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On a Construction of L. Hua for Positive Reproducing Kernels

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1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain (i.e., a connected open set). Following the general rubric of "Hilbert space with reproducing kernel" laid down by N. Aronszajn [Aro], both the Bergman space $A^2(\Omega)$ and the Hardy space $H^2(\Omega)$ have reproducing kernels. We shall provide the details of these assertions below.

The Bergman kernel (for A^2) and the Szegő kernel (for H^2) both have the advantage of being canonical. But neither is positive, and this makes them tricky to handle. The Bergman kernel can be treated with the theory of the Hilbert integral (see [PSt]), and the Szegő kernel can often be handled with a suitable theory of singular integrals (see [K7]).

It is a classical construction of Hua [H] that one can use the Szegő kernel to produce another reproducing kernel $\mathcal{P}(z, \zeta)$ that also reproduces H^2 but is positive; in this sense, it is more like the Poisson kernel of harmonic function theory. In fact, this so-called Poisson–Szegő kernel coincides with the Poisson kernel when the domain is the disc D in the complex plane $\mathbb{C}$. Furthermore, the Poisson–Szegő kernel solves the Dirichlet problem for the invariant Laplacian (i.e., the Laplace–Beltrami operator for the Bergman metric) on the ball in $\mathbb{C}^n$. Unfortunately, a similar statement about the Poisson–Szegő kernel cannot be made on any other domain (although in this paper we explore substitute results on strongly pseudoconvex domains).

Our aim is to develop these ideas with the Szegő kernel replaced by the Bergman kernel. This notion was developed independently by Berezin [Ber] in the context of quantization of Kähler manifolds. Indeed, one assigns to a bounded function on the manifold the corresponding Toeplitz operator. This process of assigning a linear operator to a function is called *quantization*. A nice exposition of the ideas appears in [Pe]; additional basic properties may be found in [Z].

Approaches to the Berezin transform may be operator-theoretic (see [E1; E2]) or geometric [Pe]. The point of view taken in this paper will be more function-theoretic. We shall repeat (in perhaps new language) some results that are known in other contexts. We shall also enunciate and prove new results; many of the

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theorems in Sections 4 and 5 appear here for the first time (at least in this formulation). We hope that the mix serves to be both informative and useful.

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2. Fundamental Ideas

For $\Omega \subseteq \mathbb{C}^n$ a bounded domain, set

$$A^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \int_\Omega |f(z)|^2 \, dV(z) < \infty \right\},$$

where dV is the standard Euclidean volume measure on Ω. As usual, A^2 is equipped with the inner product

$$\langle f, g \rangle_{A^2(\Omega)} = \int_\Omega f(z)\overline{g(z)} \, dV(z).$$

Then A^2 is a subspace of $L^2(\Omega)$, and it can be shown (see [K5]) that $A^2(\Omega)$ is a Hilbert space. The following lemma is key.

LEMMA 2.1. *Let $K \subseteq \Omega$ be a compact subset of $\Omega \subseteq \mathbb{C}^n$. There is a constant $C = C(K, n)$ such that, if $f \in A^2(\Omega)$, then*

$$\sup_{z \in K} |f(z)| \le C \cdot \|f\|_{A^2(\Omega)}.$$

We shall not prove the lemma here; see [K5] or [K6] for the details.

Now if $z \in \Omega$ is a fixed point then, by applying the lemma with $K = \{z\}$, we find that the linear functional

$$e_z \colon A^2(\Omega) \ni f \mapsto f(z)$$

is bounded. Then, by the Riesz representation theorem, there exists a function $k_z \in A^2(\Omega)$ such that

$$f(z) = e_z(f) = \langle f, k_z \rangle$$

for all $f \in A^2(\Omega)$. We set $K(z, \zeta) = K_\Omega(z, \zeta) = \overline{k_z(\zeta)}$ and write

$$f(z) = \int_\Omega K(z, \zeta) f(\zeta) \, dV(\zeta).$$

This is the Bergman reproducing formula, and $K(z, \zeta)$ is the Bergman (reproducing) kernel.

There is a similar theory for H^2. Fix a bounded domain Ω, and define

$$H^2(\Omega) = \{ f \text{ holomorphic on } \Omega : |f|^2 \text{ has a harmonic majorant on } \Omega \}.$$

This definition is equivalent to several other natural definitions of H^2; see [K3] for the details. In particular, it can be shown that an H^2 function f has an $L^2(\partial\Omega)$ boundary function $\tilde{f}$ and that f is the Poisson integral of $\tilde{f}$. It is convenient to set $\|f\|_{H^2(\Omega)} = \|\tilde{f}\|_{L^2(\partial\Omega)}$. This definition of the norm is equivalent to several other standard definitions (see [K3]).

We now have the following fundamental lemma.

LEMMA 2.2. *Let $K \subseteq \Omega$ be a compact subset of $\Omega \subseteq \mathbb{C}^n$. There is a constant $C' = C'(K, n)$ such that, if $f \in H^2(\Omega)$, then*

$$\sup_{z \in K} |f(z)| \leq C' \cdot \|f\|_{H^2(\Omega)}.$$

Again, details of the proof are omitted.

As a consequence, if a point $z \in \Omega$ is fixed, then we can be sure that the functional

$$e'_z : H^2(\Omega) \ni f \mapsto f(z)$$

is bounded. Then, by the Riesz representation theorem, there exists a function $k'_z \in A^2(\Omega)$ such that

$$f(z) = e'_z(f) = \langle f, k'_z \rangle$$

for all $f \in H^2(\Omega)$. We set $S(z, \zeta) = S_\Omega(z, \zeta) = \overline{k'_z(\zeta)}$ and write

$$f(z) = \int_{\partial\Omega} S(z, \zeta) f(\zeta) \, d\sigma(\zeta),$$

where $d\sigma$ is the standard area measure (i.e., the Hausdorff measure) on $\partial\Omega$. This is the Szegő reproducing formula, and $S(z, \zeta)$ is the Szegő (reproducing) kernel.

The projection

$$P_B : L^2(\Omega) \to A^2(\Omega)$$

is well-defined by

$$P_B f(z) = \int_\Omega K(z, \zeta) f(\zeta) \, dV(\zeta);$$

likewise, the projection

$$P_S : L^2(\partial\Omega) \to H^2(\Omega)$$

is well-defined by

$$P_S f(z) = \int_{\partial\Omega} S(z, \zeta) f(\zeta) \, d\sigma(\zeta).$$

These two facts establish the centrality and importance of the kernels K and S. But neither kernel is positive, which makes their analysis difficult.

3. Positive Kernels

In a seminal work, L. Hua [H] proposed a program for producing a positive kernel from a canonical kernel. He defined

$$\mathcal{P}(z, \zeta) = \frac{|S(z, \zeta)|^2}{S(z, z)},$$

where S is the standard Szegő kernel on a given bounded domain Ω.

