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Wiener's Positive Fourier Coefficients Theorem in Variants of L^p Spaces

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1. Introduction

In this paper we consider spaces that are "close" to $L^p(\mathbb{T})$: L^p itself; the space of functions f with positive Fourier coefficients that have $|f|^p$ integrable near 0; the space of functions whose Fourier coefficients are in $\ell^{p'}$; the space of functions whose Fourier coefficients $\{c_n\}$ satisfy $\sum |c_n|^p n^{p-2} < \infty$; and the mixed norm spaces $\ell^{p',2}$, $1 < p < 2$. We shall describe several relationships between these spaces.

Let $\mathbb{T}$ be the interval $[-\pi, \pi]$. For every $1 \le p < \infty$, we say that a measurable function f is in $L^p = L^p(\mathbb{T})$ if

$$\|f\|_p^p = \frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^p\,dx < \infty.$$

Note that $L^p \subseteq L^1$ for every $p \ge 1$. For $f \in L^1$ and for every integer n, let

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-inx}\,dx \tag{1.1}$$

be the nth Fourier coefficient of f and let $\sum \hat{f}(n)e^{inx}$ be the Fourier series of f. For each $p > 1$, let

$$L^p_{\text{loc}+} = \left\{ f : \text{all } \hat{f}(n) \ge 0 \text{ and } \int_{-\delta}^{\delta} |f|^p\,dx < \infty \text{ for some } \delta = \delta(f) > 0 \right\}.$$

An unpublished theorem of Norbert Wiener asserts that if $f \in L^2_{\text{loc}+}$ then $f \in L^2(\mathbb{T})$. The short proof involves observing that, for each n, $\hat{f}(n) \le$ a constant times $|\widehat{hf}(n)|$, where

$$h(x) = \begin{cases} 1 - |x/\delta|, & x \in [-\delta, \delta], \\ 0, & x \in \mathbb{T} \setminus [-\delta, \delta], \end{cases}$$

so that $hf \in L^2(\mathbb{T})$. Thus $\sum |\widehat{hf}(n)|^2 < \infty$, $\sum |\hat{f}(n)|^2 < \infty$, and $f \in L^2(\mathbb{T})$ by Parseval's theorem. Much later, Stephen Wainger remarked that $f \in L^{2n}_{\text{loc}+}$ implies $f \in L^{2n}_{\text{loc}+}$, $n = 1, 2, 3, \dots$, but gave examples showing that $f \in L^p_{\text{loc}+}$ does not necessarily imply that $f \in L^p(\mathbb{T})$ when $1 < p < 2$. Next, Harold Shapiro showed that if $p \in (2, \infty)$ is not an even integer then $f \in L^p_{\text{loc}+}$ does not necessarily imply

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that $f \in L^p(\mathbb{T})$. Then in [ARV], a "replacement" for the potential $L^p_{\mathrm{loc}+} \Rightarrow L^p$ theorem disallowed by Wainger's counterexamples was found—namely, that for $1 < p < 2$, if $f \in L^p_{\mathrm{loc}+}$ then $f \in \ell^{p'}$; that is, $\sum |\hat{f}(n)|^{p'} < \infty$, where $p' = \frac{p}{p-1}$ is the conjugate of p.

We believe that in Theorem 1 (see Section 3) we have found a sufficiently general setting for the appropriate theorem to fit Wiener's proof. The spaces to which Theorem 1 applies include:

(i) $\ell^{p'}$;
(ii) spaces that we will call HL^p in honor of Hardy and Littlewood;
(iii) the mixed-norm spaces $\ell^{p',2}$ for $1 < p < 2$; and
(iv) L^p for $p = 2, 4, 6, \dots$.

When $1 < p < 2$, the spaces $\ell^{p'}$ are "close" to L^p in a sense made precise by the Hausdorff–Young theorem (see Theorem A to follow). When $1 < p < \infty$, we define HL^p to be the set of all L^1 functions whose Fourier coefficients satisfy

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^p (|n| + 1)^{p-2} < \infty.$$

A result of Hardy and Littlewood connects HL^p to L^p in the same way that the Hausdorff–Young Theorem connects $\ell^{p'}$ to L^p (see Theorem B).

