$$+ \int_{\substack{D \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{2+\epsilon}} \frac{|r(\zeta)||z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2\mu}} \, dV(\zeta)$$

$$= IV_a + IV_b.$$

To estimate IV_a, we break the region of integration into E_c and $D \setminus E_c$. In the region $D \setminus E_c$ we use $\gamma(w) \lesssim |\zeta - w|$ and estimate

$$\int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta, w)|^{\mu + \epsilon'/2}|\zeta - z|^{2n-1-2\mu}} \, dV(\zeta)$$

$$\lesssim |z - w|^\alpha \int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\phi(\zeta, w)|^{\mu + \epsilon'/2 - 1/2 + \alpha/2}|\zeta - z|^{2n-1-2\mu}} \, dV(\zeta)$$

$$\lesssim |z - w|^\alpha \int_{\substack{D \setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\zeta - z|^{2n-2-2\mu+\epsilon'+\alpha}} \, dV(\zeta)$$

$$\lesssim |z - w|^\alpha,$$

where the last line follows from (25).

In the region E_c, we consider the different cases $\gamma(w) \le \gamma(z)$ and $\gamma(z) \le \gamma(w)$ separately. In the case $\gamma(w) \le \gamma(z)$ we write

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta, w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu}} \, dV(\zeta)$$

$$\lesssim \gamma(w) \int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(z)}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta, w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu+\epsilon'}} \, dV(\zeta)$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(z) \frac{1}{|\phi(\zeta, w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} \, dV(\zeta).$$

We choose a coordinate system in which $s = -r(\zeta)$, and we use the estimate on the volume element given by (20) to reduce the estimate to

$$\frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{t^{2\mu-2-\epsilon-\epsilon'}}{(s + t^2)^{\mu+1/4+\alpha/2}} \, ds \, dt \lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w),$$

which follows from (21).

In the case $\gamma(z) \le \gamma(w)$, we have

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z - w|}{|\phi(\zeta, w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu}} \, dV(\zeta)$$

$$\lesssim \frac{|z - w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

$$\int_{\substack{D \cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(w) \frac{1}{|\phi(\zeta, w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} \, dV(\zeta).$$

We then write $\gamma(w) \lesssim \gamma(z) + |\zeta - w|$. We bound

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_{\substack{D\cap E_c\\|\zeta-z|\le|\zeta-w|}}\gamma(z)\frac{1}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta-z|^{2n-2\mu+\epsilon+\epsilon'}}\,dV(\zeta)$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)$$

by (18) and then consider

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_{\substack{D\cap E_c\\|\zeta-z|\le|\zeta-w|}}\frac{1}{|\phi(\zeta,w)|^{\mu-1/4+\alpha/2}|\zeta-z|^{2n-2\mu+\epsilon+\epsilon'}}\,dV(\zeta).\tag{26}$$

The case $\mu=1$ is trivial, so we assume $\mu\ge 2$. Here we use coordinates $s=-r(\zeta),t_1,\dots,t_{2n-1}$ and bound (26) by

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V \frac{t^{2\mu-3-\epsilon-\epsilon'}}{(s+t^2)^{\mu-1/4+\alpha/2}}\,ds\,dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V \frac{1}{s^{7/8}t^{3/4+\alpha+\epsilon+\epsilon'}}\,ds\,dt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w).$$

To estimate IV_b, we use

$$\frac{|r(\zeta)|}{\gamma(\zeta)^2}\lesssim 1,$$

which follows by working in the coordinates of (13) near a critical point; thus we have

$$IV_b\lesssim \int_{\substack{D\\|\zeta-z|\le|\zeta-w|}}\frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^\epsilon}\frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2\mu}}\,dV(\zeta).\tag{27}$$

