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Symmetries of Julia Sets of Nondegenerate Polynomial Skew Products on $\mathbf{C}^2$

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1. Introduction

The Julia sets of any kind of functions or maps can have symmetries. We say that a Julia set has symmetries if some nonelementary transformations preserve it. Beardon [B] investigated the symmetries of the Julia sets of polynomials on $\mathbf{C}$. For the Julia set of a polynomial, the symmetries of the Julia set are rotations about some point. The group of symmetries is infinite if and only if the Julia set is a circle, which is equivalent to the polynomial being conjugate to $z \to z^d$. There was a problem for polynomials that had the same Julia set. Beardon [B] gave an answer to this problem in terms of a functional equation in which the symmetries of the Julia set are used. Finally, the problem was solved by [SS] and [AHu] independently: polynomials having the same Julia set are essentially the same.

We want to extend these dynamical objects and results in one dimension to those in higher dimensions. As a first step, we extend these dynamical objects and results of polynomials to those of nondegenerate polynomial skew products. Although the dynamics of polynomial skew products is a complicated dynamics in higher dimensions, it has many analogies to the dynamics of polynomials.

The paper is organized as follows. In Section 2, we recall the dynamics of a nondegenerate polynomial skew product and show the existence of the vertical Green functions and Böttcher functions of the map. In Section 3 we investigate the symmetries of the Julia set of a nondegenerate polynomial skew product. We show that suitable transformations preserving the Julia set are conjugate to rotational product maps, and we give a necessary and sufficient condition for the group of symmetries to be infinite. In Section 4 we deal with the problem case of nondegenerate polynomial skew products that have the same Julia set. We place a restriction on nondegenerate polynomial skew products and show that, except for two types, these maps having the same Julia set are essentially the same. The paper concludes with application of this result to the dynamics of regular polynomial skew products.

2. Dynamics of Polynomial Skew Products

Let us recall the dynamics of nondegenerate polynomial skew products on $\mathbf{C}^2$. Heinemann [H] and Jonsson [J] studied the dynamics of regular polynomial skew

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products. Favre and Guedj [FG] studied the dynamics of polynomial skew products. A polynomial skew product is a polynomial map of the form $f(z,w) = (p(z), q(z,w))$, where $p(z) = az^\delta + O(z^{\delta-1})$ and $q(z,w) = b(z)w^d + O_z(w^{d-1})$. In this paper we say that f is *nondegenerate* if $b(z)$ is a nonzero constant and that f is *of bidegree* (δ, d) if $\deg p = \delta$ and $\deg_w q = d$. We always assume that the degrees δ and d are at least 2. As we shall see in Section 4, regular polynomial skew products are nondegenerate.

Let $f(z,w) = (p(z), q(z,w))$ be a nondegenerate polynomial skew product of bidegree (δ, d). Roughly speaking, the dynamics of nondegenerate polynomial skew products consists of the dynamics on the base space and the dynamics on the vertical lines. We denote the nth iterate of f by f^n and the composition of maps f and g by fg; that is, $fg(z) = f(g(z))$. The first component p defines the dynamics on the base space $\mathbf{C}$. Define $K_p = \{z : \{p^n(z)\}_{n\geq 1} \text{ bounded}\}$ and $J_p = \partial K_p$. In this paper we call J_p the *base Julia set* of f. Note that f preserves the set of vertical lines in $\mathbf{C}^2$. In this sense, we often use the notation $q_z(w)$ instead of $q(z,w)$. The restriction of f^n to a line $\{z\} \times \mathbf{C}$ can be viewed as the composition of n polynomials on $\mathbf{C}$, $q_{p^{n-1}(z)} \cdots q_{p(z)} q_z$. For z in K_p, define $K_z = \{w : \{Q_z^n(w)\}_{n\geq 1} \text{ bounded}\}$ and $J_z = \partial K_z$. In this paper we call J_z the *vertical Julia set* of f at z. We define the *pre-Julia set* of f as the union of the vertical Julia sets on the base Julia set, and we define the *Julia set* of f as the closure of the pre-Julia set of f:

$$J_f = \overline{J'_f}, \quad \text{where } J'_f = \bigcup_{z \in J_p} \{z\} \times J_z.$$

By definition, J_f is compact and completely invariant under f.

A useful tool in the study of the dynamics of p on the base space is the Green function G_p of p, defined by

$$G_p(z) = \lim_{n\to\infty} \delta^{-n} \log^+ |p^n(z)|.$$

It is known that G_p is a nonnegative, continuous, and subharmonic function on $\mathbf{C}$. More precisely, G_p is harmonic on $\mathbf{C} - K_p$ and is zero on K_p, and $G_p(z) = \log|z| + O(1)$ as $z \to \infty$. By definition, $G_p(p(z)) = \delta G_p(z)$. Note that G_p coincides with the Green function for K_p with a pole at infinity, which is determined only by the compact set K_p. In a similar fashion, we consider the function

$$G_z(w) = \lim_{n\to\infty} d^{-n} \log^+ |Q_z^n(w)|,$$

where $Q_z^n(w) = q_{p^{n-1}(z)} \cdots q_{p(z)} q_z(w)$. Favre and Guedj [FG] proved that the limit function G_z is well-defined on $K_p \times \mathbf{C}$ and has similar properties to G_p without the assumption of nondegeneracy. For completeness we restate here their claim and proof with the assumption of nondegeneracy.

LEMMA 2.1 [FG, Thm. 6.1]. *For a nondegenerate polynomial skew product f, where $f(z,w) = (p(z), q(z,w))$ and $q(z,w) = bw^d + O_z(w^{d-1})$, and for every z in K_p, there exists a unique function G_z on $\{z\} \times \mathbf{C}$ such that*:

(i) *G_z is harmonic on $\mathbf{C} - K_z$ and is zero on K_z;*
(ii) *$G_z(w) = \log|w| + \frac{1}{d-1}\log|b| + o_z(1)$ as $w \to \infty$; and*
(iii) *$G_{p(z)}(q_z(w)) = dG_z(w)$.*

Proof. We may assume that $b = 1$ because a polynomial skew product $(az^\delta + O(z^{\delta-1}), bw^d + O_z(w^{d-1}))$ is conjugate to a map $(z^\delta + O(z^{\delta-1}), w^d + O_z(w^{d-1}))$ by a linear map $(a^{1/(\delta-1)}z, b^{1/(d-1)}w)$.

