Michigan Math. J. 59 (2010), 189–210

Boundedness for Commutators of Rough Hypersingular Integrals with Variable Kernels

Yanping Chen & Yong Ding

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ with area element $d\sigma(x')$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q \geq 1$, if Ω satisfies the following conditions:

(1) for any $x, z \in \mathbb{R}^n$, and $\lambda > 0$, $\Omega(x, \lambda z) = \Omega(x,z)$;

(2) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} := \sup_{x\in\mathbb{R}^n}\big(\int_{S^{n-1}}|\Omega(x,z')|^q\,d\sigma(z')\big)^{1/q} < \infty$, where $z' = z/|z|$ for any $z \in \mathbb{R}^n \setminus \{0\}$.

For $\gamma \geq 0$, we define the operator T_γ with variable kernel by

$$T_\gamma f(x) = \text{p.v.}\int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n+\gamma}} f(y)\,dy,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})$ satisfies

$$\int_{S^{n-1}} \Omega(x,z')Y_m(z')\,d\sigma(z') = 0 \quad \text{for any } x \in \mathbb{R}^n \tag{1.1}$$

for all spherical harmonic polynomials Y_m with degree $\leq [\gamma]$. In the sequel, we denote $T_0 = T$ when $\gamma = 0$ for simplicity.

Obviously, T is the singular integral operator with variable kernel, which was first studied by Calderón and Zygmund in [2]. They found that these operators connect closely to the problem about the second-order linear elliptic equations with variable coefficients. Calderón and Zygmund obtained the following result.

Theorem A (see [2] or [3]). *If $\Omega(x,z') \in L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})$, $q > 2(n-1)/n$, satisfies*

$$\int_{S^{n-1}} \Omega(x,z')\,d\sigma(z') = 0 \quad \textit{for any } x \in \mathbb{R}^n, \tag{1.2}$$

then there is a constant $C > 0$ such that $\|Tf\|_{L^2} \leq C\|f\|_{L^2}$.

In [16], for $\gamma > 0$ the operator T_γ is called the hypersingular integral operator with variable kernel. Chen, Fan, and Ying [4] extended Theorem A to the homogeneous

Received October 1, 2008. Revision received March 30, 2009.
The research was supported partly by NSFC, Grant nos. 10931001 and 10901017, and by SRFDP of China, Grant no. 20090003110018.

Sobolev space $\dot{L}^p_\gamma(\mathbb{R}^n)$. Before stating the result in [4], let us recall the definition of $\dot{L}^p_\gamma(\mathbb{R}^n)$.

DEFINITION 1. Let $\gamma \in \mathbb{R}$ and $1 < p < \infty$. The homogeneous Sobolev space $\dot{L}^p_\gamma(\mathbb{R}^n)$ is defined as the space of all tempered distributions f in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ for which the expression $(|\cdot|^\gamma \widehat{f}\,)^\vee$ is a function in $L^p(\mathbb{R}^n)$, where (and in the sequel) "$\widehat{\ }$" and "$^\vee$" denote the Fourier transform and inverse Fourier transform, respectively. For distributions f in $\dot{L}^p_\gamma(\mathbb{R}^n)$ we define $\|f\|_{\dot{L}^p_\gamma} = \|(|\cdot|^\gamma \widehat{f}\,)^\vee\|_{L^p}$.

THEOREM B (see [4]). *Let $\gamma \ge 0$. If (1.1) is satisfied by $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ with $q > \max\{1, \frac{2(n-1)}{n+2\gamma}\}$, then there is a constant $C > 0$ such that $\|T_\gamma f\|_{L^2} \le C\|f\|_{\dot{L}^2_\gamma}$.*

On the other hand, it is well known that the commutator of the Calderòn–Zygmund singular integral operator and a $\mathrm{BMO}(\mathbb{R}^n)$ function b plays an important role in characterizing the Hardy space $H^1(\mathbb{R}^n)$ and in studying the regularity of the solution of the second-order elliptic equations (see e.g. [6; 7; 8]).

To study the interior $W^{2,2}$ estimates for nondivergence elliptic second-order equation with discontinuous coefficients, Chiarenza, Frasca, and Longo [6] gave the $L^2(\mathbb{R}^n)$ boundedness of the commutator $[b,T]$ with variable kernel, which is defined by

$$\begin{aligned}[b,T]f(x) &:= b(x)Tf(x) - T(bf)(x)\\ &= \text{p.v.}\int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^n}(b(x)-b(y))f(y)\,dy,\end{aligned}$$

where $b \in \mathrm{BMO}(\mathbb{R}^n)$. That is,

$$\|b\|_* := \sup_Q \frac{1}{|Q|}\int_Q |b(y) - b_Q|\,dy < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ and where

$$b_Q = \frac{1}{|Q|}\int_Q b(x)\,dx.$$

THEOREM C (see [6]). *Suppose that $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times C^\infty(S^{n-1})$ satisfies (1.2). Then there is a constant $C > 0$ such that $\|[b,T]f\|_{L^2} \le C\|b\|_*\|f\|_{L^2}$.*

In [8], Di Fazio and Ragusa gave the weighted form of Theorem C. This was used to obtain the local regularity in Morrey spaces of the solution of the second-order elliptic equation with discontinuous coefficients in nondivergence form.

In [5] we proved that the conclusion of Theorem C holds even when the strong smoothness assumption in the second variate of $\Omega(x,z')$ is removed.

THEOREM D (see [5]). *If (1.2) is satisfied by $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ with $q > 2(n-1)/n$, then there is a constant $C > 0$ such that $\|[b,T]f\|_{L^2} \le C\|b\|_*\|f\|_{L^2}$.*

Given Theorem B and Theorem D, it is natural to ask whether the commutator $[b, T_\gamma]$ of T_γ is still from $\dot{L}^2_\gamma$ to L^2 with the same condition in Theorem B, where $\gamma \geq 0$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, and $[b, T_\gamma]$ is defined by

$$[b, T_\gamma] f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n+\gamma}} (b(x) - b(y)) f(y)\, dy.$$

The purpose of this paper is to give a positive answer to this question.

THEOREM 1. *Let* $0 < \gamma \leq \frac{n}{2}$ *and* $b \in \dot{L}^{n/\gamma}_\gamma \subset \mathrm{BMO}$. *Suppose that* (1.1) *is satisfied by* $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ *with* $q > \max\left\{1, \frac{2(n-1)}{n+2\gamma}\right\}$. *Then there is a constant* $C > 0$ *such that* $\|[b, T_\gamma] f\|_{L^2} \leq C \|b\|_{\dot{L}^{n/\gamma}_\gamma} \|f\|_{\dot{L}^2_\gamma}$.

REMARK 1.1. As shown in Section 2, the following relationships exist between the homogeneous Triebel–Lizorkin space $\dot{F}^{\gamma,2}_p(\mathbb{R}^n)$, the homogeneous Sobolev space $\dot{L}^p_\gamma(\mathbb{R}^n)$, and $\mathrm{BMO}(\mathbb{R}^n)$:

$$\dot{F}^{0,2}_\infty = \mathrm{BMO} \quad \text{and} \quad \dot{F}^{\gamma,2}_p = \dot{L}^p_\gamma \ \text{ for } 1 < p < \infty,\ \gamma \in \mathbb{R}.$$

Thus, at least formally, we may view $\mathrm{BMO}(\mathbb{R}^n)$ as a limit case of $\dot{L}^{n/\gamma}_\gamma(\mathbb{R}^n)$ as $\gamma \to 0$. In this sense, Theorem 1 is just an extension of Theorem D.

2. Definition and Some Lemmas

Let us begin by recalling the definitions of the homogeneous Triebel–Lizorkin space $\dot{F}^{s,q}_p$ and the Bony paraproduct.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function satisfying $\varphi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$ and $\varphi(\xi) = 0$ for $|\xi| > 1$. The function $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$ is in $C^\infty_c(\mathbb{R}^n)$, supported by $\left\{\frac{1}{2} \leq |\xi| \leq 2\right\}$, and satisfies the identity

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0.$$

We denote by Δ_j the convolution operator whose symbol is $\psi(2^{-j}\cdot)$. For $s \in \mathbb{R}$ and $1 < p, q < \infty$, the homogeneous Triebel–Lizorkin space is defined by

$$\|f\|_{\dot{F}^{s,q}_p} = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{sjq} |\Delta_j f|^q \right)^{1/q} \right\|_{L^p} < \infty.$$

To give the definition of the BMO–Triebel–Lizorkin space $\dot{F}^{s,q}_\infty$, let us first recall the definition of Carleson measure. A sequence of positive Borel measure $\{v_j\}_{j \in \mathbb{Z}}$ is said to be a Carleson measure in $\mathbb{R}^n \times \mathbb{Z}$ if there exists a constant $C > 0$ such that

$$\sum_{j \geq k} v_j(B) \leq C|B|$$

for all $k \in \mathbb{Z}$ and all balls B in $\mathbb{R}^n$ with radius 2^{-k}, where $|B|$ denotes the Lebesgue measure of B. The norm of the Carleson measure $v = \{v_j\}_{j \in \mathbb{Z}}$ is given by

$$|||v||| = \sup\left\{\frac{1}{|B|}\sum_{j\geq k} v_j(B)\right\},$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k}.

The homogeneous BMO–Triebel–Lizorkin space $\dot{F}^{s,q}_{\infty}$ ($1 \leq q < +\infty$) is the space of all distributions b for which the sequence $\{2^{sjq}|\Delta_j b(x)|^q dx\}_j$ is a Carleson measure (see [10]). The norm of b in $\dot{F}^{s,q}_{\infty}$ is given by

$$\|b\|_{\dot{F}^{s,q}_{\infty}} = \sup\left[\frac{1}{|B|}\sum_{j\geq k}\int_B 2^{sjq}|\Delta_j b(x)|^q\,dx\right]^{1/q},$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k}. The following facts are well known (see [10; 11]): for $1 < p, q < \infty$ and $s \in \mathbb{R}$,

(1) $\dot{F}^{0,2}_p = H^p$ for $0 < p \leq 1$ and $\dot{F}^{0,2}_p = L^p$ for $1 < p < \infty$;
(2) $\dot{F}^{s,2}_p = \dot{L}^p_s$ for $1 < p < \infty$;
(3) $\dot{F}^{0,2}_{\infty} = \text{BMO} \subset \dot{F}^{0,\infty}_{\infty}$.

