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On the Hypersurface of Lüroth Quartics

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Introduction

In his celebrated paper [18], Lüroth proved that a nonsingular quartic plane curve containing the ten vertices of a complete pentalateral contains infinitely many such 10-tuples. This implies that such curves, called *Lüroth quartics,* fill an open set of an irreducible, SL(3)-invariant hypersurface $\mathcal{L} \subset \mathbb{P}^{14}$. In his short paper [19], Morley computed the degree of the Lüroth hypersurface $\mathcal{L}$ by introducing some interesting ideas that seem to have been forgotten, maybe because a few arguments are somehow obscure. In this paper we put Morley's result and method on a solid foundation by reconstructing his proof as faithfully as possible. The main result is the following.

Theorem 0.1. *The Lüroth hypersurface $\mathcal{L} \subset \mathbb{P}^{14}$ has degree 54.*

Morley's proof uses the description of plane quartics as branch curves of the degree-2 rational self-maps of $\mathbb{P}^2$ called *Geiser involutions.* Every such involution is determined by the linear system of cubics having as base locus a 7-tuple of distinct points $Z = \{P_1, \dots, P_7\}$; let's denote by $B(Z) \subset \mathbb{P}^2$ the corresponding quartic branch curve. Morley introduces a closed condition on the space of such 7-tuples given by the vanishing of the Pfaffian of a natural skew-symmetric bilinear form between conics associated to each such Z. By this procedure one obtains an irreducible polynomial $\Psi(P_1, \dots, P_7)$ that is multihomogeneous of degree 3 in the coordinates of the points $P_1, \dots, P_7$ and skew-symmetric with respect to their permutations. We call Ψ *the Morley invariant*. The symbolic expression of Ψ is related to $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$, classically known as *the Fano plane* (see Section 4).

Then Morley proceeds to prove that the nonsingular quartics $B(Z)$ corresponding to the 7-tuples Z for which the Morley invariant vanishes are precisely the Lüroth quartics. This step of the proof uses a result of Bateman [2], which gives an explicit description of an irreducible 13-dimensional family of configurations Z such that $B(Z)$ is Lüroth: Morley shows that the Bateman configurations are precisely those making Ψ vanish. In order to gain control on the degree of $\mathcal{L}$, one must consider the full locus of configurations Z such that $B(Z)$ is a Lüroth

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quartic that contains the locus of Bateman configurations as a component. This can be realized as follows. Fix six general points $P_1, \ldots, P_6 \in \mathbb{P}^2$; the condition $\Psi(P_1, \ldots, P_6, P_7) = 0$ on the point P_7 defines a plane cubic $E_{P_1,\ldots,P_6}$ containing $P_1, \ldots, P_6$, thus corresponding to a plane section $S \cap \Xi$ of the cubic surface $S \subset \mathbb{P}^3$ associated to the linear system of plane cubics through $P_1, \ldots, P_6$. The plane Ξ can be described explicitly by means of the invariants introduced by Coble and associated to the Cremona hexahedral equations of S; we call Ξ the *Cremona plane*. By construction, the branch curve of the projection of S to $\mathbb{P}^2$ from a point of $S \cap \Xi$ is Lüroth. Conversely, given a general cubic surface $S \subset \mathbb{P}^3$ we obtain as many such plane sections as the number of double-sixes on S (i.e., 36). The final part, which relates the numbers 36 and 54, was implicitly considered by Morley to be well known. We have supplied a proof that uses vector bundle techniques (see Theorem 8.1).

In order to put our work in perspective, it is worth recalling here some recent work related to Lüroth quartics. Let $M(0,4)$ be the moduli space of stable rank-2 vector bundles on $\mathbf{P}^2$ with $(c_1, c_2) = (0,4)$. Let $E \in M(0,4)$ and let

$$J(E) = \{l \in (\mathbf{P}^2)^\vee \mid E_l \neq \mathcal{O}_l^2\}$$

be its curve of jumping lines. Barth proved in [1] the remarkable facts that $J(E)$ is a Lüroth quartic and that $\dim[M(0,4)] = 13$. The Barth map, in this case, is the morphism

$$b \colon M(0,4) \to \mathbb{P}^{14}, \quad E \mapsto J(E)$$

It is well known that b is generically finite and moreover that

$$\deg(b) \cdot \deg[\mathrm{Im}(b)] = 54. \tag{1}$$

Indeed, the value 54 corresponds to the Donaldson invariant q_{13} of $\mathbf{P}^2$, and it has been computed by Li and Qin in [17, Thm. 6.29] and independently by Le Potier, Tikhomirov, and Tyurin; see [13] and the references therein. Another proof, related to secant varieties, is in [20, Thm. 8.8]. Thanks to the result of Barth mentioned previously, the (closure of the) image of b can be identified with the Lüroth hypersurface $\mathcal{L}$, and Theorem 0.1, originally due to Morley, implies that $\deg[\mathrm{Im}(b)] = 54$. The obvious corollary is that $\deg(b) = 1$; that is, *the Barth map b is generically injective*. This last result was obtained by Le Potier and Tikhomirov in [16] with a subtle and technical degeneration argument. It also implies Theorem 0.1 via the identity (1). Our approach, which closely follows [19], is more elementary and direct. Le Potier and Tikhomirov also proved the injectivity of the Barth map for all higher values of c_2, treating the case $c_2 = 4$ as the starting point of their inductive argument.

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1. Apolarity

We will work over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let V be a $\mathbf{k}$-vector space of dimension 3 and denote by $V^\vee$ its dual. The canonical bilinear form $V \times V^\vee \to \mathbf{k}$ extends to a natural pairing:

$$S^d V \times S^n V^\vee \to S^{n-d} V^\vee, \quad (\Phi, F) \mapsto P_\Phi(F) \tag{2}$$

for each $n \geq d$, which is called *apolarity.* Φ and F will be called *apolar* if $P_\Phi(F) = 0$.

After choosing a basis of V we can identify the symmetric algebra $\mathrm{Sym}(V^\vee)$ with the polynomial algebra $\mathbf{k}[X_0, X_1, X_2]$ and $\mathrm{Sym}(V)$ with $\mathbf{k}[\partial_0, \partial_1, \partial_2]$, where $\partial_i := \frac{\partial}{\partial X_i}$, $i = 0, 1, 2$, are the *dual indeterminates.* With this notation, apolarity is the natural pairing between differential operators and polynomials. We can also identify $\mathbb{P}(V) = \mathbb{P}^2$ and $\mathbb{P}(V^\vee) = \mathbb{P}^{2\vee}$.

Elements of $S^d V$, up to a nonzero factor, are called *line curves* of degree d (line conics, line cubics, etc.) while elements of $S^d V^\vee$, up to a nonzero factor, are *point curves* of degree d (point conics, point cubics, etc.). We will be mostly interested in the case of degree $d = 2$. In this case, in coordinates, apolarity takes the form

$$P_\Phi\Big(\sum_{ij} \alpha_{ij} X_i X_j\Big) = \sum_{ij} a_{ij} \alpha_{ij}$$

if $\Phi = \sum_{ij} a_{ij} \partial_i \partial_j$. Suppose we are given a point conic defined by the polynomial

$$\theta = \sum_{ij} A_{ij} X_i X_j \in S^2 V^\vee. \tag{3}$$

Assume that θ is *nonsingular* (i.e., that its coefficient matrix (A_{ij}) is invertible) and consider its *dual curve* $\theta^* = \sum_{ij} a_{ij} \partial_i \partial_j$. We will say that *a point conic* $C = \sum_{ij} \alpha_{ij} X_i X_j \in S^2 V^\vee$ is *conjugate* to θ if it is apolar to θ^*. This gives a notion of conjugation between point conics and, dually, between line conics. Note that if C is conjugate to θ then it is not necessarily true that θ is conjugate to C; that is, this notion is not symmetric. In particular, we did not require C to be nonsingular in the definition.

Another important special case of (2) is the following. Given a point $\xi = (\xi_0, \xi_1, \xi_2) \in \mathbb{P}^2$, the corresponding linear form $\xi_0 \partial_0 + \xi_1 \partial_1 + \xi_2 \partial_2 \in (V^\vee)^\vee = V$ will be also denoted by Δ_ξ and called the *polarization operator* with pole ξ. For each $d \geq 2$ it defines a linear map

$$\Delta_\xi : S^d V^\vee \to S^{d-1} V^\vee,$$
$$\Delta_\xi F(X) = \xi_0 \partial_0 F(X) + \xi_1 \partial_1 F(X) + \xi_2 \partial_2 F(X)$$

associating to a homogeneous polynomial F of degree d a homogeneous polynomial of degree $d - 1$ called *the* (*first*) *polar* of ξ with respect to F. Higher polars are defined similarly by iteration.

Consider the case $d = 2$. Given a nonsingular point conic θ, polarity associates to each point $\xi \in \mathbb{P}^2$, the *pole,* its polar line $\Delta_\xi\theta$, and this gives an isomorphism $\mathbb{P}^2 \cong \mathbb{P}^{2\vee}$. Two points will be called *conjugate with respect to* θ if each of them belongs to the polar line of the other. Two lines are called *conjugate with respect to* θ if each of them contains the pole of the other. We will need the following elementary properties of apolarity, the proofs of which we leave to the reader.

PROPOSITION 1.1. *Let θ be a nonsingular point conic.*

(i) *A point conic C reducible in two distinct lines $\ell_1\ell_2$ is conjugate to θ if and only if the two lines are conjugate with respect to θ or, equivalently, if and only if ℓ_1 and ℓ_2 are conjugate in the involution on the pencil of lines through the point $\ell_1 \cap \ell_2$ having as fixed points the tangent lines to θ.*
(ii) *A point conic C consisting of a double line is conjugate to θ if and only if the line is tangent to θ.*
(iii) *Every point conic C reducible in the tangent line to θ at a point $\xi \in \theta$ and in any other line through ξ is conjugate to θ.*
(iv) *If a point conic C is conjugate to θ then for each $\xi \in C$ the reducible point conic consisting of the lines joining ξ with $C \cap \Delta_\xi\theta$ is also conjugate to θ.*

Given a nonsingular point conic θ, we will call a point cubic $D \in S^3V^\vee$ *apolar* to θ if $P_{\theta^*}(D) = 0$—that is, if θ^* and D are apolar. Note that, since $P_{\theta^*}(D) \in V^\vee$, the condition of apolarity to θ is equivalent to three linear conditions on point cubics. We will need the following.

PROPOSITION 1.2. *Let θ be a nonsingular point conic and D a point cubic apolar to θ. Then for every point $\xi \in \mathbb{P}^2$ the polar conic $\Delta_\xi D$ is conjugate to θ.*

Proof. From $P_{\theta^*}(D) = 0$ it follows that for any $\xi \in \mathbb{P}^2$ we have

$$0 = P_{\Delta_\xi\theta^*}(D) = P_{\theta^*\Delta_\xi}(D) = P_{\theta^*}(\Delta_\xi D). \qquad \square$$

PROPOSITION 1.3. *Let θ be a nonsingular point conic. Then the following statements hold.*

(i) *For every line L the point cubic θL is not apolar to θ.*
(ii) *For every effective divisor $\sum_{i=1}^6 P_i$ of degree 6 on θ there is a unique point cubic D such that D is apolar to θ and $D \cdot \theta = \sum_{i=1}^6 P_i$. If $\sum_{i=1}^6 P_i$ is general then D is irreducible.*
(iii) *For every effective divisor $\sum_{i=1}^5 P_i$ of degree 5 on θ and for a general point $P_6 \notin \theta$ there is a unique point cubic D containing P_6 such that D is apolar to θ and $D \cdot \theta > \sum_{i=1}^5 P_i$.*

Proof. (i) Let $\xi \in \theta$ but $\xi \notin L$. Then

$$\begin{aligned}\Delta_\xi[P_{\theta^*}(\theta L)] &= P_{\theta^*}(\Delta_\xi(\theta L)) = P_{\theta^*}[\Delta_\xi(\theta)L + \theta\Delta_\xi(L)] \\ &= P_{\theta^*}(\Delta_\xi(\theta)L) + P_{\theta^*}(\theta\Delta_\xi(L)) = 0 + 3\Delta_\xi(L) \neq 0,\end{aligned}$$

where the last equality can be checked in a coordinate system. Therefore

$$P_{\theta^*}(\theta L) \neq 0.$$

(ii) If C is a point cubic such that $C \cdot \theta = \sum_{i=1}^{6} P_i$, then all other point cubics with this property are of the form $D = C - \theta L$ for some line L. Taking $\xi \in \theta$, by the previous computation we obtain

$$\Delta_\xi(P_{\theta^*} D) = \Delta_\xi[P_{\theta^*}(C - \theta L)] = \Delta_\xi P_{\theta^*}(C) - \Delta_\xi(3L).$$

This expression is equal to zero for all $\xi \in \theta$ if and only if $P_{\theta^*}(D) = 0$ if and only if $3L = P_{\theta^*}(C)$. Finally, the 6-dimensional linear system of cubics apolar to θ cannot consist of reducible cubics.