We break the regions of integration in (27) into E_c and $D\setminus E_c$. The estimates for IV_b in the region E_c are handled in the same manner as for IV_a. In the region $D\setminus E_c$, we use $\gamma(w)\lesssim|\zeta-w|$ to bound (27) by

$$\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}}\frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^\epsilon}\frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2\mu}}\,dV(\zeta)$$

$$\lesssim |z-w|^\alpha\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}}\frac{1}{\gamma(\zeta)^\epsilon}\frac{1}{|\phi(\zeta,w)|^{\mu-1/2+\epsilon'/2+\alpha/2}|\zeta-z|^{2n-2\mu}}\,dV(\zeta)$$

$$\lesssim |z-w|^\alpha\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}}\frac{1}{\gamma(\zeta)^\epsilon}\frac{1}{|\zeta-z|^{2n-1+\epsilon'+\alpha}}\,dV(\zeta)$$

$$\lesssim |z-w|^\alpha.$$

This completes the proof of part (i) of Theorem 3.10.

(ii) For T^z a smooth, first-order tangential differential operator on D with respect to the z variable, we have

$$T^z r = 0,$$

$$T^z r^* = \mathcal{E}_{0,0} r,$$

$$T^z P = \mathcal{E}_{1,0} + \mathcal{E}_{0,0}^* \frac{r}{\gamma} \frac{r^*}{(\gamma^*)^2}$$

$$= \mathcal{E}_{1,0} + \frac{\mathcal{E}_{0,0}^*}{\gamma^*}(P + \mathcal{E}_{2,0}),$$

$$T^z \phi = \mathcal{E}_{0,1} + \mathcal{E}_{1,0}.$$

We consider first the case in which the kernel of A is of double type $(1,3)$ and of the form $A_{(3)}(\zeta, z)$, where the subscript (3) refers to the smooth type. Thus we write

$$\gamma^* T^z A_{(3)} = \gamma^* A_{(1)} \gamma + \gamma^* A_{(2)} + A_{(3)} \tag{28}$$

and estimate integrals involving the various forms that the integral kernels of different types assume.

We insert (28) into

$$\gamma^* T A_{(3)} f = \int_D f(\zeta) \gamma^* T^z A_{(3)}(\zeta, z) \, dV(\zeta),$$

and we change the factors of γ^* through the equality $\gamma(z) = \gamma(\zeta) + \mathcal{E}_{1,0}$. Part (ii) will then follow in this case by the estimates

$$\int_D \frac{\gamma^{\epsilon'}(z)}{\gamma^\epsilon(\zeta)} |A_{(1)}(\zeta, z)| \, dV(\zeta) \lesssim \frac{1}{|r(z)|^\delta},$$

$$\int_D \frac{\gamma^{\epsilon'}(z)}{\gamma^{1+\epsilon}(\zeta)} |A_{(2)}(\zeta, z)| \, dV(\zeta) \lesssim \frac{1}{|r(z)|^\delta},$$

$$\int_D \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} |A_{(3)}(\zeta, z)| \, dV(\zeta) \lesssim \frac{1}{|r(z)|^\delta}. \tag{29}$$

We will prove the case of (29) in which $A_{(3)}$ satisfies

$$|A_{(3)}| \lesssim \frac{1}{P^{n-3/2-\mu} |\phi|^{\mu+1}}, \quad \mu \geq 1;$$

the other cases are handled similarly.

Using the notation from part (i), we choose coordinates $u_{j_1}, \ldots, u_{j_m}, v_{j_{m+1}}, \ldots, v_{j_{2n}}$ such that

$$-r(\zeta) = u_{j_1}^2 + \cdots + u_{j_m}^2 - v_{j_{m+1}}^2 - \cdots - v_{j_{2n}}^2$$

and let $U_\varepsilon = \bigcup_{j=1}^k U_\varepsilon(p_j)$. We break the problem into subcases depending on whether $z \in U_\varepsilon$.