PROPOSITION 3.1. *Let Ω be a bounded domain with C^2 boundary and S its Szegő kernel. With $\mathcal{P}(z, \zeta)$ as just defined and with $f \in C(\overline{\Omega})$ holomorphic on Ω, we have*

$$f(z) = \int_{\partial\Omega} \mathcal{P}(z,\zeta) f(\zeta)\, d\sigma(\zeta)$$

for all $z \in \Omega$.

Proof. Fix $z \in \Omega$. Define $g(\zeta) = \overline{S(z,\zeta)} \cdot f(\zeta)/S(z,z)$. Then it is easy to see that $g \in H^2(\Omega)$ as a function of ζ. As a result,

$$\begin{aligned}\int_{\partial\Omega} f(\zeta)\mathcal{P}(z,\zeta)\, d\sigma(\zeta) &= \int_{\partial\Omega} \left[f(\zeta) \cdot \frac{\overline{S(z,\zeta)}}{S(z,z)} \right] \cdot S(z,\zeta)\, d\sigma(\zeta) \\ &= \int_{\partial\Omega} g(\zeta) \cdot S(z,\zeta)\, d\sigma(\zeta) \\ &= g(z) \\ &= f(z). \end{aligned}$$

□

Observe that the continuity of f on $\overline{\Omega}$ is used to guarantee that $g \in H^2$. It is natural to ask whether the result of the proposition extends to all functions $f \in H^2(\Omega)$. For this, it would suffice to show that $C(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ is dense in $H^2(\Omega)$. This density result is known to be true, because of the regularity theory for the $\bar{\partial}_b$ operator, when Ω is either strongly pseudoconvex or of finite type in the sense of Catlin/D'Angelo/Kohn. One can reason as follows (we thank Harold Boas for this argument). Let $f \in H^2(\Omega)$. Then certainly $f \in L^2(\partial\Omega)$ and, just by measure theory, one can approximate f in L^2 norm by a function $\varphi \in C^\infty(\partial\Omega)$. Let $\Phi = P_S\varphi$, the Szegő projection of φ. Then, since P_S is a continuous operator on $L^2(\partial\Omega)$, it follows that the function Φ is an $L^2(\partial\Omega)$ approximant of f. But it is also the case, by regularity theory of the $\bar{\partial}_b$ operator, that $\Phi = P_S\varphi$ is in $C^\infty(\overline{\Omega})$. This proves the needed approximation result. Naturally, a similar argument would apply to any domain on which the Szegő projection maps smooth functions to smooth functions. See [St] for some observations about this matter.

Hua did not consider his construction for the Bergman kernel, but in fact it is no less valid in that context. Define

$$\mathcal{B}(z,\zeta) = \frac{|K(z,\zeta)|^2}{K(z,z)},$$

which we call the *Poisson–Bergman kernel.*

PROPOSITION 3.2. *Let Ω be a bounded domain and K its Bergman kernel. With $\mathcal{B}(z,\zeta)$ as just defined and with $f \in C(\overline{\Omega})$ holomorphic on Ω, we have*

$$f(z) = \int_{\partial\Omega} \mathcal{B}(z,\zeta) f(\zeta)\, dV(\zeta)$$

for all $z \in \Omega$.

The proof is the same as that for Proposition 3.1, and we omit the details. One of our purposes in this paper is to study properties of the Poisson–Bergman kernel $\mathcal{B}$.

Because the Poisson–Bergman kernel is real, it will also reproduce the real parts of holomorphic functions. Thus, in one complex variable, the integral reproduces harmonic functions. In several complex variables, it reproduces pluriharmonic functions.

Again, it is natural to ask under what circumstances Proposition 3.2 holds for all functions in the Bergman space $A^2(\Omega)$. The question is virtually equivalent to asking when the elements that are continuous on $\overline{\Omega}$ are dense in A^2. Catlin [Ca] gave an affirmative answer to this query on any smoothly bounded pseudoconvex domain.

One of the features that makes the Bergman kernel both important and useful is its invariance under biholomorphic mappings. This fact is useful in conformal mapping theory, and it also gives rise to the Bergman metric. The fundamental result is as follows.

PROPOSITION 3.3. *Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$, and let $f\colon \Omega_1 \to \Omega_2$ be biholomorphic. Then*

$$\det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), f(\zeta)) \det \overline{J_{\mathbb{C}} f(\zeta)} = K_{\Omega_1}(z, \zeta).$$

Here $J_{\mathbb{C}} f$ is the complex Jacobian matrix of the mapping f (see [K3; K6] for more on this topic).

It is useful to know that the Poisson–Bergman kernel satisfies a similar transformation law.

PROPOSITION 3.4. *Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$, and let $f\colon \Omega_1 \to \Omega_2$ be biholomorphic. Then*

$$\mathcal{B}_{\Omega_2}(f(z), f(\zeta)) |\det J_{\mathbb{C}} f(\zeta)|^2 = \mathcal{B}_{\Omega_1}(z, \zeta).$$

Proof. Given the result of Proposition 3.3, we have

$$\begin{aligned}
\mathcal{B}_{\Omega_1}(z, \zeta) &= \frac{|K_{\Omega_1}(z, \zeta)|^2}{K_{\Omega_1}(z, z)} \\
&= \frac{|\det J_{\mathbb{C}} f(z) \cdot K_{\Omega_2}(f(z), f(\zeta)) \cdot \overline{\det J_{\mathbb{C}} f(\zeta)}|^2}{\det J_{\mathbb{C}} f(z) \cdot K_{\Omega_2}(f(z), f(z)) \cdot \overline{\det J_{\mathbb{C}} f(z)}} \\
&= \frac{|\det J_{\mathbb{C}} f(\zeta)|^2 \cdot |K_{\Omega_2}(f(z), f(\zeta))|^2}{K_{\Omega_2}(f(z), f(z))} \\
&= |\det J_{\mathbb{C}} f(\zeta)|^2 \cdot \mathcal{B}_{\Omega_2}(f(z), f(\zeta)). \qquad \square
\end{aligned}$$

We conclude this section with an interesting observation about the Berezin transform (see [Z]).

PROPOSITION 3.5. *The operator*

$$\mathcal{B} f(z) = \int_B \mathcal{B}(z, \zeta) f(\zeta)\, dV(\zeta),$$

acting on $L^1(B)$, is univalent.

Proof. It is useful to take advantage of the symmetry of the ball. We can rewrite the Poisson–Bergman integral as

$$\int_B f \circ \Phi_z(\zeta)\, dV(\zeta),$$

where Φ_z is a suitable automorphism of the ball. Then it is clear that this integral can be identically zero in z only if $f \equiv 0$, which completes the proof.