(iii) This follows easily from (ii). □

We refer the reader to [8] for a more detailed treatment of polarity and apolarity. From now on, by a *conic* (resp. a *cubic,* etc.) we will mean a point conic (resp. point cubic, etc.) unless otherwise specified.

2. The Morley Form

Consider seven distinct points $P_1, \dots, P_7 \in \mathbb{P}^2$ and let $Z = \{P_1, \dots, P_7\}$. Let $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaf of Z and let

$$I_Z = \bigoplus_k I_{Z,k} = \bigoplus_k H^0(\mathbb{P}^2, \mathcal{I}_Z(k)) \subset \mathbf{k}[X_0, X_1, X_2]$$

be the homogeneous ideal of Z.

Proposition 2.1. *Assume that Z is not contained in a conic.*

(i) *There is a matrix of homogeneous polynomials*

$$A = \begin{pmatrix} L_0(X) & L_1(X) & L_2(X) \\ \theta_0(X) & \theta_1(X) & \theta_2(X) \end{pmatrix},$$

where $\deg(L_i(X)) = 1$ *and* $\deg(\theta_i(X)) = 2$ *for* $i = 0,1,2$ *such that I_Z is generated by the maximal minors of A.*

(ii) *Six of the seven points $P_1, \dots, P_7$ are on a conic if and only if, for any matrix A as in* (i), *the linear forms $L_0(X), L_1(X), L_2(X)$ are linearly dependent.*

Proof. (i) By the Hilbert–Burch theorem, the homogeneous ideal of any finite set of points in $\mathbb{P}^2$ is generated by the maximal minors of a $t \times (t+1)$ matrix A of homogeneous polynomials of positive degrees for some $t \geq 1$ [12, Thm. 3.2]. Since Z is contained in at least three linearly independent cubics, it must be that $t \leq 2$. Since moreover Z is not a complete intersection of two curves, we must have $t = 2$. The numerical criterion of [4] (see also [12, Cor. 3.10]) shows that the only possibility is the one stated.

(ii) Clearly it suffices to prove the assertion for one matrix A as in (i). Assume that $P_1, \dots, P_6$ are on a conic θ_0 and that $P_7 \notin \theta_0$. Let L_1, L_2 be two distinct lines through P_7. Then $\langle C_0, L_2\theta_0, -L_1\theta_0 \rangle = H^0(\mathcal{I}_Z(3))$ for some cubic C_0, and since $P_7 \in C_0$ there are conics θ_1, θ_2 such that $C_0 = L_1\theta_2 - L_2\theta_1$ so that we can take

$$A = \begin{pmatrix} 0 & L_1(X) & L_2(X) \\ \theta_0(X) & \theta_1(X) & \theta_2(X) \end{pmatrix}, \tag{4}$$

and $L_0 = 0, L_1(X), L_2(X)$ are linearly dependent. Conversely, assume that $L_0(X), L_1(X), L_2(X)$ are linearly dependent for some A as in (i). After multiplying to the right by a suitable element of $\mathrm{SL}(3)$ we may assume that $L_0(X) = 0$—in other words, that A has the form (4). It immediately follows that one of the seven points is $L_1 \cap L_2$ and that the other six are contained in θ_0. $\square$

Unless otherwise specified, *we will always assume that* $Z = \{P_1, \dots, P_7\}$ *consists of distinct points not on a conic.*

Let $\xi = (\xi_0, \xi_1, \xi_2)$ be new indeterminates, and consider the polynomial

$$\begin{aligned} S(\xi, X) :=& \begin{vmatrix} L_0(\xi) & L_1(\xi) & L_2(\xi) \\ L_0(X) & L_1(X) & L_2(X) \\ \theta_0(X) & \theta_1(X) & \theta_2(X) \end{vmatrix} \\ &= L_0(\xi)C_0(X) + L_1(\xi)C_1(X) + L_2(\xi)C_2(X), \end{aligned}$$

where the L_j and the θ_j are the entries of a matrix A as in (i) of Proposition 2.1. Here $S(\xi, X)$ is bihomogeneous of degrees 1 and 3 in ξ and X, respectively.

Given points $P = (x_0, x_1, x_2) \in \mathbb{P}^2$ and $Q = (y_0, y_1, y_2) \in \mathbb{P}^2$, we will denote by

$$|PQX| = \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ X_0 & X_1 & X_2 \end{vmatrix}.$$

If $P \neq Q$ then $|PQX| = 0$ is the line containing P and Q.

LEMMA 2.2. *Let Z and $S(\xi, X)$ be as before. Then*:

(i) *Up to a constant factor, $S(\xi, X)$ depends only on Z and not on the particular choice of the matrix A.*

(ii) *If no six of the points of Z are on a conic then, for every choice of $\xi \in \mathbb{P}^2$, the cubic $S(\xi, X)$ is not identically zero, contains ξ and Z, and is singular at ξ if $\xi \in Z$. All nonzero cubics in $H^0(\mathbb{P}^2, \mathcal{I}_Z(3))$ are obtained as ξ varies in $\mathbb{P}^2$.*

(iii) *If $\{P_1, \dots, P_6\}$ are on a nonsingular conic θ and $P_7 \notin \theta$ then*

$$S(\xi, X) = |P_7 \xi X| \theta.$$

In particular, $S(P_7, X) \equiv 0$ and only the 2-dimensional vector space of reducible cubics in $H^0(\mathcal{I}_Z(3))$ is represented in the form $S(\xi, X)$.

Proof. (i) A different choice of the matrix A can be obtained by multiplying it on the right by some $M \in \mathrm{GL}(3)$, and this has the effect of changing $S(\xi, X)$ into $S(\xi, X)\det(M)$. Also, left action is possible but it does not change $S(\xi, X)$.

(ii) Since $L_0(\xi), L_1(\xi), L_2(\xi)$ are linearly independent (Lemma 2.1), $S(\xi, X)$ cannot be identically zero, and it follows that all of $H^0(\mathbb{P}^2, \mathcal{I}_Z(3))$ is obtained in this way. Clearly $S(\xi, X)$ contains ξ. From the identity

$$0 = \frac{\partial\left[\sum_j L_j(X)C_j(X)\right]}{\partial X_h} = \sum_j \frac{\partial L_j(X)}{\partial X_h} C_j(X) + \sum_j L_j(X) \frac{\partial C_j(X)}{\partial X_h}$$

we deduce

$$\frac{\partial S(\xi, X)}{\partial X_h} = \sum_j L_j(\xi) \frac{\partial C_j(X)}{\partial X_h} = -\sum_j \frac{\partial L_j(\xi)}{\partial \xi_h} C_j(X).$$

The last expression for the partials of $S(\xi, X)$ shows that

$$\frac{\partial S(\xi, X)}{\partial X_h}(\xi) = 0$$

for $h = 0, 1, 2$ if $\xi \in Z$, so that $S(\xi, X) = 0$ is singular at ξ in this case.

(iii) As in the proof of Lemma 2.1, we can choose $L_0 = 0$ and L_1 and L_2 linearly independent and containing P_7 and $\{C_0, L_2(X)\theta, -L_1(X)\theta\}$ as a basis of $H^0(\mathbb{P}^2, \mathcal{I}_Z(3))$. Then

$$S(\xi, X) = [L_1(\xi)L_2(X) - L_2(\xi)L_1(X)]\theta.$$

From this expression, (iii) follows immediately. □

Lemma 2.2 shows that $S(\xi, X)$ is uniquely determined by Z up to a constant factor. More precisely, we have the following.

PROPOSITION 2.3. *The coefficients of $S(\xi, X)$ can be expressed as multihomogeneous polynomials of degree 5 in the coordinates of the points $P_1, \dots, P_7$ that are symmetric with respect to permutations of $P_1, \dots, P_7$.*

Proof. On $\mathbb{P}^2 \times \mathbb{P}^2$ with homogeneous coordinates ξ and X, consider the exact sequence

$$0 \to \mathcal{I}_\Delta(1,3) \to \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,3) \to \mathcal{O}_\Delta(4) \to 0,$$

where $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the diagonal. Since $h^0(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,3)) = 30$ and $h^0(\mathcal{O}_\Delta(4)) = 15$, from the exact sequence we deduce that $h^0(\mathcal{I}_\Delta(1,3)) = 15$ and that $S(\xi, X) \in H^0(\mathcal{I}_\Delta(1,3))$. Given a polynomial

$$P(\xi, X) = \sum_j \xi_j D_j(X) \in H^0(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,3)),$$

the condition that it belongs to $H^0(\mathcal{I}_\Delta(1,3))$ corresponds to the vanishing of the fifteen coefficients of $P(X, X) \in H^0(\mathcal{O}_\Delta(4))$, and these are fifteen linear homogeneous conditions with constant coefficients on the thirty coefficients of $P(\xi, X)$. The condition that $P(\xi, X) = S(\xi, X)$ up to a constant factor is that, moreover,

$$\sum_j \xi_j D_j(P_i) = 0, \quad i = 1, \dots, 7, \tag{5}$$

because this means that the cubic $P(\xi, X) = 0$ contains Z for all $\xi \in \mathbb{P}^2$. For each $i = 1, \dots, 7$, condition (5) means that

$$D_0(P_i) = D_1(P_i) = D_2(P_i) = 0, \tag{6}$$

and these are three linear homogeneous conditions on the thirty coefficients of $P(\xi, X)$ with coefficients that are homogeneous of degree 3 in P_i. Since $P(\xi, X) \in H^0(\mathcal{I}_\Delta(1,3))$, we also have

$$\sum_j x_{ij} D_j(P_i) = 0, \tag{7}$$

where $P_i = (x_{i0}, x_{i1}, x_{i2})$. This condition implies that only two of the three conditions (6) are independent: if (say) $x_{i0} \neq 0$, then we can choose $D_1(P_i) = D_2(P_i) = 0$. Moreover, whenever either one is satisfied, the remaining one is divisible by x_{i0} thanks to the relation (7). Therefore, for each $i = 1, \dots, 7$ we obtain two linear homogeneous conditions on the thirty coefficients of $P(\xi, X)$, with coefficients that are homogeneous of degree 3 and 2 respectively in P_i. Altogether we obtain $29 = 15 + 14$ linear homogeneous conditions on the thirty coefficients of $P(\xi, X)$. The maximal minors of their coefficient matrix are the coefficients of $S(\xi, X)$, and they are multihomogeneous of degree 5 in the P_i by what we have shown. Any transposition of $P_1, \dots, P_7$ permutes two pairs of adjacent rows of the matrix so that the maximal minors remain unchanged. □

DEFINITION 2.4. The *Morley form* of Z is the biquadratic homogeneous polynomial in ξ, X:

$$M(\xi, X) := \Delta_\xi S(\xi, X) = \begin{vmatrix} L_0(\xi) & L_1(\xi) & L_2(\xi) \\ L_0(X) & L_1(X) & L_2(X) \\ \Delta_\xi \theta_0(X) & \Delta_\xi \theta_1(X) & \Delta_\xi \theta_2(X) \end{vmatrix}.$$

For every $\xi \in \mathbb{P}^2$ such that $S(\xi, X)$ is not identically zero, $M(\xi, X)$ represents in $\mathbb{P}^2 = \operatorname{Proj}(\mathbf{k}[X_0, X_1, X_2])$ the polar conic of ξ with respect to the cubic $S(\xi, X)$. Clearly it contains ξ and, if $\xi \in Z$, it is reducible into the principal tangent lines at ξ of $S(\xi, X)$ by Lemma 2.2. Note that, by Lemma 2.2, $S(\xi, X) \equiv 0$ (and consequently $M(\xi, X) \equiv 0$) if and only if six of the seven points of Z are on a conic and ξ is the seventh point. Since $M(\xi, \xi) = 0$, the Morley form is skew-symmetric in ξ, X. Therefore its 6×6 matrix of coefficients (M_{hk}) has a determinant that is the square of its Pfaffian.