Subcase a: $z \in U_\varepsilon(p_j)$. We estimate

$$\int_{U_{2\varepsilon}(p_j)} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1} P^{n-3/2-\mu}} \, dV(\zeta) \tag{30}$$

and

$$\int_{D_\epsilon \setminus U_{2\epsilon}} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1} P^{n-3/2-\mu}} \, dV(\zeta). \tag{31}$$

We break up the integral in (30) into integrals over $E_c(z)$ and its complement, where c is as in Lemma 3.6. We also choose $c < 1$ so that we also have the estimate $|\zeta - z| \lesssim \gamma(\zeta)$. We set $\theta = -r(z)$.

In the case $U_{2\epsilon}(p_j) \cap E_c(z)$, we use a coordinate system $s = -r(\zeta), t_1, \ldots, t_{2n-1}$ and estimate

$$\int_{U_{2\epsilon}(p_j) \cap E_c(z)} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1} P^{n-3/2-\mu}} \, dV(\zeta)$$

$$\lesssim \int_V \frac{t^{2n-2}}{\gamma^{1-\epsilon'}(z)(\theta + s + t^2)^{\mu+1}(s+t)^{2n-1-2\mu+\epsilon}} \, ds \, dt$$

$$\lesssim \int_V \frac{t^{2\mu-2+\epsilon'-\epsilon}}{(\theta + s + t^2)^{\mu+1}} \, ds \, dt$$

$$\lesssim \frac{1}{\theta^\delta} \int_V \frac{t^{2\mu-2+\epsilon'-\epsilon}}{(s+t^2)^{\mu+1-\delta}} \, ds \, dt$$

$$\lesssim \frac{1}{\theta^\delta} \int_0^M \frac{1}{s^{3/2-\delta}} \, ds \int_0^\infty \frac{\tilde{t}^{2\mu-2+\epsilon'-\epsilon}}{(1+\tilde{t}^2)^{\mu+1-\delta}} \, d\tilde{t}$$

$$\lesssim \frac{1}{\theta^\delta},$$

where $M > 0$ is some constant and we have made the substitution $t = s^{1/2}\tilde{t}$.

We now estimate the integral

$$\int_{U_{2\epsilon}(p_j) \setminus E_c(z)} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1} P^{n-3/2-\mu}} \, dV(\zeta). \tag{32}$$

Defining $u = \sqrt{u_{j_1}^2 + \cdots + u_{j_m}^2}$ and $v = \sqrt{v_{j_{m+1}}^2 + \cdots + v_{j_{2n}}^2}$ and then using the estimates

$$|w(\zeta)| \lesssim |\zeta - z| \quad \text{and} \quad |w(\zeta)| \lesssim \gamma(\zeta),$$

where $w(\zeta)$ is defined as in (14), we can bound the integral in (32) by

$$\int_{U_{2\epsilon}(p_j) \setminus E_c(z)} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1} P^{n-3/2-\mu}} \, dV(\zeta)$$

$$\lesssim \int_V \frac{u^{m-1} v^{2n-m-1}}{(u+v)^{2n-1+\epsilon-\epsilon'}(\theta + u^2 + v^2)} \, du \, dv$$

$$\lesssim \int_V \frac{1}{(u+v)^{1+\epsilon-\epsilon'}(\theta + u^2 + v^2)} \, du \, dv$$

$$\lesssim \frac{1}{\theta^\delta} \int_V \frac{1}{(u+v)^{3-2\delta+\epsilon-\epsilon'}} \, du \, dv$$

$$\lesssim \frac{1}{\theta^\delta}, \tag{33}$$

where V is a bounded region. We have therefore bounded (30), and we turn now to (31).

In $D \setminus U_{2\varepsilon}$ we have that $|\zeta - z|$ and $\gamma(\zeta)$ are bounded from below, so

$$\int_{D \setminus U_{2\varepsilon}} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1} P^{n-3/2-\mu}} \, dV(\zeta) \lesssim 1.$$

This finishes Subcase a.

Subcase b: $z \notin U_\varepsilon$. We divide D into the regions $D \cap E_c(z)$ and $D \setminus E_c(z)$. In $D \cap E_c(z)$ the same coordinates and estimates work here as when we established the estimates for the integral in (33).