Another aim of this paper is to give a collection of counterexamples that make explicit the relations between the spaces L^p, $L^p_{\mathrm{loc}+}$, $\ell^{p'}$, and HL^p. These results are summarized in three Venn diagrams. In particular, standard examples showing the sharpness of both sides of the Hausdorff–Young theorem have oscillating coefficients; here these are replaced by examples with positive coefficients.

2. History

The following theorems give conditions for membership of a function in L^p in terms of its Fourier coefficients (see [B; Z, Chap. XII, 2.3, 3.19]).

Theorem A (Hausdorff–Young). *Let $1 < p < \infty$, and let p' denote the conjugate index of p.*

(i) *For $1 < p \le 2$, if $f \in L^p$ then*

$$\left(\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^{p'} \right)^{1/p'} \le \|f\|_p.$$

(ii) *For $2 \le p < \infty$, if $\{c_n\}$ is a set of numbers such that $\sum |c_n|^{p'} < \infty$, then there is an $f \in L^p$ with $\hat{f}(n) = c_n$ and*

$$\|f\|_p \le \left(\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^{p'} \right)^{1/p'}.$$

Theorem B (Hardy–Littlewood).

(i) *For $1 < p \le 2$, if $f \in L^p$ then*

$$\left(\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^p (|n| + 1)^{p-2} \right)^{1/p} \le C \|f\|_p.$$

(ii) *For $2 \le p < \infty$, if $\{c_n\}$ is a set of numbers such that $\sum |c_n|^p n^{p-2} < \infty$, then there is an $f \in L^p$ with $\hat{f}(n) = c_n$ and*

$$\|f\|_p \le C\Big(\sum_{n\in\mathbb{Z}} |c_n|^p (|n|+1)^{p-2}\Big)^{1/p}.$$

THEOREM C (Wiener). *Let $f \in L^1$ be such that $\hat{f}(n) \ge 0$ for every integer n, and suppose there exists some $\delta > 0$ such that*

$$\int_{-\delta}^{\delta} |f(x)|^2\, dx < \infty.$$

Then $f \in L^2$.

These theorems motivate the definitions of the following subclasses of L^1. For $f \in L^1$, we say that $f \in \ell^p$ if

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^p < \infty$$

and we say $f \in HL^p$ if

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^p (|n|+1)^{p-2} < \infty.$$

Finally, we say that $f \in L^p_{\text{loc}+}$ if $\hat{f}(n) \ge 0$ for every integer n and if, for some $\delta > 0$,

$$\int_{-\delta}^{\delta} |f(x)|^p\, dx < \infty.$$

With these definitions, Theorems A, B, and C can be succinctly stated as follows.

THEOREM A′ (Hausdorff–Young).

(i) *For $1 < p \le 2$, $L^p \subset \ell^{p'}$.*
(ii) *For $2 \le p < \infty$, $\ell^{p'} \subset L^p$.*

THEOREM B′ (Hardy–Littlewood).

(i) *For $1 < p \le 2$, $L^p \subset HL^p$.*
(ii) *For $2 \le p < \infty$, $HL^p \subset L^p$.*

THEOREM C′ (Wiener). $L^2_{\text{loc}+} \subset L^2$.

Note that for $p = 2$ we have this tidy state of affairs:

$$L^2_{\text{loc}+} \subset L^2 = \ell^2 = HL^2. \tag{2.1}$$

If $p \ne 2$, then equality fails to hold in all cases. The purpose of this paper is to explore the relationships between the various subclasses of L^1 just defined, for the cases $p \ne 2$, and to provide explicit examples of functions with positive coefficients in these spaces. We will treat the cases $1 < p < 2$ and $p > 2$ separately.

3. The Case $p < 2$

For $1 < p < 2$, Wainger [W] proved the following theorem by constructing an explicit example.

THEOREM D. *For* $1 < p < 2$,

$$L^p_{\text{loc}+} \not\subset L^p.$$

For more positive results, Ash, Rains, and Vági [ARV] have shown that the containment holds.

THEOREM E. *For* $1 < p < 2$,

$$L^p_{\text{loc}+} \subset \ell^{p'}.$$

The following theorem generalizes Theorems C and E. Recall that a space of functions X is called *solid* if it satisfies the following property: For every $f = \sum c_n e^{in\theta}$ in X, if another function $g = \sum d_n e^{in\theta}$ satisfies $|d_n| \le |c_n|$ for every n then g is also in X.