Let $G_n(z,w) = d^{-n}\log^+|Q_z^n(w)|$ and $W = K_p \times \{w : |w| > R\}$ for large $R > 0$. We prove the uniform convergence of $G_n(z,w)$ on W. Since K_p is compact, there exists a $c > 0$ such that, for any (z,w) in W,

$$||q_z(w)| - |w|^d| < cR^{-1}|w|^d.$$

It follows that f maps W into itself from this inequality, which thus induces the following inequality for any (z,w) in W and for any positive integer n:

$$\begin{aligned}|G_{n+1}(z,w) - G_n(z,w)| &= \frac{1}{d^{n+1}}|\log|Q_z^{n+1}(w)| - \log|Q_z^n(w)|^d| \\ &\leq \frac{1}{d^{n+1}}\log\left\{1 + \left|\frac{Q_z^{n+1}(w) - (Q_z^n(w))^d}{(Q_z^n(w))^d}\right|\right\} \\ &< \frac{1}{d^{n+1}}\log\left(1 + \frac{c}{R}\right).\end{aligned}$$

Therefore, $G_n(z,w)$ converges uniformly to $G_z(w)$ on W. Since $G_n(z,w)$ is harmonic with respect to w, it follows that the limit function $G_z(w)$ is also harmonic with respect to w. By the preceding equation, we have

$$\begin{aligned}|G_{n+1}(z,w) - \log|w|| &= |G_{n+1}(z,w) - G_0(z,w)| \\ &< \sum_{j=0}^{n}\frac{1}{d^{j+1}}\log\left(1 + \frac{c}{R}\right) \leq \frac{1}{d-1}\log\left(1 + \frac{c}{R}\right)\end{aligned}$$

for any (z,w) in W and for any positive integer n. Hence $G_z(w) = \log|w| + o(1)$ as $w \to \infty$. By definition, $G_{p(z)}(q_z(w)) = dG_z(w)$.

Define $K = \{(z,w) : \{f^n(z,w)\}_{n\geq 1} \text{ bounded}\}$. We can extend the domain of $G_z(w)$ to $(K_p \times \mathbf{C}) - K$ by using the equation $G_{p(z)}(q_z(w)) = dG_z(w)$, since for any (z,w) in $(K_p \times \mathbf{C}) - K$ there exists a positive integer n such that $f^n(z,w)$ belongs to W. Finally, let $G_z(w)$ be zero on K; then clearly the extension satisfies all the required properties. Uniqueness follows from properties (i) and (ii). □

In this paper we call G_z the *vertical Green function* of f at z. By (i) and (ii) of Lemma 2.1 we know that, for every z in the base Julia set, G_z coincides with the Green function for K_z with a pole at infinity, which is determined only by the compact set K_z.

A polynomial p of degree δ is conjugate to $z \to z^\delta$ near infinity by the Böttcher function of p. One can construct the Böttcher function of p from the Green function of p. For a nondegenerate polynomial skew product, similar Böttcher functions exist on vertical lines. Lemma 2.1 induces the following proposition.

PROPOSITION 2.2. *For a nondegenerate polynomial skew product f, where $f(z,w) = (p(z), q(z,w))$ and $q(z,w) = bw^d + O_z(w^{d-1})$, and for every z in K_p, there exists a unique conformal function φ_z defined near infinity such that*:

(i) $\varphi_z(w) = w + O_z(1)$ *as* $w \to \infty$;
(ii) $\log|c\varphi_z(w)| = G_z(w)$, *where* $c = b^{1/(d-1)}$; *and*
(iii) $\varphi_{p(z)}(q_z(w)) = b(\varphi_z(w))^d$.

Proof. Define $u_z(w) = G_z(w) - \log|w| - \frac{1}{d-1}\log|b|$. Then u_z is a harmonic function defined near infinity and maps ∞ to 0. Hence there exists a harmonic function v_z near infinity such that $u_z + iv_z$ is holomorphic and maps ∞ to 0. Then the function $\varphi_z(w) = we^{u_z(w)+iv_z(w)}$ satisfies all the required conditions. Uniqueness follows from properties (i) and (ii). □

In this paper we call φ_z the *vertical Böttcher function* of f at z. These functions may not correspond to the usual one, $c\varphi_z(w)$. However, our vertical Böttcher functions are useful for investigating the symmetries of the Julia set of f and for dealing with the problem case when nondegenerate polynomial skew products have the same Julia set (see e.g. Lemma 4.1).

In Section 3 we use the vertical Green functions and Böttcher functions over the base Julia set in order to investigate the symmetries of the Julia set of f. Moreover, we need the vertical Böttcher functions over the whole base space when solving, in Section 4, the problem when nondegenerate polynomial skew products have the same Julia set. Let us now show the existence of the vertical Green functions and Böttcher functions over the whole base space.

LEMMA 2.3. *Let f be a nondegenerate polynomial skew product of bidegree (δ, d), where $f(z,w) = (p(z), q(z,w))$ and $q(z,w) = bw^d + O_z(w^{d-1})$. If $\delta \le d$ then, for every z in $\mathbf{C}$, there exists a harmonic function G_z defined near infinity such that*

(i) $G_z(w) = \log|w| + \frac{1}{d-1}\log|b| + o_z(1)$ *as* $w \to \infty$ *and*
(ii) $G_{p(z)}(q_z(w)) = dG_z(w)$.

Proof. We may assume that $b = 1$ as the proof of Lemma 2.1. Let $G_n(z,w) = d^{-n}\log|Q_z^n(w)|$. We prove the uniform convergence of $G_n(z,w)$ on some subset of $(\mathbf{C} - K_p) \times \mathbf{C}$. Let $k = \deg_z q$ and $W = \{(z,w) : |z| > R, |w| > R|z|^k\}$ for large $R > 0$. Then there exists a $c > 0$ such that, for any (z,w) in W,

$$\left|\frac{q_z(w) - w^d}{w^{d-1}z^k}\right| \le c.$$

Hence we get the following inequality for any (z,w) in W:

$$||q_z(w)| - |w|^d| \le c \cdot \left|\frac{z^k}{w}\right| \cdot |w|^d < cR^{-1}|w|^d. \tag{2.1}$$

We claim that f maps W into itself. It is enough to show that $|q_z(w)| > R|p(z)|^k$ for any (z,w) in W. Let (z,w) be a point in W. It is clear that if R is large enough then there exists a $c_1 > 0$ such that $|p(z)| < c_1|z^\delta|$ for any $|z| > R$. Since $|q_z(w) - w^d| < cR^{-1}|w|^d$, there exists a $c_2 > 0$ such that

$$\left|\frac{q_z(w)}{w^d}\right| \geq 1 - \left|\frac{q_z(w) - w^d}{w^d}\right| \geq c_2.$$

From these inequalities, it follows that

$$\begin{aligned}\left|\frac{q_z(w)}{p(z)^k}\right| = \left|\frac{z^\delta}{p(z)}\right|^k \cdot \left|\frac{q_z(w)}{z^{\delta k}}\right| &> \frac{1}{c_1^k} \cdot \left|\frac{q_z(w)}{w^d} \cdot \frac{w^d}{z^{dk}} \cdot \frac{z^{dk}}{z^{\delta k}}\right| \\ &> \frac{c_2}{c_1^k} R^d \cdot \left|\frac{z^{dk}}{z^{\delta k}}\right| = \frac{c_2}{c_1^k} R^{d-1} \cdot R \cdot \left|\frac{z^{dk}}{z^{\delta k}}\right|.\end{aligned}$$

If R is large enough, then $c_2 c_1^{-k} R^{d-1} > 1$. Finally, the condition $\delta \leq d$ implies that $|q_z(w)| > R|p(z)|^k$.