In $D \setminus E_c(z)$ we have $|\zeta - z| \gtrsim \gamma(z)$, but $\gamma(z)$ is bounded from below since $z \notin U_\varepsilon$. We therefore have to estimate

$$\int_D \frac{1}{\gamma^{2+\epsilon}(\zeta)} \, dV(\zeta),$$

which is easily done by working with the coordinates $w_1, \ldots, w_{2n}$.

(iii) The proof of Theorem 3.10(iii) follows the same steps as those in the proof of part (ii); we leave the details to the reader. $\qquad\square$

THEOREM 3.11. *Let X be a smooth tangential vector field. Then*

$$\gamma^* X^z E_{1-2n} = -E_{1-2n} \tilde{X}^\zeta \gamma + E_{1-2n}^{(0)} + \sum_{\nu=1}^{l} E_{1-2n}^{(\nu)},$$

where $\tilde{X}$ is the adjoint of X and the $E_{1-2n}^{(\nu)}$ are isotropic operators.

Proof. The proof follows the line of argument used in proving Case 1 of Theorem 3.4 and makes use of $(\gamma X^\zeta + \gamma^* X^z)\mathcal{E}_{1-2n}^i = \mathcal{E}_{1-2n}^i$. $\qquad\square$

THEOREM 3.12. *Let T be a smooth tangential vector field. Set E to be an operator with kernel of the form $\mathcal{E}_{1-2n}^i(\zeta, z)R_1(\zeta)$ or $\mathcal{E}_{2-2n}^i(\zeta, z)$. Then, for any $1 \leq p \leq s \leq \infty$ with $1/s > 1/p - 1/2n$, we have the following properties:*

(i) $E_{1-2n} \colon L^p(D) \to L^s(D)$;
(ii) $E \colon L^{\infty,2+\epsilon,0}(D) \to \Lambda_{\alpha,2-\epsilon'}(D)$ *with* $0 < \epsilon, \epsilon'$ *and* $\alpha + \epsilon + \epsilon' < 1$;
(iii) $\gamma^* TE \colon \Lambda_{\alpha,2+\epsilon}(D) \to L^{\infty,\epsilon',0}(D)$ *with* $\epsilon < \epsilon'$;
(iv) $E \colon L^{\infty,\epsilon,\delta}(D) \to L^{\infty,\epsilon',0}(D)$ *with* $\epsilon < \epsilon'$ *and* $\delta < 1/2 + (\epsilon' - \epsilon)/2$.

Proof. Part (i) is proved in [3]. The proof of (ii) follows that of Theorem 3.10(i).

For (iii), we let $\mathcal{E}(\zeta, z)$ be the kernel of E and calculate

$$(\gamma^*)^{1+\epsilon'} TEf = \int_D f(\zeta)\gamma^* T^z \mathcal{E}(\zeta, z) \, dV(\zeta)$$

$$= \int_D (\gamma^*)^{\epsilon'} \gamma^{2+\epsilon} f(\zeta) \frac{\gamma^* T^z \mathcal{E}(\zeta, z)}{\gamma^{2+\epsilon}} \, dV(\zeta)$$

$$= \int_D (\gamma^*)^{\epsilon'} (\gamma^{2+\epsilon} f(\zeta) - (\gamma^*)^{2+\epsilon} f(z)) \frac{\gamma^* T^z \mathcal{E}(\zeta, z)}{\gamma^{2+\epsilon}} \, dV(\zeta)$$

$$+ (\gamma^*)^{2+\epsilon} f(z) \int_D (\gamma^*)^{\epsilon'} \frac{\gamma^* T^z \mathcal{E}(\zeta, z)}{\gamma^{2+\epsilon}} \, dV(\zeta). \qquad (34)$$