THEOREM 1. *Let* X *be a space of functions. If* $L^p \subset X$ *and if* X *is solid, then*

$$L^p_{\text{loc}+} \subset X.$$

Proof. Let $f = \sum c_n e^{in\theta} \in L^p_{\text{loc}+}$. Let $h(\theta)$ be the 2π-periodic function that for $|t| \le \pi$ is defined by

$$h(\theta) = \begin{cases} 1 - |\theta|/\delta, & |\theta| \le \delta; \\ 0, & \delta < |\theta| \le \pi. \end{cases}$$

Then $|hf| \le \chi_{[-\delta,\delta]}|f| \in L^p$, so that $hf \in L^p \subset X$.

For every n we have

$$\widehat{hf}(n) = \sum_{k+\ell=n} \hat{h}(k)c_\ell,$$

where $c_\ell \ge 0$ by hypothesis and $\hat{h}(k) \ge 0$ by direct calculation. Drop all terms (except for the $k = 0$ term) from the right side of the last equation to obtain

$$c_n \le \frac{\widehat{hf}(n)}{\hat{h}(0)} = \frac{2\pi}{\delta}\widehat{hf}(n).$$

Since X is solid, it follows that $f \in X$. □

COROLLARY 2. *For* $1 < p < 2$,

$$L^p_{\text{loc}+} \subset HL^p.$$

Proof. By Theorem B, we know that $L^p \subset HL^p$ for $1 < p < 2$. The definition of HL^p shows it is solid, and the corollary follows from Theorem 1. □

COROLLARY 3. *For* $1 < p < 2$,

$$L^p_{\text{loc}+} \subset \ell^{p',2}.$$

Proof. We refer the reader to [K] for the precise definition of the mixed-norm spaces $\ell^{p',2}$, which shows that they are solid. □

REMARK 1. Combining Theorem 1 with Wainger's theorem shows that Wainger's function is an example of a function *with positive coefficients* that shows the converse of the Hardy–Littlewood theorem to be false. The same function shows that the converse of the Hausdorff–Young theorem is false when we combine Wainger's theorem and the result of [ARV].

REMARK 2. Kellogg proved the following result:

$$L^p \subset \ell^{p',2}.$$

Since $\ell^{p',2} \subset \ell^{p'}$, Kellogg's result is an improvement to Hausdorff–Young. Similarly, Corollary 3 improves the result of Ash, Rains, and Vági.

We summarize the situation for $1 < p < 2$ in Figure 1. Points on the diagram above the line X correspond to Fourier series with positive coefficients; points below X correspond to those series with some or all negative coefficients.

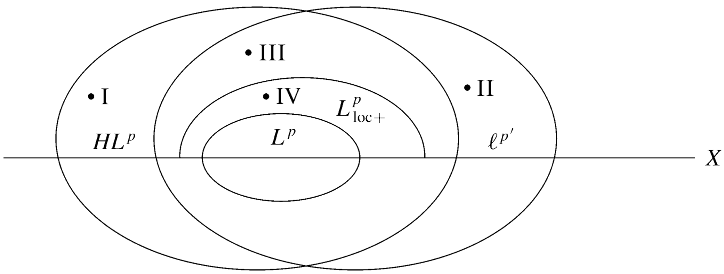

Figure 1

The goal now is to provide examples of functions with positive coefficients in each labeled region of the diagram. Note that Wainger's function is an example in region IV. We first give examples of functions in regions I and II.

THEOREM 4 (Example I and II). *For $1 < p < 2$, we have $\ell^{p'} \not\subset HL^p$ and $HL^p \not\subset \ell^{p'}$.*

Proof. Let

$$\alpha_k = k^{-1/p'} \ln^{-1/p} k, \quad k = 2, 3, \ldots;$$

then $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{p'}{p} = \frac{1}{p-1}$, and

$$\sum_{k=2}^{\infty} \alpha_k^{p'} = \sum_{k=2}^{\infty} k^{-1} \ln^{-1/(p-1)} k < \infty,$$

since $\frac{1}{p-1} > 1$ if and only if $2 > p$. However,

$$\sum_{k=2}^{\infty} \alpha_k^p k^{p-2} = \sum_{k=2}^{\infty} k^{-p+1} k^{p-2} \ln^{-1} k = \sum_{k=2}^{\infty} k^{-1} \ln^{-1} k = \infty.$$