By the same argument used in the proof of Lemma 2.1, (2.1) induces the uniform convergence of $G_n(z, w)$ on W. The limit function $G_z(w)$ satisfies all the required properties. We can extend the domain of $G_z(w)$ with respect to z to $\mathbf{C} - K_p$ by using the equation $G_{p(z)}(q_z(w)) = dG_z(w)$, since for any z in $\mathbf{C} - K_p$ there exists a positive integer n such that $p^n(z)$ belongs to $\{|z| > R\}$. Clearly, the extension satisfies all the required properties. $\square$

The existence of vertical Green functions of f implies the existence of vertical Böttcher functions of f, as in the proof of Proposition 2.2.

PROPOSITION 2.4. *Let f be a nondegenerate polynomial skew product of bidegree (δ, d), where $f(z, w) = (p(z), q(z, w))$ and $q(z, w) = bw^d + O_z(w)$. If $\delta \leq d$ then, for every z in $\mathbf{C}$, there exists a conformal function φ_z defined near infinity such that*:

(i) $\varphi_z(w) = w + O_z(1)$ *as* $w \to \infty$;
(ii) $\log|c\varphi_z(w)| = G_z(w)$, *where* $c = b^{1/(d-1)}$; *and*
(iii) $\varphi_{p(z)}(q_z(w)) = b(\varphi_z(w))^d$.

3. Symmetries of a Julia Set

In this section we investigate the symmetries of the Julia set of a nondegenerate polynomial skew product f of bidegree (δ, d),

$$f\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} p(z) \\ q(z, w) \end{pmatrix} = \begin{pmatrix} a_\delta z^\delta + a_{\delta-1} z^{\delta-1} + \cdots + a_0 \\ b_d w^d + b_{d-1}(z) w^{d-1} + \cdots + b_0(z) \end{pmatrix}.$$

We use this notation for f (including p, q_z, b_d, and $b_j(z)$) in the proofs that follow. We omit the proofs of results in one dimension, but one can reconstruct the proofs for p from our proofs for f.

First, let us recall objects and results of the symmetries of the Julia sets of polynomials on $\mathbf{C}$ (for further detail, see [B]). We view conformal functions as the symmetries of the Julia set of a polynomial p. Because the Julia set of p is compact, such functions are conformal Euclidean isometries. Hence the group of the symmetries of the Julia set of p is defined by

$$\Sigma = \Sigma(J_p) = \{\sigma(z) = c_1 z + c_2 : |c_1| = 1,\ \sigma(J_p) = J_p\},$$

where c_1 and c_2 are complex numbers.

The centroid of p is defined by

$$\zeta = \frac{-a_{\delta-1}}{\delta a_\delta}.$$

If the solutions of $p(z) = Z$ are $z_1, z_2, \ldots, z_\delta$, then

$$p(z) = a_\delta (z - z_1)(z - z_2) \cdots (z - z_\delta) + Z$$

and so the center of gravity of the points z_j coincides with ζ. Each symmetry σ is a rotation about the centroid of p; that is, $\sigma(z) = \mu(z - \zeta) + \zeta$ for some μ in the unit circle S^1. We can normalize p by a conjugation function $z \to z - \zeta$ so that the centroid is at the origin, and the group $\Sigma(J_p)$ can be identified with a subgroup of the unit circle S^1.

Let us generalize these dynamical objects and results of polynomials to those of nondegenerate polynomial skew products. We consider polynomial automorphisms whose first components depend only on the first coordinate as the symmetries of the Julia set of a nondegenerate polynomial skew product. Let $\gamma(z, w) = (\sigma(z), \gamma_z(w))$ be a polynomial automorphism that preserves the Julia set of f. Since σ is homeomorphic and since the base Julia set is compact, it follows that σ is a conformal Euclidean isometry: $\sigma(z) = c_1 z + c_2$ for some complex numbers c_1 and c_2 with $|c_1| = 1$. Note that γ preserves the set of vertical lines. Since γ_z is homeomorphic, it is an affine function in w: $\gamma_z(w) = c_3(z) w + c_4(z)$ for some polynomials $c_3(z)$ and $c_4(z)$. Since γ is homeomorphic, $c_3(z)$ is a constant c_3. Moreover, since γ preserves the compact set J_f, it follows that γ_z is also a conformal Euclidean isometry: $|c_3| = 1$. Hence we may define the group of the symmetries of the Julia set of f as

$$\Gamma = \Gamma(J_f) = \{\gamma \in S : \gamma(J_f) = J_f\},$$

where

$$S = \left\{ \gamma \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} c_1 z + c_2 \\ c_3 w + c_4(z) \end{pmatrix} : |c_1| = |c_3| = 1 \right\}.$$

In the same manner, we can define $\Gamma(J_f')$. Let J_z^* be the intersection of the Julia set of f and the vertical line $\{z\} \times \mathbb{C}$ for any z in the base Julia set. Because J_z^* is included in K_z, it follows that $\Gamma(J_f) = \Gamma(J_f')$. Hence γ in $\Gamma(J_f)$ preserves J_f' and so it sends J_z onto $J_{\sigma(z)}$ for any z in the base Julia set.

As in the one-dimensional case, we define the centroid of q_z by

$$\zeta_z = \frac{-b_{d-1}(z)}{d b_d}.$$

If the solutions of $q_z(w) = W$ are $w_1, w_2, \ldots, w_d$, then the center of gravity of the points w_j coincides with ζ_z. We can normalize f by the conjugation map $(z, w) \to (z - \zeta, w - \zeta_z)$ so that all centroids ζ and ζ_z are at the origin. We say that a polynomial skew product f is *in normal form* if all its centroids ζ and ζ_z are at the origin and if the coefficients of the leading terms of p and q_z are both 1. Before normalizing the polynomial skew product, we express symmetries by using the centroids ζ and ζ_z.

PROPOSITION 3.1. *Let $f(z, w) = (p(z), q(z, w))$ be a nondegenerate polynomial skew product. Then any symmetry γ in Γ can be written as*

$$\gamma\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu(z - \zeta) + \zeta \\ \nu(w - \zeta_z) + \zeta_{\sigma(z)} \end{pmatrix}$$

for some μ and ν in S^1, where $\sigma(z) = \mu(z - \zeta) + \zeta$ belongs to Σ.