Finally, let us recall the Bony decomposition. For functions f and g, the Bony paraproduct $\pi_f(g)$ of f and g is defined by

$$\pi_f(g) = \sum_{j\in\mathbb{Z}}(\Delta_j f)(G_{j-3}g),$$

where G_j is the convolution operator whose symbol is $\varphi(2^{-j}\xi)$. The following Bony decomposition is well known (see [1]):

$$fg = \pi_f(g) + \pi_g(f) + R(f,g), \tag{2.1}$$

where $R(f,g) = \sum_{|i-j|\leq 2}(\Delta_j f)(\Delta_i g)$.

For $\delta \in (0,1)$, the Riesz potential operator of order δ is defined on the space of tempered distributions modulo polynomials by setting

$$\widehat{I_\delta f}(\xi) = |\xi|^{-\delta}\hat{f}(\xi).$$

The Sobolev space $I_\delta(\text{BMO})$ is the image of BMO under I_δ. Strichwarz [14] has shown that

$$I_\delta(\text{BMO}) \subsetneq \text{Lip}_\delta.$$

Now we give some lemmas that will play an important role in the proof of Theorem 1.

LEMMA 2.1 (see [13]). *Let $n \geq 2$ and let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ have the form $f(x) = f_0(|x|)P(x)$, where $P(x)$ is a solid spherical harmonic of degree m. Then the Fourier transform of f has the form $\hat{f}(\xi) = F_0(|\xi|)P(\xi)$, where*

$$F_0(r) = 2\pi i^{-m} r^{-[(n+2m-2)/2]}\int_0^\infty f_0(s)J_{(n+2m-2)/2}(2\pi rs)s^{(n+2m)/2}\,ds,$$

$r = |\xi|$, and J_v is the Bessel function.

LEMMA 2.2. *Suppose that* $\gamma \ge 0$, $0 < \beta < 1$, $\alpha \in \mathbb{Z}$, *and* $m \in \mathbb{N}$. *Let* $\mathcal{H}_m$ *denote the space of surface spherical harmonics of degree* m *on* S^{n-1} *with its dimension* D_m, *and let* $\{Y_{m,j}\}_{j=1}^{D_m}$ *denote the normalized complete system in* $\mathcal{H}_m$. *Let*

$$\sigma_{\gamma,\alpha,m,j}(x) = \frac{Y_{m,j}(x')}{|x|^{n+\gamma}} \chi_{\{2^\alpha \le |x| \le 2^{\alpha+1}\}}(x).$$

Then

$$|\widehat{\sigma_{\gamma,\alpha,m,j}}(\xi)| \le C 2^{-\alpha\gamma} m^{-\lambda-1+\beta/2} \min\{|2^\alpha \xi|^{[\gamma]+1}, |2^\alpha \xi|^{-\beta/2}\} |Y_{m,j}(\xi')|, \tag{2.2}$$

$$|\widehat{\sigma_{\gamma,\alpha,m,j}}(\xi)| \le C |\xi|^\gamma m^{-\lambda-1} |Y_{m,j}(\xi')|, \tag{2.3}$$

$$|\nabla \widehat{\sigma_{\gamma,\alpha,m,j}}(\xi)| \le C 2^{(-\gamma+1)\alpha}, \tag{2.4}$$

where $\lambda = (n-2)/2$ *and* $\xi' = \xi/|\xi|$.

Proof. First we give the estimate (2.4). Since

$$\widehat{\sigma_{\gamma,\alpha,m,j}}(\xi) = \int_{2^\alpha}^{2^{\alpha+1}} \int_{S^{n-1}} Y_{m,j}(x') e^{-2\pi i r x' \cdot \xi} \, d\sigma(x') \, \frac{dr}{r^{1+\gamma}},$$

it follows from $\|Y_{m,j}\|_{L^2(S^{n-1})} = 1$ that

$$|\nabla \widehat{\sigma_{\gamma,\alpha,m,j}}(\xi)| \le C \int_{2^\alpha}^{2^{\alpha+1}} \int_{S^{n-1}} |Y_{m,j}(x')| \, d\sigma(x') \, \frac{dr}{r^\gamma} \le C 2^{(-\gamma+1)\alpha}.$$

To show (2.2) and (2.3), we set $P_{m,j}(x) = Y_{m,j}(x')|x|^m$. Then $P_{m,j}$ is a solid spherical harmonic of degree m and $\sigma_{\gamma,\alpha,m,j}(x) = |x|^{-n-\gamma-m} P_{m,j}(x) \chi_{\{2^\alpha \le |x| \le 2^{\alpha+1}\}}(x)$. Using Lemma 2.1 and noting that $\psi_0(|x|) := |x|^{-n-\gamma-m} \chi_{\{2^\alpha \le |x| \le 2^{\alpha+1}\}}(x)$ is a radial function in x, we have

$$\widehat{\sigma_{\gamma,\alpha,m,j}}(\xi) = \Psi_0(|\xi|) P_{m,j}(\xi) = Y_{m,j}(\xi') |\xi|^m \Psi_0(|\xi|),$$

where

$$\begin{aligned}
\Psi_0(r) &= 2\pi i^{-m} r^{-[(n+2m-2)/2]} \int_0^\infty \psi_0(s) J_{(n+2m-2)/2}(2\pi r s) s^{(n+2m)/2} \, ds \\
&= 2\pi i^{-m} r^{-[(n+2m-2)/2]} \int_{2^\alpha}^{2^{\alpha+1}} s^{-n-\gamma-m} J_{(n+2m-2)/2}(2\pi r s) s^{(n+2m)/2} \, ds \\
&= (2\pi)^{n/2+\gamma} i^{-m} r^{-m+\gamma} \int_{2\pi 2^\alpha r}^{2\pi 2^{\alpha+1} r} \frac{J_{(n+2m-2)/2}(t)}{t^{(n-2)/2+1+\gamma}} \, dt.
\end{aligned}$$

From this it follows that

$$\begin{aligned}
\widehat{\sigma_{\gamma,\alpha,m,j}}(\xi) &= (2\pi)^{n/2+\gamma} i^{-m} Y_{m,j}(\xi') |\xi|^\gamma \int_{2\pi 2^\alpha |\xi|}^{2\pi 2^{\alpha+1} |\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1+\gamma}} \, dt \\
&= (2\pi)^{n/2} i^{-m} Y_{m,j}(\xi') \Upsilon(\xi),
\end{aligned} \tag{2.5}$$

where

$$\Upsilon(\xi) = |\xi|^\gamma \int_{2\pi 2^\alpha |\xi|}^{2\pi 2^{\alpha+1} |\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1+\gamma}} \, dt.$$

Now we consider $\Upsilon(\xi)$ under three cases: (1) $2^\alpha|\xi| \le 1$; (2) $1 < 2^\alpha|\xi| < m+\lambda$; (3) $2^\alpha|\xi| \ge m+\lambda$.

Case 1. By a classical formula of the Bessel function (see [15, p. 48]),

$$\begin{aligned}|J_{m+\lambda}(t)| &= \left|\frac{(t/2)^{m+\lambda}}{\Gamma(m+\lambda+1/2)\Gamma(1/2)}\int_{-1}^{1}(1-r^2)^{m+\lambda-1/2}e^{itr}\,dr\right| \\ &\le C\frac{(t/2)^{m+\lambda}}{\Gamma(m+\lambda+1/2)}.\end{aligned}$$

Applying Stirling's formula for $x > 1$ yields

$$\sqrt{2\pi}x^{x-1/2}e^{-x} \le \Gamma(x) \le 2\sqrt{2\pi}x^{x-1/2}e^{-x}.$$

Thus, since $2^\alpha|\xi| \le 1$, we have

$$\begin{aligned}|\Upsilon(\xi)| &\le |\xi|^\gamma \int_{2\pi 2^\alpha|\xi|}^{2\pi 2^{\alpha+1}|\xi|} \frac{|J_{m+\lambda}(t)|}{t^{\lambda+1+\gamma}}\,dt \\ &\le \frac{C2^{-\alpha\gamma}}{2^{m+\lambda}\Gamma(m+\lambda+1/2)}\int_{2\pi 2^\alpha|\xi|}^{2\pi 2^{\alpha+1}|\xi|} t^{m-1}\,dt \\ &\le \frac{C2^{-\alpha\gamma}}{2^{m+\lambda}\Gamma(m+\lambda+1/2)}\cdot\frac{1}{m}(2\pi 2^{\alpha+1}|\xi|)^m \\ &\le C2^{-\alpha\gamma}\frac{(4\pi)^{[\gamma]+1}(2^\alpha|\xi|)^{[\gamma]+1}}{m}\frac{(2\pi 2^{\alpha+1}|\xi|)^{m-[\gamma]-1}}{2^{m+\lambda}\sqrt{2\pi}(m+\lambda+1/2)^{m+\lambda}e^{-m-\lambda}} \\ &\le C2^{-\alpha\gamma}(2^\alpha|\xi|)^{[\gamma]+1}m^{-\lambda-1}\frac{(4\pi)^m}{2^{m+\lambda}}\cdot\frac{e^{m+\lambda}}{(m+\lambda+1/2)^m} \\ &\le C2^{-\alpha\gamma}m^{-\lambda-1}(2^\alpha|\xi|)^{[\gamma]+1}. \end{aligned} \tag{2.6}$$

Before considering Case 2 and Case 3, we state the following fact:

$$\Upsilon(\xi) = C2^{-\alpha\gamma}\int_{2\pi 2^\alpha|\xi|}^{h}\frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\,dt + C2^{-(\alpha+1)\gamma}\int_{h}^{2\pi 2^{\alpha+1}|\xi|}\frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\,dt, \tag{2.7}$$

where $2\pi 2^\alpha|\xi| \le h \le 2\pi 2^{\alpha+1}|\xi|$. Moreover, by [2, Lemma 2] there exists a $C > 0$ such that, for any $0 \le a, b \le \infty$,

$$\left|\int_a^b \frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\,dt\right| \le Cm^{-\lambda-1}. \tag{2.8}$$

Case 2. By (2.8) and noting that $1 < 2^\alpha|\xi| < m+\lambda$, we have

$$\begin{aligned}\left|\int_h^{2\pi 2^{\alpha+1}|\xi|}\frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\,dt\right| &\le Cm^{-\lambda-1} \\ &\le C\left(1+\frac{\lambda}{m}\right)^{\beta/2}m^{-1-\lambda+\beta/2}(m+\lambda)^{-\beta/2} \\ &\le Cm^{-1-\lambda+\beta/2}(2^\alpha|\xi|)^{-\beta/2}.\end{aligned}$$