We use Theorem 3.11 in the last integral to bound the last term of (34) by

$$(\gamma^*)^{2+\epsilon} f(z) \int_D \mathcal{E}_{1-2n,0}(\gamma^*)^{\epsilon'} \left(\frac{1}{\gamma^{1+\epsilon}} + \frac{\mathcal{E}_{1,0}}{\gamma^{2+\epsilon}} \right) dV(\zeta) \lesssim (\gamma^*)^{2+\epsilon} |f(z)|$$
$$\lesssim \|f\|_{L^{\infty,2+\epsilon,0}},$$
$$\lesssim \|f\|_{\Lambda_{\alpha,2+\epsilon}},$$

where the first inequality can be proved by breaking the integrals into the regions $U_{2\varepsilon}$ and $D \setminus U_{2\varepsilon}$ and by using, in the region $D \setminus U_{2\varepsilon}$, the same coordinates as in the proof of Theorem 3.10(ii).

For the first integral in (34), we note that if $f \in \Lambda_\alpha$ then $\gamma^{2+\epsilon} f \in \Lambda_\alpha$. We have

$$\int_D (\gamma^*)^{\epsilon'} \left| (\gamma^{2+\epsilon} f(\zeta) - (\gamma^*)^{2+\epsilon} f(z)) \frac{\gamma^* T^z \mathcal{E}(\zeta,z)}{\gamma^{2+\epsilon}} \right| dV(\zeta)$$
$$\lesssim \|\gamma^{2+\epsilon} f\|_{\Lambda_\alpha} \int_D |\zeta - z|^\alpha (\gamma^*)^{\epsilon'} \left| \frac{\gamma^* T^z \mathcal{E}(\zeta,z)}{\gamma^{2+\epsilon}} \right| dV(\zeta)$$
$$\lesssim \|\gamma^{2+\epsilon} f\|_{\Lambda_\alpha}.$$

The proof of part (iv) follows as in the case of Theorem 3.10(iii).   □

4. C^k Estimates

We define Z_1 operators to be those that take the form

$$Z_1 = A_{(1,1)} + E_{1-2n} \circ \gamma,$$

and we write Theorem 2.3 as

$$\gamma^3 f = Z_1 \gamma^2 \bar{\partial} f + Z_1 \gamma^2 \bar{\partial}^* f + Z_1 f. \tag{35}$$

We define Z_j operators to be those operators of the form

$$Z_j = \underbrace{Z_1 \circ \cdots \circ Z_1}_{j \text{ times}}.$$

We establish mapping properties for Z_j operators as follows.

LEMMA 4.1.   *For $0 < \epsilon' < \epsilon$,*

$$\|Z_j f\|_{L^{\infty,\epsilon,0}} \lesssim \|f\|_{L^{\infty,j+\epsilon',0}}. \tag{36}$$

The proof follows arguments similar to those used to prove Theorem 3.10.

LEMMA 4.2.   *Let T be a tangential vector field and $\varepsilon > 0$. For $\epsilon > 0$ sufficiently small, we have*:

(i) $Z_{n+2}: L^2(D) \to L^\infty(D)$;
(ii) $\|\gamma T Z_4 f\|_{C^{1/4-\varepsilon}} \lesssim \|f\|_{L^{\infty,3+\epsilon,0}}.$

Proof. For part (i), apply Corollary 3.1 and Theorem 3.12(i) $n + 2$ times.

For part (ii) we let $\alpha < 1/4$, apply the commutator theorem (Theorem 3.4), and consider the two compositions $Z_1 \circ Z_1 \circ \gamma T A_1 \circ Z_1$ and $Z_1 \circ Z_1 \circ \gamma T E \circ Z_1$. From

Theorems 3.10 and 3.12 we can find $\epsilon_1, \ldots, \epsilon_4$ such that $0 < \epsilon_{j+1} < \epsilon_j$ and such that in the first composition we have

$$\|Z_1 \circ Z_1 \circ \gamma TA_1 \circ Z_1 f\|_{\Lambda_\alpha} \lesssim \|Z_1 \circ \gamma TA_1 \circ Z_1 f\|_{L^{\infty, \epsilon_1, 0}} \lesssim \|\gamma TA_1 \circ Z_1 f\|_{L^{\infty, \epsilon_2, \delta}}$$
$$\lesssim \|Z_1 f\|_{L^{\infty, 2 + \epsilon_3, 0}} \lesssim \|f\|_{L^{\infty, 3 + \epsilon_4, 0}}$$