Proof. Let us denote γ in Γ by $(\sigma(z), \gamma_z(w))$. It is known that $\sigma(z) = \mu(z-\zeta)+\zeta$ for some μ in S^1, which is proved by an argument similar to the following.

First, note that the Böttcher function φ_z has a relationship with the centroid ζ_z of q_z for any z in the base Julia set. Combining (i) and (iii) of Proposition 2.2 yields

$$b_d w^d + b_{d-1}(z) w^{d-1} + \cdots = b_d(w + c_z + \cdots)^d.$$

Comparing the second terms in this equation shows that $c_z = -\zeta_z$ and so $\varphi_z(w) = w - \zeta_z + o_z(1)$.

Next, let us show that the assumption $\gamma(J_f) = J_f$ induces the required formula. Fix any z in the base Julia set. Then $\gamma_z(J_z) = J_{\sigma(z)}$ and so $\gamma_z(K_z) = K_{\sigma(z)}$. Hence $G_{\sigma(z)}\gamma_z$ and G_z are the Green functions for K_z with a pole at infinity. By the uniqueness property of the Green functions, $G_{\sigma(z)}\gamma_z = G_z$. From the relation between the vertical Green functions and Böttcher functions it follows that $\varphi_{\sigma(z)}\gamma_z(w) = \nu\varphi_z(w)$ for some ν in S^1. By comparing the regular terms in this equation, we obtain $\gamma_z(w) - \zeta_{\sigma(z)} = \nu(w - \zeta_z)$. By the uniqueness theorem of holomorphic functions on horizontal lines, this equation holds on $\mathbf{C}^2$. □

We can now identify the group of symmetries with a subgroup of the torus. For a nondegenerate polynomial skew product f in normal form,

$$\begin{aligned} \Gamma &= \{\gamma_{\mu,\nu}(z, w) = (\mu z, \nu w) : \gamma_{\mu,\nu}(J_f) = J_f\} \\ &\simeq \{(\mu, \nu) \in S^1 \times S^1 : \gamma_{\mu,\nu} \in \Gamma\} \subset S^1 \times S^1. \end{aligned}$$

The following lemma helps us to investigate the symmetries of the Julia set of f. Although the idea comes from [B], we give a slightly different statement.

LEMMA 3.2. *Let $f(z, w) = (p(z), q(z, w))$ be a nondegenerate polynomial skew product of bidegree (δ, d) in normal form. Then*

$$\Gamma = \{\gamma(z, w) = (\mu z, \nu w) : f^n\gamma = \gamma^{(\delta^n, d^n)} f^n \text{ for any } n \geq 1\},$$

where μ and ν belong to S^1 and where $\gamma^{(\delta^n, d^n)}(z, w) = (\mu^{\delta^n} z, \nu^{d^n} w)$.

Proof. Since f is in normal form, any γ in Γ is a rotational product map. Let $\gamma(z,w) = (\sigma(z), \gamma_z(w))$, where $\sigma(z) = \mu z$ and $\gamma_z(w) = \nu w$.

Assume that γ belongs to Γ. Since σ belongs to Σ, it is known that $p(\mu z) = \mu^\delta p(z)$, which is proved by an argument similar to the following. Fix any z in the base Julia set. Since γ preserves the Julia set, $G_{\sigma(z)}\gamma_z = G_z$; that is, $G_{\mu z}(\nu w) = G_z(w)$. Therefore, $\varphi_{\mu z}(\nu w) = \nu\varphi_z(w)$. By Lemma 2.2, $\varphi_{p(\mu z)}(q_{\mu z}(\nu w)) = (\varphi_{\mu z}(\nu w))^d = (\nu\varphi_z(w))^d = \nu^d(\varphi_z(w))^d = \nu^d\varphi_{p(z)}(q_z(w))$. A comparison of the regular terms in this equation yields $q_{\mu z}(\nu w) = \nu^d q_z(w)$. By the uniqueness theorem of holomorphic functions on horizontal lines, this equation holds on $\mathbf{C}^2$. Consequently, $f\gamma = \gamma^{(\delta,d)}f$ and so $\gamma^{(\delta,d)}$ belongs to Γ. It follows similarly that $f\gamma^{(\delta,d)} = \gamma^{(\delta^2,d^2)}f$ and so $\gamma^{(\delta^2,d^2)}$ belongs to Γ. Therefore, $f^2\gamma = \gamma^{(\delta^2,d^2)}f^2$. By continuing this argument, we arrive at $f\gamma^{(\delta^{n-1},d^{n-1})} = \gamma^{(\delta^n,d^n)}f$ and so $f^n\gamma = \gamma^{(\delta^n,d^n)}f^n$ for any positive integer n.

Assume that $f^n\gamma = \gamma^{(\delta^n,d^n)}f^n$ for any positive integer n. Since $p(\mu z) = \mu^\delta p(z)$, it is known that σ belongs to Σ, which is proved by an argument similar to the following. Fix any z in the base Julia set. Since $Q^n_{\mu z}(\nu w) = \nu^{d^n}Q^n_z(w)$, it follows that w belongs to K_z if and only if νw belongs to $K_{\mu z}$. Therefore, γ_z maps K_z onto $K_{\mu z}$ and hence it maps J_z onto $J_{\mu z}$. Consequently, γ preserves the Julia set and so it belongs to Γ. $\square$

REMARK 3.3. If $\delta = d$, then one can replace the conditions $f^n\gamma = \gamma^{d^n}f^n$ for every positive integer n with the condition $f\gamma = \gamma^d f$.

Let us now give three examples of the symmetries of the Julia sets of nondegenerate polynomial skew products in normal form. Using $f\gamma = \gamma^d f$, one can calculate the group of symmetries from the given map.

EXAMPLE 3.4. Let $f(z,w) = (z^3+1, w^3+w)$ be a polynomial product. Then $\Gamma \simeq \{(\mu,\nu) : \mu^3 = \nu^2 = 1\}$.

EXAMPLE 3.5. Let $f(z,w) = (z^3, w^3+zw+1)$. Then it follows that $\Gamma \simeq \{(\mu,\nu) : \mu^3 = \mu\nu = 1\} = \{(1,1), (\rho,\rho^2), (\rho^2,\rho) \text{ for } \rho^3 = 1\}$.

EXAMPLE 3.6. Let $f(z,w) = (z^2, w^2+z)$. Then $\Gamma \simeq \{(\mu,\nu) : \mu = \nu^2 \in S^1\}$. In particular, Γ is an infinite group. Moreover, f is semiconjugate to $(z,w) \to (z^2, w^2+1)$ by $\pi(z,w) = (z^2, zw)$.

Next we consider when the group of symmetries is infinite. A nondegenerate polynomial skew product is conjugate to a map that is in normal form. Hence we may assume, without loss of generality, that the polynomial skew product is in normal form.