Another, slightly more abstract, way to look at this matter is as follows (we thank Richard Rochberg for this idea; see also [E1]). Let f be any L^1 function on B. For $w \in B$, define

$$g_w(\zeta) = \frac{1}{(1 - \bar{w} \cdot \zeta)^{n+1}}.$$

If f is bounded on the ball, let

$$T_f : g \mapsto P_B(fg).$$

Then we may write the Berezin transform as

$$\Lambda f(w, z) = \frac{\langle T_f g_z, g_w \rangle}{\langle g_w, g_w \rangle}.$$

This function is holomorphic in z and conjugate holomorphic in w. The statement that the Berezin transform $\mathcal{B}f(\cdot) \equiv 0$ is the same as $\Lambda f(z, z) = 0$. But it is a standard fact (see [K3]) that we may thus conclude $\Lambda f(w, z) \equiv 0$. But then $T_f g_z \equiv 0$ and so $f \equiv 0$; therefore, the Berezin transform is univalent. □

4. Boundary Behavior

It is natural to want information about the boundary limits of potentials of the form $\mathcal{B}f$ for $f \in L^2(\Omega)$. We begin with a simple lemma.

LEMMA 4.1. *Let Ω be a bounded domain and $\mathcal{B}$ its Poisson–Bergman kernel. If $z \in \Omega$ is fixed, then*

$$\int_\Omega \mathcal{B}(z, \zeta)\, dV(\zeta) = 1.$$

Proof. Certainly the function $f(\zeta) \equiv 1$ is an element of the Bergman space on Ω. Consequently,

$$1 = f(z) = \int_\Omega \mathcal{B}(z, \zeta) f(\zeta)\, dV(\zeta) = \int_\Omega \mathcal{B}(z, \zeta)\, dV(\zeta)$$

for any $z \in \Omega$. □

Our first result is as follows.

PROPOSITION 4.2. *Let Ω be the ball B in $\mathbb{C}^n$. Then the mapping*

$$f \mapsto \int_\Omega \mathcal{B}(z, \zeta) f(\zeta)\, dV(\zeta)$$

sends $L^p(\Omega)$ to $L^p(\Omega)$ for $1 \le p \le \infty$.

Proof. We know from Lemma 4.1 that

$$\|\mathcal{B}(z,\cdot)\|_{L^1(\Omega)} = 1$$

for each fixed z. An even easier estimate shows that

$$\|\mathcal{B}(\cdot,\zeta)\|_{L^1(\Omega)} \le 1$$

for each fixed ζ. Now Schur's lemma, or the generalized Minkowski inequality, gives the desired conclusion. □

PROPOSITION 4.3. *Let $\Omega \subseteq \mathbb{C}^n$ be the unit ball B. Let $f \in C(\overline{\Omega})$, and let $F = \mathcal{B}f$. Then F extends to a function that is continuous on $\overline{\Omega}$. Moreover, if $P \in \partial\Omega$ then*

$$\lim_{\Omega \ni z \to P} F(z) = f(P).$$

Proof. Let $\varepsilon > 0$. Choose $\delta > 0$ such that, if $z, w \in \overline{\Omega}$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$. Let $M = \sup_{\zeta \in \overline{\Omega}} |f(\zeta)|$. Now, for $z \in \Omega$, $P \in \partial\Omega$, and $|z - P| < \varepsilon$, we have

$$\begin{aligned}
|F(z) - f(P)| &= \left| \int_\Omega \mathcal{B}(z,\zeta) f(\zeta)\, dV(\zeta) - f(P) \right| \\
&= \left| \int_\Omega \mathcal{B}(z,\zeta) f(\zeta)\, dV(\zeta) - \int_\Omega \mathcal{B}(z,\zeta) f(P)\, dV(\zeta) \right| \\
&\le \int_{\substack{\zeta\in\Omega \\ |\zeta - P| < \delta}} \mathcal{B}(z,\zeta) |f(\zeta) - f(P)|\, dV(\zeta) \\
&\quad + \int_{\substack{\zeta\in\Omega \\ |\zeta - P| \ge \delta}} \mathcal{B}(z,\zeta) |f(\zeta) - f(P)|\, dV(\zeta) \\
&\le \int_{\substack{\zeta\in\Omega \\ |\zeta - P| < \delta}} \mathcal{B}(z,\zeta) \cdot \varepsilon\, dV(\zeta) + \int_{\substack{\zeta\in\Omega \\ |\zeta - P| \ge \delta}} \mathcal{B}(z,\zeta) \cdot 2M\, dV(\zeta) \\
&\equiv \mathrm{I} + \mathrm{II}.
\end{aligned}$$

According to Lemma 4.1, $\mathrm{I} = \varepsilon$. We also know that the Poisson–Bergman kernel for the ball is

$$\mathcal{B}(z,\zeta) = c_n \frac{(1 - |z|^2)^{n+1}}{|1 - z \cdot \bar{\zeta}|^{2n+2}}.$$

Thus, by inspection, $\mathcal{B}(z,\zeta) \to 0$ as $z \to P$ for $|\zeta - P| \ge \delta$. Hence II is smaller than ε as soon as z is close enough to P.

In summary, for z sufficiently close to P, we have $|F(z) - f(P)| < 2\varepsilon$. This is what we wished to prove. □

Arazy and Engliš [ArE] showed that the last result is true on any pseudoconvex domain for which each boundary point is a peak point (for the algebra $A(\Omega)$ of functions continuous on the closure and holomorphic inside). Thus the result is true in particular on strongly pseudoconvex domains (see [K3]) and finite type domains in $\mathbb{C}^2$ (see [BeFo]).

Here is another way to look at the matter on strongly pseudoconvex domains. Our observation, at the end of the proof of Proposition 4.3, about the vanishing of $\mathcal{B}(z,\zeta)$ for $z \to P$ and $|\zeta - P| \geq \delta$ is actually a tricky point that is not generally known. On a strongly pseudoconvex domain Ω we have Fefferman's asymptotic expansion [Fe]. This states that, in suitable local holomorphic coordinates near a boundary point P,

$$K_\Omega(z,\zeta) = \frac{c_n}{(1 - z \cdot \bar{\zeta})^{n+1}} + k(z,\zeta) \cdot \log|1 - z \cdot \bar{\zeta}|.$$

Thus, using an argument quite similar to the one carried out in Section 5 for the Poisson–Szegő kernel, one can obtain an asymptotic expansion for the Poisson–Bergman kernel. It is clear that, in local coordinates near the boundary.

$$\mathcal{B}_\Omega(z,\zeta) = c_n \cdot \frac{(1 - |z|^2)^{n+1}}{|1 - z \cdot \bar{\zeta}|^{2n+2}} + \mathcal{E}(z,\zeta),$$

where $\mathcal{E}$ is a kernel that induces a smoothing operator. In particular, the singularity of $\mathcal{E}$ will be measurably less than the singularity of the lead term. Thus it will still be the case that $\mathcal{B}(z,\zeta) \to 0$ as $z \to P \in \partial\Omega$ and $|\zeta - P| \geq \delta$. This leads to the following result.