COROLLARY 2.5. *The Pfaffian of $M(\xi, X)$ can be expressed as a polynomial $F(P_1, \dots, P_7)$ multihomogeneous of degree 15 in the coordinates of the points $P_1, \dots, P_7$ and symmetric with respect to permutations of the points.*

Proof. By Proposition 2.3, the coefficients M_{hk} are multihomogeneous of degree 5 in the coordinates of the P_i. Therefore the determinant is multihomogeneous of degree 30 in the P_i; thus the Pfaffian has degree 15 in each of them. The symmetry follows from that of the coefficients M_{hk}, which holds by Proposition 2.3. □

The Morley form $M(\xi, X)$ defines a bilinear skew-symmetric form

$$S^2 V^\vee \times S^2 V^\vee \to \mathbf{k}.$$

If $F(P_1, \dots, P_7) = 0$ then this form is degenerate. The 7-tuples $\{P_1, \dots, P_7\}$ of points in $\mathbb{P}^2$ for which this happens are such that, when ξ varies in $\mathbb{P}^2$, all the conics $M(\xi, X)$ are contained in a hyperplane of $\mathbb{P}(S^2 V^\vee)$. The search for such 7-tuples is our next goal.

3. The Morley Invariant

PROPOSITION 3.1. *If $Z = \{P_1, \dots, P_7\}$ consists of distinct points not on a conic, six of which are on a conic, then $F(P_1, \dots, P_7) = 0$.*

Proof. Let θ be the conic containing six of the seven points, say $P_1, \dots, P_6$. From Lemma 2.2(iii) it follows that $S(\xi, X) = \theta|P_7\xi X|$. Therefore, all the conics $M(\xi, X)$, $P_7 \neq \xi \in \mathbb{P}^2$, are contained in the hyperplane $H_{P_7} \subset S^2V^\vee$ of conics that contain P_7. This implies that the skew-symmetric form $M\colon S^2V^\vee \times S^2V^\vee \to \mathbf{k}$ is degenerate; hence its Pfaffian vanishes. $\square$

Given $p_1, \dots, p_6 \in \mathbb{P}^2$, define as in [6, p. 136] (see also [11, p. 191])

$$\mathcal{Q}(p_1, \dots, p_6) = |134||156||235||246| - |135||146||234||256|,$$

where we use the symbolic notation

$$|ijk| := \begin{vmatrix} p_{i0} & p_{i1} & p_{i2} \\ p_{j0} & p_{j1} & p_{j2} \\ p_{k0} & p_{k1} & p_{k2} \end{vmatrix}.$$

Here $\mathcal{Q}(p_1, \dots, p_6)$ is a multihomogeneous polynomial of degree 2 in the coordinates of the points $p_1, \dots, p_6$, skew-symmetric with respect to them and that vanishes if and only if $p_1, \dots, p_6$ are on a conic. Moreover, $\mathcal{Q}(p_1, \dots, p_6)$ is irreducible because any factorization would involve invariants of lower degree for the group $\mathrm{SL}(3) \times \mathrm{Alt}_6$ that do not exist.

PROPOSITION 3.2. *Consider distinct points $P_1, \dots, P_7$. The polynomial*

$$\prod_i \mathcal{Q}(P_1, \dots, \hat{P}_i, \dots, P_7) \tag{8}$$

is multihomogeneous of degree 12 in the coordinates of each point P_i, $i = 1, \dots, 7$, and skew-symmetric with respect to permutations of $P_1, \dots, P_7$. It vanishes precisely on the 7-tuples that contain six points on a conic and divides the Pfaffian polynomial $F(P_1, \dots, P_7)$.

Proof. Since each polynomial $\mathcal{Q}(P_1, \dots, \hat{P}_i, \dots, P_7)$ has degree 2 in the coordinates of each of the six points $P_1, \dots, \hat{P}_i, \dots, P_7$, it follows that the product (8) is multihomogeneous of degree 12 in the coordinates of each point P_i, $i = 1, \dots, 7$. Let $1 \le i < j \le 7$. Then each $\mathcal{Q}(P_1, \dots, \hat{P}_k, \dots, P_7)$, $k \neq i, j$, is skew-symmetric with respect to P_i and P_j. On the other hand,

$$\mathcal{Q}(P_1, \dots, \hat{P}_i, \dots, P_7)\mathcal{Q}(P_1, \dots, \hat{P}_j, \dots, P_7)$$

is symmetric with respect to P_i and P_j because

$$\mathcal{Q}(P_1, \dots, \hat{P}_i, \dots, P_i, \dots, P_7) = (-1)^{i-j+1}\mathcal{Q}(P_1, \dots, \hat{P}_j, \dots, P_7),$$

where on the left side P_j has been replaced by P_i at the jth place. Therefore (8) is skew-symmetric. It is clear that (8) vanishes precisely at those 7-tuples that include six points on a conic. The last assertion follows at once from Proposition 3.1. □

We will denote by $\Psi(P_1,\dots,P_7)$ the polynomial such that

$$F(P_1,\dots,P_7)=\Psi(P_1,\dots,P_7)\prod_i \mathcal{Q}(P_1,\dots,\hat{P}_i,\dots,P_7). \tag{9}$$

We call $\Psi(P_1,\dots,P_7)$ the *Morley invariant* of the seven points $P_1,\dots,P_7$.

COROLLARY 3.3. *The Morley invariant $\Psi(P_1,\dots,P_7)$ is multihomogeneous of degree 3 in the coordinates of the points P_i and skew-symmetric with respect to $P_1,\dots,P_7$.*

Proof. This follows from Corollary 2.5 and from the fact that the polynomial (8) is multihomogeneous of degree 12 and skew-symmetric. □

Let $Z=\{P_1,\dots,P_7\}$ be given consisting of distinct points not on a conic as before. The net of cubic curves $|H^0(\mathcal{I}_Z(3))|$ contains a unique cubic singular at P_i for each $i=1,\dots,7$; we denote by $M^Z_{P_i}$, or simply by M_{P_i} when no confusion is possible, the reducible conic of its principal tangents at P_i.

As remarked after Definition 2.4, if no six of the points of Z are on a conic then $M_{P_i}=M(P_i,X)$ for all $1\le i\le 7$. If instead six of the points, say $P_1,\dots,P_6$, are on a conic θ then $M_{P_i}=M(P_i,X)$ for $i=1,\dots,6$ but M_{P_7} *is not obtained from* $M(\xi,X)$. By Lemma 2.2 we have $S(\xi,X)=\theta|P_7\xi X|$ and therefore, if $\xi\ne P_7$, then $M(\xi,X)$ is reducible in the line $|P_7\xi X|$ and in the polar line of ξ with respect to θ.

PROPOSITION 3.4. *Assume that $Z=\{P_1,\dots,P_7\}$ are such that six of them, say $P_1,\dots,P_6$, are on a nonsingular conic θ. Then the following statements hold.*

(i) *The conics $M(\xi,X)$, as ξ varies in $\mathbb{P}^2\setminus\{P_7\}$, depend only on θ and P_7 and not on the points $P_1,\dots,P_6$. They generate a vector subspace of dimension 4 of $S^2V^\vee$ that is the intersection of the hyperplane H_θ of conics conjugate to θ with the hyperplane H_{P_7} of conics containing P_7. Moreover,*

$$H_\theta\cap H_{P_7}=\langle M_{P_1},\dots,M_{P_6}\rangle \tag{10}$$

for a general choice of $P_1,\dots,P_6\in\theta$.

(ii) *For a general choice of $P_1,\dots,P_6\in\theta$ and $P_7\notin\theta$, the reducible conic M_{P_7} is not conjugate to θ. In particular,*

$$\langle M_{P_1},\dots,M_{P_6},M_{P_7}\rangle=H_{P_7}$$

has dimension 5.

Proof. (i) We can assume that $\theta=X_0^2+2X_1X_2$ and that $P_7=(1,0,0)$. Then $\theta X_1,\theta X_2\in H^0(\mathcal{I}_Z(3))$ and therefore

$$S(\xi,X)=\theta(\xi_1X_2-\xi_2X_1),$$

so that

$$M(\xi, X) = (\Delta_\xi \theta)(\xi_1 X_2 - \xi_2 X_1) = 2(\xi_0 X_0 + \xi_1 X_2 + \xi_2 X_1)(\xi_1 X_2 - \xi_2 X_1).$$

Clearly this expression does not depend on the points $P_1, \dots, P_6$.

The dual of θ is the line conic $\theta^* = \partial_0^2 + 2\partial_1\partial_2$. Therefore the hyperplane H_θ of conics conjugate to θ consists of the conics $C\colon \sum_{ij} \alpha_{ij} X_i X_j$ such that $\alpha_{00} + \alpha_{12} = 0$. The hyperplane H_{P_7} of conics containing P_7 is given by the condition $\alpha_{00} = 0$. Therefore $H_\theta \cap H_{P_7}$ has equations $\alpha_{00} = \alpha_{12} = 0$. Since $M(\xi, X)$ does not contain the terms X_0^2 and $X_1 X_2$, it follows that $M(\xi, X) \in H_\theta \cap H_{P_7}$ for all $\xi \neq P_7$. Now observe that

$$M(\xi, X) = \begin{cases} X_1^2 & \text{if } \xi = (0,0,1), \\ X_2^2 & \text{if } \xi = (0,1,0), \\ X_0 X_1 - X_1^2 & \text{if } \xi = (1,0,1), \\ X_0 X_2 + X_2^2 & \text{if } \xi = (1,1,0), \end{cases}$$

which are linearly independent. From this it follows that the conics $M(\xi, X)$ generate $H_\theta \cap H_{P_7}$.

Equation (10) can be proved by a direct computation as follows. The conics $M(\xi, X)$ corresponding to the points

$$\xi = (0,0,1), (0,1,0), (2,-2,1), (2i,-2i,-1) \in \theta$$

are (respectively)

$$X_1^2,\ X_2^2,\ 2X_0X_1 + 4X_0X_2 + X_1^2 - 4X_2^2,\ -2iX_0X_1 + 4iX_0X_2 + X_1^2 - 4X_2^2,$$

and they are linearly independent.