THEOREM 3.7. *Let $f(z,w) = (p(z), q(z,w))$ be a nondegenerate polynomial skew product of bidegree (δ,d) in normal form. Then Γ is infinite if and only if one of the following holds for a one-dimensional Julia set J and for some positive integers n and m:*

(i) $J_f = S^1 \times S^1$;
(ii) $J_f = J_p \times S^1$;
(iii) $J_f = S^1 \times J$;
(iv) $J_f = \bigcup_{z \in S^1} \{z\} \times z^{m/n} J$.

These conditions are equivalent to the following counterparts:

(i′) $f(z, w) = (z^\delta, w^d)$;
(ii′) $f(z, w) = (p(z), w^d)$;
(iii′) $f(z, w) = (z^\delta, q(w))$;
(iv′) $\delta = d$, $p(z) = z^d$, *and* f *is semiconjugate to a polynomial product* $(z^d, q(1, w))$ *by* $\pi(z, w) = (z^n, z^m w)$ *for some positive integers* n *and* m.

Proof. Each condition implies that Γ is infinite. We prove the converse. Let Γ be infinite. We identify the group $\Gamma = \{\gamma_{\mu,\nu}(z, w) = (\mu z, \nu w) : \gamma_{\mu,\nu}(J_f) = J_f\}$ with the subgroup of the torus, $\{(\mu, \nu) \in S^1 \times S^1 : \gamma_{\mu,\nu} \in \Gamma\}$.

If Γ has only finitely many different μ, then it must have infinitely many different ν. Because Γ is a closed subgroup of $S^1 \times S^1$, it includes $\{1\} \times S^1$. Hence each vertical Julia set J_z is a circle with center at the origin for any z in the base Julia set. We can use vertical Böttcher functions to show that $q_z(w) = w^d$. Let $r_z(w) = c_z w^d$ so that it maps the circle J_z to the circle $J_{p(z)}$. Then $d^{-1} G_{p(z)} r_z$ and G_z are the Green functions for K_z with a pole at infinity. From the uniqueness property of the Green functions, $G_{p(z)} r_z = d G_z$; hence $\varphi_{p(z)}(r_z(w)) = c_z (\varphi_z(w))^d$. By combining this equation and the equation $\varphi_{p(z)}(q_z(w)) = (\varphi_z(w))^d$, we establish that $\varphi_{p(z)}(r_z(w)) = c_z \varphi_{p(z)}(q_z(w))$. Therefore, $q_z(w) = (c_z)^{-1} r_z(w) = w^d$. Thus we get (ii′), which implies (ii). One can use the same argument to show that (ii) implies (ii′).

Assume that Γ has infinitely many different μ. Because the projection of Γ to the first coordinate is a closed subgroup of S^1, it coincides with S^1. Hence J_p is a circle. In a similar way as described previously, one can show that J_p being a circle is equivalent to p being conjugate to $z \to z^d$. By assumption, $p(z) = z^d$ and $J_p = S^1$.

In addition, assume that Γ has only finitely many different ν. Then Γ includes $S^1 \times \{1\}$. Hence $J_{\mu z} = J_z$ for any z in the base Julia set and for any μ in S^1; that is, the vertical Julia sets over the base Julia set are all the same. Thus we get (iii). Arguing similarly (or from Theorem 4.6 in the next section), we can show that polynomials of degree d that map J to itself differ only in terms of the symmetries of J. Since the coefficient of the leading term of q_z is a constant, $q_z(w)$ is independent of z. Thus we get (iii′). Clearly, (iii′) implies (iii).

Assume that Γ has infinitely many different μ and ν. We show that the condition $\delta \neq d$ implies (i) and (i′). Because the projection of Γ to the second coordinate coincides with S^1, there exists a $\gamma = (\mu, \nu)$ in Γ such that $\nu^n \neq 1$ for any nonzero integer n. Then, by Lemma 3.2, (μ^δ, ν^d) also belongs to Γ. On the other hand, $\gamma^\delta = (\mu^\delta, \nu^\delta)$ belongs to Γ. Thus the difference $(1, \nu^{d-\delta})$ belongs to Γ, which implies that the vertical Julia set J_z is invariant under the function $w \to \nu^{d-\delta} w$ for any z in the base Julia set. Since the group $\{\nu^{(d-\delta)n} : n \in Z\}$ is dense in S^1, it follows that Γ includes $\{1\} \times S^1$. Hence the vertical Julia sets over the base Julia set are circles with centers at the origin. Analogous arguments show that $q_z(w) = w^d$. Thus we get (i′), which implies (i). One can show that (i) implies (i′) by the same argument used previously. Finally, we consider the case where $\delta = d$ and Γ has infinitely many different μ and ν. If f is a polynomial product, then we get (i) and (i′). Thus we may assume that f is not a polynomial product. Proposition 3.9 (to follow) then completes the proof. □

REMARK 3.8. We can classify the types of an infinite group Γ by Theorem 3.7. Case (i) occurs if Γ is isomorphic to the torus. Case (ii) occurs if Γ is isomorphic to the product of a finite group and the unit circle, and case (iii) occurs if Γ is isomorphic to the product of the unit circle and a finite group. Case (iv) occurs if the projections of Γ to the first and the second coordinates are both the unit circle and if Γ is not isomorphic to the torus.

Let us now complete the proof of Theorem 3.7.

PROPOSITION 3.9. *Let $f(z, w) = (z^d, q(z, w))$ be a nondegenerate polynomial skew product of bidegree (d, d) in normal form. Assume that f is not a polynomial product. Then the following statements are equivalent.*

(i) Γ *is infinite.*
(ii) $f\tau = \tau^d f$ *for some* $\tau(z, w) = (\lambda z, \kappa w)$ *with* $|\lambda| \neq 1$.
(iii) $q(z^n, z^m w) = z^{md} q(1, w)$ *for some positive integers n and m.*
(iv) f *is semiconjugate to a polynomial product* $(z^d, q(1, w))$ *by the projection* $\pi(z, w) = (z^n, z^m w)$ *for some positive integers n and m.*
(v) $J_f = \bigcup_{z \in S^1} \{z\} \times z^{m/n} J$ *for a one-dimensional Julia set J and for some positive integers n and m.*

