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Borel–Moore Homology and K-theory on the Steinberg Variety

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1. Introduction

Let G be a simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, $\mathcal{B}$ the flag variety of G, $\mathcal{N}$ the nilpotent cone in $\mathfrak{g}$, and $\widetilde{\mathcal{N}}$ the Springer resolution $\{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}$ of $\mathcal{N}$. We also let

$$Z = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b} \cap \mathfrak{b}'\}$$

be the Steinberg variety. Since its inception, the Steinberg variety has proved to be an important object in the representation theory of Weyl groups and affine Hecke algebras. In particular, if we let $H_*(Z, \mathbb{C}) = \bigoplus_{i \geq 0} H_i(Z, \mathbb{C})$ be the complex graded Borel–Moore homology algebra equipped with a convolution product, then the knowledge of the graded algebra structure of $H_*(Z, \mathbb{C})$ is a key ingredient for obtaining all irreducible representations of Weyl groups through the decomposition theorem of Bellinson, Bernstein, and Deligne.

In this paper, we study more explicitly the $\mathbb{C}$-algebra structure of $H_*(Z, \mathbb{C})$. To do so, we will examine the convolution product of $H_*(Z, \mathbb{C})$ through multiplications on other objects, such as the Grothendieck group of a graph variety, (co)homology of the flag variety, and a certain crossed product algebra. We prove that the convolution product on $H_*(Z, \mathbb{C})$ is compatible with all of these multiplications (see Theorem 3.7). As an application of our compatibility theorem, we construct a $\mathbb{C}$-algebra isomorphism between $H_*(Z, \mathbb{C})$ and a certain crossed product algebra (see Theorem 3.9).

After this paper was written, the author was informed that Douglass and Röhrle [DR1; DR2] proved a result similar to Corollary 3.10. However, our approach differs from that of Douglass and Röhrle.

2. Preliminaries

Borel–Moore Homology. Let X be a complex algebraic variety, and let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of X. Then the ith Borel–Moore homology space is defined as $H_i(X) := H_i^{\mathrm{ord}}(\hat{X}, \infty)$, where H_i^{ord} denotes the ith singular relative homology over the complex coefficients (see [BoMo] for details). Throughout the paper, we will consider all Borel–Moore homology spaces to be over the complex numbers $\mathbb{C}$.

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One of the most important properties of Borel–Moore homology is the existence of a fundamental class $[X]$ for *any* complex algebraic variety X. In particular, let X be a complex algebraic variety of complex dimension n, and let $\{X_1, \dots, X_m\}$ be the n-dimensional irreducible components of X. Then it is known that the fundamental classes $[X_1], \dots, [X_m]$ form a basis for the vector space $H_{2n}(X)$.

Let M be a smooth variety of complex dimension n. Then we can consider the following standard $\cup$-product in cohomology:

$$\cup\colon H^{2n-i}(M) \times H^{2n-j}(M) \to H^{4n-i-j}(M). \tag{2.1}$$

Applying the Poincaré duality to (2.1) yields the following bilinear pairing, which is called the *intersection pairing*:

$$\cap\colon H_i(M) \times H_j(M) \to H_{i+j-2n}(M). \tag{2.2}$$

CONVOLUTIONS IN BOREL–MOORE HOMOLOGY AND K-THEORY. We first review the construction of the convolution product in Borel–Moore homology that is due to Ginzburg [CG].

Let M_1, M_2, and M_3 be connected, oriented smooth manifolds with the (i, j)-projections $p_{ij}\colon M_1 \times M_2 \times M_3 \to M_i \times M_j$. We suppose that $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ are closed subsets. We also assume that

$$p_{13}\colon (p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})) \to M_1 \times M_3 \tag{2.3}$$

is proper, and we write $Z_{12} \circ Z_{23}$ for the image of p_{13}.