(ii) Keeping the same notation as before, observe that a reducible conic with double point P_7 and not conjugate to θ is of the form $\alpha_{11}X_1^2 + \alpha_{22}X_2^2 + \alpha_{12}X_1X_2$ for coefficients $\alpha_{11}, \alpha_{22}, \alpha_{12}$ such that $\alpha_{12} \neq 0$. It follows that any cubic of the form $D = X_0X_1X_2 + F(X_1, X_2)$, where $F(X_1, X_2)$ is a general cubic polynomial, is singular at P_7, and has the conic of principal tangents equal to X_1X_2 and therefore not conjugate to θ. Now it suffices to take $\{P_1, \dots, P_6\} = D \cap \theta$ to have a configuration $Z = \{P_1, \dots, P_7\}$ satisfying the desired conditions. □

Corollary 3.5. *The Morley invariant $\Psi(P_1, \dots, P_7)$ is not identically zero.*

Proof. Since $M(\xi, X)$ is skew-symmetric, for a given $Z = \{P_1, \dots, P_7\}$ the subspace $\Sigma_Z \subset S^2V^\vee$ generated by the conics $M(\xi, X)$ when ξ varies in $\mathbb{P}^2$ has even dimension. Moreover, if no six of the points of Z are on a conic then $\langle M_{P_1}, \dots, M_{P_7} \rangle \subset \Sigma_Z$. If moreover $P_1, \dots, P_7$ are general points then the space on the left-hand side has dimension ≥ 5 because this happens for the special choice of $P_1, \dots, P_6$ on a nonsingular conic and P_7 general (Proposition 3.4). Therefore we conclude that $\Sigma_Z = S^2V^\vee$ if $P_1, \dots, P_7$ are general points, and this means that the skew-symmetric form $M(\xi, X)$ is nondegenerate—equivalently, its Pfaffian does not vanish—and, a fortiori, $\Psi(P_1, \dots, P_7) \neq 0$. □

REMARK 3.6. From Proposition 3.5 it follows that Ψ defines a hypersurface

$$V(\Psi) = \{(P_1, \dots, P_7) \in (\mathbb{P}^2)^7 : \Psi(P_1, \dots, P_7) = 0\}$$

in the seventh Cartesian product of $\mathbb{P}^2$. Since Ψ is not divisible by

$$\prod_i \mathcal{Q}(P_1, \dots, \hat{P}_i, \dots, P_7),$$

the general element $(P_1, \dots, P_7)$ of each irreducible component of $V(\Psi)$ consists of points no six of which are on a conic. Moreover, it follows from the proof of Corollary 3.5 that if $(P_1, \dots, P_7) \in V(\Psi)$ has such a property then $\dim\langle M_{P_1}, \dots, M_{P_7}\rangle \leq 4$.

Since Ψ is skew-symmetric with respect to the action of S_7 on $(\mathbb{P}^2)^7$, the hypersurface $V(\Psi)$ is S_7-invariant and therefore defines a hypersurface in the symmetric product $(\mathbb{P}^2)^{(7)}$.

DEFINITION 3.7. We will denote by $\mathcal{W}$ the image of $V(\Psi)$ in the symmetric product $(\mathbb{P}^2)^{(7)}$.

4. Cremona Hexahedral Equations

In this section we want to show the relations between some classical work by Cremona and Coble and the objects we have considered and to clarify their invariant-theoretic significance.

In $\mathbb{P}^5$ with coordinates $(Z_0, \dots, Z_5)$, consider the equations

$$\begin{cases} Z_0^3 + Z_1^3 + Z_2^3 + Z_3^3 + Z_4^3 + Z_5^3 = 0, \\ Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 0, \\ \beta_0 Z_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \beta_4 Z_4 + \beta_5 Z_5 = 0, \end{cases} \tag{11}$$

where the β_s are constants. These equations define a cubic surface S in a $\mathbb{P}^3$ contained in $\mathbb{P}^5$. If S is nonsingular then the equations (11) are called *Cremona hexahedral equations* of S after [7]. They have several remarkable properties, the most important for us being that these equations also determine a double-six of lines on the surface S.

Recall that a *double-six* of lines on a nonsingular cubic surface $S \subset \mathbb{P}^3$ consists of two sets of six lines $\Delta = (A_1, \dots, A_6; B_1, \dots, B_6)$ such that the lines A_j are mutually skew as well as the lines B_j; moreover, each A_i meets each B_j except when $i = j$.

If the cubic surface S is given as the image of the rational map $\mu\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by the linear system of plane cubic curves through six points $\{P_1, \dots, P_6\}$, then a double-six is implicitly defined by such a representation by letting A_j be the transform of the point P_j and letting B_j be the image of the plane conic θ_j containing $P_1, \dots, \hat{P}_j, \dots, P_6$. We have two morphisms:

$$\mathbb{P}^2 \xleftarrow{\pi_A} S \xrightarrow{\pi_B} \mathbb{P}^2.$$

Here π_A is the contraction of the lines $A_1, \ldots, A_6$ and is the inverse of μ. Similarly, π_B contracts $B_1, \ldots, B_6$ to points $R_1, \ldots, R_6 \in \mathbb{P}^2$ and is the inverse of the rational map defined by the linear system of plane cubics through $R_1, \ldots, R_6$. We have the following theorem.

THEOREM 4.1. *Each system of Cremona hexahedral equations of a nonsingular cubic surface S defines a double-six of lines on S. Conversely, the choice of a double-six of lines on S defines a system of Cremona hexahedral equations* (11) *of S, which is uniquely determined up to replacing the coefficients $(\beta_0, \ldots, \beta_5)$ by $(a + b\beta_0, \ldots, a + b\beta_5)$ for some $a, b \in \mathbf{k}$, $b \neq 0$.*

We refer to [8, Thm. 9.4.6] for the proof. We need to point out the following.

COROLLARY 4.2. *To a pair (S, Δ) consisting of a nonsingular cubic surface $S \subset \mathbb{P}^3$ and a double-six of lines Δ on S there is canonically associated a plane $\Xi \subset \mathbb{P}^3$ that is given by the equations*

$$\begin{cases} Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 0, \\ \beta_0 Z_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \beta_4 Z_4 + \beta_5 Z_5 = 0, \\ \beta_0^2 Z_0 + \beta_1^2 Z_1 + \beta_2^2 Z_2 + \beta_3^2 Z_3 + \beta_4^2 Z_4 + \beta_5^2 Z_5 = 0. \end{cases} \tag{12}$$

Proof. By replacing in the third equation β_i by $a + b\beta_i$ with $b \neq 0$, the plane Ξ remains the same. Therefore this plane depends only on the equations (11), which in turn depend only on (S, Δ). $\square$

DEFINITION 4.3. The plane $\Xi \subset \mathbb{P}^3$ will be called the *Cremona plane* associated to the pair (S, Δ).

If a cubic surface $S \subset \mathbb{P}^3$ is given as the image of a linear system of plane cubic curves through six points $\{P_1, \ldots, P_6\}$, then a double-six is implicitly selected by such a representation, and therefore S can be given in $\mathbb{P}^5$ by equations (11). In the first of the two papers [5], Coble found a parameterization of the cubic surface and of the constants β_s depending on the points $\{P_1, \ldots, P_6\}$ such that Cremona hexahedral equations are satisfied. Coble defined six cubic polynomials $z_0, z_1, \ldots, z_5 \in H^0(\mathcal{I}_{\{P_1,\ldots,P_6\}}(3))$ whose coefficients are multilinear in $P_1, \ldots, P_6$ and such that

$$z_0 + z_1 + z_2 + z_3 + z_4 + z_5 = 0 = z_0^3 + z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 \tag{13}$$

identically. In modern language, the cubics z_j span the 5-dimensional representation of S_6, which is called *outer automorphism representation* (see [8, Sec. 9.4.3]). Moreover, Coble defined certain multilinear polynomials $g_0, g_1, \ldots, g_5$ in the P_i satisfying the identity $g_0 + g_1 + \cdots + g_5 = 0$. The representation of S_6 spanned by the g_j is the transpose of the outer automorphism representation, and it is obtained by tensoring with the sign representation. In fact, Coble proves that the following identity holds:

$$g_0 z_0 + g_1 z_1 + \cdots + g_5 z_5 = 0. \tag{14}$$

Putting together the identities (13) and (14), he then obtains the following.

THEOREM 4.4. *The cubic surface $S \subset \mathbb{P}^3$ image of the rational map $\mu\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by the linear system $|H^0(\mathcal{I}_{\{P_1,\dots,P_6\}}(3))|$ satisfies the Cremona hexahedral equations* (11) *with $\beta_s = g_s$.*

Now consider

$$C(P_1,\dots,P_6,X) = g_0^2 z_0 + g_1^2 z_1 + \cdots + g_5^2 z_5.$$

It is a multihomogeneous polynomial of degree 3 in $P_1,\dots,P_6,X$. The equation $C(P_1,\dots,P_6,X) = 0$ defines a cubic plane curve that is the pullback by μ of the Cremona plane Ξ considered in Corollary 4.2.

THEOREM 4.5. *There is a constant $\lambda \neq 0$ such that the identity*

$$\Psi(P_1,\dots,P_6,P_7) = \lambda C(P_1,\dots,P_6,P_7) \tag{15}$$

holds for each 7-tuple of distinct points $P_1,\dots,P_6,P_7 \in \mathbb{P}^2$.

Proof. Both $\Psi(P_1,\dots,P_6,P_7)$ and $C(P_1,\dots,P_6,P_7)$ are cubic SL(3)-invariants of $P_1,\dots,P_6,P_7$ and skew-symmetric with respect to the points. Therefore it is enough to show that there is only one skew-symmetric cubic SL(3)-invariant of seven points, up to a constant factor. This is proved in [5]. We sketch a different approach to the proof.

Let S_n be the symmetric group of permutations on n objects. We denote by $\alpha = \alpha_1,\dots,\alpha_k$ the Young diagram with α_i boxes in the ith row and with F^α the corresponding representation of S_n. Denote by Γ^α the Schur functor corresponding to the Young diagram α. The product group $\mathrm{SL}(V) \times S_7$ acts in a natural way on the vector space $S^3V \otimes \cdots \otimes S^3V$ (seven times). We have the formula

$$S^3V \otimes \cdots \otimes S^3V = \bigoplus_\alpha \Gamma^\alpha(S^3V) \otimes F^\alpha,$$

where we sum over all Young diagrams α with seven boxes; see [21, Chap. 9, Thm. 3.1.4]. The skew invariants are contained in the summand where $\alpha = 1^7$; indeed, only in this case is F^α the sign representation of dimension 1. Correspondingly we can check, with a plethysm computation, that $\Gamma^{1^7}(S^3V) = \bigwedge^7(S^3V)$ contains just one trivial summand of dimension 1. This proves our result. $\square$

Let us mention also that the same method gives another proof of the well-known fact that every symmetric cubic invariant of seven points is zero.

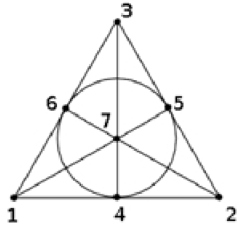

Figure 1 The Fano plane

The symbolic expression of the Morley invariant is

$$|142||253||361||175||276||374||456|.$$

Indeed, skew-symmetrizing the previous expression over S_7 yields the Morley invariant Ψ. The seven factors correspond to the seven lines of the Fano plane $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$, which is the smallest projective plane and consists of seven points (see Figure 1). Since the order of the automorphism group of the Fano plane is 168, it is enough to consider just $7!/168 = 30$ summands, 15 of them corresponding to even permutations and the remaining 15 corresponding to odd permutations. For another approach, which uses Gopel functions, see [11, Chap. IX].

5. The Cubic of the Seventh Point

The analysis of Section 3 suggests the following. If we are given six distinct points $P_1, \dots, P_6 \in \mathbb{P}^2$, no three of which are on a line, then we can consider the condition $\Psi(P_1, \dots, P_7) = 0$ as defining a plane cubic curve $E_{P_1,\dots,P_6} \subset \mathbb{P}^2$ described by the seventh point P_7.

Proposition 5.1. *In the situation just described, the cubic $E_{P_1,\dots,P_6}$ contains $P_1, \dots, P_6$.*

Proof. From the skew-symmetry of $\Psi(P_1, \dots, P_7)$ (Corollary 3.3) it follows that $\Psi(P_1, \dots, P_6, P_i) = 0$ for all $i = 1, \dots, 6$. □

From this proposition it follows that $E_{P_1,\dots,P_6}$ corresponds to a plane section $\Xi \cap S$ of the cubic surface $S \subset \mathbb{P}^3$ determined by the linear system of cubics through $P_1, \dots, P_6$. A description of the plane Ξ will be given in Section 6. We will now look more closely at the curve $E_{P_1,\dots,P_6}$.