Thus

$$\ell^{p'} \not\subset HL^p.$$

Conversely, let

$$\beta_k = \begin{cases} j^{-1/p'}, & k = 2^j; \\ 0, & k \text{ is not a power of } 2. \end{cases}$$

Then, since
$$\sum_k \beta_k^{p'} = \sum_j j^{-1} = \infty,$$
it follows that
$$\sum_k \beta_k e^{ikx} \notin \ell^{p'};$$
however,
$$\sum_k \beta_k^p (k+1)^{p-2} = \sum_j j^{-p/p'}(2^j+1)^{p-2} < \sum_j \frac{1}{(2^{2-p})^j} < \infty$$
so that
$$HL^p \not\subset \ell^{p'}. \qquad \square$$

Remark 3. Similar results have been observed for analytic functions in Fock spaces (see [T]).

The following function is an example in region III of the figure.

Theorem 5 (Example III). *There is a function with positive Fourier coefficients in $HL^p \cap \ell^{p'}$ but not in $L^p_{\mathrm{loc}+}$.*

Proof. Let
$$g(\theta) = \sum_{n=1}^{\infty} \frac{e^{in^\alpha}}{n^\gamma} e^{in\theta},$$
where γ and $0 < \alpha < 1$ are parameters to be determined. From [Z, V.5] we know that
$$g(\theta) = O\Big(|\theta|^{\frac{-1+\gamma+\alpha/2}{1-\alpha}}\Big)$$
as $|\theta| \to 0$.

Because $1 < p < 2$, we can choose γ so that $\frac{1}{p'} < \gamma < \frac{1}{2}$. Then the coefficients $a_n = e^{in^\alpha}/n^\gamma$ satisfy
$$\sum |a_n|^{p'} = \sum \frac{1}{n^{\gamma p'}} < \infty,$$
so that $g \in \ell^{p'}$. Similarly,
$$\sum |a_n|^p n^{p-2} = \sum \frac{1}{n^{-p+2+\gamma p}} < \infty,$$
and we have $g \in HL^p$.

On the other hand, we can choose α sufficiently close to 1 so that
$$p\left(\frac{-1+\gamma+\alpha/2}{1-\alpha}\right) < p\left(\frac{\gamma-1/2}{1-\alpha}\right) < -1.$$
This shows that g is not locally integrable. We shall now construct another function with positive coefficients that is not in $L^p_{\mathrm{loc}+}$.

With the parameters γ and α chosen as before, consider
$$g(\theta) = \sum_{n=1}^{\infty} \frac{\cos n^\alpha}{n^\gamma} e^{in\theta} + \sum_{n=1}^{\infty} \frac{i \sin n^\alpha}{n^\gamma} e^{in\theta} = g_1(\theta) + i g_2(\theta).$$
Since g is not locally integrable, it follows that either g_1 or g_2 is not locally integrable. If g_1 is not locally integrable, consider the series

$$h(\theta) = \sum_{n=1}^{\infty} \frac{2}{n^\gamma} e^{in\theta} + g_1(\theta) = \sum_{n=1}^{\infty} \frac{2 + \cos n^\alpha}{n^\gamma} e^{in\theta}.$$

Since $\gamma > \frac{1}{p'}$, the series $\sum \frac{2}{n^\gamma} e^{in\theta}$ is in L^p [Z, Chap. V]. Thus h is not locally integrable, has positive coefficients, and belongs to both $\ell^{p'}$ and HL^p.

If g_2 is not integrable, then the following function has the same desired properties:

$$k(\theta) = \sum_{n=1}^{\infty} \frac{2}{n^\gamma} e^{in\theta} + g_2(\theta) = \sum_{n=1}^{\infty} \frac{2 + \sin n^\alpha}{n^\gamma} e^{in\theta}. \qquad \square$$

4. The Case $p > 2$

For $2 < p < \infty$, it can be easily shown that $L^p_{\text{loc}+} \subset L^p$ when p is an even integer. Thus we have $L^p_+ = L^p_{\text{loc}+}$, where L^p_+ denotes L^p functions with nonnegative coefficients. Figure 2 depicts the situation when $p > 2$ and p is even.