PROPOSITION 4.4. *Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded and strongly pseudoconvex domain in $\mathbb{C}^n$. Let $f \in C(\overline{\Omega})$. Then the function $\mathcal{B}f$ extends to be continuous on $\overline{\Omega}$. Moreover, if $P \in \partial\Omega$ then*

$$\lim_{\Omega \ni z \to P} \mathcal{B}f(z) = f(P).$$

It is natural, from the point of view of measure theory and harmonic analysis, to want to extend the result of Proposition 4.4 to a broader class of functions. Toward this end, we introduce a maximal function to use as a tool.

DEFINITION 4.1. Let Ω be a smoothly bounded and strongly pseudoconvex domain in $\mathbb{C}^n$. If $z, \zeta \in \overline{\Omega}$ then we set

$$\rho(z,\zeta) = |1 - z \cdot \bar{\zeta}|^{1/2}.$$

PROPOSITION 4.5. *If $\Omega = B$, the unit ball, then the function ρ is a metric on ∂B. For a more general smoothly bounded and strongly pseudoconvex domain, the function ρ is a pseudometric. In other words, there is constant $C \geq 1$ such that*

$$\rho(z,\zeta) \leq C(\rho(z,\xi) + \rho(\xi,\zeta)).$$

Proof. The first assertion is [K2, Prop. 6.5.1]; the second assertion is proved in [K3, pp. 357–358]. □

PROPOSITION 4.6. *The balls*

$$\beta_2(z,r) = \{\zeta \in \Omega : \rho(z,\zeta) < r\},$$

together with the ordinary Euclidean volume measure dV, form a space of homogeneous type in the sense of Coifman and Weiss [CoW].

Proof. This is almost immediate from the preceding proposition; see [K3, Sec. 8.6] for details. □

DEFINITION 4.2. For $z \in \Omega$ and $f \in L^1_{\mathrm{loc}}(\Omega)$, we define

$$\mathcal{M}f(z) = \sup_{r>0} \frac{1}{V(\beta_2(z,r))} \int_{\beta_2(z,r)} |f(\zeta)|\, dV(\zeta).$$

THEOREM 4.7. *The operator $\mathcal{M}$ is of weak type $(1,1)$ and of strong type (p,p) for $1 < p \leq \infty$.*

Proof. This theorem is also a standard consequence of Proposition 4.6 in the context of spaces of homogeneous type; see [CoW]. □

THEOREM 4.8. *Let Ω be the unit ball B in $\mathbb{C}^n$, and let f be a locally integrable function on Ω. Then there is a constant $C > 0$ such that, for $z \in \Omega$,*

$$|\mathcal{B}f(z)| \leq C \cdot \mathcal{M}f(z).$$

Proof. It is easy to see that $|1 - z \cdot \bar{\zeta}| \geq (1/2)(1 - |z|^2)$. As a result, we may perform the following standard estimates:

$$\begin{aligned}
|\mathcal{B}f(z)| &= \left| \int_\Omega \mathcal{B}(z,\zeta) f(\zeta)\, dV(\zeta) \right| \\
&\leq \sum_{j=-1}^{\infty} \int_{2^j(1-|z|^2) \leq |1-z\cdot\bar{\zeta}| \leq 2^{j+1}(1-|z|^2)} \mathcal{B}(z,\zeta) |f(\zeta)|\, dV(\zeta) \\
&\leq \sum_{j=-1}^{\infty} \int_{|1-z\cdot\bar{\zeta}| \leq 2^{j+1}(1-|z|^2)} \frac{(1-|z|^2)^{n+1}}{[2^j(1-|z|^2)]^{2n+2}}\, dV(\zeta) \\
&\leq C \cdot \sum_{j=-1}^{\infty} 2^{-j(n+1)} \\
&\quad \cdot \left[\frac{1}{(1-|z|^2)^{n+1} 2^{(j+1)(n+1)}} \right] \int_{|1-z\cdot\bar{\zeta}| \leq 2^{j+1}(1-|z|^2)} |f(\zeta)|\, dV(\zeta) \\
&\leq C \cdot \sum_{j=-1}^{\infty} 2^{-j(n+1)} \\
&\quad \cdot \left[\frac{1}{V\big(\beta_2\big(z, \sqrt{2^{j+1}(1-|z|^2)}\,\big)\big)} \right] \int_{\beta_2\left(z, \sqrt{2^{j+1}(1-|z|^2)}\right)} |f(\zeta)|\, dV(\zeta).
\end{aligned}$$

The last line is majorized by

$$\begin{aligned}
&\leq C' \cdot \sum_{j=-1}^{\infty} 2^{-j(n+1)} \mathcal{M}f(z) \\
&\leq C \cdot \mathcal{M}f(z).
\end{aligned}$$

□

THEOREM 4.9. *Let Ω be the unit ball B in $\mathbb{C}^n$. Let f be an $L^p(\Omega, dV)$ function, $1 \leq p \leq \infty$. Then $\mathcal{B}f$ has radial boundary limits almost everywhere on $\partial\Omega$.*

Proof. The proof follows standard lines, using Theorems 4.6 and 4.7. See the detailed argument in [3, Thm. 8.6.11]. □

In fact, a slight emendation of the arguments just presented allows a more refined result.

DEFINITION 4.3. Let $P \in \partial B$ and $\alpha > 1$. Define the *admissible approach region of aperture* α by

$$\mathcal{A}_\alpha(P) = \{z \in B : |1 - z \cdot \bar{\zeta}| < \alpha(1 - |z|^2)\}.$$

Admissible approach regions are a new type of region for Fatou-type theorems. These were first introduced in [Ko1; Ko2] and then generalized and developed in [St] and later in [K1].

THEOREM 4.10. *Let f be an $L^p(B)$ function, $1 \le p \le \infty$. Then, for almost every $P \in \partial B$,*

$$\lim_{\mathcal{A}_\alpha(P) \ni z \to P} \mathcal{B}f(z)$$

exists.

In fact, using the Fefferman asymptotic expansion (discussed in Section 5), we may imitate the development of Theorems 4.6 and 4.7 and prove a result analogous to Theorem 4.8 on any smoothly bounded, strongly pseudoconvex domain. We omit the details, since they repeat ideas presented elsewhere in the paper for slightly different purposes.