Proposition 5.2. *Assume that $P_1, \dots, P_6$ are on a nonsingular conic θ. Then $E_{P_1,\dots,P_6}$ is the closure of the locus of points $P_7 \notin \theta$ such that the reducible conic $M^Z_{P_7}$, where $Z = \{P_1, \dots, P_7\}$, is conjugate to θ.*

Proof. By Remark 3.6 and by lower semicontinuity, the condition $P_7 \in E_{P_1,\dots,P_6}$ is equivalent to $\dim(\langle M^Z_{P_1}, \dots, M^Z_{P_7}\rangle) \le 4$. By Proposition 3.4, this is the condition that the seven reducible conics $M^Z_{P_i}$ are conjugate to θ. But when $i = 1, \dots, 6$ this condition is automatically satisfied. Therefore, the only condition for $P_7 \in E_{P_1,\dots,P_6}$ is that $M^Z_{P_7}$ is conjugate to θ; that is, this condition defines $E_{P_1,\dots,P_6}$. □

We have another description of $E_{P_1,\dots,P_6}$, as follows.

Proposition 5.3. *Assume that $P_1, \dots, P_6$ are on a nonsingular conic θ. Then $E_{P_1,\dots,P_6}$ is the cubic passing through $P_1, \dots, P_6$ and apolar to θ.*

Proof. Note that the cubic D passing through $P_1, \dots, P_6$ and apolar to θ is unique by Proposition 1.3. Since both D and $E_{P_1,\dots,P_6}$ are cubics, it suffices to show that $D \subset E_{P_1,\dots,P_6}$. By Proposition 5.2, for this purpose it suffices to show that, for each $P \in D$ and $P \neq P_1, \dots, P_6$, the cubic G containing $P_1, \dots, P_6$ and singular at P has the conic M_P of principal tangents at P conjugate to θ.

We may assume that $\theta = X_0^2 + 2X_1X_2$ and $P = (1, 0, 0)$. Let $L = aX_1 + bX_2$ be any line containing P. From Proposition 1.3 it follows that θL is not apolar to θ. This means that $D \notin \langle \theta X_1, \theta X_2 \rangle$ so that $\langle D, \theta X_1, \theta X_2 \rangle$ is the net of cubics through $P_1, \dots, P_6, P$.

If D is singular at P then $D = G$. In this case $M_P = \Delta_P D$, and this is conjugate to θ by Proposition 1.2. Otherwise,

$$D = \alpha_1(X_1, X_2)X_0^2 + \alpha_2(X_1, X_2)X_0 + \alpha_3(X_1, X_2)$$

with $\alpha_1 \neq 0$. Then

$$G = D - \alpha_1\theta = \alpha_2(X_1, X_2)X_0 + \alpha_3 - 2\alpha_1X_1X_2$$

so that $M_P = \alpha_2$. On the other hand, $\Delta_P D = 2\alpha_1(X_1, X_2)X_0 + \alpha_2(X_1, X_2)$ and the reducible conic joining P to $\Delta_P D \cap \Delta_P\theta$ is α_2. This is conjugate to θ by Proposition 1.1. □

COROLLARY 5.4. *The Morley invariant $\Psi(P_1, \dots, P_7)$ is irreducible and therefore the hypersurface $\mathcal{W} \subset (\mathbb{P}^2)^{(7)}$ is irreducible.*

Proof. If Ψ is reducible then the cubic $E_{P_1,\dots,P_6}$ is reducible for every choice of $P_1, \dots, P_6$. But for a general choice of $P_1, \dots, P_6$ on a nonsingular conic θ, the cubic D passing through $P_1, \dots, P_6$ and apolar to θ is irreducible by Proposition 1.3, and $D = E_{P_1,\dots,P_6}$ by Proposition 5.3. □

PROPOSITION 5.5. *Assume that $P_1, \dots, P_6$ are not on a conic. For each $i = 1, \dots, 6$, let θ_i be the conic containing all the P_j except P_i and let D_i be the cubic containing $P_1, \dots, P_6$ and apolar to θ_i. Denote by $Q_i \in \theta_i$ the sixth point of $D_i \cap \theta_i$. Then $E_{P_1,\dots,P_6}$ contains $Q_1, \dots, Q_6$.*

Proof. Let $1 \leq i \leq 6$. Then D_i is apolar to θ_i and contains the six points $P_1, \dots, \hat{P_i}, \dots, P_6, Q_i$, which are on θ_i. From Proposition 5.3 it follows that $D_i = E_{P_1,\dots,\hat{P_i},\dots,P_6,Q_i}$ and therefore

$$\Psi(P_1, \dots, P_6, Q_i) = 0, \quad i = 1, \dots, 6.$$

This means that $E_{P_1,\dots,P_6}$ contains $Q_1, \dots, Q_6$. □

6. A Geometrical Interpretation of the Cremona Planes

Consider a nonsingular cubic surface $S \subset \mathbb{P}^3$ and two skew lines $A, B \subset S$. Denote by $f\colon A \to B$ the double cover associating to $p \in A$ the point $f(p) := T_pS \cap B$, where T_pS is the tangent plane to S at p. Define $g\colon B \to A$ similarly. We call f and g the *involutory morphisms* relative to the pair of lines A and B. Let $p_1, p_2 \in A$ (resp. $q_1, q_2 \in B$) be the ramification points of f (resp. g). Consider the pairs of branch points $f(p_1), f(p_2) \in B$ and $g(q_1), g(q_2) \in A$ as well as the new morphisms

$$f' \colon A \to \mathbb{P}^1 \quad \text{and} \quad g' \colon B \to \mathbb{P}^1$$

defined by the conditions that $g(q_1), g(q_2)$ are ramification points of f' and $f(p_1), f(p_2)$ are ramification points of g'. Let $Q_1 + Q_2$ (resp. $P_1 + P_2$) be the common divisor of the two g^1_2 on A (resp. on B) defined by f and f' (resp. by g and g'). The points

$$\bar{P} = g(P_1) = g(P_2) \in A \quad \text{and} \quad \bar{Q} = f(Q_1) = f(Q_2) \in B$$

are called the *involutory points* (relative to the pair of lines A and B).

Consider six distinct points $P_1, \dots, P_6 \in \mathbb{P}^2$ not on a conic. Denote by $\mathcal{A} = \{P_1, \dots, P_6\}$ and $\mu_{\mathcal{A}} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ the rational map defined by the linear system of cubics through $\mathcal{A}$. On the nonsingular cubic surface $S = \mathrm{Im}(\mu_{\mathcal{A}}) \subset \mathbb{P}^3$, consider the double-six of lines

$$\Delta = (A_1, \dots, A_6; B_1, \dots, B_6),$$

where $A_1, \dots, A_6 \subset S$ are the lines that are proper transforms under $\mu_{\mathcal{A}}$ of $P_1, \dots, P_6$ (respectively) and $B_i \subset S$ is the line that is the proper transform of the conic $\theta_i \subset \mathbb{P}^2$ containing $P_1, \dots, \hat{P}_i, \dots, P_6$. Consider the diagram

$$\mathbb{P}^2 \xleftarrow{\pi_A} S \xrightarrow{\pi_B} \mathbb{P}^2,$$

where π_A (resp. π_B) is the morphism that contracts the lines $A_1, \dots, A_6$ (resp. the lines $B_1, \dots, B_6$). Let $R_i = \pi_B(B_i) \in \mathbb{P}^2$. Then π_A is the inverse of $\mu_{\mathcal{A}}$ and π_B is the inverse of $\mu_{\mathcal{B}} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$, where $\mathcal{B} = \{R_1, \dots, R_6\}$. Let $\bar{P}_i \in A_i$ and $\bar{Q}_i \in B_i$ be the involutory points relative to the pair A_i and B_i. We obtain twelve points,

$$\bar{P}_1, \dots, \bar{P}_6, \bar{Q}_1, \dots, \bar{Q}_6 \in S,$$

which are canonically associated to the double-six Δ.

THEOREM 6.1. *There is a unique plane $\Xi \subset \mathbb{P}^3$ containing the involutory points*

$$\bar{P}_1, \dots, \bar{P}_6, \bar{Q}_1, \dots, \bar{Q}_6.$$

Moreover, Ξ coincides with the Cremona plane (12) *associated to the pair* (S, Δ) *and* $\mu^*_{\mathcal{A}}(\Xi) = E_{P_1, \dots, P_6}$.

Proof. We will keep the notation just introduced. Fix $1 \le i \le 6$ and denote by $f_i \colon A_i \to B_i$ and $g_i \colon B_i \to A_i$ the involutory morphisms. The cubics D through $P_1, \dots, P_6$ and singular at P_i form a pencil Λ_i and are mapped by $\mu_{\mathcal{A}}$ on S to the conics cut by the planes containing A_i. Similarly, the pencil L_i of lines through P_i is mapped by $\mu_{\mathcal{A}}$ on S to the pencil of conics cut by the planes containing B_i. It follows that f_i can be interpreted in $\mathbb{P}^2$ as the map sending a line $\ell \in L_i$ to the sixth point of $D \cap \theta_i$, where $D \in \Lambda_i$ is the cubic having ℓ as a principal tangent. Therefore the ramification points of f_i are the images $p_1, p_2 \in A_i$ of the lines λ_1, λ_2, which are principal tangents of the two cuspidal cubics $D_1, D_2 \in \Lambda_i$, and $f_i(p_1), f_i(p_2) \in \theta_i$ are the two sixth points of intersection of D_1 (resp. D_2) with θ_i.

On the other hand, g_i can be interpreted as associating to a point $q \in \theta_i$ the line $\langle P_i, q\rangle \in L_i$. The ramification points $q_1, q_2 \in \theta_i$ are the tangency points on the two lines $\ell_1, \ell_2 \in L_i$ that are tangent to θ_i.

It follows that $g_i(q_1) = \ell_1$ and $g_i(q_2) = \ell_2$. Then clearly $\bar{Q}_i = \mu_{\mathcal{A}}(Q_i) \in B_i$, where $Q_i \in \theta_i$ is the sixth point of $D \cap \theta_i$ and $D \in \Lambda_i$ is the cubic whose principal tangents are conjugate with respect to ℓ_1 and ℓ_2 or, equivalently, such that the reducible conic of its principal tangents is conjugate to θ_i (Proposition 1.1).

From Propositions 5.1 and 5.5 it follows that $E_{P_1,\dots,P_6}$ contains $\mathcal{A}$ and $Q_1, \dots, Q_6$, and clearly it is the only cubic curve with this property. Therefore there is a unique plane $\Xi \subset \mathbb{P}^3$ containing $\bar{Q}_1, \dots, \bar{Q}_6$ and $\mu^*_{\mathcal{A}}(\Xi) = E_{P_1,\dots,P_6}$.

Reversing the roles of $A_1, \dots, A_6$ and $B_1, \dots, B_6$, we can argue similarly using the rational map $\mu_{\mathcal{B}}$ instead of $\mu_{\mathcal{A}}$ to conclude that the points $\bar{P}_1, \dots, \bar{P}_6$ are contained in a unique plane Π and that $\mu^*_{\mathcal{B}}(\Pi) = E_{R_1,\dots,R_6}$.

From Theorem 4.5 and the remarks before it we get that both Ξ and Π coincide with the plane (12), which is canonically associated to the double-six Δ. In particular, $\Xi = \Pi$, and this concludes the proof. □

REMARK 6.2. The same proof as just given shows that the point $\bar{P}_i \in A_i$ corresponds to the line $\tau_i := \langle P_i, z\rangle \in L_i$ joining P_i with the pole z with respect to θ_i of the line $\langle f_i(p_1), f_i(p_2)\rangle$. Therefore the theorem implies that the plane cubic curve $E_{P_1,\dots,P_6} \in H^0(\mathbb{P}^2, \mathcal{I}_S(3))$ contains the points $Q_1, \dots, Q_6$ and that its tangent lines at the points $P_1, \dots, P_6$ are $\tau_1, \dots, \tau_6$, respectively.