and, in the second,

$$\|Z_1 \circ Z_1 \circ \gamma TE \circ Z_1 f\|_{\Lambda_\alpha} \lesssim \|Z_1 \circ \gamma TE \circ Z_1 f\|_{L^{\infty, \epsilon_1, 0}} \lesssim \|\gamma TE \circ Z_1 f\|_{L^{\infty, 1 + \epsilon_2, 0}}$$
$$\lesssim \|Z_1 f\|_{\Lambda_{\alpha, 3 + \epsilon_3}} \lesssim \|f\|_{L^{\infty, 3 + \epsilon_4, 0}}.$$

The second and third inequalities are proved in the same way as in parts (ii) and (iii) of Theorem 3.12. $\qquad \square$

We now iterate (35) to get

$$\gamma^{3j} f = (Z_1 \gamma^{3(j-1)+2} + Z_2 \gamma^{3(j-2)+2} + \cdots + Z_j \gamma^2) \bar{\partial} f$$
$$+ (Z_1 \gamma^{3(j-1)+2} + Z_2 \gamma^{3(j-2)+2} + \cdots + Z_j \gamma^2) \bar{\partial}^* f + Z_j f. \quad (37)$$

Then we can prove the following statement.

THEOREM 4.3. *For $f \in L^2_{0,q}(D) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, $q \geq 1$, and $\varepsilon > 0$,*

$$\|\gamma^{3(n+3)} f\|_{C^{1/4 - \varepsilon}} \lesssim \|\gamma^2 \bar{\partial} f\|_\infty + \|\gamma^2 \bar{\partial}^* f\|_\infty + \|f\|_2.$$

Proof. Use Theorems 3.10(i) and 3.12(ii) and Lemma 4.2(i) in (37) with $j = n + 3$. $\qquad \square$

We use D^k to denote a kth-order differential operator, which is a sum of terms that are composites of k vector fields.
We define

$$Q_k(f) = \sum_{j=0}^{k} \|\gamma^{j+2} D^j \bar{\partial} f\|_\infty + \sum_{j=0}^{k} \|\gamma^{j+2} D^j \bar{\partial}^* f\|_\infty + \|f\|_2.$$

We shall use T^k to denote a kth-order tangential differential operator, which is a sum of terms that are composites of k tangential vector fields.

LEMMA 4.4. *Let T^k be a tangential operator of order k. For $\varepsilon, \epsilon > 0$,*

$$\|\gamma^{3(n+6) + 8k + \epsilon} T^k f\|_{C^{1/4 - \varepsilon}} \lesssim Q_k(f).$$

Proof. We first prove

$$\|\gamma^{3(n+2) + 9 + 8k + \epsilon} T^k f\|_{L^\infty} \lesssim Q_k(f). \quad (38)$$

The proof is by induction in which the first step is proved as was Theorem 4.3. We choose $j = 3$ in (37) and then apply (37) to $\gamma^{3(n+2) + 7k} f$ to get

$$\gamma^{3(n+2) + 9 + 7k} f = Z_1 \gamma^2 \bar{\partial} f + Z_1 \gamma^2 \bar{\partial}^* f + Z_3 \gamma^{3(n+2) + 7k} f.$$

We then apply $\gamma^\epsilon (\gamma T)^k$, where T is a tangential operator. We use the commutator theorem, Theorem 3.4, to show that

$$\gamma^{3(n+2)+9+8k+\epsilon}T^k f$$

$$= \gamma^\epsilon \sum_{j=0}^{k-1} Z_3 \gamma^{3(n+2)+7k+j} T^j f + \gamma^\epsilon \gamma T Z_3 \gamma^{3(n+2)+8k-1} T^{k-1} f$$

$$+ \gamma^\epsilon \sum_{j=0}^{k} Z_1 \gamma^{j+2} T^j \bar{\partial} f + \gamma^\epsilon \sum_{j=0}^{k} Z_1 \gamma^{j+2} T^j \bar{\partial}^* f. \tag{39}$$

By Lemma 4.1 and the induction hypothesis, we conclude that the L^∞ norm of the first term on the right-hand side of (39) is bounded by $Q_{k-1}(f)$.