We define a convolution product in Borel–Moore homology as

$$*\colon H_i(Z_{12}) \otimes H_j(Z_{23}) \to H_{i+j-d}(Z_{12} \circ Z_{23}), \quad c_{12} \otimes c_{23} \mapsto c_{12} * c_{23}, \tag{2.4}$$

where $c_{12} * c_{23} = (p_{13})_*(p_{12}^* c_{12} \cap p_{23}^* c_{23})$ and $d = \dim_{\mathbb{R}} M_2$.

In particular, if we let $M_1 = M_2 = M_3 = M$ be a smooth manifold of dimension n and $Z_{12} = Z_{23} = M$, then we have the convolution map

$$*\colon H_i(M) \otimes H_j(M) \to H_{i+j-2n}(M). \tag{2.5}$$

In this case, the convolution product in (2.5) reduces to the intersection pairing in (2.2).

A similar convolution construction works for the algebraic K-theory. More explicitly, let X be a smooth algebraic variety and let Z_1, Z_2 be closed subsets of X. We also let $K(X)$ be the Grothendieck group of all coherent sheaves on X.

For given $\mathcal{F}_1 \in K(Z_1)$ and $\mathcal{F}_2 \in K(Z_2)$, we write $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in K(Z_1 \times Z_2)$ for the external tensor product $p_{Z_1}^* \mathcal{F}_1 \otimes_{\mathcal{O}_{Z_1 \times Z_2}} p_{Z_2}^* \mathcal{F}_2$ of $\mathcal{F}_1$ and $\mathcal{F}_2$. Here p_{Z_1} and p_{Z_2} denote the projections of $Z_1 \times Z_2$ to corresponding factors. Next, let $\Delta\colon X \hookrightarrow X \times X$ be the diagonal embedding. Then we have $\Delta^{-1}(Z_1 \times Z_2) = Z_1 \cap Z_2$. So the restriction map $\Delta\colon Z_1 \cap Z_2 \to Z_1 \times Z_2$ induces the map

$$\Delta^*\colon K(Z_1 \times Z_2) \to K(Z_1 \cap Z_2). \tag{2.6}$$

By combining Δ^* with the external tensor product, we obtain the following tensor product with supports:

$$\otimes\colon K(Z_1) \otimes K(Z_2) \to K(Z_1 \cap Z_2), \quad (\mathcal{F}_1, \mathcal{F}_2) \mapsto \Delta^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2). \tag{2.7}$$

With the tensor product in (2.7), we define the convolution product in the algebraic K-theory as follows.

Let M_1, M_2, M_3 be smooth algebraic varieties. Let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be closed subvarieties satisfying condition (2.3). Then we define a convolution product

$$*\colon K(Z_{12}) \otimes K(Z_{23}) \to K(Z_{12} \circ Z_{23}) \tag{2.8}$$

to be $\mathcal{F}_{12} * \mathcal{F}_{23} := (p_{13})_*((p_{12})^*\mathcal{F}_{12} \otimes (p_{23})^*\mathcal{F}_{23})$.

In particular, the convolution product $*\colon K(X) \otimes K(X) \to K(X)$ coincides with the tensor product $\otimes\colon K(X) \otimes K(X) \to K(X)$ defined in (2.7).

Specializations in Borel–Moore Homology and K-theory. Let (X, o) be a smooth manifold with base point o. Given a possibly singular space E, suppose that we have a map $p\colon E \to X$. We also assume that the restriction map $p\colon P^{-1}(X - o) \to X$ is a locally trivial fibration with possibly singular fibers. However, we do not assume that $p\colon E \to X$ is locally trivial around o. According to Fulton and MacPherson [FM], there is a homomorphism

$$\lim_{X \to o}\colon H_i(p^{-1}(X - o)) \to H_{i - \dim_{\mathbb{R}}(X)}(p^{-1}(o)),$$

which is called a *specialization map.*

One of the essential features of the specialization map in Borel–Moore homology is that it commutes with the convolution product of Borel–Moore homology. We can similarly define the specialization homomorphism in K-theory. In more detail, let X be a variety and C a smooth algebraic curve with a base point o. Further, let $p\colon X \to C$ be a morphism. Then there exists a specialization homomorphism $\lim_{C \to o}\colon K(p^{-1}(X - o)) \to K(p^{-1}(o))$. Like the specialization map in Borel–Moore homology, the specialization map in K-theory commutes with the convolution product in K-theory.