Since there are 36 double-six configurations of lines on a nonsingular cubic surface, we obtain 36 Cremona planes in $\mathbb{P}^3$ and, correspondingly, 36 cubic curves belonging to the linear system $|H^0(\mathbb{P}^2, \mathcal{I}_S(3))|$. One of them is $E_{P_1,\dots,P_6}$.

PROPOSITION 6.3. *The 36 Cremona planes and, consequently, the corresponding 36 cubic curves in* $H^0(\mathbb{P}^2, \mathcal{I}_S(3))$, *are pairwise distinct.*

Proof. We first remark that two double-sixes always have a common line, which we call A_1. The two corresponding skew lines B_1 and B_1' (one for each double-six) are different. This follows from the explicit list of all the double-sixes; see [8]. It is well known that, given a line A_1 on S, there are exactly sixteen lines on S that are skew with A_1. Then it is enough to prove that the sixteen involutory points on A_1 determined by the sixteen lines skew with A_1 are all distinct. Indeed, if two Cremona planes corresponding to two different double-sixes coincide, then on their common line A_1 we get that two points, among the sixteen involutory points, should coincide.

We computed explicitly these sixteen points in a particular case, and we obtained sixteen distinct points, as we wanted. Let us sketch how this computation works. We start from a set $\mathcal{A} = \{P_1, \dots, P_6\}$ of six general points in the plane, and we consider the cubic surface $S = \mathrm{Im}(\mu_{\mathcal{A}}) \subset \mathbb{P}^3$. Denote by A_i the exceptional divisor on S corresponding to P_i. The sixteen lines on S that are skew with A_1 correspond to:

(a) the five exceptional divisors A_i for $i \geq 2$;
(b) the ten lines joining two points among $P_2, \ldots, P_6$;
(c) the conic θ_1 passing through $P_2, \ldots, P_6$.

We have constructed the (five) involutory points on A_1 in case (a), as explained in the proof of Theorem 6.1, starting from the points $P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$, $P_3 = (0, 0, 1)$, $P_4 = (1, 1, 1)$, $P_5 = (2, 3, 5)$, $P_6 = (11, 13, 29)$. With the same points P_i, we have computed also the other ten points of case (b) and the eleventh of case (c). This last one corresponds to τ_1 of Remark 6.2. The resulting sixteen points on A_1 are all distinct. □

7. The Geiser Involution

Our interest in configurations Z of seven distinct points in $\mathbb{P}^2$ comes from the classical Aronhold construction of plane quartics starting from such a Z. We refer to [8] for details and proofs. See also [14, p. 319] and [15, p. 783] for classical expositions and [24] for recent applications to vector bundles.

Let $\mathbb{P}$ be a projective plane and let $B \subset \mathbb{P}$ be a nonsingular quartic. Recall that an unordered 7-tuple $\mathrm{T} = \{t_1, \ldots, t_7\}$ of bitangent lines of B is called an *Aronhold system* if for all triples of distinct indices $1 \leq i, j, k \leq 7$ the six points $(t_i \cup t_j \cup t_k) \cap B$ do not lie on a conic. Every nonsingular plane quartic has 288 distinct Aronhold systems of bitangents.

Let $Z = \{P_1, \ldots, P_7\} \subset \mathbb{P}(V)$ be seven distinct points such that no six of them are on a conic. The rational map

$$\gamma_Z \colon \mathbb{P}(V) \dashrightarrow |H^0(\mathcal{I}_Z(3))|^\vee$$

defined by the net of cubics through Z is called the *Geiser involution* defined by Z. It associates to a point $P \in \mathbb{P}(V)$ the pencil of cubics of the net containing P. We have $\gamma_Z(P) = \gamma_Z(P')$ if and only if P and P' are base points of the same pencil of cubics. Therefore γ_Z has degree 2 and is not defined precisely at the points of Z. We can identify the target of γ_Z with $\mathbb{P}(V)^\vee$ by associating to P the line $\langle P, P' \rangle$ so that γ_Z can be viewed as a rational map $\gamma_Z \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V)^\vee$. Note that, conversely, from any general line $\ell \subset \mathbb{P}(V)$ we can recover $\{P, P'\} = \gamma_Z^{-1}(\ell)$ as the unique pair of points that are identified by the g^2_3 defined on ℓ by the net of cubics.

After choosing a basis of V, we obtain a parameterization of the cubics of the net $|H^0(\mathcal{I}_Z(3))|$ by associating to each $\xi \in \mathbb{P}(V)$ the cubic $S(\xi, X)$ (Lemma 2.2(ii)). This defines an isomorphism:

$$\mathbb{P}(V) \xrightarrow{\sim} |H^0(\mathcal{I}_Z(3))|. \tag{16}$$

The map γ_Z can be described explicitly as follows. Let $P \in \mathbb{P}(V)$, $P \notin Z$. Then $\gamma_Z(P) \in \mathbb{P}(V)^\vee$ is the line of $\mathbb{P}(V)$ given by the equation $S(\xi, P) = 0$ in coordinates ξ. This line parameterizes the pencil of cubics of the net containing P via the isomorphism (16).

Set $S(\xi, X) = L_0(\xi)C_0(X) + L_1(\xi)C_1(X) + L_2(\xi)C_2(X)$. Then the sextic

$$\Sigma : \left| \frac{\partial C_j}{\partial X_h} \right| = 0$$

is the *Jacobian curve* of the net $|H^0(\mathcal{I}_Z(3))|$—that is, the locus of double points of curves of the net.

All the properties of γ_Z can be easily deduced by considering the del Pezzo surface of degree 2 that is the blow-up of $\mathbb{P}(V)$ at Z. The following proposition summarizes the main properties that we will need.

PROPOSITION 7.1. (i) *The branch curve of the Geiser involution γ_Z is a nonsingular quartic $B(Z) \subset \mathbb{P}(V)^\vee$.*

(ii) *The Jacobian curve Σ is the ramification curve of γ_Z.*

(iii) *The points $P_1, \dots, P_7$ are transformed into seven bitangent lines $t_1, \dots, t_7$ of $B(Z)$, which form an Aronhold system. The other 21 bitangent lines of $B(Z)$ are the transforms of the conics through five of the seven points of Z.*

If we order the points $P_1, \dots, P_6, P_7$, we can consider the birational map $\mu : \mathbb{P}^2 \dashrightarrow S$ of $\mathbb{P}^2$ onto a nonsingular cubic surface $S \subset \mathbb{P}^3$ defined by the linear system of cubics through $\{P_1, \dots, P_6\}$. The point P_7 is mapped to a point $O \in S$, and the ramification sextic $\Sigma \subset \mathbb{P}^2$ is transformed into a sextic $\mu(\Sigma) \subset S$ of genus 3 with a double point at O. Then γ_Z is the composition of μ with the projection of S from O onto the projective plane of lines through O. The quartic $B(Z)$ is the image of the sextic $\mu(\Sigma)$ under this projection.

Consider the following space:

$$\mathcal{A} := \{(B, T) : B \subset \mathbb{P}(V^\vee) \text{ is a n.s. quartic and } T \text{ is an Aronhold system of bitangents of } B\}.$$

It is nonsingular and irreducible of dimension 14. We have a commutative diagram of generically finite rational maps:

$$\begin{tikzcd}
(\mathbb{P}^2)^{(7)} \arrow[r, dashed, "\mathcal{T}"] \arrow[dr, dashed, "B"'] & \mathcal{A} \arrow[d, "\varphi"] \\
& \mathbb{P}(S^4V),
\end{tikzcd} \tag{17}$$

where B is the rational map associating to a 7-tuple Z of distinct points, no six of which are on a conic, the quartic $B(Z)$. The map $\mathcal{T}$ associates to Z the pair $\mathcal{T}(Z) = (B(Z), \{t_1, \dots, t_7\})$, where t_i is the bitangent that is the image of P_i. Since B has finite fibres, the image $B(\mathcal{W}) \subset \mathbb{P}(S^4V)$, with $\mathcal{W} \subset (\mathbb{P}^2)^{(7)}$ as defined in Definition 3.7, is an open set of an irreducible hypersurface whose elements will be called *Morley quartics.*

8. Morley Quartics

Consider the irreducible hypersurface of Morley quartics

$$\mathcal{M} := \overline{B(\mathcal{W})} \subset \mathbb{P}(S^4V),$$

which is the closure of the image of the hypersurface $\mathcal{W} \subset \mathbb{P}(V)^{(7)}$ under the rational map B. Clearly $\mathcal{M}$ is SL(3)-invariant. In this section we will compute its degree.

THEOREM 8.1. *The hypersurface of Morley quartics $\mathcal{M} \subset \mathbb{P}(S^4V)$ has degree 54.*

Proof. Consider the projective bundle $\pi : \mathbb{P}(\mathbf{Q}) \to \mathbb{P}^3$, where $\mathbf{Q} = T_{\mathbb{P}^3}(-1)$ is the tautological quotient bundle that appears in the twisted Euler sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 4} \to \mathbf{Q} \to 0.$$

For each $z \in \mathbb{P}^3$, the fibre $\pi^{-1}(z)$ is the projective plane of lines through z. Also consider the projective bundle $\beta : \mathbb{P}(S^4\mathbf{Q}^\vee) \to \mathbb{P}^3$. For each $z \in \mathbb{P}^3$, the fibre $\beta^{-1}(z)$ is the linear system of quartics in $\pi^{-1}(z)$. The Picard group of $\mathbb{P} = \mathbb{P}(S^4\mathbf{Q}^\vee)$ is generated by $H = \mathcal{O}_{\mathbb{P}}(1)$ and by the pullback F of a plane in $\mathbb{P}^3$. Let $\widetilde{\mathcal{M}} \subset \mathbb{P}(S^4\mathbf{Q}^\vee)$ be the β-relative hypersurface of Morley quartics, and assume that it is given by a section of $aH + bF$. Since it is invariant under the natural action of $\mathrm{SL}(4)$ on $\mathbb{P}(S^4\mathbf{Q}^\vee)$, $\widetilde{\mathcal{M}}$ corresponds to a trivial summand of

$$H^0(\mathbb{P}(S^4\mathbf{Q}^\vee), aH + bF) = H^0(\mathbb{P}^3, [S^a(S^4\mathbf{Q})](b)),$$

which exists if and only if $[S^a(S^4\mathbf{Q})](b)$ contains $\mathcal{O}$ as a summand. Since $\mathbf{Q}$ is homogeneous and indecomposable, it follows that all the indecomposable summands of $[S^a(S^4\mathbf{Q})](b)$ have the same slope. This is well known and can be easily deduced from the discussion at 5.2 of [22]. We get that $c_1([S^a(S^4\mathbf{Q})](b)) = 0$. Computing the slope,

$$\mu([S^a(S^4\mathbf{Q})](b)) = \frac{4a}{3} + b,$$

we deduce that $0 = 4a + 3b$ and therefore $\widetilde{\mathcal{M}} = k(3H - 4F)$ for some k. On the other hand, $\widetilde{\mathcal{M}}$ has relative degree d, where d is the degree of $\mathcal{M} \subset \mathbb{P}(S^4V)$. Therefore, $3k = d$ and it follows that $\widetilde{\mathcal{M}}$ has class $dH - \frac{4d}{3}F$.