Similarly,

$$\left|\int_{2\pi 2^\alpha|\xi|}^{h} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\,dt\right| \le Cm^{-1-\lambda+\beta/2}(2^\alpha|\xi|)^{-\beta/2}.$$

Then by (2.7) we obtain

$$|\Upsilon(\xi)| \le C2^{-\alpha\gamma}m^{-1-\lambda+\beta/2}(2^\alpha|\xi|)^{-\beta/2}. \tag{2.9}$$

Case 3. Since $|J'_{m+\lambda}(t)| \le 1$ for $t > 0$, we can use the second mean-valued theorem and the following differential equation of $J_{m+\lambda}$ (see [15]),

$$\frac{J_{m+\lambda}(t)}{t^{\lambda+1}} = -\frac{J'_{m+\lambda}(t)}{t^{\lambda}(t^2-(m+\lambda)^2)} - \frac{J''_{m+\lambda}(t)}{t^{\lambda-1}(t^2-(m+\lambda)^2)},$$

to show that there exists $2\pi 2^\alpha|\xi| \le h_1 \le h \le 2\pi 2^{\alpha+1}|\xi|$ such that

$$\begin{aligned}\left|\int_{2\pi 2^\alpha|\xi|}^{h} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\,dt\right| &\le C\int_{2\pi 2^\alpha|\xi|}^{h} t^{-2-\lambda}\,dt + \frac{C}{(2^\alpha|\xi|)^{\lambda+1}}\left|\int_{2\pi 2^\alpha|\xi|}^{h_1} J''_{m+\lambda}(t)\,dt\right| \\ &\quad + \frac{C}{(2^\alpha|\xi|)^{\lambda+1}}\left|\int_{h_1}^{h} J''_{m+\lambda}(t)\,dt\right| \\ &\le C(2^\alpha|\xi|)^{-\lambda-1} \le Cm^{-1-\lambda+\beta/2}(2^\alpha|\xi|)^{-\beta/2},\end{aligned}$$

where we assume $2\pi 2^\alpha|\xi| \ge 2\pi(m+\lambda)$. Similarly, we can get

$$\int_{h}^{2\pi 2^{\alpha+1}|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\,dt \le Cm^{-1-\lambda+\beta/2}(2^\alpha|\xi|)^{-\beta/2}.$$

Then by (2.7) we have

$$|\Upsilon(\xi)| \le C2^{-\alpha\gamma}m^{-1-\lambda+\beta/2}(2^\alpha|\xi|)^{-\beta/2}. \tag{2.10}$$

Thus, (2.2) follows from (2.5), (2.6), (2.9), and (2.10). On the other hand, it is easy to check that

$$|\Upsilon(\xi)| \le Cm^{-\lambda-1}|\xi|^{\gamma} \tag{2.11}$$

by (2.6) (for $2^\alpha|\xi| \le 1$) and by (2.7) and (2.8) (for $2^\alpha|\xi| > 1$). Hence, (2.3) follows from (2.5) and (2.11), completing the proof of Lemma 2.2. □

LEMMA 2.3. *For* $\gamma \ge 0$, $0 < \delta < \infty$, $m \in \mathbb{N}$, *and* $j = 1, \dots, D_m$, *take* $B_{\gamma,\delta,m,j} \in C_0^\infty(\mathbb{R}^n)$ *such that* $\mathrm{supp}(B_{\gamma,\delta,m,j}) \subset \{\delta/2 \le |\xi| \le 2\delta\}$. *Let* $T_{\gamma,\delta,m,j}$ *be the multiplier operators defined by*

$$\widehat{T_{\gamma,\delta,m,j}f}(\xi) = B_{\gamma,\delta,m,j}(\xi)\widehat{f}(\xi), \quad j = 1, \dots, D_m.$$

Moreover, for $b \in \mathrm{BMO}$, *denote by* $[b, T_{\gamma,\delta,m,j}]$ *the commutator of* $T_{\gamma,\delta,m,j}$ *and* b. *Define* $T_{\gamma,\delta,m;b}$ *by*

$$T_{\gamma,\delta,m;b}f(x) = \left(\sum_{j=1}^{D_m}([b, T_{\gamma,\delta,m,j}]f(x))^2\right)^{1/2}.$$

If for some constants $0 < \beta < 1$, $\alpha \in \mathbb{N}$, *and* $0 < \tau \le 1$, $B_{\gamma,\delta,m,j}$ *satisfies the conditions*

$$|B_{\gamma,\delta,m,j}(\xi)| \le C(2^{-\alpha}\delta)^{\gamma} m^{-\lambda-1+\beta/2} \min\{\delta^{\tau}, \delta^{-\beta/2}\}|Y_{m,j}(\xi')|, \tag{2.12}$$

$$|B_{\gamma,\delta,m,j}(\xi)| \le C(2^{-\alpha}\delta)^{\gamma} m^{-\lambda-1}|Y_{m,j}(\xi')|, \tag{2.13}$$

$$|\nabla B_{\gamma,\delta,m,j}(\xi)| \le C(2^{-\alpha}\delta)^{\gamma}\delta^{-\gamma}, \tag{2.14}$$

then for any fixed $0 < v < 1$ *there exists a positive constant* $C = C(n, v)$ *such that*

$$\|T_{\gamma,\delta,m;b} f\|_{L^2} \le C(2^{-\alpha}\delta)^{\gamma} m^{(-1+\beta/2)v} \min\{\delta^{\tau v}, \delta^{-\beta v/2}\}\|b\|_*\|f\|_{L^2}. \tag{2.15}$$

Proof. Assume that $\|b\|_* = 1$. Take a $C_c^\infty(\mathbb{R}^n)$ radial function ϕ such that $\operatorname{supp}\phi \subset \{1/2 \le |x| \le 2\}$ and $\sum_{l\in\mathbb{Z}} \phi(2^{-l}|x|) = 1$ for any $|x| > 0$. Denote $\phi_0(x) = \sum_{l=-\infty}^{0} \phi(2^{-l}|x|)$ and $\phi_l(x) = \phi(2^{-l}|x|)$ for a positive integer l. Then $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp}\phi_0 \subset \{x : 0 < |x| \le 2\}$. Let $K_{\gamma,\delta,m,j}(x) = (B_{\gamma,\delta,m,j})^\vee(x)$, the inverse Fourier transform of $B_{\gamma,\delta,m,j}$, and let $K^l_{\gamma,\delta,m,j}(x) = K_{\gamma,\delta,m,j}(x)\phi_l(x)$ for $l = 0, 1, \ldots$; then

$$K_{\gamma,\delta,m,j}(x) = \sum_{l=0}^{\infty} K^l_{\gamma,\delta,m,j}(x).$$

Denote by $T^l_{\gamma,\delta,m,j}$ the convolution operator with kernel $K^l_{\gamma,\delta,m,j}$. Then the Minkowski inequality implies that

$$\begin{aligned}
\|T_{\gamma,\delta,m;b} f\|_{L^2} &= \left(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \left|\sum_{l=0}^{\infty} [b, T^l_{\gamma,\delta,m,j}] f(x)\right|^2 dx\right)^{1/2} \\
&\le \sum_{l=0}^{\infty} \left(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \left|[b, T^l_{\gamma,\delta,m,j}] f(x)\right|^2 dx\right)^{1/2} \\
&:= \sum_{l=0}^{\infty} \|T^l_{\gamma,\delta,m;b} f\|_{L^2},
\end{aligned} \tag{2.16}$$

where $T^l_{\gamma,\delta,m;b} f(x) = \left(\sum_{j=1}^{D_m} |[b, T^l_{\gamma,\delta,m,j}] f(x)|^2\right)^{1/2}$. By (2.16), to derive (2.15) it suffices to show that, for any fixed $0 < v < 1$, there exists a $\kappa > 0$ such that

$$\|T^l_{\gamma,\delta,m;b} f\|_{L^2} \le C(2^{-\alpha}\delta)^{\gamma} m^{(-1+\beta/2)v} 2^{-\kappa l} \min\{\delta^{\tau v}, \delta^{-\beta v/2}\}\|b\|_*\|f\|_{L^2}. \tag{2.17}$$

We shall prove (2.17) by an almost orthogonality decomposition. For $l \ge 0$, we decompose $\mathbb{R}^n = \bigcup_{d=-\infty}^{\infty} Q_d$, where the Q_d are nonoverlapping cubes with side length 2^l. Set $f_d = f\chi_{Q_d}$. Then

$$f(x) = \sum_{d=-\infty}^{\infty} f_d(x) \text{ a.e.,} \quad x \in \mathbb{R}^n.$$

Since $\operatorname{supp}(K^0_{\gamma,\delta,m,j}) \subset \{x : |x| \le 2\}$ and $\operatorname{supp}(K^l_{\gamma,\delta,m,j}) \subset \{x : 2^{l-1} \le |x| \le 2^{l+2}\}$, it is obvious that $\operatorname{supp}([b, T^l_{\gamma,\delta,m,j}] f_d) \subset 10nQ_d$ and that the supports of $\{[b, T^l_{\gamma,\delta,m,j}] f_d\}_{d=-\infty}^{\infty}$ have bounded overlaps. We thus have the following almost orthogonality property:

$$\|[b, T^l_{\gamma,\delta,m,j}]f\|^2_{L^2} \le C \sum_{d=-\infty}^{\infty} \|[b, T^l_{\gamma,\delta,m,j}]f_d\|^2_{L^2}.$$

Therefore,

$$\begin{aligned}\|T^l_{\gamma,\delta,m;b}f\|^2_{L^2} &= \sum_{j=1}^{D_m} \|[b, T^l_{\gamma,\delta,m,j}]f\|^2_{L^2}\\ &\le C \sum_{d=-\infty}^{\infty} \sum_{j=1}^{D_m} \|[b, T^l_{\gamma,\delta,m,j}]f_d\|^2_{L^2} = C \sum_{d=-\infty}^{\infty} \|T^l_{\gamma,\delta,m;b}f_d\|^2_{L^2}.\end{aligned}$$