5. Results on the Invariant Laplacian

If $g = (g_{jk})$ is a Riemannian metric on a domain Ω in complex Euclidean space, then there is a second-order partial differential operator, known as the *Laplace–Beltrami operator,* that is invariant under isometries of the metric. In fact, if g denotes the determinant of the metric matrix g and if $\left(g^{jk}\right)$ denotes the inverse matrix, then this partial differential operator is defined to be

$$\mathcal{L} = \frac{2}{g} \sum_{j,k} \left\{ \frac{\partial}{\partial \bar{z}_j} \left(g g^{jk} \frac{\partial}{\partial z_k} \right) + \frac{\partial}{\partial z_k} \left(g g^{jk} \frac{\partial}{\partial \bar{z}_k} \right) \right\}.$$

At this point we are interested in artifacts of the Bergman theory. If $\Omega \subseteq \mathbb{C}^n$ is a bounded domain and $K = K_\Omega$ its Bergman kernel, then it is well known [K3] that $K(z, z) > 0$ for all $z \in \Omega$. Then it makes sense to define

$$g_{jk}(z) = \frac{\partial^2}{\partial z_k \partial \bar{z}_k} \log K(z, z)$$

for $j, k = 1, \ldots, n$. Now Proposition 3.2 can be used to demonstrate that this metric—which is, in fact, a Kähler metric on Ω—is invariant under biholomorphic mappings of Ω. In other words, any biholomorphic $\Phi : \Omega \to \Omega$ is an isometry in the metric g. This is the celebrated *Bergman metric.*

If $\Omega \subseteq \mathbb{C}^n$ is the unit ball B, then the Bergman kernel is given by

$$K_B(z,\zeta) = \frac{1}{V(B)} \cdot \frac{1}{(1 - z \cdot \bar{\zeta})^{n+1}},$$

where $V(B)$ denotes the Euclidean volume of the domain B. Then

$$\log K(z,z) = -\log V(B) - (n+1)\log(1 - |z|^2).$$

Furthermore,

$$\frac{\partial}{\partial z_j}(-(n+1)\log(1-|z|^2)) = (n+1)\frac{\bar{z}_j}{1-|z|^2}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial z_j \partial \bar{z}_k}(-(n+1)\log(1-|z|^2)) &= (n+1)\left[\frac{\delta_{jk}}{1-|z|^2} + \frac{\bar{z}_j z_k}{(1-|z|^2)^2}\right] \\ &= \frac{n+1}{(1-|z|^2)^2}[\delta_{jk}(1-|z|^2) + \bar{z}_j z_k] \\ &\equiv g_{jk}(z).\end{aligned}$$

When $n = 2$ we have

$$g_{jk}(z) = \frac{3}{(1-|z|^2)^2}[\delta_{jk}(1-|z|^2) + \bar{z}_j z_k].$$

Therefore

$$\big(g_{jk}(z)\big) = \frac{3}{(1-|z|^2)^2}\begin{pmatrix} 1-|z_2|^2 & \bar{z}_1 z_2 \\ \bar{z}_2 z_1 & 1-|z_1|^2 \end{pmatrix}.$$

Let

$$\big(g^{jk}(z)\big)_{j,k=1}^2$$

represent the inverse of the matrix

$$\big(g_{jk}(z)\big)_{j,k=1}^2.$$

Then an elementary computation shows that

$$\big(g^{jk}(z)\big)_{j,k=1}^2 = \frac{1-|z|^2}{3}\begin{pmatrix} 1-|z_1|^2 & -z_2\bar{z}_1 \\ -z_1\bar{z}_2 & 1-|z_2|^2 \end{pmatrix} = \frac{1-|z|^2}{3}(\delta_{jk} - \bar{z}_j z_k)_{j,k}.$$

Let

$$g \equiv \det\big(g_{jk}(z)\big).$$

Then

$$g = \frac{9}{(1-|z|^2)^3}.$$

Now let us calculate. If $\big(g_{jk}\big)_{j,k=1}^2$ is the Bergman metric on the ball in $\mathbb{C}^2$, then

$$\sum_{j,k} \frac{\partial}{\partial \bar{z}_j}\big(g g^{jk}\big) = 0$$

and

$$\sum_{j,k} \frac{\partial}{\partial z_j}\big(gg^{jk}\big) = 0.$$

We shall verify these assertions in dimension 2. We have

$$\begin{aligned} gg^{jk} &= \frac{9}{(1-|z|^2)^3} \cdot \frac{1-|z|^2}{3}(\delta_{jk} - \bar{z}_j z_k) \\ &= \frac{3}{(1-|z|^2)^2}(\delta_{jk} - \bar{z}_j z_k). \end{aligned}$$

It follows that

$$\frac{\partial}{\partial \bar{z}_j}\big[gg^{jk}\big] = \frac{6z_j}{(1-|z|^2)^3}(\delta_{jk} - \bar{z}_j z_k) - \frac{3z_k}{(1-|z|^2)^2}.$$

Therefore

$$\begin{aligned} \sum_{j,k=1}^{2} \frac{\partial}{\partial \bar{z}_j}\big[gg^{jk}\big] &= \sum_{j,k=1}^{2}\left[\frac{6z_j(\delta_{jk} - \bar{z}_j z_k)}{(1-|z|^2)^3} - \frac{3z_j}{(1-|z|^2)^2}\right] \\ &= 6\sum_k \frac{z_k}{(1-|z|^2)^3} - 6\sum_{j,k} \frac{|z_j|^2 z_k}{(1-|z|^2)^3} - 6\sum_k \frac{z_k}{(1-|z|^2)^2} \\ &= 6\sum_j \frac{z_j}{(1-|z|^2)^2} - 6\sum_k \frac{z_k}{(1-|z|^2)^2} \\ &= 0. \end{aligned}$$

The other derivative is calculated similarly.

Our calculations show that, on the ball in $\mathbb{C}^2$,

$$\begin{aligned} \mathcal{L} &\equiv \frac{2}{g}\sum_{j,k}\left\{\frac{\partial}{\partial \bar{z}_j}\left(gg^{jk}\frac{\partial}{\partial z_k}\right) + \frac{\partial}{\partial z_k}\left(gg^{jk}\frac{\partial}{\partial \bar{z}_j}\right)\right\} \\ &= 4\sum_{j,k} g^{jk}\frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_k} \\ &= 4\sum_{j,k} \frac{1-|z|^2}{3}(\delta_{jk} - \bar{z}_j z_k)\frac{\partial^2}{\partial z_k \partial \bar{z}_j}. \end{aligned}$$

Now the interesting fact for us is encapsulated in the following proposition.

PROPOSITION 5.1. *The Poisson–Szegő kernel on the ball B solves the Dirichlet problem for the invariant Laplacian $\mathcal{L}$. That is to say, if f is a continuous function on ∂B then the function*

$$u(z) = \begin{cases} \int_{\partial B} \mathcal{P}(z,\zeta)\cdot f(\zeta)\,d\sigma(\zeta) & \text{if } z \in B, \\ f(z) & \text{if } z \in \partial B \end{cases}$$

is continuous on $\bar{B}$ and is annihilated by $\mathcal{L}$ on B.