Let $S \subset \mathbb{P}^3$ be a nonsingular cubic surface. To each $z \in S$ there is associated the quartic branch curve of the rational projection $S \dashrightarrow \pi^{-1}(z)$ with center z. This defines a section s of β over S:

$$\begin{tikzcd} \mathbb{P}(S^4\mathbf{Q}^\vee)|_S \arrow[r, "\beta"'] & S. \arrow[l, bend right, "s"'] \end{tikzcd}$$

The pullback $s^*\widetilde{\mathcal{M}} \subset S$ is the divisor of points $z \in S$ such that the branch curve of the projection of S from z is a Morley quartic. From Theorem 6.1 and Proposition 6.3 it follows that $s^*\widetilde{\mathcal{M}}$ is a section of $\mathcal{O}_S(\Xi_1 + \cdots + \Xi_{36}) = \mathcal{O}_S(36)$, where $\Xi_1, \ldots, \Xi_{36}$ are the Cremona planes of S.

Let's write an equation of S as $f(X, X, X) = 0$, where f is a symmetric trilinear form, and let $z \in S$. The line through z and X is parameterized by $z + tX$ and meets S where $f(z + tX, z + tX, z + tX)$ vanishes. We get

$$3tf(z, z, X) + 3t^2 f(z, X, X) + t^3 f(X, X, X) = 0,$$

which has a root for $t = 0$ and a residual double root when

$$3f(z, X, X)^2 - 4f(z, z, X)f(X, X, X) = 0,$$

which is a quartic cone with vertex in z. From this expression we see that the section s is quadratic in the coordinates of z. It follows that $s^*H = \mathcal{O}_S(2)$. Therefore,

$$\mathcal{O}_S(36) = s^*\widetilde{\mathcal{M}} = s^*\left(dH - \frac{4d}{3}F\right) = \mathcal{O}_S\left(2d - \frac{4d}{3}\right) = \mathcal{O}_S\left(\frac{2d}{3}\right),$$

and we get $d = 54$. □

REMARK 8.2. The same proof shows that every invariant of a plane quartic of degree d gives a covariant of the cubic surface of degree $\frac{2d}{3}$. A classical reference for this statement is [6, p. 189].

9. Bateman Configurations

The configurations Z of seven points in $\mathbb{P}(V) = \mathbb{P}^2$, no six of which are on a conic and for which the Morley invariant vanishes (i.e., belonging to the irreducible hypersurface $\mathcal{W}$), have a simple description that is due to Bateman [2].

Consider a nonsingular conic θ, a cubic D, and the matrix

$$A(\theta, D) = \begin{pmatrix} \partial_0\theta & \partial_1\theta & \partial_2\theta \\ \partial_0 D & \partial_1 D & \partial_2 D \end{pmatrix}. \tag{18}$$

The 7-tuple of points $Z = Z(\theta, D)$ defined by the maximal minors of this matrix is called the *Bateman configuration* defined by θ and D. Note that $Z(\theta, D)$ consists of the points that have the same polar line with respect to θ and D.

LEMMA 9.1. *Let $Z = Z(\theta, D)$ be the Bateman configuration defined by a nonsingular conic θ and a cubic D. If D is general then Z consists of seven distinct points no six of which are on a conic.*

Proof. We may assume that $\theta = X_0^2 + X_1^2 + X_2^2$. It suffices to prove the assertion in a special case. If we take $D = X_0X_1X_2$ then the maximal minors of $A(\theta, D)$ are

$$C_0 = X_0(X_1^2 - X_2^2), \quad C_1 = X_1(X_2^2 - X_0^2), \quad C_2 = X_2(X_0^2 - X_1^2),$$

and

$$Z(\theta, D) = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1), (1,-1,1), (1,1,-1), (-1,1,1)\}.$$

One easily checks directly that no six of the points of $Z(\theta, D)$ are on a conic. This also follows from Lemma 2.1, because $(\partial_0\theta, \partial_1\theta, \partial_2\theta) = (X_0, X_1, X_2)$ is the first row of $A(\theta, D)$ and has linearly independent entries. □

The Geiser involution associated to a general Bateman configuration $Z = Z(\theta, D)$ can be described as follows. Given a point $P \in \mathbb{P}(V)$, $P \notin Z$, then $Q = (C_0(P), C_1(P), C_2(P))$ is the point of intersection of the polar lines of P with respect to θ and D. The rational map

$$\mathbb{P}(V) \dashrightarrow \mathbb{P}(V), \quad P \mapsto Q$$

coincides with the Geiser involution γ_Z composed with the identification $\mathbb{P}(V^\vee) = \mathbb{P}(V)$ obtained thanks to the polarity with respect to θ. In particular, *the quartic* $B(Z(\theta, D))$ *lies in* $\mathbb{P}(V)$. For each point $Q \in \mathbb{P}(V)$ we have $\gamma_Z^{-1}(Q) = \{P, P'\}$, where P, P' are the points of intersection of the polar line of Q with respect to θ with the polar conic of Q with respect to D.

We have

$$S(\xi, X) = \begin{vmatrix} \partial_0\theta(\xi) & \partial_1\theta(\xi) & \partial_2\theta(\xi) \\ \partial_0\theta(X) & \partial_1\theta(X) & \partial_2\theta(X) \\ \partial_0 D(X) & \partial_1 D(X) & \partial_2 D(X) \end{vmatrix} \tag{19}$$

and

$$M(\xi, X) = \begin{vmatrix} \partial_0\theta(\xi) & \partial_1\theta(\xi) & \partial_2\theta(\xi) \\ \partial_0\theta(X) & \partial_1\theta(X) & \partial_2\theta(X) \\ \Delta_\xi\partial_0 D(X) & \Delta_\xi\partial_1 D(X) & \Delta_\xi\partial_2 D(X) \end{vmatrix}.$$

THEOREM 9.2. *Let $Z(\theta, D)$ be a Bateman configuration. Then, for every $\xi \in \mathbb{P}^2$, the conic $M(\xi, X)$ is conjugate to θ.*

Proof. We can change coordinates and assume that $\theta = X_0^2 + 2X_1X_2$, so that its dual is $\theta^* = \partial_0^2 + 2\partial_1\partial_2$, and we must show that

$$P_{\theta^*}(M(\xi, X)) = 0 \tag{20}$$

identically, where

$$\begin{aligned} M(\xi, X) &= \begin{vmatrix} \xi_0 & \xi_2 & \xi_1 \\ X_0 & X_2 & X_1 \\ \Delta_\xi\partial_0 D(X) & \Delta_\xi\partial_1 D(X) & \Delta_\xi\partial_2 D(X) \end{vmatrix} \\ &= \xi_0(X_2\Delta_\xi\partial_2 D - X_1\Delta_\xi\partial_1 D) - \xi_2(X_0\Delta_\xi\partial_2 D - X_1\Delta_\xi\partial_0 D) \\ &\quad + \xi_1(X_0\Delta_\xi\partial_1 D - X_2\Delta_\xi\partial_0 D). \end{aligned}$$

If we write the cubic polynomial defining D as

$$\sum_{0\le i\le j\le k\le 2} \beta_{ijk} X_i X_j X_k,$$

then an easy computation shows that

$$\begin{aligned} M(\xi, X) = (X_1X_2 - X_0^2)(\beta_{002}\xi_0\xi_2 - \beta_{001}\xi_0\xi_1 + \beta_{022}\xi_2^2 - \beta_{011}\xi_1^2) \\ + \text{terms not involving } X_0^2 \text{ and } X_1X_2, \end{aligned}$$

and (20) follows immediately. □

COROLLARY 9.3. *If $Z(\theta, D) = \{P_1, \dots, P_7\}$ is a Bateman configuration of distinct points no six of which are on a conic, then $\Psi(P_1, \dots, P_7) = 0$. In other words, the image of the rational map*

$$Z\colon \mathbb{P}(S^2V^\vee) \times \mathbb{P}(S^3V^\vee) \dashrightarrow \mathbb{P}(V)^{(7)} = (\mathbb{P}^2)^{(7)},$$

which associates to a general pair (θ, D) the Bateman configuration $Z(\theta, D)$, is contained in $\mathcal{W} \subset (\mathbb{P}^2)^{(7)}$ (see Definition 3.7).

Proof. From the theorem it follows that all the conics $M(\xi, X)$, $\xi \in \mathbb{P}^2$, are contained in the hyperplane of conics conjugate to the conic θ. This implies that the skew-symmetric form

$$M\colon S^2V^\vee \times S^2V^\vee \to \mathbf{k}$$

is degenerate; hence its Pfaffian vanishes. But since no six of the points of $Z(\theta, D)$ are on a conic, we have $Q(P_1, \dots, \hat{P}_i, \dots, P_7) \neq 0$ for all $i = 1, \dots, 7$. Then the conclusion follows from the factorization (9). □

DEFINITION 9.4. Identity (20) is called *Morley's differential identity.*

Corollary 9.3 shows in particular that Bateman configurations are not the most general 7-tuples of points because they are in $\mathcal{W}$. The corollary does not exclude that $\mathrm{Im}(Z)$ (i.e., the locus of Bateman configurations) is contained in a proper closed subset of $\mathcal{W}$. We will show in Section 10 that the Bateman configurations actually fill a dense open subset of $\mathcal{W}$; that is, they depend on thirteen parameters and not fewer.

10. Lüroth Quartics

A configuration consisting of five lines in $\mathbb{P}(V)$, three by three linearly independent together with the ten double points of their union, will be called a *complete pentalateral*. The ten nodes of their union are called *vertices* of the complete pentalateral.

DEFINITION 10.1. A *Lüroth quartic* is a nonsingular quartic $B \subset \mathbb{P}(V)$ that is circumscribed to a complete pentalateral—in other words, that contains its ten vertices.

Consider the incidence relation $\widetilde{\mathcal{L}} \subset \mathbb{P}(S^4V^\vee) \times \mathbb{P}(V^\vee)^{(5)}$ described as

$$\widetilde{\mathcal{L}} := \{(B, \{\ell_0, \dots, \ell_4\}) : \{\ell_0, \dots, \ell_4\} \text{ is a complete pentalateral and } B \text{ is a n.s. quartic circumscribed to it}\}$$

and consider the projections

$$\mathbb{P}(S^4V^\vee) \xleftarrow{q_1} \widetilde{\mathcal{L}} \xrightarrow{q_2} \mathbb{P}(V^\vee)^{(5)}.$$

Clearly $q_1(\widetilde{\mathcal{L}}) \subset \mathbb{P}(S^4V^\vee)$ is the locus of Lüroth quartics. The following facts are well known (see [20]).

(i) q_2 is dominant with general fibre of dimension 4, and $\widetilde{\mathcal{L}}$ is irreducible of dimension 14.

(ii) The general fibre of q_1 has dimension 1. This means that every Lüroth quartic has infinitely many inscribed pentalaterals. Moreover,

$$\mathcal{L} := \overline{q_1(\widetilde{\mathcal{L}})} \subset \mathbb{P}(S^4V^\vee)$$

is an $\mathrm{SL}(3)$-invariant irreducible hypersurface that is called *the Lüroth hypersurface*.

Given a complete pentalateral $\{\ell_0,\dots,\ell_4\}\in\mathbb{P}(V^\vee)^{(5)}$, the fibre $q_2^{-1}(\{\ell_0,\dots,\ell_4\})$ is the linear system of quartics circumscribed to it. It consists of the quartics of the form

$$\sum_{k=0}^{4}\lambda_k\ell_0\cdots\hat{\ell}_k\cdots\ell_4=0 \tag{21}$$

as $(\lambda_0,\dots,\lambda_4)\in\mathbb{P}^4$.

Another way of describing a general element of $q_2^{-1}(\{\ell_0,\dots,\ell_4\})$ is under the form

$$\sum_{k=0}^{4}\frac{1}{l_k\ell_k}=0 \tag{22}$$

as $(l_0,\dots,l_4)\in\mathbb{P}^4$. The two descriptions are of course related by a Cremona transformation of $\mathbb{P}^4$. We have moreover the following elementary property.

(iii) Any given $(B,\{\ell_0,\dots,\ell_4\})\in\widetilde{\mathcal{L}}$ is uniquely determined by any of the five pairs

$$(B,\{\ell_0,\dots,\hat{\ell}_k,\dots,\ell_4\})\in\mathbb{P}(S^4V^\vee)\times\mathbb{P}(V^\vee)^{(4)}$$

consisting of the quartic B and of four of the five lines of the pentalateral.