Proof. (i) ⇒ (iii). We identify the group

$$\Gamma = \{\gamma_{\mu,\nu}(z, w) = (\mu z, \nu w) : \gamma_{\mu,\nu}(J_f) = J_f\}$$

with the subgroup of the torus, $\{(\mu, \nu) \in S^1 \times S^1 : \gamma_{\mu,\nu} \in \Gamma\}$. Observe that Γ has infinitely many different μ. Otherwise, it follows (from the same argument used in the proof of Theorem 3.7) that f is a polynomial product, which contradicts the assumption. Because the projection of Γ to the first coordinate is a closed subset of S^1, it coincides with S^1. Thus there exists $\gamma(z, w) = (\mu z, \nu w)$ in Γ such that $\mu^n \neq 1$ for any nonzero integer n. Lemma 3.2 implies that $q(\mu z, \nu w) = \nu^d q(z, w)$. Therefore, if q contains the term $z^{m_i} w^{l_i}$ with a nonzero coefficient for $l_i < d$, then μ and ν are related by $\mu^{m_i} \nu^{l_i} = \nu^d$. The equations $\mu^{m_i} = \nu^{d-l_i}$ and $\mu^{m_j} = \nu^{d-l_j}$ imply that $\mu^{m_i(d-l_j)-m_j(d-l_i)} = 1$. By the property of μ, we have $m_i(d-l_j) - m_j(d-l_i) = 0$; hence the ratio of $d - l_i$ and m_i is independent of i. Let n and m be, for example, the minimum positive integers whose ratio is equal to that of $d - l_i$ and m_i. Then these integers n and m satisfy (iii). Similarly, (ii) implies (iii) because $\lambda^n \neq 1$ for any nonzero integer n.

(iii) ⇒ (iv). Let $f_0(z, w) = (z^d, q(1, w))$. Then f is semiconjugate to f_0 by π: $\pi f_0 = f\pi$.

(iv) $\Rightarrow$ (i). Let $f_0(z,w) = (z^d, q_0(w))$ be a polynomial product such that $\pi f_0 = f\pi$. Then $\gamma_0(z,w) = (\mu z, w)$ belongs to $\Gamma(J_{f_0})$ for any μ in S^1. A rotational product map γ_0 projects to $\gamma(z,w) = (\mu^n z, \mu^m w)$ by π. The equation $f_0\gamma_0 = \gamma_0^d f_0$ implies that $f\gamma = \gamma^d f$. By Lemma 3.2, γ belongs to $\Gamma(J_f)$. Similarly, (iv) implies (ii) because $\tau_0(z,w) = (\lambda z, w)$ satisfies $f_0\tau_0 = \tau_0^d f_0$ for any λ in $\mathbf{C} - \{0\}$.

(iii) $\Rightarrow$ (v) $\Rightarrow$ (i). Let J be the Julia set of a polynomial $q(1,w)$. Then (iii) implies that $J_z = z^{m/n}J$ for any z in the base Julia set S^1. Moreover, (v) implies that the linear map $(z,w) \to (\mu^n z, \mu^m w)$ preserves J_f for any μ in S^1. Thus Γ is infinite. $\square$

4. Polynomial Skew Products with Same Julia Set

In this section we consider the case where nondegenerate polynomial skew products have the same Julia set. We give partial answers to this question. First, we generalize Beardon's answer in terms of a functional equation. Next, we place a restriction on the maps and generalize the answer in [SS] and [AHu].

Let $f(z,w) = (p(z), q(z,w))$ and $g(z,w) = (r(z), s(z,w))$ be nondegenerate polynomial skew products having the same Julia set. We often use the notation $f = (p,q)$ and $g = (r,s)$ for simplicity. We denote the vertical Julia set, Green function, and Böttcher function of f (resp. g) by J_z^f, G_z^f, and φ_z^f (resp. J_z^g, G_z^g, and φ_z^g). The following two lemmas are useful for dealing with this case.

LEMMA 4.1. *Let f and g be nondegenerate polynomial skew products. If $J_f = J_g$, then $\varphi_z^f = \varphi_z^g$ for any z in the base Julia set.*

Proof. Fix any z in the base Julia set. Then $J_z^f = J_z^g$ and $K_z^f = K_z^g$. By the uniqueness property of the Green functions, $G_z^f = G_z^g$. The relation between the vertical Green functions and Böttcher functions implies the identity $\varphi_z^f = \varphi_z^g$, since the coefficients of the leading terms of φ_z^f and φ_z^g are both 1. $\square$

LEMMA 4.2. *Let f and g be nondegenerate polynomial skew products. If $J_f = J_g$ and if the bidegrees of f and g are the same, then $f = \gamma g$ for some γ in Γ.*

Proof. Let $f = (p,q)$ and $g = (r,s)$, where $q_z(w) = bw^d + O_z(w^{d-1})$ and $s_z(w) = b'w^d + O_z(w^{d-1})$. Since $J_p = J_r$ and $\deg p = \deg r$, it is known that $p = \sigma r$ for some σ in Σ, which is proved by an argument similar to the following. We start by comparing the second components of f and g. Fix any z in the base Julia set. By Lemma 4.1, $\varphi_z^f = \varphi_z^g$; we denote it by φ_z for simplicity. By Proposition 2.2,

$$\varphi_{p(z)}(q_z(w)) = b(\varphi_z(w))^d \quad \text{and} \quad \varphi_{r(z)}(s_z(w)) = b'(\varphi_z(w))^d.$$

Comparing the regular terms in $\varphi_{p(z)}(q_z(w)) = \nu\varphi_{r(z)}(s_z(w))$ then yields

$$q_z(w) - \zeta_{p(z)} = \nu(s_z(w) - \zeta_{r(z)}), \quad \text{where } \nu = \frac{b}{b'} \in S^1.$$

Hence $q_z = \gamma_{r(z)} s_z$, where $\gamma_z(w) = \nu(w - \zeta_z) + \zeta_{\sigma(z)}$. By the uniqueness theorem of holomorphic functions on horizontal lines, $q_z(w) = \gamma_{r(z)} s_z(w)$ on $\mathbf{C}^2$. Consequently, $f = \gamma g$ for $\gamma(z, w) = (\sigma(z), \gamma_z(w))$ and so γ belongs to Γ. $\square$

We next recall Beardon's answer to the problem when polynomials have the same Julia set. We assume that the degrees of the polynomials are at least 2.

THEOREM 4.3 [B, Thm. 1]. *Let P and Q be polynomials. Then $J_P = J_Q$ if and only if $PQ = \sigma QP$ for some σ in $\Sigma(J_Q)$.*

REMARK 4.4. The condition $\sigma \in \Sigma(J_Q)$ is equivalent to the condition $\sigma \in \Sigma(J_P)$, since the equation $PQ = \sigma QP$ for some σ in $\Sigma(J_Q)$ implies $QP = \sigma^{-1} PQ$, where σ^{-1} also belongs to $\Sigma(J_Q)$.

We can generalize Theorem 4.3 to the case of nondegenerate polynomial skew products. We use Montel's theorem as in [B], from which the idea of the proof comes, but we give a slightly different proof.

THEOREM 4.5. *Let f and g be nondegenerate polynomial skew products. Then $J_f = J_g$ if and only if $fg = \gamma gf$ for some γ in $\Gamma(J_f)$.*

Proof. Assume that $J_f = J_g$. Since the bidegrees of fg and gf are the same, it follows from Lemma 4.2 that $fg = \gamma gf$ for some γ in Γ.