The Chern Character. Let X be a smooth quasi-projective variety. Then there is a ring homomorphism $\mathrm{ch}^*\colon K(X) \to H^*(X)$, called the *cohomological Chern character map,* from the Grothendieck ring $K(X)$ to the cohomology ring $H^*(X)$. If we apply the Poincaré duality between the cohomology ring $H^*(X)$ and the graded Borel–Moore homology $H_*(X)$, then we obtain the *homological Chern character map* $\mathrm{ch}_*\colon K(X) \to H_*(X)$. The cohomological Chern character map $\mathrm{ch}^*\colon K(X) \to H^*(X)$ is a ring homomorphism, so we see that the homological Chern character map $\mathrm{ch}_*\colon K(X) \to H_*(X)$ becomes a ring homomorphism for a smooth quasi-projective variety X (see Corollary 3.4). Construction of the Chern character maps over a smooth quasi-projective variety can be extended to those over any closed subvariety of a smooth quasi-projective variety. However, for a nonsmooth variety such as the Steinberg variety, we can no longer guarantee that the homological Chern character map is a ring homomorphism.

NOTATION. Throughout the paper, the following notation will be in force:

- $K_{\mathbb{C}}(X)$ denotes the tensor product $\mathbb{C} \otimes_{\mathbb{Z}} K(X)$ for any algebraic variety X;
- $\mathcal{B}$ is the flag variety of a complex semisimple Lie algebra $\mathfrak{g}$;
- $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$;
- Z denotes the Steinberg variety $\{(x, \mathfrak{b}_1, \mathfrak{b}_1) \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b}_1 \cap \mathfrak{b}_2\}$.

The following theorem shows why we are interested in the homological Chern character map.

THEOREM 2.1. *For a variety $X = \mathcal{B}$, $T^*(\mathcal{B})$, or Z, the homological Chern character map yields an isomorphism $K_{\mathbb{C}}(X) \to H_*(X)$ of abelian groups.*

Proof. See [CG, Thm. 6.2.4]. □

We conclude this section with the following theorem.

THEOREM 2.2. (1) *The homological Chern character map commutes with the specialization homomorphism in K-theory and also with Borel–Moore homology.*

(2) *Let X be a smooth variety. Then the following diagram commutes*:

$$\begin{array}{ccc} K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(X) & \xrightarrow{*=\otimes} & K_{\mathbb{C}}(X) \\ {\scriptstyle \mathrm{ch}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_*} \\ H_*(X) \otimes H_*(X) & \xrightarrow{*=\cap} & H_*(X). \end{array} \tag{2.9}$$

Proof. See [BFM]. In fact, in [BFM] we can find a proof of a more generalized version of (2). Yet the statement (2) is enough for our purpose, and we give a simple proof in Corollary 3.4. □

3. $\mathbb{C}$-algebra Structure of $H_*(Z)$

In the rest of this paper we fix a complex, semisimple, simply connected Lie group G with the Lie algebra $\mathfrak{g}$. We also fix a maximal torus T of G.