We will need the following result, which is due to Roberts [23].

THEOREM 10.2. *Let θ be a nonsingular conic and D a general cubic. Then there are lines $\ell_1,\ell_2,\ell_3,\ell_4$, three by three linearly independent, and constants $a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4$ such that*

$$\begin{aligned}\theta&=a_1\ell_1^2+a_2\ell_2^2+a_3\ell_3^2+a_4\ell_4^2,\\ D&=b_1\ell_1^3+b_2\ell_2^3+b_3\ell_3^3+b_4\ell_4^3.\end{aligned}\tag{23}$$

The four lines are uniquely determined, and each of the two 4-tuples of constants is uniquely determined up to a constant factor.

Proof. The line conics that are simultaneously apolar to θ and D form at least a pencil because being apolar to θ (resp. to D) is one condition (resp. three conditions) for a line conic. Moreover, for a general choice of (θ,D), these conditions are independent. In fact, taking (θ,D) as in (23) and letting Σ be a line conic belonging to the pencil whose base consists of the lines $\ell_1,\ell_2,\ell_3,\ell_4$, we have

$$P_\Sigma(\theta)=2a_1\Sigma(\ell_1)+2a_2\Sigma(\ell_2)+2a_3\Sigma(\ell_3)+2a_4\Sigma(\ell_4)=0.$$

Similarly, $P_\Sigma(D)=0$. In other words, Σ is apolar to both θ and D. On the other hand, it is clear that there are no other line conics apolar to θ and D. Now the theorem follows from Proposition 4.3 of [10]. $\square$

Theorem 10.2 can be conveniently rephrased as follows.

COROLLARY 10.3. *By associating to a pair $(\theta,D)\in\mathbb{P}(S^2V^\vee)\times\mathbb{P}(S^3V^\vee)$ consisting of a nonsingular conic and a general cubic the data*

$$((\ell_1,\ell_2,\ell_3,\ell_4),(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4))$$

given by Theorem 10.2, one obtains a birational map:

$$R\colon \mathbb{P}(S^2V^\vee)\times\mathbb{P}(S^3V^\vee) \dashrightarrow (\mathbb{P}^{2\vee})^4\times\mathbb{P}^3\times\mathbb{P}^3.$$

We have the following remarkable result due to Bateman [2].

THEOREM 10.4. *Consider a nonsingular conic θ and a general cubic D, and represent them as*

$$\theta = a_1\ell_1^2 + a_2\ell_2^2 + a_3\ell_3^2 + a_4\ell_4^2,$$
$$D = b_1\ell_1^3 + b_2\ell_2^3 + b_3\ell_3^3 + b_4\ell_4^3,$$

according to Theorem 10.2. Then the plane quartic $B = B(Z(\theta, D))$ associated to the Bateman configuration $Z(\theta, D)$ is a Lüroth quartic, and $\ell_1, \ell_2, \ell_3, \ell_4$ are four lines of a complete pentalateral inscribed in B.

Proof. Let $X_1, \dots, X_4$ be homogeneous coordinates in $\mathbb{P}^3$. We may identify $\mathbb{P}(V)$ with the plane $H \subset \mathbb{P}^3$ of equations $\sum_i X_i = 0$. After a change of coordinates in H, we may further assume that $\ell_1, \ell_2, \ell_3, \ell_4$ are respectively the lines $X_i = 0$, $i = 1, \dots, 4$. With this convention we have

$$\theta = \sum_{i=1}^{4} a_i X_i^2, \quad D = \sum_{i=1}^{4} b_i X_i^3,$$

and we may assume that the constants a_i, b_i are all nonzero. Let $Q = (y_1, \dots, y_4) \in H$ be a (variable) point. The polar line of Q with respect to θ is

$$\sum_k a_k y_k X_k. \tag{24}$$

Similarly, the polar conic of Q with respect to D is

$$\sum_k b_k y_k X_k^2. \tag{25}$$

Assume that the line (24) is tangent to the conic (25) at the point $P = (z_1, \dots, z_4)$. Then its equation must be equivalent to the equation $\sum_k b_k y_k z_k X_k = 0$. This means that there are constants $(\lambda, \mu) \neq (0, 0)$ such that

$$\sum_k b_k y_k z_k X_k = \lambda\Big[\sum_k a_k y_k X_k\Big] + \mu\Big[\sum_k X_k\Big]$$

or, equivalently,

$$b_k y_k z_k = \lambda a_k y_k + \mu, \quad k = 1, 2, 3, 4.$$

Since $P \in H$, we find

$$0 = \sum_k z_k = \lambda\Big[\sum_k \frac{a_k}{b_k}\Big] + \mu\Big[\sum_k \frac{1}{b_k y_k}\Big].$$

Using the fact that P belongs to the polar line (24), we also deduce that

$$0 = \sum_k a_k y_k z_k = \lambda \left[\sum_k \frac{a_k^2 y_k}{b_k} \right] + \mu \left[\sum_k \frac{a_k}{b_k} \right].$$

These two identities imply that

$$\begin{vmatrix} \sum \frac{a_k}{b_k} & \sum \frac{1}{b_k y_k} \\ \sum \frac{a_k^2 y_k}{b_k} & \sum \frac{a_k}{b_k} \end{vmatrix} = 0$$

or, equivalently,

$$\left(\sum_{k=1}^{4} \frac{a_k}{b_k} \right)^2 = \left(\sum_{k=1}^{4} \frac{a_k^2 y_k}{b_k} \right) \left(\sum_{k=1}^{4} \frac{1}{b_k y_k} \right). \tag{26}$$

Now let's define

$$L = - \left(\sum_{k=1}^{4} \frac{a_k}{b_k} \right)^{-2} \left(\sum_{k=1}^{4} \frac{a_k^2 y_k}{b_k} \right). \tag{27}$$

Then L is a linear form in the coordinates $y_1, \dots, y_4$ of Q, and the identity (26) is equivalent to

$$\sum_{k=1}^{4} \frac{1}{b_k y_k} + \frac{1}{L} = 0. \tag{28}$$

This is the equation of a Lüroth quartic in the coordinates of Q. $\square$

COROLLARY 10.5. *There is a dominant, generically finite, rational map*

$$\widetilde{Z} \colon \mathbb{P}(S^2 V^\vee) \times \mathbb{P}(S^3 V^\vee) \dashrightarrow \widetilde{\mathcal{L}}$$

such that $q_1(\widetilde{Z}(\theta, D)) = B(Z(\theta, D))$. *In particular*:

(i) *the general Lüroth quartic is of the form* $B(Z(\theta, D))$ *for some* (θ, D);
(ii) *the rational map* $Z \colon \mathbb{P}(S^2 V^\vee) \times \mathbb{P}(S^3 V^\vee) \dashrightarrow \mathcal{W}$ *of Corollary 9.3 is dominant.*

Proof. Consider a pair (θ, D) consisting of a nonsingular conic and a general cubic. By Theorem 10.4, the quartic $B(Z(\theta, D))$ is Lüroth. Moreover, again by Theorem 10.4, the lines $\ell_1, \dots, \ell_4$ associated to (θ, D) by Theorem 10.2 are components of a complete pentalateral inscribed in $B(Z(\theta, D))$. Then the map $\widetilde{Z}$ is defined by associating to (θ, D) the pair $(B(Z(\theta, D)), \{\ell_0, \ell_1, \dots, \ell_4\}) \in \widetilde{\mathcal{L}}$, where ℓ_0 is the fifth line of the complete pentalateral inscribed in $B(Z(\theta, D))$ having $\ell_1, \dots, \ell_4$ as components.

Consider a general $(\theta, D) \in \mathbb{P}(S^2 V^\vee) \times \mathbb{P}(S^3 V^\vee)$ and let

$$(B, \{\ell_0, \dots, \ell_4\}) = \widetilde{Z}(\theta, D).$$

By the definition of $\widetilde{Z}$ it follows that for some $0 \le k \le 4$ the lines $\ell_0, \dots, \hat{\ell}_k, \dots, \ell_4$ are simultaneously apolar to θ and D. We may assume that $k = 0$ and choose coordinates so that $\ell_1 + \ell_2 + \ell_3 + \ell_4 = 0$. Then from the proof of Theorem 10.4 it follows that B has equations of the form (28), which in our notation takes the form

$$\frac{1}{b_1\ell_1}+\frac{1}{b_2\ell_2}+\frac{1}{b_3\ell_3}+\frac{1}{b_4\ell_4}+\frac{1}{L}=0,$$

where $b_1,\dots,b_4$ are the uniquely defined nonzero coefficients such that $D = b_1\ell_1^3+\cdots+b_4\ell_4^3$ and L is a linear combination of $\ell_1,\dots,\ell_4$ given by (27), which now takes the form

$$L=-\left(\sum_{k=1}^{4}\frac{a_k}{b_k}\right)^{-2}\left(\sum_{k=1}^{4}\frac{a_k^2\ell_k}{b_k}\right), \tag{29}$$

where $a_1,\dots,a_4$ are the uniquely determined nonzero coefficients such that $\theta = a_1\ell_1^2+\cdots+a_4\ell_4^2$. From these expressions it follows that D is uniquely determined by $(B,\{\ell_0,\dots,\ell_4\})$. The coefficients of the linear combination (29) are rational functions of $a_1,\dots,a_4$, homogeneous of degree 0, which can be interpreted as follows. Let $\mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ be defined by sending

$$(a_1,\dots,a_4)\mapsto\left(\left(\sum_{k=1}^{4}\frac{a_k}{b_k}\right)^2,\frac{a_1^2}{b_1},\dots,\frac{a_4^2}{b_4}\right).$$

Because this is the composition of a Veronese map with a projection, it is finite on its set of definition. From this remark it follows that, given $b_1,\dots,b_4$, there are finitely many $a_1,\dots,a_4$ defining L. This shows that (θ,D) is isolated in $\widetilde{Z}^{-1}(B,\{\ell_0,\dots,\ell_4\})$. Since its domain and range are irreducible of dimension 14, this proves that $\widetilde{Z}$ is dominant and generically finite. The assertions (i) and (ii) are now obvious once we recall that $\mathcal{W}$ is irreducible (see Corollary 5.4). □

Our final and main result is as follows.

THEOREM 10.6. *The hypersurface $\mathcal{L}\subset\mathbb{P}(S^4V^\vee)$ has degree 54.*

Proof. From Corollary 10.5 we deduce that the hypersurface of Lüroth quartics can be identified with the hypersurface $\mathcal{M}$ of Morley quartics. In particular, we have $\deg(\mathcal{L})=\deg(\mathcal{M})=54$. □

REMARK 10.7. We recall the following construction from [8]. Given an Aronhold system $\{t_1,\dots,t_7\}$ of bitangents for a nonsingular plane quartic B, consider them as odd theta characteristics, and let H be the divisor on C cut by a line. The 35 divisors $t_i+t_j+t_k-H$, $1\le i<j<k\le 7$, define 35 distinct even theta characteristics. Since there are 36 even theta characteristics, we may denote the remaining one by $t(t_1,\dots,t_7)$.

We come back to the Cremona planes of Section 6. Given a nonsingular cubic surface $S\subset\mathbb{P}^3$ and a double-six $\Delta=(A_1,\dots,A_6;B_1,\dots,B_6)$, a point $P\in S$ belongs to the corresponding Cremona plane Ξ if and only if the quartic branch curve B of the rational projection with center P is Lüroth and the vertices $v_1,\dots,v_{10}\in B$ of an inscribed pentalateral satisfy $v_1+\cdots+v_{10}\in|2H+t(t_1,\dots,t_7)|$, where $\{t_1,\dots,t_7\}$ is the Aronhold system consisting of the bitangents that are projections of $A_1,\dots,A_6,P$. It is natural to call $t(t_1,\dots,t_7)$ *the pentalateral even theta characteristic* on B.

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