Assume that $fg = \gamma gf$ for some γ in $\Gamma(J_f)$. We may also assume that f is in normal form and that $\gamma(z, w) = (\mu z, \nu w)$ for some μ and ν in S^1. Let $f = (p, q)$ and $g = (r, s)$. Since $pr = \sigma rp$, it is known that $J_p = J_r$, which is proved by an argument similar to the following. Combining the equations $fg = \gamma gf$ and $f\gamma^{(\delta^{n-1}, d^{n-1})} = \gamma^{(\delta^n, d^n)} f$ in the proof of Lemma 3.2 yields $f^{n+1} g = \gamma^{(\delta_{n+1}, d_{n+1})} g f^{n+1}$ for any positive integer n, where $\delta_{n+1} = \delta^n + \cdots + \delta + 1$ and $d_{n+1} = d^n + \cdots + d + 1$. Fix any z in the base Julia set. Then $Q^n_{r(z)} s_z = \nu^{d_n} s_{p^n(z)} Q^n_z$, where $Q^n_z = q_{p^{n-1}(z)} \cdots q_{p(z)} q_z$. Hence w belongs to $\mathbf{C} - K^f_z$ if and only if $s_z(w)$ belongs to $\mathbf{C} - K^f_{r(z)}$. In particular, s_z maps $\mathbf{C} - K^f_z$ to $\mathbf{C} - K^f_{r(z)}$ and so S^n_z maps $\mathbf{C} - K^f_z$ to $\mathbf{C} - K^f_{r^n(z)}$, where $S^n_z = s_{r^{n-1}(z)} \cdots s_{r(z)} s_z$. Note that K^f_z contains infinitely many points and is uniformly bounded for any z in the base Julia set. Thus it contains different points a_z and b_z, which are uniformly bounded for any z in the base Julia set.

Let us define affine functions h^n_z as

$$h^n_z(w) = \frac{w - a_{r^n(z)}}{b_{r^n(z)} - a_{r^n(z)}}.$$

Then $h^n_z S^n_z(\mathbf{C} - K^f_z)$ omits 0 and 1. By Montel's theorem, the family of polynomials $\{h^n_z S^n_z\}_{n \ge 1}$ is normal in $\mathbf{C} - K^f_z$. We may assume that $b_{r^n(z)} - a_{r^n(z)}$ is uniformly bounded away from 0. Hence $\{S^n_z\}_{n \ge 1}$ is also normal in $\mathbf{C} - K^f_z$ and so $\mathbf{C} - K^f_z \subset \mathbf{C} - K^g_z$. Again, the equation $Q^n_{r(z)} s_z = \nu^{l_n} s_{p^n(z)} Q^n_z$ implies that w belongs to K^f_z if and only if $s_z(w)$ belongs to $K^f_{r(z)}$. In particular, s_z maps K^f_z to

$K^f_{r(z)}$ and so S^n_z maps K^f_z to $K^f_{r^n(z)}$. Hence $K^f_z \subset K^g_z$. Consequently, $K^f_z = K^g_z$ and so $J^f_z = J^g_z$. □

We now recall the answer in [SS] and [AHu] to this problem, which will be used to prove our Theorem 4.10.

THEOREM 4.6 [SS; AHu]. *For any Julia set J of a polynomial that is not a circle or a straight line segment, there exists a polynomial R such that any polynomial with the Julia set J can be written in the form σR^k for some positive integer k and σ in Σ:*

$$\{Q : \textit{polynomial},\ J_Q = J\} = \{\sigma R^k : \sigma \in \Sigma,\ k \in N\}.$$

REMARK 4.7. A polynomial P is conjugate to $z \to z^d$ if and only if J_P is a circle. A polynomial P is conjugate to a Chebyshev polynomial if and only if J_P is a straight line segment. Combined, these results imply that polynomials having the same Julia set are essentially the same.

Let us generalize Theorem 4.6 to the case of nondegenerate polynomial skew products. For this purpose, we need to place a restriction on the maps. We shall say that a nondegenerate polynomial skew product f of bidegree (δ, d) is *of iso-bidegree* if $\delta = d$. Important tools are the one-dimensional theorem (Theorem 4.6) and the vertical Böttcher functions over the whole base space (Proposition 2.4).

PROPOSITION 4.8. *Let f and g be nondegenerate polynomial skew products of iso-bidegree. If $J_f = J_g$ and if the base Julia set is not a circle or a straight line segment, then $f^n = \gamma g^m$ for some positive integers n and m and for some γ in Γ.*

Proof. Let $f = (p, q)$ and $g = (r, s)$. By Theorem 4.6, there exists a polynomial R such that $p = \sigma_1 R^m$ and $r = \sigma_2 R^n$ for some positive integers m and n and for some σ_1 and σ_2 in Σ. Hence the bidegrees of f^n and g^m are the same, so Lemma 4.2 now completes the proof. □

LEMMA 4.9. *Let f and g be nondegenerate polynomial skew products of iso-bidegree. If $J_f = J_g$ and if the base Julia set is not a circle or a straight line segment, then $\varphi^f_z = \varphi^g_z$ for any z in $\mathbf{C}$.*

Proof. We may assume that f is in normal form and that $\gamma(z, w) = (\mu z, \nu w)$ for some μ and ν in S^1. By Proposition 4.8, $f^n = \gamma g^m$ for some positive integers n and m and for some γ in Γ. Let $Q^n_z = q_{p^{n-1}(z)} \cdots q_{p(z)} q_z$ and $S^n_z = s_{r^{n-1}(z)} \cdots s_{r(z)} s_z$, where $f = (p, q)$ and $g = (r, s)$. Then $Q^n_z = \nu S^m_z$ and so $Q^{nj}_z = \nu^{l_j} S^{mj}_z$ for any positive integer j, where $l_j = d^{(j-1)m} + \cdots + d^m + 1$. Hence, for $d_1 = \deg_w q$ and $d_2 = \deg_w s$,

$$\begin{aligned} G^f_z(w) &= \lim_{j\to\infty} \frac{1}{d_1^j} \log|Q^j_z(w)| = \lim_{j\to\infty} \frac{1}{d_1^{nj}} \log|Q^{nj}_z(w)| \\ &= \lim_{j\to\infty} \frac{1}{d_2^{mj}} \log|S^{mj}_z(w)| = \lim_{j\to\infty} \frac{1}{d_2^j} \log|S^j_z(w)| = G^g_z(w). \end{aligned}$$

The relation between the vertical Green functions and Böttcher functions implies the identity $\varphi_z^f = \varphi_z^g$ for any z in $\mathbf{C}$, since the coefficients of the leading terms of φ_z^f and φ_z^g are both 1. $\square$

Now, let us generalize the answer in [SS] and [AHu].