Hence it suffices to verify (2.17) for the function f with $\operatorname{supp} f \subset Q$, where Q has side length 2^l. Choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset 100nQ$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $50nQ$. For $\tilde{Q} =: 200nQ$ let $\tilde{b} = (b - b_{\tilde{Q}})\varphi$, where $b_{\tilde{Q}} = |\tilde{Q}|^{-1}\int_{\tilde{Q}} b(y)\,dy$. It is easy to see that

$$[b, T^l_{\gamma,\delta,m,j}]f(x) = \tilde{b}(x)T^l_{\gamma,\delta,m,j}f(x) - T^l_{\gamma,\delta,m,j}(\tilde{b}f)(x).$$

Denoting $T^l_{\gamma,\delta,m}f(x) = \big(\sum_{j=1}^{D_m}|T^l_{\gamma,\delta,m,j}f(x)|^2\big)^{1/2}$, we then have

$$\begin{aligned}&\|T^l_{\gamma,\delta,m;b}f\|^2_{L^2}\\ &\quad\le C\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\Big(|\tilde{b}(x)T^l_{\gamma,\delta,m,j}f(x)|^2 + |T^l_{\gamma,\delta,m,j}(\tilde{b}f)(x)|^2\Big)\,dx\\ &\quad= C\int_{\mathbb{R}^n}|\tilde{b}(x)|^2\sum_{j=1}^{D_m}|T^l_{\gamma,\delta,m,j}(f)(x)|^2\,dx + \int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|T^l_{\gamma,\delta,m,j}(\tilde{b}f)(x)|^2\,dx\\ &\quad= C\|\tilde{b}T^l_{\gamma,\delta,m}f\|^2_{L^2} + \|T^l_{\gamma,\delta,m}(\tilde{b}f)\|^2_{L^2}.\end{aligned} \tag{2.18}$$

Thus by (2.18) we need only show that, for the function f supported in Q with side length 2^l, the following estimates hold for $\mu = 0$ and 1:

$$\begin{aligned}&\|\tilde{b}^\mu T^l_{\gamma,\delta,m}(\tilde{b}^{1-\mu}f)\|_{L^2}\\ &\qquad\le C(2^{-\alpha}\delta)^\gamma m^{(-1+\beta/2)v}\min\{\delta^{\tau v}, \delta^{-\beta v/2}\}2^{-\kappa l}\|b\|_*\|f\|_{L^2}.\end{aligned} \tag{2.19}$$

We shall first show that (2.19) is a consequence of the following statement: For $g \in L^{q'}(\mathbb{R}^n)$, $1 < q' \le 2$ (hence $2 \le q < \infty$), and $0 < t < 1$,

$$\begin{aligned}\|T^l_{\gamma,\delta,m}g\|_{L^q} &\le C(2^{-\alpha}\delta)^\gamma 2^{-2tl/q} m^{(-2+\beta)(1-t)/q-(1-2/q)+2t\lambda/q}\delta^{n(1-2/q)-2t\gamma/q}\\ &\quad\times(\min\{\delta^\tau, \delta^{-\beta/2}\})^{2(1-t)/q}\|g\|_{L^{q'}}.\end{aligned} \tag{2.20}$$

In fact, for $2 < q_1, q_2 < \infty$ with $1/q_1 + 1/q_2 = 1/2$, by (2.20) and the obvious fact

$$\|\tilde{b}\|_{L^\sigma} \le C\|b\|_*|\tilde{Q}|^{1/\sigma} \le C2^{nl/\sigma}\|b\|_* \quad \text{for } 1 < \sigma < \infty$$

we have

$$\begin{aligned}
&\|\tilde{b}T^l_{\gamma,\delta,m}f\|_{L^2}\\
&\quad\le \|\tilde{b}\|_{L^{q_1}}\|T^l_{\gamma,\delta,m}f\|_{L^{q_2}}\\
&\quad\le C(2^{-\alpha}\delta)^{\gamma}2^{-2tl/q_2}\delta^{n(1-2/q_2)-2t\gamma/q_2}m^{(-2+\beta)(1-t)/q_2-(1-2/q_2)+2t\lambda/q_2}\\
&\qquad\times(\min\{\delta^{\tau},\delta^{-\beta/2}\})^{2(1-t)/q_2}\|\tilde{b}\|_{L^{q_1}}\|f\|_{L^{q_2'}}\\
&\quad\le C(2^{-\alpha}\delta)^{\gamma}2^{-2tl/q_2+nl(1-2/q_2)}m^{(-2+\beta)(1-t)/q_2-(1-2/q_2)+2t\lambda/q_2}\delta^{n(1-2/q_2)-2t\gamma/q_2}\\
&\qquad\times(\min\{\delta^{\tau},\delta^{-\beta/2}\})^{2(1-t)/q_2}\|b\|_*\|f\|_{L^2} \qquad (2.21)
\end{aligned}$$

and

$$\begin{aligned}
&\|T^l_{\gamma,\delta,m}(\tilde{b}f)\|_{L^2}\\
&\quad\le C|Q|^{1/q_1}\|T^l_{\gamma,\delta,m}(\tilde{b}f)\|_{L^{q_2}}\\
&\quad\le C(2^{-\alpha}\delta)^{\gamma}2^{-2tl/q_2}\delta^{n(1-2/q_2)-2t\gamma/q_2}m^{(-2+\beta)(1-t)/q_2-(1-2/q_2)+2t\lambda/q_2}\\
&\qquad\times(\min\{\delta^{\tau},\delta^{-\beta/2}\})^{2(1-t)/q_2}\|\tilde{b}f\|_{L^{q_2'}}\\
&\quad\le C|Q|^{1/q_1}(2^{-\alpha}\delta)^{\gamma}2^{-2tl/q_2}\delta^{n(1-2/q_2)-2t\gamma/q_2}m^{(-2+\beta)(1-t)/q_2-(1-2/q_2)+2t\lambda/q_2}\\
&\qquad\times(\min\{\delta^{\tau},\delta^{-\beta/2}\})^{2(1-t)/q_2}\|\tilde{b}\|_{L^{2q_2/(q_2-2)}}\|f\|_{L^2}\\
&\quad\le C(2^{-\alpha}\delta)^{\gamma}2^{-2tl/q_2+nl(1-2/q_2)}m^{(-2+\beta)(1-t)/q_2-(1-2/q_2)+2t\lambda/q_2}\delta^{n(1-2/q_2)-2t\gamma/q_2}\\
&\qquad\times(\min\{\delta^{\tau},\delta^{-\beta/2}\})^{2(1-t)/q_2}\|b\|_*\|f\|_{L^2}. \qquad (2.21')
\end{aligned}$$

Now, for any fixed $0 < v < 1$, we choose $q_2 > 2$ sufficiently close to 2 and $t > 0$ sufficiently close to 0 such that q_2 and t satisfy

$$\begin{aligned}
2t/q_2 &> n(1-2/q_2),\\
\tau\beta(1-t)/q_2 &> n(1-2/q_2)+v\tau\beta/2+2t\lambda\beta/q_2+2t\gamma\beta/q_2. \qquad (*)
\end{aligned}$$

Then we take $\kappa := 2t/q_2 - n(1-2/q_2) > 0$, and by $(*)$ we have $(1-t)/q_2 > v/2 + 2t\lambda/q_2$. Hence $(-2+\beta)(1-t)/q_2 < (-1+\beta/2)v - 2t\lambda/q_2$, so

$$m^{(-2+\beta)(1-t)/q_2-(1-2/q_2)+2t\lambda/q_2} \le m^{(-1+\beta/2)v}.$$

By $(*)$, we have $\beta(1-t)/q_2 - n(1-2/q_2) > v\beta/2$. Thus, for $\delta \ge 1$, by (2.21) and (2.21′) we have

$$\begin{aligned}
&\max\{\|\tilde{b}T^l_{\gamma,\delta,m}f\|_{L^2},\|T^l_{\gamma,\delta,m}(\tilde{b}f)\|_{L^2}\}\\
&\quad\le C(2^{-\alpha}\delta)^{\gamma}m^{(-1+\beta/2)v}2^{-\kappa l}\delta^{n(1-2/q_2)-2t\gamma/q_2}\delta^{-\beta(1-t)/q_2}\|f\|_{L^2}\\
&\quad\le C(2^{-\alpha}\delta)^{\gamma}m^{(-1+\beta/2)v}2^{-\kappa l}\delta^{-\beta v/2}\|b\|_*\|f\|_{L^2}.
\end{aligned}$$

By $(*)$ again we have $2\tau(1-t)/q_2 - 2t\gamma/q_2 > v\tau$ if $0 < \delta < 1$. Then, by (2.21) and (2.21′),

$$\begin{aligned}
&\max\{\|\tilde{b}T^l_{\gamma,\delta,m}f\|_{L^2},\|T^l_{\gamma,\delta,m}(\tilde{b}f)\|_{L^2}\}\\
&\quad\le C(2^{-\alpha}\delta)^{\gamma}m^{(-1+\beta/2)v}2^{-\kappa l}\delta^{n(1-2/q_2)-2t\gamma/q_2}\delta^{2\tau(1-t)/q_2}\|f\|_{L^2}\\
&\quad\le C(2^{-\alpha}\delta)^{\gamma}m^{(-1+\beta/2)v}2^{-\kappa l}\delta^{\tau v}\|b\|_*\|f\|_{L^2}.
\end{aligned}$$

Thus we obtain (2.19). Therefore, to finish the proof of Lemma 2.3, it remains to verify (2.20).