In the same way that we proved Lemma 4.2 we have

$$\gamma T Z_3 : L^{\infty, 3+\epsilon', 0}(D) \to L^{\infty, \epsilon, 0}(D)$$

for some $0 < \epsilon' < \epsilon$, so the L^∞ norm of the second term is bounded by

$$\|\gamma^{3(n+2)+8k+2+\epsilon'} T^{k-1} f\|_{L^\infty} \lesssim \|\gamma^{3(n+2)+9+8(k-1)} T^{k-1} f\|_{L^\infty} \lesssim Q_{k-1}(f).$$

The last two terms on the right-hand side of (39) are obviously bounded by $Q_k(f)$, and thus we are done with the proof of (38).

To finish the proof of the lemma, we follow the proof of (38) and choose $k = 4$ in (37). We then apply (37) to $\gamma^{3(n+2)+7k} f$ and again apply the operators $\gamma^\epsilon (\gamma T)^k$, where T is a tangential operator. In this way, we show that

$$\gamma^{3(n+2)+12+8k+\epsilon} T^k f$$

$$= \gamma^\epsilon \sum_{j=0}^{k-1} Z_4 \gamma^{3(n+2)+7k+j} T^j f + \gamma^\epsilon \gamma T Z_4 \gamma^{3(n+2)+8k-1} T^{k-1} f$$

$$+ \gamma^\epsilon \sum_{j=0}^{k} Z_1 \gamma^{j+2} T^j \bar{\partial} f + \gamma^\epsilon \sum_{j=0}^{k} Z_1 \gamma^{j+2} T^j \bar{\partial}^* f. \tag{40}$$

By Theorems 3.10(i) and 3.12(ii), for some $\epsilon' > 0$ the first sum on the right-hand side of (40) has its $C^{1/4-\varepsilon}$ norm bounded by

$$\|Z_3 \gamma^{3(n+2)+7k+\epsilon'+j} T^j f\|_{L^\infty} \lesssim Q_{k-1}(f).$$

We can use Lemma 4.2(ii) to show that the $C^{1/4-\varepsilon}$ norm of the second term is bounded by

$$\|\gamma^{3(n+2)+10+8(k-1)+\epsilon'} T^{k-1} f\|_\infty \lesssim Q_{k-1}(f)$$

as before.

The last two terms on the right-hand side of (40) are easily seen to be bounded by $Q_k(f)$, and this finishes the proof of Lemma 4.4. $\qquad \square$

In order to generalize Lemma 4.4 to include nontangential operators, we use the familiar argument of utilizing the ellipticity of $\bar{\partial} \oplus \bar{\partial}^*$ to express a normal derivative of a component of a $(0, q)$-form f in terms of tangential operators acting on components of f and on components of $\bar{\partial} f$ and $\bar{\partial}^* f$. With the $(0, q)$-form f written as

$$f = \sum_{|J|=q} f_J \bar{\omega}^J$$

locally, we have the decomposition in the following form:

$$\gamma N f_J = \sum_{jK} a_{JjK} \gamma T_j f_K + \sum_L b_{JL} f_L$$
$$+ \sum_M c_{JM} \gamma (\bar{\partial} f)_M + \sum_P d_{JP} \gamma (\bar{\partial}^* f)_P, \tag{41}$$