THEOREM 4.10. *For any Julia set J of a nondegenerate polynomial skew product of iso-bidegree whose base Julia set is not a circle or a straight line segment, there exists a nondegenerate polynomial skew product h of iso-bidegree such that any nondegenerate polynomial skew product of iso-bidegree with the Julia set J can be written in the form γh^k for some positive integer k and γ in Γ:*

$$\{g : \textit{polynomial skew product},\ J_g = J\} = \{\gamma h^k : \gamma \in \Gamma,\ k \in N\},$$

where g is nondegenerate and of iso-bidegree.

Proof. Let h be a nondegenerate polynomial skew product of the minimum iso-bidegree such that $J_h = J$. If the bidegree of g coincides with that of h then, by Lemma 4.2, $g = \gamma h$ for some γ in Γ. If the bidegree of g is larger than that of h, then it follows from Theorem 4.6 and the property of iso-bidegree that the bidegree of g is divisible by that of h. Let $h = (p, q)$ and $g = (r, s)$, where $q_z(w) = bw^d + O_z(w^{d-1})$ and $s_z(w) = b'w^{ds} + O_z(w^{ds-1})$.

We shall construct a nondegenerate polynomial skew product f of iso-bidegree such that $g = fh$; in a similar way, one can construct a polynomial t such that $r = tp$. By Lemma 4.9, $\varphi_z^f = \varphi_z^g$ for any z in $\mathbf{C}$. We denote it simply by φ_z and define the following function for any z in $\mathbf{C}$:

$$\tilde{u}_z(w) = \varphi_{t(z)}^{-1}(c(\varphi_z(w))^s), \quad \text{where } c = \frac{b'}{b^s}.$$

Then $\tilde{u}_z$ is a holomorphic function defined near infinity and $s_z = \tilde{u}_{p(z)} q_z$ near infinity. Since $\tilde{u}_z \sim cw^s$ as $w \to \infty$, one can put $\tilde{u}_z(w) = u_z(w) + o_z(1)$, where $u_z(w) = cw^s + O_z(w^{s-1})$ are the regular terms of $\tilde{u}_z$. From the equation $s_z = \tilde{u}_{p(z)} q_z$ it follows that $s_z(w) - u_{p(z)}(q_z(w)) = o_{p(z)}(q_z(w))$ near infinity. Thus $o_{p(z)}(q_z(w)) = 0$ and so $o_{p(z)}(1) = 0$. Hence $\tilde{u}_z$ coincides with u_z, which is a polynomial in w, and $g = fh$ for $f = (t, u)$. Let us show that f is a polynomial skew product.

We must show that all the coefficients in u_z are polynomials in z. Let $u(z, w) = u_z(w) = cw^s + c_{s-1}(z)w^{s-1} + c_{s-2}(z)w^{s-2} + \cdots + c_0(z)$. Since $g = fh$, it follows that all the coefficients in uh with respect to w are polynomials in z. The coefficient of w^{s-1} in uh is the sum of $c_{s-1}(p(z))$ and a polynomial in z. Thus $c_{s-1}(p(z))$ is a polynomial in z, and so is $c_{s-1}(z)$. The coefficient of w^{s-2} in uh is the sum of $c_{s-2}(p(z))$ and a polynomial in z that contains $c_{s-1}(p(z))$. Hence $c_{s-2}(p(z))$ is a polynomial in z, and so is $c_{s-2}(z)$. By continuing the same argument, it follows that $c_j(z)$ is a polynomial in z for any j.

We have shown that g is divisible by h; that is, $g = fh$. It then follows that J is completely invariant under f and so $J_f = J$. Again we divide f by h—in other words, we divide g by h^2. We continue dividing g by h until the bidegree of g coincides with that of h^k for some positive integer k. Then Lemma 4.2 completes the proof. □

We end the paper by applying the results in this section to the dynamics of regular polynomial skew products. We say that a polynomial skew product on $\mathbf{C}^2$ is *regular* if it extends to a holomorphic map on the two-dimensional complex projective space $\mathbf{P}^2$. For a polynomial skew product $f(z,w) = (p(z), q(z,w))$ of bidegree (δ, d), where $q(z,w) = b_d(z)w^d + b_{d-1}(z)w^{d-1} + \cdots + b_0(z)$, it follows that f is regular if and only if $\delta = d$ and $\deg b_j(z) \le d - j$ for any j. Clearly, a regular polynomial skew product is nondegenerate and of iso-bidegree.

Corollary 4.11. *Let J be the Julia set of a regular polynomial skew product whose base Julia set is not a circle or a straight line segment. Then all nondegenerate polynomial skew products of iso-bidegree with the Julia set J are regular.*

Proof. Let f be a regular polynomial skew product with the Julia set J, and let g be a nondegenerate polynomial skew product of iso-bidegree with the Julia set J. By Proposition 4.8, $f^n = \gamma g^m$ for some integers n and m and for some γ in Γ. Since f^n and γ are regular, g^m is also regular. Hence g is regular. □

Let f be a regular polynomial skew product, and let $\hat{f}$ be the extension of f to a holomorphic map on $\mathbf{P}^2$. We define the first Julia set $J_1(f)$ of f as the support of the Green *current* of $\hat{f}$ and define the second Julia set $J_2(f)$ of f as the support of the Green *measure* of $\hat{f}$. In [J] it was shown that the second Julia set $J_2(f)$ coincides with the Julia set J_f. One can denote the first Julia set in terms of Green functions. There exists the Green function G_f of f on $\mathbf{C}^2$, defined by

$$G_f(z) = \lim_{n\to\infty} d^{-n} \log^+|f^n(z,w)|,$$

where d is the degree of f and $|(z,w)| = \max\{|z|,|w|\}$ is a norm on $\mathbf{C}^2$. Define $K_z = \{w : G_f(z,w) = G_p(z)\}$ and $J_z = \partial K_z$, where p is the first component of f. Then

$$J_1(f) = \overline{\bigcup_{z\in\mathbf{C}} \{z\} \times J_z \cup \bigcup_{z\in J_p} \{z\} \times K_z},$$

where the closure is taken in $\mathbf{P}^2$.

Corollary 4.12. *For a regular polynomial skew product whose base Julia set is not a circle or a straight line segment, the first Julia set is uniquely determined by the second Julia set.*

Proof. Let $f = (p,q)$ and $g = (r,s)$ be regular polynomial skew products having the same Julia set. If the base Julia set is not a circle or a straight line segment, then by Proposition 4.8 it follows that $f^n = \gamma g^m$ for some positive integers n and m and for some γ in Γ. Hence $G_p = G_r$ and $G_f = G_g$, so $J_1(f) = J_1(g)$. □

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