By the definition of $T^l_{\gamma,\delta,m,j}$, we have

$$\begin{aligned}|T^l_{\gamma,\delta,m}g(x)| &\le \left(\sum_{j=1}^{D_m}\left(\int_{\mathbb{R}^n}|K^l_{\gamma,\delta,m,j}(x-y)||g(y)|\,dy\right)^2\right)^{1/2}\\ &\le \int_{\mathbb{R}^n}\left(\sum_{j=1}^{D_m}|K^l_{\gamma,\delta,m,j}(x-y)|^2\right)^{1/2}|g(y)|\,dy\\ &\le \|g\|_{L^1}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{D_m}|\widehat{K^l_{\gamma,\delta,m,j}}(\xi)|^2\right)^{1/2}d\xi.\end{aligned}$$

Since

$$\widehat{K^l_{\gamma,\delta,m,j}}(\xi)=\widehat{K_{\gamma,\delta,m,j}}*\widehat{\phi_l}(\xi)=\int_{\mathbb{R}^n}B_{\gamma,\delta,m,j}(\xi-y)\widehat{\phi_l}(y)\,dy, \tag{2.22}$$

it follows from (2.13) and from $\sum_{j=1}^{D_m}|Y_{m,j}(x')|^2\sim m^{2\lambda}$ (see [3, p. 225, (2.6)]) that

$$\begin{aligned}|T^l_{\gamma,\delta,m}g(x)| &\le \int\left(\sum_{j=1}^{D_m}\left|\int_{\mathbb{R}^n}B_{\gamma,\delta,m,j}(\xi-y)\widehat{\phi_l}(y)\,dy\right|^2\right)^{1/2}d\xi\|g\|_{L^1}\\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{D_m}|B_{\gamma,\delta,m,j}(\xi-y)|^2\right)^{1/2}\widehat{\phi_l}(y)\,dy\,d\xi\|g\|_{L^1}\\ &\le \int_{\delta/2<|\xi|<2\delta}\left(\sum_{j=1}^{D_m}|B_{\gamma,\delta,m,j}(\xi)|^2\right)^{1/2}d\xi\|\widehat{\phi_l}\|_{L^1}\|g\|_{L^1}\\ &\le (2^{-\alpha}\delta)^{\gamma}m^{-\lambda-1}\int_{\delta/2<|\xi|<2\delta}\left(\sum_{j=1}^{D_m}|Y_{m,j}(\xi')|^2\right)^{1/2}d\xi\|g\|_{L^1}\\ &\le C(2^{-\alpha}\delta)^{\gamma}m^{-1}\delta^n\|g\|_{L^1}.\end{aligned}$$

That is,

$$\|T^l_{\gamma,\delta,m}g\|_{L^\infty}\le C(2^{-\alpha}\delta)^{\gamma}m^{-1}\delta^n\|g\|_{L^1}. \tag{2.23}$$

On the other hand, observe that $\int_{\mathbb{R}^n}\widehat{\phi}(\eta)\,d\eta=\phi(0)=0$; then, by (2.22) and (2.14),

$$\begin{aligned}|\widehat{K^l_{\gamma,\delta,m,j}}(x)| &\le \int_{\mathbb{R}^n}|(B_{\gamma,\delta,m,j}(x-2^{-l}y)-B_{\gamma,\delta,m,j}(x))||\widehat{\phi}(y)|\,dy\\ &\le C2^{-l}\|\nabla B_{\gamma,\delta,m,j}\|_{L^\infty}\int_{\mathbb{R}^n}|y||\widehat{\phi}(y)|\,dy\\ &\le C(2^{-\alpha}\delta)^{\gamma}\delta^{-\gamma}2^{-l}.\end{aligned}$$

Thus, by the Plancherel theorem and the fact that $D_m\sim m^{2\lambda}$ (see [3, p. 226, (2.8)]), we have

$$\begin{aligned}\|T^l_{\gamma,\delta,m}g\|_{L^2} &\le \left(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|\widehat{K^l_{\gamma,\delta,m,j}}(\xi)|^2|\widehat{g}(\xi)|^2\,d\xi\right)^{1/2}\\ &\le C(2^{-\alpha}\delta)^{\gamma}\delta^{-\gamma}2^{-l}m^{\lambda}\|g\|_{L^2}.\end{aligned} \tag{2.24}$$

Applying the Plancherel theorem again and then using (2.22), (2.12), and

$$\sum_{j=1}^{D_m} |Y_{m,j}(x')|^2 \sim m^{2\lambda},$$

we obtain

$$\begin{aligned}
\|T^l_{\gamma,\delta,m} g\|^2_{L^2} &\le \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |\widehat{K^l_{\gamma,\delta,m,j}}(\xi)|^2 |\widehat{g}(\xi)|^2 \, d\xi \\
&\le C \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |B_{\gamma,\delta,m,j} * \widehat{\phi_l}(\xi)|^2 |\widehat{g}(\xi)|^2 \, d\xi \\
&\le C \int_{\mathbb{R}^n} \left\{ \left(\sum_{j=1}^{D_m} \left| \int_{\mathbb{R}^n} B_{\gamma,\delta,m,j}(\xi - y) \widehat{\phi_l}(y)\, dy \right|^2 \right)^{1/2} \right\}^2 |\widehat{g}(\xi)|^2 \, d\xi \\
&\le C \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left(\sum_{j=1}^{D_m} |B_{\gamma,\delta,m,j}(\xi - y)|^2 \right)^{1/2} |\widehat{\phi_l}(y)|\, dy \right\}^2 |\widehat{g}(\xi)|^2 \, d\xi \\
&\le (2^{-\alpha}\delta)^{2\gamma} m^{-2\lambda-2+\beta} (\min\{\delta^\tau, \delta^{-\beta/2}\})^2 \\
&\quad \times \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{D_m} |Y_{m,j}((\xi - y)')|^2 \right)^{1/2} |\widehat{\phi_l}(y)|\, dy \right)^2 |\widehat{g}(\xi)|^2 \, d\xi \\
&\le C(2^{-\alpha}\delta)^{2\gamma} m^{-2+\beta} (\min\{\delta^\tau, \delta^{-\beta/2}\})^2 \|\widehat{\phi_l}\|^2_{L^1} \|g\|^2_{L^2}.
\end{aligned}$$

In other words,

$$\|T^l_{\gamma,\delta,m} g\|_{L^2} \le C(2^{-\alpha}\delta)^\gamma m^{-1+\beta/2} \min\{\delta^\tau, \delta^{-\beta/2}\} \|g\|_{L^2}. \tag{2.25}$$

Hence, by (2.24) and (2.25), for any $0 < t < 1$ we have

$$\begin{aligned}
&\|T^l_{\gamma,\delta,m} g\|_{L^2} \\
&\quad \le C(2^{-\alpha}\delta)^\gamma \delta^{-t\gamma} 2^{-tl} m^{t\lambda} m^{(-1+\beta/2)(1-t)} (\min\{\delta^\tau, \delta^{-\beta/2}\})^{1-t} \|g\|_{L^2}.
\end{aligned} \tag{2.26}$$

Thus we obtain (2.20) by interpolating between (2.23) and (2.26), completing the proof of Lemma 2.3. □

REMARK 2.1. If we denote

$$T_{\gamma,\delta,m} f(x) = \left(\sum_{j=1}^{D_m} [T_{\gamma,\delta,m,j} f(x)]^2 \right)^{1/2},$$

then it is easy to see that (2.15) still holds for $T_{\gamma,\delta,m}$ under the same conditions of Lemma 2.3.

REMARK 2.2. For $b \in \mathrm{BMO}$ and $k \in \mathbb{N}$, the kth-order commutator of $T_{\gamma,\delta,m,j}$ and b is defined by

$$T_{\gamma,\delta,m,j;b,k} f(x) := \underbrace{[b, \dots [b, T_{\gamma,\delta,m,j}]]}_{k} f(x) = T_{\gamma,\delta,m,j}((b(x) - b(\cdot))^k f)(x).$$

Let

$$T_{\gamma,\delta,m;b,k}f(x) = \left(\sum_{j=1}^{D_m}(T_{\gamma,\delta,m,j;b,k}f(x))^2\right)^{1/2}.$$

Then, using methods in the proof of Lemma 2.3, (2.15) holds for $T_{\gamma,\delta,m;b,k}$ under the conditions of that lemma.

LEMMA 2.4 (see [12]). *Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a radial function such that* $\operatorname{supp}(\phi) \subset \{1/2 \le |\xi| \le 2\}$ *and* $\sum_{l\in\mathbb{Z}}\phi^3(2^{-l}\xi) = 1$ $(\xi \ne 0)$. *Define the multiplier S_l by* $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi)$. *For* $b \in \mathrm{BMO}$ *denote by $[b, S_l]$ the commutator of S_l and b, which is defined by $[b, S_l]f(x) := b(x)S_l f(x) - S_l(bf)(x)$. Then, for $\{f_l\} \in L^2(l^2)$,*

(i) $\left\|\left(\sum_{l\in\mathbb{Z}}|[b,S_l]f_l|^2\right)^{1/2}\right\|_{L^2} \le C\left\|\left(\sum_{l\in\mathbb{Z}}|f_l|^2\right)^{1/2}\right\|_{L^2}$ *and*

(ii) $\left\|\sum_{l\in\mathbb{Z}}[b,S_l]f_l\right\|_{L^2} \le C\|b\|_*\left\|\left(\sum_{l\in\mathbb{Z}}|f_l|^2\right)^{1/2}\right\|_{L^2}$.

3. Proof of Theorem 1

As before, we denote the space of surface spherical harmonics of degree m on S^{n-1} by $\mathcal{H}_m$ and its dimension by D_m. By a limit argument (see [3] for the details), we may reduce the proof of Theorem 1 to the case of $f \in C_c^\infty(\mathbb{R}^n)$ and

$$\Omega(x,z') = \sum_{m\ge 0}\sum_{j=1}^{D_m} a_{m,j}(x)Y_{m,j}(z')$$

a finite sum, where $\{Y_{m,j}\}_{j=1}^{D_m}$ denotes the complete system of normalized surface spherical harmonics in $\mathcal{H}_m$ (see [3] or [13]) and

$$a_{m,j}(x) = \int_{S^{n-1}}\Omega(x,z')\overline{Y_{m,j}(z')}\,d\sigma(z').$$

Notice that Ω satisfies (1.1), so $a_{m,j} \equiv 0$ for $m = 0,\dots,[\gamma]$. Therefore, we actually have

$$\Omega(x,z') = \sum_{m=[\gamma]+1}^{\infty}\sum_{j=1}^{D_m} a_{m,j}(x)Y_{m,j}(z').$$