This fact is of more than passing interest. In one complex variable, the study of holomorphic functions on the disc and the study of harmonic functions on the disc are inextricably linked because the real part of a holomorphic function is harmonic and conversely. Such is not the case in several complex variables. Certainly the real part of a holomorphic function is harmonic. But in fact it is more: such a function is *pluriharmonic*. For the converse direction, any real-valued pluriharmonic function is locally the real part of a holomorphic function. This assertion is false if "pluriharmonic" is replaced by "harmonic".

The result of Proposition 5.1 should not be surprising: the invariant Laplacian is invariant under isometries of the Bergman metric and hence invariant under automorphisms of the ball. Moreover, the Poisson–Szegő kernels behave nicely under automorphisms. Stein [St] took advantage of these invariance properties to prove Proposition 5.1 using Godement's theorem that any function satisfying a suitable mean-value property must be harmonic (i.e., annihilated by the relevant Laplace operator).

Sketch of the Proof of Proposition 5.1. We have

$$\mathcal{L}u = \mathcal{L}\int_{\partial B} \mathcal{P}(z,\zeta)\cdot f(\zeta)\,d\sigma(\zeta) = \int_{\partial B} [\mathcal{L}_z\mathcal{P}(z,\zeta)]\cdot f(\zeta)\,d\sigma(\zeta).$$

Hence it behooves us to calculate $\mathcal{L}_z\mathcal{P}(z,\zeta)$. Now we shall calculate this quantity for each fixed ζ. Thus, without loss of generality, we may compose with a unitary rotation and suppose that $\zeta = (1+i0, 0+i0)$ so that (in complex dimension 2)

$$\mathcal{P} = c_2 \cdot \frac{(1-|z|^2)^2}{|1-z_1|^4}.$$

This will make our calculations considerably easier.

By brute force, we find that

$$\begin{aligned}
\frac{\partial \mathcal{P}}{\partial \bar{z}_1} &= -2(1-z_1)(1-|z|^2)\cdot\left[\frac{-1+z_1+|z_2|^2}{|1-z_1|^6}\right],\\
\frac{\partial^2 \mathcal{P}}{\partial \bar{z}_1\partial z_1} &= \frac{-2}{|1-z_1|^6}\cdot\big[-|z_1|^2-|z_1|^2|z_2|^2+3|z_2|^2-z_1|z_2|^2\\
&\qquad -2|z_2|^4-1+z_1+\bar{z}_1-\bar{z}_1|z_2|^2\big],\\
\frac{\partial^2 \mathcal{P}}{\partial \bar{z}_1\partial z_2} &= \frac{-2(1-z_1)}{|1-z_1|^6}\cdot\big[2\bar{z}_2-\bar{z}_2 z_1-2\bar{z}_2|z_2|^2-\bar{z}_2|z_1|^2\big],\\
\frac{\partial^2 \mathcal{P}}{\partial z_1\partial \bar{z}_2} &= \frac{-2(1-\bar{z}_1)}{|1-z_1|^6}\cdot\big[2z_2-z_2\bar{z}_1-2z_2|z_2|^2-z_2|z_1|^2\big],\\
\frac{\partial \mathcal{P}}{\partial z_2} &= \frac{-2z_2+2|z_1|^2z_2+2|z_2|^2z_2}{|1-z_1|^4}, \quad \text{and}\\
\frac{\partial^2 \mathcal{P}}{\partial z_2\partial \bar{z}_2} &= \frac{-2+2|z_1|^2+4|z_2|^2}{|1-z_1|^4}.
\end{aligned} \tag{1}$$

We know that, in complex dimension 2,

$$
\begin{aligned}
&\mathcal{L}_z\mathcal{P}(z,\zeta)\\
&\quad = \frac{4}{3}(1-|z|^2)\cdot(1-|z_1|^2)\cdot\frac{\partial^2\mathcal{P}_z}{\partial z_1\partial\bar z_1} + \frac{4}{3}(1-|z|^2)\cdot(-\bar z_1 z_2)\cdot\frac{\partial^2\mathcal{P}_z}{\partial z_2\partial\bar z_1}\\
&\qquad + \frac{4}{3}(1-|z|^2)\cdot(-\bar z_2 z_1)\cdot\frac{\partial^2\mathcal{P}_z}{\partial z_1\partial\bar z_2} + \frac{4}{3}(1-|z|^2)\cdot(1-|z_2|^2)\cdot\frac{\partial^2\mathcal{P}_z}{\partial z_2\partial\bar z_2}.
\end{aligned}
$$

Plugging the values from (1) into this last equation gives

$$
\begin{aligned}
&\mathcal{L}_z\mathcal{P}(z,\zeta)\\
&\quad = \frac{4}{3}(1-|z|^2)\cdot(1-|z_1|^2)\cdot\frac{-2}{|1-z_1|^6}\\
&\qquad \times\left[-|z_1|^2-|z_1|^2|z_2|^2+3|z_2|^2-z_1|z_2|^2-2|z_2|^4-1+z_1+\bar z_1-\bar z_1|z_2|^2\right]\\
&\qquad + \frac{4}{3}(1-|z|^2)\cdot(-\bar z_1 z_2)\\
&\qquad \times\frac{-2(1-z_1)}{|1-z_1|^6}\cdot\left[2\bar z_2-\bar z_2 z_1-2\bar z_2|z_2|^2-\bar z_2|z_1|^2\right]\\
&\qquad + \frac{4}{3}(1-|z|^2)\cdot(-\bar z_2 z_1)\\
&\qquad \times\frac{-2(1-\bar z_1)}{|1-z_1|^6}\cdot\left[2z_2-z_2\bar z_1-2z_2|z_2|^2-z_2|z_1|^2\right]\\
&\qquad + \frac{4}{3}(1-|z|^2)\cdot(1-|z_2|^2)\cdot|1-z_1|^2\cdot\frac{-2+2|z_1|^2+4|z_2|^2}{|1-z_1|^6}.
\end{aligned}
$$