where $N = L_n + \bar{L}_n$ is the normal vector field and $T_1, \ldots, T_{2n-1}$ are the tangential fields described in Section 3. The coefficients a_{JjK}, b_{JL}, c_{JM}, and d_{JP} are all of the form $\mathcal{E}_{0,0}$, and the index sets are strictly ordered with $J, K, L, M, P \subset \{1, \ldots, n\}$, $|J| = |K| = |L| = q$, $|M| = q + 1$, $|P| = q - 1$, and $j = 1, \ldots, 2n - 1$. The decomposition is well known in the smooth case (see [3]), and to verify (41) in a neighborhood of $\gamma = 0$ one may use the coordinates $u_{j_1}, \ldots, u_{j_m}, v_{j_{m+1}}, \ldots, v_{j_{2n}}$ as in (13). For instance, integrating by parts to compute $\bar{\partial}^* f$ leads to terms of the form $\mathcal{E}_{0,-1} f_J$, where multiplication by γ allows us to absorb these terms into b_{JL}.

It is then straightforward to generalize Lemma 4.4. Suppose D^k is a kth-order differential operator that contains the normal field at least once. In $\gamma^k D^k$ we commute γN with terms of the form γT, where T is tangential, and consider the operator $D^k = D^{k-1} \circ \gamma N$, where D^{k-1} is of order $k - 1$. The error terms due to the commutation involve differential operators of order $\leq k - 1$. By (41) we need only consider $D^{k-1} \gamma T f$, $D^{k-1} \bar{\partial} f$, and $D^{k-1} \bar{\partial}^* f$. The last two terms are bounded by $Q_{k-1}(f)$, and we repeat the process with $D^{k-1} \gamma T f$ until we are left with k tangential operators for which we can apply Lemma 4.4.

We thus obtain the weighted C^k estimates described in the following theorem.

THEOREM 4.5.   *Let* $f \in L^2_{0,q}(D) \cap \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$, $q \geq 1$, $\alpha < 1/4$, *and* $\epsilon > 0$. *Then*

$$\|\gamma^{3(n+6)+8k+\epsilon} f\|_{C^{k+\alpha}} \lesssim Q_k(f).$$

As an immediate consequence we obtain weighted C^k estimates for the canonical solution to the $\bar{\partial}$-equation.

COROLLARY 4.6.   *Let* $q \geq 2$ *and let* N_q *denote the* $\bar{\partial}$-*Neumann operator for* $(0, q)$-*forms. Let* f *be a* $\bar{\partial}$-*closed* $(0, q)$-*form. Then, for* $\alpha < 1/4$ *and* $\epsilon > 0$, *the canonical solution* $u = \bar{\partial}^* N_q f$ *to* $\bar{\partial} u = f$ *satisfies*

$$\|\gamma^{3(n+6)+8k+\epsilon} u\|_{C^{k+\alpha}} \lesssim \|\gamma^{k+2} f\|_{C^k} + \|f\|_2.$$

Using more efficient definitions and notation developed by Lieb and the author, one can show that the left-hand side of the relation may be replaced with $\gamma^2 f$. This would imply an improvement for the estimates in Theorem 4.5 as well, with the $3(n + 6)$ term being replaced by $2(n + 6)$.

References

[1] D. Ehsani, *Integral representations on non-smooth domains,* Illinois J. Math. 53 (2010), 1127–1156.
[2] G. Henkin and J. Leiterer, *Theory of functions on complex manifolds,* Monogr. Math., 79, Birkhäuser, Basel, 1984.

[3] I. Lieb and J. Michel, *The Cauchy–Riemann complex,* Aspects Math., E34, Vieweg, Braunschweig, 2002.

[4] I. Lieb and R. Range, *Estimates for a class of integral operators and applications to the $\bar{\partial}$-Neumann problem,* Invent. Math. 85 (1986), 415–438.

[5] ———, *Integral representations and estimates in the theory of the $\bar{\partial}$-Neumann problem,* Ann. of Math. (2) 123 (1986), 265–301.

[6] G. Schmalz, *Solution of the $\bar{\partial}$-equation with uniform estimates on strictly q-convex domains with non-smooth boundary,* Math. Z. 202 (1989), 409–430.

Humboldt-Universität
Institut für Mathematik
10099 Berlin
Germany

dehsani.math@gmail.com