Writing

$$T_{\gamma,m,j}f(x) = \text{p.v.}\int_{\mathbb{R}^n}\frac{Y_{m,j}(x-y)}{|x-y|^{n+\gamma}}f(y)\,dy$$

and denoting by $[b, T_{\gamma,m,j}]$ the commutator of $T_{\gamma,m,j}$ and b, by Hölder's inequality we have

$$\begin{aligned}([b,T_\gamma]f(x))^2 \\ \le \left\{\sum_{m=[\gamma]+1}^{\infty}\sum_{j=1}^{D_m} a_{m,j}^2(x)m^{-\varepsilon(1+2\gamma)}\right\}\left\{\sum_{m=[\gamma]+1}^{\infty} m^{\varepsilon(1+2\gamma)}\sum_{j=1}^{D_m}([b,T_{\gamma,m,j}]f(x))^2\right\},\end{aligned} \tag{3.1}$$

where $0 < \varepsilon < 1$. According to [4, p. 527], for each x fixed, the series in the first set of braces on the RHS of (3.1) is equal to $\|\Omega(x,\cdot)\|^2_{\dot{L}^2_{-\rho}(S^{n-1})}$, where $\dot{L}^2_{-\rho}(S^{n-1})$ is the homogeneous Sobolev space on S^{n-1} with $\rho = \varepsilon(1/2+\gamma)$. If we take ε sufficiently close to 1 then, by the Sobolev imbedding theorem, $L^q \hookrightarrow \dot{L}^2_{-\rho}$ for $q > \max\{1, \frac{2(n-1)}{n+2\gamma}\}$ and

$$\sup_{x\in\mathbb{R}^n} \|\Omega(x,\cdot)\|_{\dot{L}^2_{-\rho}(S^{n-1})} \le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})}.$$

Let

$$T_{\gamma,m;b}f(x) = \left(\sum_{j=1}^{D_m} |[b, T_{\gamma,m,j}]f(x)|^2\right)^{1/2}; \tag{3.2}$$

then by (3.1) and (3.2) we have

$$\|[b,T_\gamma]f\|^2_{L^2} \le C\|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} \sum_{m=[\gamma]+1} m^{\varepsilon(1+2\gamma)}\|T_{\gamma,m;b}f\|^2_{L^2}. \tag{3.3}$$

Let $\eta = \varepsilon(1+2\gamma)$. If we can show that there exists $0 < \beta < (1-\eta)/2$ such that

$$\|T_{\gamma,m;b}f\|^2_{L^2} \le Cm^{-2+2\beta}\|b\|^2_{\dot{L}^{n/\gamma}_\gamma}\|f\|^2_{\dot{L}^2_\gamma}, \tag{3.4}$$

then the conclusion of Theorem 1 follows immediately from (3.3) and (3.4). Hence, to prove Theorem 1, it remains to show (3.4).

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a radial function such that $0 \le \phi \le 1$, $\operatorname{supp}\phi \subset \{1/2 \le |\xi| \le 2\}$, and $\sum_{l\in\mathbb{Z}} \phi^4(2^{-l}\xi) = 1$ for $|\xi| \ne 0$. Define the multiplier S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi)$. Let

$$\sigma_{\gamma,\alpha,m,j}(x) = \frac{Y_{m,j}(x')}{|x|^{n+\gamma}}\chi_{\{2^\alpha<|x|\le 2^{\alpha+1}\}}(x)$$

for $\alpha \in \mathbb{Z}$, $m = 1,2,\dots$, and $j = 1,\dots,D_m$. Set

$$B_{\gamma,\alpha,m,j}(\xi) = \widehat{\sigma_{\gamma,\alpha,m,j}}(\xi) \quad \text{and} \quad B^l_{\gamma,\alpha,m,j}(\xi) = B_{\gamma,\alpha,m,j}(\xi)\phi(2^{\alpha-l}\xi).$$

Define the operators $T_{\gamma,\alpha,m,j}$ and $T^l_{\gamma,\alpha,m,j}$ by

$$T_{\gamma,\alpha,m,j}f(x) = (\sigma_{\gamma,\alpha,m,j} * f)(x) \quad \text{and} \quad \widehat{T^l_{\gamma,\alpha,m,j}f}(\xi) = B^l_{\gamma,\alpha,m,j}(\xi)\widehat{f}(\xi),$$

respectively. Define the operator $V_{\gamma,l,j}$ by

$$V_{\gamma,l,j}f(x) = \sum_{\alpha\in\mathbb{Z}}[b, S_{l-\alpha}T^l_{\gamma,\alpha,m,j}S^2_{l-\alpha}]f(x),$$

where $[b, S_{l-\alpha}T^l_{\gamma,\alpha,m,j}S^2_{l-\alpha}]$ denotes the commutator of $S_{l-\alpha}T^l_{\gamma,\alpha,m,j}S^2_{l-\alpha}$ and the function b. Then it is easy to check that, for $f, g \in C_c^\infty(S^{n-1})$,

$$\int_{\mathbb{R}^n}[b,T_{\gamma,m,j}]f(x)g(x)\,dx = \int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}} V_{\gamma,l,j}f(x)g(x)\,dx. \tag{3.5}$$

By (3.2) and the Minkowski inequality, we have

$$\|T_{\gamma,m;b}f\|_{L^2}=\left(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\left|\sum_{l\in\mathbb{Z}}V_{\gamma,l,j}f(x)\right|^2dx\right)^{1/2}$$
$$\leq\sum_{l\in\mathbb{Z}}\left(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|V_{\gamma,l,j}f(x)|^2dx\right)^{1/2}. \tag{3.6}$$

So by (3.6), to prove (3.4) it suffices to show that there exists $0<\beta<(1-\eta)/2$ such that

$$\sum_{l\in\mathbb{Z}}\left(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|V_{\gamma,l,j}f(x)|^2dx\right)^{1/2}\leq Cm^{-1+\beta}\|b\|_{\dot{L}^{n/\gamma}_{\gamma}}\|f\|_{\dot{L}^2_{\gamma}}, \tag{3.7}$$

where C is independent of l and f. Then, by Lemma 2.4(ii) and Littlewood–Paley theory,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|V_{\gamma,l,j}f(x)|^2dx\\
&\quad=\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\left(\sum_{\alpha\in\mathbb{Z}}[b,S_{l-\alpha}T^l_{\gamma,\alpha,m,j}S^2_{l-\alpha}]f(x)\right)^2dx\\
&\quad=\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\Bigg[\sum_{\alpha\in\mathbb{Z}}([b,S_{l-\alpha}](T^l_{\gamma,\alpha,m,j}S^2_{l-\alpha})f(x)+S_{l-\alpha}[b,T^l_{\gamma,\alpha,m,j}]S^2_{l-\alpha}f(x)\\
&\qquad\qquad+S_{l-\alpha}T^l_{\gamma,\alpha,m,j}[b,S_{l-\alpha}]S_{l-\alpha}f(x)\\
&\qquad\qquad+S_{l-\alpha}T^l_{\gamma,\alpha,m,j}S_{l-\alpha}[b,S_{l-\alpha}]f(x))\Bigg]^2dx\\
&\quad\leq C\sum_{\alpha\in\mathbb{Z}}\sum_{j=1}^{D_m}\Bigg(\|b\|_*^2\int_{\mathbb{R}^n}|T^l_{\gamma,\alpha,m,j}S^2_{l-\alpha}f(x)|^2dx\\
&\qquad\qquad+\int_{\mathbb{R}^n}|[b,T^l_{\gamma,\alpha,m,j}]S^2_{l-\alpha}f(x)|^2dx\\
&\qquad\qquad+\int_{\mathbb{R}^n}|T^l_{\gamma,\alpha,m,j}[b,S_{l-\alpha}]S_{l-\alpha}f(x)|^2dx\\
&\qquad\qquad+\int_{\mathbb{R}^n}|T^l_{\gamma,\alpha,m,j}S_{l-\alpha}[b,S_{l-\alpha}]f(x)|^2dx\Bigg).
\end{aligned}$$

Define $T^l_{\gamma,\alpha,m}$ and $T^l_{\gamma,\alpha,m;b}$ by

$$T^l_{\gamma,\alpha,m}h(x)=\left(\sum_{j=1}^{D_m}|T^l_{\gamma,\alpha,m,j}h(x)|^2\right)^{1/2}$$

and

$$T^l_{\gamma,\alpha,m;b}h(x) = \left(\sum_{j=1}^{D_m} |[b, T^l_{\gamma,\alpha,m,j}]h(x)|^2\right)^{1/2},$$

respectively. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |V_{\gamma,l,j} f(x)|^2\,dx \\
&\quad \le C\|b\|_*^2 \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} |T^l_{\gamma,\alpha,m} S^2_{l-\alpha} f(x)|^2\,dx + \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} |T^l_{\gamma,\alpha,m;b} S^2_{l-\alpha} f(x)|^2\,dx \\
&\qquad + \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} |T^l_{\gamma,\alpha,m}[b, S_{l-\alpha}] S_{l-\alpha} f(x)|^2\,dx \\
&\qquad + \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} |T^l_{\gamma,\alpha,m} S_{l-\alpha}[b, S_{l-\alpha}] f(x)|^2\,dx \\
&\quad := I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Obviously, (3.7) will follow if we can show that, for $0 < \beta < (1-\eta)/2$, there exists $0 < v_0 < 1$ such that

$$\max_{1\le i\le 4}\{I_i\} \le Cm^{-2+2\beta} \min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\} \|b\|^2_{\dot{L}^{n/\gamma}_\gamma} \|f\|^2_{\dot{L}^2_\gamma}. \tag{3.8}$$

Our proof of (3.8) requires the following fact: For $0 < \beta < (1-\eta)/2$, there exists $0 < v_0 < 1$ such that

$$\|T^l_{\gamma,\alpha,m} h\|_{L^2} \le C 2^{(l-\alpha)\gamma} m^{-1+\beta} \min\{2^{\tau v_0 l}, 2^{-\beta v_0 l/2}\} \|h\|_{L^2} \tag{3.9}$$

and

$$\|T^l_{\gamma,\alpha,m;b} h\|_{L^2} \le C 2^{(l-\alpha)\gamma} m^{-1+\beta} \min\{2^{\tau v_0 l}, 2^{-\beta v_0 l/2}\} \|b\|_* \|h\|_{L^2}. \tag{3.10}$$