Multiplying out the terms, we find that

$$
\begin{aligned}
&\mathcal{L}_z\mathcal{P}(z,\zeta)\\
&\quad = \frac{-2}{|1-z_1|^6}\cdot\big[-|z_1|^2-4|z_1|^2|z_2|^2+3|z_2|^2-z_1|z_2|^2-2|z_2|^4-1\\
&\qquad\qquad +z_1+\bar z_1-\bar z_1|z_2|^2+|z_1|^4+|z_1|^4|z_2|^2+z_1|z_1|^2|z_2|^2\\
&\qquad\qquad +2|z_1|^2|z_2|^4+|z_1|^2-z_1|z_1|^2-\bar z_1|z_1|^2+\bar z_1|z_1|^2|z_2|^2\big]\\
&\qquad - \frac{2}{|1-z_1|^6}\cdot\big[-2\bar z_1|z_2|^2+3|z_1|^2|z_2|^2+2|z_2|^4\bar z_1+\bar z_1|z_2|^2|z_1|^2\\
&\qquad\qquad -z_1|z_1|^2|z_2|^2-2|z_1|^2|z_2|^4-|z_2|^2|z_1|^4\big]\\
&\qquad - \frac{2}{|1-z_1|^6}\cdot\big[-2z_1|z_2|^2+3|z_1|^2|z_2|^2+2|z_2|^4 z_1+z_1|z_2|^2|z_1|^2\\
&\qquad\qquad -\bar z_1|z_1|^2|z_2|^2-2|z_1|^2|z_2|^4-|z_2|^2|z_1|^4\big]\\
&\qquad - \frac{2}{|1-z_1|^6}\cdot\big[1-|z_1|^2-3|z_2|^2+|z_1|^2|z_2|^2+2|z_2|^4-z_1+z_1|z_1|^2\\
&\qquad\qquad +3z_1|z_2|^2-z_1|z_1|^2|z_2|^2-2z_1|z_2|^4-\bar z_1+\bar z_1|z_1|^2\\
&\qquad\qquad +3\bar z_1|z_2|^2-\bar z_1|z_1|^2|z_2|^2-2\bar z_1|z_2|^4+|z_1|^2-|z_1|^4\\
&\qquad\qquad -3|z_1|^2|z_2|^2+|z_1|^4|z_2|^2+2|z_1|^2|z_2|^4\big].
\end{aligned}
$$

And now, if we combine all the terms in brackets, a small miracle happens: everything cancels. The result is

$$\mathcal{L}_z \mathcal{P}(z, \zeta) \equiv 0. \qquad \square$$

Thus, in some respects, it is inappropriate to study holomorphic functions on the ball in $\mathbb{C}^n$ using the Poisson kernel. The classical Poisson integral does *not* create pluriharmonic functions, and it does not create functions that are annihilated by the invariant Laplacian. In view of Proposition 5.1, the Poisson–Szegő kernel is much more apposite. For instance, Koranyi [Ko2] made decisive use of this observation in his study (proving boundary limits of H^2 functions through admissible approach regions $\mathcal{A}_\alpha$) of the boundary behavior of $H^2(B)$ functions.

It is known that the property described in Proposition 5.1 is special to the ball—it is simply untrue on any other domain (see [Gr1; Gr2] for details). In this section we demonstrate that the result of the proposition can be extended—in an approximate sense—to a broader class of domains.

PROPOSITION 5.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded, strongly pseudoconvex domain, and let $\mathcal{P}$ be its Poisson–Szegő kernel. Then, if $f \in C(\partial\Omega)$ we may write*

$$\mathcal{P}f(z) = \mathcal{P}_1 f(z) + \mathcal{E}f(z),$$

where

(i) *the term $\mathcal{P}_1 f$ is "approximately annihilated" by the invariant Laplacian on Ω and*
(ii) *the operator $\mathcal{E}$ is smoothing in the sense of pseudodifferential operators.*

We shall explain the meaning of (i) and (ii) in the course of the proofs of these statements.

Proof of Proposition 5.2. We shall utilize the asymptotic expansion for the Szegő kernel on a smoothly bounded and strongly pseudoconvex domain (see [BoS; Fe]). Thus, for z and ζ near a boundary point P, we have (in suitable biholomorphic local coordinates)

$$S_\Omega(z, \zeta) = \frac{c_n}{(1 - z \cdot \bar{\zeta})^n} + h(z, \zeta) \cdot \log|1 - z \cdot \bar{\zeta}|. \tag{2}$$

Here h is a smooth function on $\bar{\Omega} \times \bar{\Omega}$.

Now we calculate $\mathcal{P}(z, \zeta)$ in the usual fashion:

$$\mathcal{P}_\Omega(z, \zeta) = \frac{|S(z, \zeta)|^2}{S(z, z)} = \frac{\left| \dfrac{c_n}{(1 - z \cdot \bar{\zeta})^n} + h(z, \zeta) \cdot \log|1 - z \cdot \bar{\zeta}| \right|^2}{\dfrac{c_n}{(1 - |z|^2)^n} + h(z, z) \cdot \log(1 - |z|^2)}. \tag{3}$$

One can use elementary algebra to simplify this expression and obtain that, in suitable local coordinates near the boundary,

$$\begin{aligned}
\mathcal{P}_\Omega(z,\zeta) &= c_n \cdot \frac{(1-|z|^2)^n}{|1-z\cdot\bar{\zeta}|^{2n}} \\
&\quad + \frac{2(1-|z|^2)^n}{|1-z\cdot\bar{\zeta}|^n}\log|1-z\cdot\bar{\zeta}| + \mathcal{O}[(1-|z|^2)^n\cdot\log|1-z\cdot\bar{\zeta}|] \\
&\equiv c_n\cdot\frac{(1-|z|^2)^n}{|1-z\cdot\bar{\zeta}|^{2n}} + \mathcal{E}(z,\zeta). \qquad (4)
\end{aligned}$$

The first expression on the right-hand side of (4) is (in the local coordinates in which we are working) the usual Poisson–Szegő kernel for the unit ball in $\mathbb{C}^n$. The second is an error term that we now analyze.

We claim that the error term is integrable in ζ, uniformly in z, and that the same can be said for the gradient (in the z variable) of the error term. The first of these statements is obvious, since both parts of the error term are clearly majorized by the Poisson–Szegő kernel itself. As for the second statement, we note that the gradient of the error gives rise to three types of terms:

$$\begin{aligned}
\nabla\mathcal{E} &\approx \frac{(1-|z|^2)^{n-1}}{|1-z\cdot\bar{\zeta}|^n}\cdot\log|1-z\cdot\bar{\zeta}| \\
&\quad + \frac{(1-|z|^2)^n}{|1-z\cdot\bar{\zeta}|^{n+1}}\cdot\log|1-z\cdot\bar{\zeta}| \\
&\quad + \frac{(1-|z|^2)^n}{|1-z\cdot\bar{\zeta}|^{n+1}} \\
&\equiv \mathrm{I}+\mathrm{II}+\mathrm{III}. \qquad (5)
\end{aligned}$$