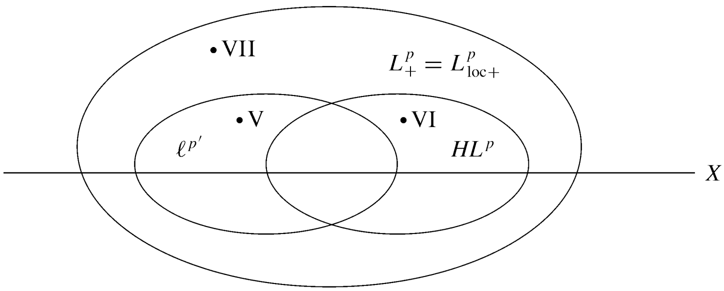

Figure 2

For p not an even integer, Shapiro [S] proved the following theorem.

THEOREM F. *Let $2 < p < \infty$. If p is not even, then there is a function in $L^p_{\text{loc}+}$ that is not in L^p.*

Thus we have Figure 3, which is similar to Figure 2 but has an added region VIII. An example in region VIII has been found by Shapiro. We now give examples in all remaining regions, beginning with regions V and VI.

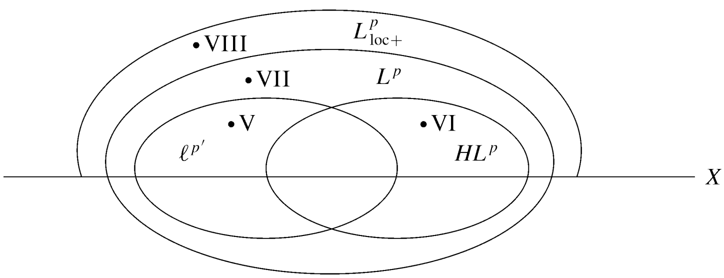

Figure 3

THEOREM 6 (Example V and VI). *For* $2 < p < \infty$, *we have* $HL^p \not\subset \ell^{p'}$ *and* $\ell^{p'} \not\subset HL^p$.

Proof. Let

$$a_k = k^{-1/p'} \ln^{-1/2} k, \quad k = 2, 3, \ldots;$$

then

$$\sum_{k=2}^{\infty} a_k^{p'} = \sum_{k=2}^{\infty} k^{-1} \ln^{-p'/2} k = \infty,$$

since $p' < 2$. However, from $\frac{1}{p} + \frac{1}{p'} = 1$ it follows from $-\frac{p}{p'} = 1 - p$ and

$$\sum_{k=2}^{\infty} a_k^p k^{p-2} = \sum_{k=2}^{\infty} k^{1-p} k^{p-2} \ln^{-p/2} k = \sum_{k=2}^{\infty} k^{-1} \ln^{-p/2} k < \infty.$$

Therefore,

$$\ell^{p'} \not\subset HL^p.$$

Conversely, let

$$b_k = \begin{cases} j^{-2/p'}, & k = 2^j; \\ 0, & k \text{ is not a power of } 2. \end{cases}$$

Then, since

$$\sum_k b_k^{p'} = \sum_j j^{-2} < \infty,$$

it follows that

$$\sum b_k e^{ikx} \in \ell^{p'};$$

however,

$$\sum_k b_k^p (k+1)^{p-2} = \sum_j j^{-2p/p'} (2^j + 1)^{p-2} = \infty$$

and so

$$HL^p \not\subset \ell^{p'}. \qquad \square$$

The next is an example in region VII.

THEOREM 7. *For* $2 < p < \infty$, *there is a function with positive Fourier coefficients in* L^p *but not in* $HL^p \cup \ell^{p'}$.

Proof. Fix $p > 2$ so that $\frac{1}{p'} > \frac{1}{2}$. Choose α with $\frac{1}{2} < \alpha < \frac{1}{p'}$. Consider the series

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta},$$

where

$$c_n = \begin{cases} 1/k^{\alpha}, & n = \pm 2^k \text{ for some integer } k; \\ 0, & n \neq \pm 2^k. \end{cases}$$

For any $\alpha > \frac{1}{2}$ we have

$$\sum_{n \in \mathbb{Z}} c_n^2 < \infty,$$

so $f \in L_p$ (see [Z, (8.4)]).

Observe that

$$\sum_{n\in\mathbb{Z}} |c_n|^{p'} = \infty$$

and

$$\sum_{n\in\mathbb{Z}} |c_n|^p |n|^{p-2} = 2\sum_{n=1}^{\infty} \frac{1}{k^{\alpha p}} 2^{k(p-2)} = \infty.$$

Thus we have shown that f is a function in L^p with positive coefficients yet $f \notin l^{p'} \cup HL^p$. □

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