The verification of (3.9) and (3.10) will be postponed until the end of this section. For now, we estimate (3.8) by applying (3.9) and (3.10). Given (3.9) and

$$\left\| \left(\sum_{\alpha\in\mathbb{Z}} |2^{\alpha\gamma} S_\alpha f|^2\right)^{1/2} \right\|_{L^2} \le C\|f\|_{\dot{L}^2_\gamma}, \tag{3.11}$$

and noting that S_α is a bounded operator on L^2 uniformly in α by the definition of S_α, we have

$$\begin{aligned}
I_1 &= C\|b\|_*^2 \sum_{\alpha\in\mathbb{Z}} \|T^l_{\gamma,\alpha,m} S^2_{l-\alpha} f\|^2_{L^2} \\
&\le C\|b\|_*^2 m^{-2+2\beta} \min\{2^{2\tau v_0 l}, 2^{-\beta v_0}\} \sum_{\alpha\in\mathbb{Z}} 2^{2(l-\alpha)\gamma} \|S_{l-\alpha} f\|^2_{L^2} \\
&= Cm^{-2+2\beta} \min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\} \|b\|_*^2 \left\| \left(\sum_{\alpha\in\mathbb{Z}} |2^{\alpha\gamma} S_\alpha f|^2\right)^{1/2} \right\|^2_{L^2} \\
&\le Cm^{-2+2\beta} \min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\} \|b\|^2_{\dot{L}^{n/\gamma}_\gamma} \|f\|^2_{\dot{L}^2_\gamma},
\end{aligned} \tag{3.12}$$

where (and in the sequel) we use that $\dot{L}_\gamma^{n/\gamma} \subset \mathrm{BMO}$ and $\|b\|_* \le \|b\|_{\dot{L}_\gamma^{n/\gamma}}$ for $b \in \dot{L}_\gamma^{n/\gamma}$. Now we give the estimate of I_2. By (3.10) and (3.11), we have

$$\begin{aligned} I_2 &= C\sum_{\alpha\in\mathbb{Z}} \|T^l_{\gamma,\alpha,m;\,b} S^2_{l-\alpha} f(x)\|^2_{L^2} \\ &\le Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\|b\|_*^2 \sum_{\alpha\in\mathbb{Z}} 2^{2(l-\alpha)\gamma}\|S_{l-\alpha} f\|^2_{L^2} \\ &\le Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\|b\|^2_{\dot{L}_\gamma^{n/\gamma}}\|f\|^2_{\dot{L}^2_\gamma}. \end{aligned} \tag{3.13}$$

Next, we estimate I_3. By (3.9), Lemma 2.4(i), and (3.11), it follows that

$$\begin{aligned} I_3 &\le Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\sum_{\alpha\in\mathbb{Z}} 2^{2(l-\alpha)\gamma}\|[b, S_{l-\alpha}]S_{l-\alpha} f\|^2_{L^2} \\ &= Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\|b\|_*^2 \sum_{\alpha\in\mathbb{Z}} 2^{2(l-\alpha)\gamma}\|S_{l-\alpha} f\|^2_{L^2} \\ &\le Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\|b\|^2_{\dot{L}_\gamma^{n/\gamma}}\|f\|^2_{\dot{L}^2_\gamma}. \end{aligned} \tag{3.14}$$

Now we consider I_4. By (3.9), we have

$$\begin{aligned} I_4 &\le Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\sum_{\alpha\in\mathbb{Z}} 2^{2(l-\alpha)\gamma}\int_{\mathbb{R}^n} |S_{l-\alpha}[b, S_{l-\alpha}]f(x)|^2\,dx \\ &= Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\sum_{\alpha\in\mathbb{Z}} 2^{2\alpha\gamma}\int_{\mathbb{R}^n} |S_\alpha[b, S_\alpha]f(x)|^2\,dx. \end{aligned} \tag{3.15}$$

Using the Bony decomposition (2.1) to estimate I_4, we have $fg = \pi_f g + \pi_g f + R(f,g)$, where $\pi_f g$ and $\pi_g f$ are Bony paraproducts (see Section 2) and

$$R(f,g) = \sum_{i\in\mathbb{Z}}\sum_{|k-i|\le 2} \Delta_i f \Delta_k g.$$

Thus,

$$\begin{aligned} [b, S_\alpha]f(x) = {} & [\pi_{S_\alpha f}(b)(x) - S_\alpha(\pi_f b)(x)] + [R(b, S_\alpha f)(x) - S_\alpha(R(b,f))(x)] \\ & + [\pi_b(S_\alpha f)(x) - S_\alpha(\pi_b f)(x)]. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we obtain

$$\begin{aligned} I_4 = {} & Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\Big(\sum_{\alpha\in\mathbb{Z}} 2^{2\alpha\gamma}\|S_\alpha(\pi_{S_\alpha f}(b) - S_\alpha(\pi_f b))\|^2_{L^2} \\ & + \sum_{\alpha\in\mathbb{Z}} 2^{2\alpha\gamma}\Big\|S_\alpha\big(R(b, S_\alpha f) - S_\alpha(R(b,f))\big)\Big\|^2_{L^2} \\ & + \sum_{\alpha\in\mathbb{Z}} 2^{2\alpha\gamma}\|S_\alpha(\pi_b(S_\alpha f) - S_\alpha(\pi_b f))\|^2_{L^2}\Big) \\ := {} & Cm^{-2+2\beta}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}(J_1 + J_2 + J_3). \end{aligned} \tag{3.17}$$

Regarding J_1, note that $(\Delta_i S_\alpha f)(G_{i-3}b) = 0$ and $S_\alpha((\Delta_i f)(G_{i-3}b)) = 0$ if $|i-\alpha| \ge 5$, so we can write

$$
\begin{aligned}
&\pi_{S_\alpha f}(b)(x) - S_\alpha(\pi_f b)(x)\\
&\quad = \sum_{|i-\alpha|\le 4} \Delta_i S_\alpha f(x) G_{i-3} b(x) - \sum_{|i-\alpha|\le 4} S_\alpha(\Delta_i f G_{i-3} b)(x)\\
&\quad = \sum_{|i-\alpha|\le 4} [G_{i-3}b, S_\alpha]\Delta_i f(x).
\end{aligned}
$$

Recall that $S_\alpha f(x) = \Phi_\alpha * f(x)$, where $\Phi(x) = \widehat{\phi}(x)$ and $\Phi_\alpha(x) = 2^{\alpha n}\Phi(2^\alpha x)$. We have

$$
\begin{aligned}
&|[G_{i-3}b, S_\alpha]\Delta_i f(x)|\\
&= \left| \int_{\mathbb{R}^n} 2^{\alpha n}\Phi(2^\alpha(x-y))(G_{i-3}b(x) - G_{i-3}b(y))\Delta_i f(y)\,dy \right|\\
&= 2^{-\alpha\delta}\left| \int_{\mathbb{R}^n} 2^{\alpha n}\Phi(2^\alpha(x-y))|2^\alpha(x-y)|^\delta \frac{(G_{i-3}b(x) - G_{i-3}b(y))}{|x-y|^\delta}\Delta_i f(y)\,dy \right|\\
&= 2^{-\alpha\delta}\left| \int_{\mathbb{R}^n} \eta_\alpha(x-y)\frac{(G_{i-3}b(x) - G_{i-3}b(y))}{|x-y|^\delta}\Delta_i f(y)\,dy \right|\\
&\le 2^{-\alpha\delta}\int_{\mathbb{R}^n} |\eta_\alpha(x-y)|\frac{|G_{i-3}b(x) - G_{i-3}b(y)|}{|x-y|^\delta}|\Delta_i f(y)|\,dy, \qquad (3.18)
\end{aligned}
$$

where $\eta(x) = \Phi(x)|x|^\delta \in \mathcal{S}(\mathbb{R}^n)$ and $\eta_\alpha(x) = 2^{\alpha n}\eta(2^\alpha x)$. On the other hand, since $G_k\Delta_i b = 0$ for $i \ge k-1$, $I_\delta(\mathrm{BMO}) \subsetneq \mathrm{Lip}_\delta$ for $0 < \delta < 1$, and $L^\infty \subset \mathrm{BMO}$, for $k \in \mathbb{Z}$ we can obtain

$$
\begin{aligned}
|G_k b(x) - G_k b(y)| &\le C|x-y|^\delta \|D^\delta G_k b\|_{\mathrm{BMO}}\\
&\le C|x-y|^\delta \|D^\delta G_k b\|_{L^\infty}\\
&= C|x-y|^\delta \left\| D^\delta G_k\Big(\sum_{u\in\mathbb{Z}} \Delta_u b\Big)\right\|_{L^\infty}\\
&\le C|x-y|^\delta \sum_{u\le k-1} \|D^\delta G_k \Delta_u b\|_{L^\infty}\\
&\le C|x-y|^\delta \sum_{u\le k-1} \|D^\delta \Delta_u b\|_{L^\infty},
\end{aligned}
$$

where $\widehat{D^\delta f}(x) = |x|^\delta \widehat{f}(x)$. Introduce a C_c^∞ function $\widetilde{\psi}$ supported in a shell and such that $\widetilde{\psi} = 1$ in $\operatorname{supp}\psi$, where $\psi(2^{-j}\xi)$ is the symbol of Δ_j (see Section 2). Then

$$
\widehat{D^\delta \Delta_u b}(\xi) = 2^{u\delta}\widetilde{\psi}(2^{-u}\xi)|2^{-u}\xi|^\delta \widehat{\Delta_u b}(\xi).
$$

Since $\widetilde{\psi}(\xi)|\xi|^\delta \in C_c^\infty$, we get

$$
\|D^\delta \Delta_u b\|_{L^\infty} \le C 2^{u\delta}\|\Delta_u b\|_{L^\infty};
$$

thus, from $\sup_{u\in\mathbb{Z}}\|\Delta_u b\|_{L^\infty} \le C\|b\|_*$ it follows that

$$
\begin{aligned}
|G_k b(x) - G_k b(y)| &\le C|x-y|^{\delta} \sum_{u \le k-1} 2^{u\delta} \|\Delta_u b\|_{L^\infty} \\
&\le C|x-y|^{\delta} 2^{k\delta} \sup_{u\in\mathbb{Z}} \|\Delta_u b\|_{L^\infty} \sum_{u\le k-1} 2^{(u-k)\delta} \\
&\le C|x-y|^{\delta} 2^{k\delta} \|b\|_{*}. \qquad (3.19)
\end{aligned}
$$

Now we return to the estimate of J_1. By Littlewood–Paley theory, (3.18), (3.19), and the convolution inequalities for series, we have

$$
\begin{aligned}
J_1 &\le \sum_{\alpha\in\mathbb{Z}} \sum_{|i-\alpha|\le 4} 2^{2(\alpha-i)\gamma} \|[G_{i-3}b, S_\alpha] 2^{i\gamma} \Delta_i f\|_{L^2}^2 \\
&\le C \sum_{\alpha\in\mathbb{Z}} \sum_{|i-\alpha|\le 4} \|b\|_*^2 2^{2(\alpha-i)(\gamma+\delta)} 2^{2i\gamma} \| |\eta_\alpha| * |\Delta_i f| \|_{L^2}^2 \\
&\le C \sum_{\alpha\in\mathbb{Z}} \sum_{|i-\alpha|\le 4} \|b\|_*^2 2^{2(\alpha-i)(\gamma+\delta)} \|\eta_\alpha\|_{L^1}^2 2^{2i\gamma} \|\Delta_i f\|_{L^2}^2 \\
&\le C \|b\|_{\dot{L}_\gamma^{n/\gamma}}^2 \|f\|_{\dot{L}_\gamma^2}^2. \qquad (3.20)
\end{aligned}
$$