It is clear by inspection that I and II are each majorized by the ordinary Poisson–Szegő kernel, so they are both integrable in ζ as claimed. As for III, we must calculate

$$\begin{aligned}
&\int_{\zeta\in\partial\Omega}\frac{(1-|z|^2)^{n-1}}{|1-z\cdot\bar{\zeta}|^{n+1}}\,d\sigma(\zeta) \\
&\quad\le \sum_{j=-1}^{\infty}\int_{2^j(1-|z|^2)\le|1-z\cdot\bar{\zeta}|\le 2^{j+1}(1-|z|^2)}\frac{(1-|z|^2)^{n-1}}{[2^j(1-|z|^2)]^{n+1}}\,d\sigma(\zeta) \\
&\quad\le \sum_{j=-1}^{\infty}\frac{1}{(1-|z|^2)^2}\int_{|1-z\cdot\bar{\zeta}|\le 2^{j+1}(1-|z|^2)} 2^{-j(n+1)}\,d\sigma(\zeta) \\
&\quad\le \sum_{j=-1}^{\infty} C\cdot\frac{2^{-j(n+1)}}{(1-|z|^2)^2}\cdot\left[\sqrt{2^{j+1}(1-|z|^2)}\,\right]^{2n-2}\cdot[2^{j+1}\cdot(1-|z|^2)] \\
&\quad\le \sum_{j=-1}^{\infty}\frac{1}{(1-|z|^2)^2}\cdot(1-|z|^2)^{n-1}\cdot(1-|z|^2)\cdot 2^{-j(n+1)}\cdot 2^{(j+1)(n-1)}\cdot 2^{j+1} \\
&\quad\le C\cdot 2^n(1-|z|^2)^{n-2}\cdot\sum_{j=-1}^{\infty}2^{-j} \\
&\quad< \infty.
\end{aligned}$$

Thus we see that the Poisson–Szegő kernel for our strongly pseudoconvex domain Ω can be expressed, in suitable local coordinates, as the Poisson–Szegő kernel for the ball plus an error term whose gradient induces a bounded operator on L^p. This means that the error term itelf maps L^p to a Sobolev space. In other words, it is a smoothing operator (and hence negligible from our point of view).

In fact, there are several fairly well-known results about the interaction between the Poisson–Bergman kernel and the invariant Laplacian. We summarize some of the basic ones here.

PROPOSITION 5.3. *Let f be a C^2 function on the unit ball that is annihilated by the invariant Laplacian $\mathcal{L}$. Then, for any $0 < r < 1$ and S the unit sphere,*

$$\int_S f(r\zeta)\, d\sigma(\zeta) = c(r) \cdot f(0).$$

Here $d\sigma$ is a rotationally invariant measure on the sphere S.

Proof. Once we replace f with the average of f over the orthogonal group, this just becomes a calculation to determine the exact value of the constant $c(r)$; see [R, p. 51]. □

PROPOSITION 5.4. *Suppose that f is a C^2 function on the unit ball B that is annihilated by the invariant Laplacian $\mathcal{L}$. Then f satisfies the identity $\mathcal{B}f = f$. In other words, for any $z \in B$,*

$$f(z) = \int_B \mathcal{B}(z, \zeta) f(\zeta)\, dV(\zeta).$$

Proof. We checked the result when $z = 0$ in Proposition 5.3. For a general z, compose with a Möbius transformation and use the biholomorphic invariance of the kernel and the differential operator $\mathcal{L}$. □

REMARK 5.5. It is a curious fact (see [AFR]) that the converse of Proposition 5.4 is true only in complex dimensions $1, 2, \ldots, 11$. It is false in dimensions 12 and higher.

Finally, we must address the question of whether the invariant Laplacian *for the domain* Ω annihilates the principal term on the right-hand side of (4). Observe that the biholomorphic change of variable that makes (4) valid is *local*; thus it is valid only on a small, smoothly bounded subdomain $\Omega' \subseteq \Omega$ that shares a piece of boundary with $\partial\Omega$. According to [Fe] (see also [GK1; GK2]), there is a smaller subdomain $\Omega'' \subseteq \Omega'$ (which also shares a piece of boundary with $\partial\Omega$ and $\partial\Omega'$) such that the Bergman metric of Ω' is close—in the C^2 topology—to the Bergman metric of Ω *on the smaller domain* Ω''. It follows that the Laplace–Beltrami operator $\mathcal{L}_{\Omega'}$ for the Bergman metric of Ω' will be close to the Laplace–Beltrami operator $\mathcal{L}_\Omega$ of Ω on the smaller subdomain Ω''. Now, on Ω', the operator $\mathcal{L}_{\Omega'}$ certainly annihilates the principal term of (4); therefore, on Ω'', the operator $\mathcal{L}_\Omega$ *nearly* annihilates the principal term of (4). We shall not calculate the exact sense in which this last statement is true, leaving the details for the interested reader to pursue.

This discussion completes the proof of Proposition 5.2. □

It is natural to wonder whether the Poisson–Bergman kernel $\mathcal{B}$ has any favorable properties with respect to important partial differential operators. We have the following positive result.

PROPOSITION 5.6. *Let $\Omega = B$, the unit ball in $\mathbb{C}^n$, and let $\mathcal{B} = \mathcal{B}_B(z, \zeta)$ be its Poisson–Bergman kernel. Then $\mathcal{B}$ is plurisubharmonic in the ζ variable.*

Proof. Fix a point $\zeta \in B$, and let Φ be an automorphism of B such that $\Phi(\zeta) = 0$. Then it follows from Proposition 3.4 that

$$\mathcal{B}_B(z,\zeta) = \mathcal{B}_B(\Phi(z),\Phi(\zeta)) \cdot |\det J_{\mathbb{C}}\Phi(\zeta)|^2 = \mathcal{B}_B(\Phi(z),0) \cdot |\det J_{\mathbb{C}}\Phi(\zeta)|^2. \tag{6}$$

We see that the right-hand side is an expression that is independent of ζ multiplied by a plurisubharmonic function. A formula similar to (6) appears in [H].

The same argument shows that $\mathcal{B}(\zeta,\zeta)$ is plurisubharmonic. □

6. Concluding Remarks

The idea of reproducing kernels in harmonic analysis is an old one. In fact, the Poisson and Cauchy kernels date back to the mid-nineteenth century.

The Cauchy integral formula is special in that its kernel, which is

$$\frac{1}{2\pi i} \cdot \frac{1}{\zeta - z},$$

is the same on any domain. A similar statement is *not* true for the Poisson kernel (but see [K1] for a study of the asymptotics of this kernel).

The complex reproducing kernels that are indigenous to several complex variables are much more subtle. It was only in 1974 that Fefferman was able to calculate Bergman kernel asymptotics on strongly pseudoconvex domains. Prior to that, the very specific calculations of Hua [H] on concrete domains with a great deal of symmetry was the standard in the subject. A variant of Fefferman's construction also applies to the Szegő kernel (see also [BoS]). Carrying out an analogous program on a more general class of domains has proved to be challenging.

The present paper is an invitation to study yet another kernel: the Poisson–Bergman kernel. This is a positive reproducing kernel for the Bergman space whose advancement here was inspired by the ideas in [H]. There are many questions about the role of this new kernel that remain unanswered. We hope to investigate these matters in future work.

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