Next, we consider J_2. By the definition of $R(f,g)$ (see Section 2), we have

$$
\begin{aligned}
&R(b, S_\alpha f)(x) - S_\alpha(R(b,f))(x) \\
&\quad = \sum_{i\in\mathbb{Z}} \sum_{|k|\le 2} \Delta_i b(x) \Delta_{i+k} S_\alpha f(x) - S_\alpha\Bigg(\sum_{i\in\mathbb{Z}} \sum_{|k|\le 2} \Delta_i b \Delta_{i+k} f \Bigg)(x).
\end{aligned}
$$

Note that both $\Delta_{i+k} S_\alpha f$ and $S_\alpha(\Delta_i b \Delta_{i+k} f)$ are zero for $|i-\alpha| \ge 6$ and $|k| \le 2$; then, applying the Littlewood–Paley theory, $\sup_{i\in\mathbb{Z}} \|\Delta_i(b)\|_{L^\infty} \le C\|b\|_*$, and (3.11) yields

$$
\begin{aligned}
J_2 &\le C \sum_{\alpha\in\mathbb{Z}} \sum_{k=-2}^{2} \sum_{|i-\alpha|\le 5} 2^{2\alpha\gamma} (\|(\Delta_i b)(S_\alpha \Delta_{i+k} f)\|_{L^2}^2 + \|(\Delta_i b)(\Delta_{i+k} f)\|_{L^2}^2) \\
&\le C \sup_{i\in\mathbb{Z}} \|\Delta_i b\|_{L^\infty}^2 \sum_{i\in\mathbb{Z}} 2^{2i\gamma} \|\Delta_i f\|_{L^2}^2 \\
&\le C \|b\|_{\dot{L}_\gamma^{n/\gamma}}^2 \|f\|_{\dot{L}_\gamma^2}^2. \qquad (3.21)
\end{aligned}
$$

Finally, we estimate J_3. Since $S_\alpha(\Delta_i b G_{i-3} S_\alpha f) = 0$ and $S_\alpha^2(\Delta_i b G_{i-3} f) = 0$ for $|i-\alpha| \ge 5$, we can derive

$$
\begin{aligned}
&S_\alpha(\pi_b(S_\alpha f) - S_\alpha(\pi_b f)) \\
&\quad = \sum_{|i-\alpha|\le 4} S_\alpha(\Delta_i b G_{i-3} S_\alpha f)(x) - \sum_{|i-\alpha|\le 4} S_\alpha^2(\Delta_i b G_{i-3} f)(x) \\
&\quad := K_1 + K_2. \qquad (3.22)
\end{aligned}
$$

Using Littlewood–Paley theory, $\|G_i f\|_{L^2} \le C\|f\|_{L^2}$ for $i \in \mathbb{Z}$,

$$
\sup_{i\in\mathbb{Z}} \|\Delta_i b\|_{L^\infty} \le C\|b\|_* \le C\|b\|_{\dot{L}_\gamma^{n/\gamma}},
$$

and (3.11), we have

$$\sum_{\alpha\in\mathbb{Z}} 2^{2\alpha\gamma}\|K_1\|_{L^2}^2 \le C\sup_i\|\Delta_i(b)\|_{L^\infty}^2 \sum_{\alpha\in\mathbb{Z}}\|2^{\alpha\gamma}S_\alpha f\|_{L^2}^2 \le C\|b\|_{\dot{L}_\gamma^{n/\gamma}}^2\|f\|_{\dot{L}_\gamma^2}. \tag{3.23}$$

Given Hölder's inequality, $\|\sup_{i\in\mathbb{Z}}|G_i f|\|_{L^p} \le C\|f\|_{L^p}$ for any $1 < p < \infty$, and $\dot{L}_\gamma^2(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2\gamma)}(\mathbb{R}^n)$ for $0 < \gamma < n/2$, we obtain

$$\begin{aligned}
\sum_{\alpha\in\mathbb{Z}} 2^{2\alpha\gamma}\|K_2\|_{L^2}^2 &\le C\sum_{i\in\mathbb{Z}}\sum_{|i-\alpha|\le 4} 2^{2(\alpha-i)\gamma}\|2^{i\gamma}\Delta_i b G_{i-3}f\|_{L^2}^2 \\
&\le C\left\|\left(\sum_{i\in\mathbb{Z}}|2^{i\gamma}\Delta_i b|^2\right)^{1/2}\right\|_{L^{n/\gamma}}^2 \left\|\sup_{i\in\mathbb{Z}}|G_{i-3}f|\right\|_{L^{2n/(n-2\gamma)}}^2 \\
&\le C\|b\|_{\dot{L}_\gamma^{n/\gamma}}^2\|f\|_{L^{2n/(n-2\gamma)}}^2 \\
&\le C\|b\|_{\dot{L}_\gamma^{n/\gamma}}^2\|f\|_{\dot{L}_\gamma^2}.
\end{aligned} \tag{3.24}$$

By (3.22)–(3.24), we can get

$$J_3 \le C\|b\|_{\dot{L}_\gamma^{n/\gamma}}^2\|f\|_{\dot{L}_\gamma^2}^2. \tag{3.25}$$

Combining this with (3.17), (3.20), (3.21), (3.25), and $\dot{L}_\gamma^{n/\gamma} \subset \mathrm{BMO}$ yields

$$I_4 \le Cm^{(-2+\beta)v_0}\min\{2^{2\tau v_0 l}, 2^{-\beta v_0 l}\}\|b\|_{\dot{L}_\gamma^{n/\gamma}}^2\|f\|_{\dot{L}_\gamma^2}^2. \tag{3.26}$$

Thus, when (3.9) and (3.10) hold, (3.8) follows from (3.12), (3.13), (3.14), and (3.26).

Therefore, to finish the proof of Theorem 1, it remains to show (3.9) and (3.10). Toward this end, we define the operator $\widetilde{T}_{\gamma,\alpha,m,j}^l$ by

$$\widehat{\widetilde{T}_{\gamma,\alpha,m,j}^l h}(\xi) = B_{\gamma,\alpha,m,j}^l(2^{-\alpha}\xi)\widehat{h}(\xi)$$

and denote

$$\widetilde{T}_{\gamma,\alpha,m;b}^l h(\xi) = \left(\sum_{j=1}^{D_m}|[b,\widetilde{T}_{\gamma,\alpha,m,j}^l]h(\xi)|^2\right)^{1/2}.$$

On the other hand, for any $0 < \beta < 1$, $\widehat{\sigma_{\gamma,\alpha,m,j}}$ satisfies (2.2)–(2.4) by Lemma 2.2. Note that, since

$$B_{\gamma,\alpha,m,j}(\xi) = \widehat{\sigma_{\gamma,\alpha,m,j}}(\xi) \quad \text{and} \quad \operatorname{supp}(B_{\gamma,\alpha,m,j}^l(2^{-\alpha}\cdot)) \subset \{2^{l-1} \le |\xi| \le 2^{l+1}\},$$

we have

$$\begin{aligned}
|B_{\gamma,\alpha,m,j}^l(2^{-\alpha}\xi)| &\le C2^{(l-\alpha)\gamma}m^{-\lambda-1+\beta/2}\min\{2^{\tau l}, 2^{-\beta l/2}\}|Y_{m,j}(\xi')|, \\
|B_{\gamma,\alpha,m,j}^l(2^{-\alpha}\xi)| &\le C2^{(l-\alpha)\gamma}m^{-\lambda-1}|Y_{m,j}(\xi')|, \\
|\nabla B_{\gamma,\alpha,m,j}^l(2^{-\alpha}\xi)| &\le 2^{(l-\alpha)\gamma}2^{-l\gamma},
\end{aligned}$$

where $\tau = [\gamma] + 1 - \gamma$. By Lemma 2.3 and Remark 2.1 with $\delta = 2^l$ we know that, for any fixed $0 < v < 1$,

$$\|\widetilde{T}_{\gamma,\alpha,m}^l h\|_{L^2} \le C2^{(l-\alpha)\gamma}m^{(-1+\beta)v}\min\{2^{\tau vl}, 2^{-\beta vl/2}\}\|h\|_{L^2}$$

and

$$\|\widetilde{T}^l_{\gamma,\alpha,m;\,b}h\|_{L^2} \le C2^{(l-\alpha)\gamma}m^{(-1+\beta)v}\min\{2^{\tau vl}, 2^{-\beta vl/2}\}\|b\|_*\|h\|_{L^2},$$

where C is independent of α and l. Hence, for $0 < \beta < (1-\eta)/2$, we can find $0 < v_0 < 1$ such that $v_0(-1+\beta/2) \le -1+\beta$ and such that

$$\|\widetilde{T}^l_{\gamma,\alpha,m}h\|_{L^2} \le C2^{(l-\alpha)\gamma}m^{-1+\beta}\min\{2^{\tau v_0 l}, 2^{-\beta v_0 l/2}\}\|h\|_{L^2} \tag{3.27}$$

and

$$\|\widetilde{T}^l_{\gamma,\alpha,m;\,b}h\|_{L^2} \le C2^{(l-\alpha)\gamma}m^{-1+\beta}\min\{2^{\tau v_0 l}, 2^{-\beta v_0 l/2}\}\|b\|_*\|h\|_{L^2}. \tag{3.28}$$

However, (3.27) and (3.28) imply, respectively, (3.9) and (3.10) by dilation-invariance. This completes the proof of Theorem 1.

ACKNOWLEDGMENT. The authors would like to express their deep gratitude to the referee for valuable comments and suggestions.

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Y. Chen
Department of Mathematics and Mechanics
Applied Science School
University of Science and Technology Beijing
Beijing 100083
China

yanpingch@126.com

Y. Ding
School of Mathematical Sciences
Laboratory of Mathematics and Complex Systems (BNU)
Ministry of Education of China
Beijing Normal University
Beijing 100875
China

dingy@bnu.edu.cn