UNIVERSAL RESOLUTION OF $\mathfrak{g}$. Let $\tilde{\mathfrak{g}}$ be the universal resolution of $\mathfrak{g}$ (i.e., $\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$), and let μ be the first projection $\mu\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$. We denote by $\tilde{N}$ the set $\mu^{-1}(N)$. Since all quotient spaces $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ are isomorphic for all $\mathfrak{b} \in \mathcal{B}$, we denote by $\mathfrak{H}$ the the resulting space. We now define a map $\nu\colon \tilde{\mathfrak{g}} \to \mathfrak{H}$ as the quotient map

$$(x, \mathfrak{b}) \mapsto x + [\mathfrak{b}, \mathfrak{b}] \in \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \simeq \mathfrak{H}.$$

We write $\mathfrak{g}^{\mathrm{sr}}$ for the set of semisimple regular elements in $\mathfrak{g}$. Here a *regular* element $x \in \mathfrak{g}$ means that $\dim Z_{\mathfrak{g}}(x) = \operatorname{rank} \mathfrak{g}$. We also denote by $\tilde{\mathfrak{g}}^{\mathrm{sr}}$ the inverse image $\mu^{-1}(\mathfrak{g}^{\mathrm{sr}})$. Then there is a natural action of the Weyl group $W = N(T)/T$ on $\tilde{\mathfrak{g}}^{\mathrm{sr}}$ that preserves fibers $\mu^{-1}(x)$.

Borel Isomorphism. Let $W = N_G(T)/T$ be the Weyl group with respect to the maximal torus T. Then the Weyl group W acts on T. This action yields the algebra $\mathbb{C}[T]^W$ of W-invariants, where $\mathbb{C}[T]$ is the algebra of regular functions on T. There is a natural evaluation homomorphism $\mathbb{C}[T]^W \to \mathbb{C}$ given by evaluation at $1 \in T$. We denote by $\mathcal{I}$ the ideal in $\mathbb{C}[T]$ generated by the kernel of this evaluation homomorphism. It is known from the Borel isomorphism that an algebra $\mathbb{C}[T]/\mathcal{I}$ is isomorphic to the cohomology ring $H^*(\mathcal{B})$ of the flag variety $\mathcal{B}$ as a $\mathbb{C}$-algebra. We now let $X(T)$ be the character group of T. Then the group algebra $\mathbb{C}[X(T)]$ is identified with $\mathbb{C}[T]$. Through this identification, we will write e^λ for the element of $\mathbb{C}[T]$ corresponding to a weight $\lambda \in X(T)$.

Let $\mathbb{C}[T]/\mathcal{I} \sharp \mathbb{C}[W]$ be the crossed product of the $\mathbb{C}$-algebra $\mathbb{C}[T]/\mathcal{I}$ and the group algebra $\mathbb{C}[W]$. This product coincides with the tensor product $\mathbb{C}[T]/\mathcal{I} \otimes_{\mathbb{C}} \mathbb{C}[W]$ (as a vector space) and is equipped with multiplication:

$$(e^{\lambda_1} \otimes w_1) \cdot (e^{\lambda_2} \otimes w_2) = e^{\lambda_1}(w_1 e^{\lambda_2}) \otimes w_1 w_2,$$

where $w_1 e^{\lambda_2}$ denotes the action of W on $\mathbb{C}[T]$.

Let us now consider the direct sum $\bigoplus_{w \in W} (\mathbb{C}[T]/I)w$ and define multiplication on it as follows:

$$(e^{\lambda_1} \cdot w_1)(e^{\lambda_2} \cdot w_2) = e^{\lambda_1}(w_1 e^{\lambda_2}) \cdot w_1 w_2.$$

Then it is clear that

$$\bigoplus_{w \in W} (\mathbb{C}[T]/\mathcal{I})w \simeq \mathbb{C}[T]/\mathcal{I} \sharp \mathbb{C}[W] \tag{3.1}$$

as $\mathbb{C}$-algebras.

Graph Variety Λ_w^h. From now on, we fix a semisimple regular element h in $\mathfrak{g}^{\mathrm{sr}}$. In [CG], the authors introduced the graph variety $\Lambda_w^h \subset \nu^{-1}(h) \times \nu^{-1}(w \cdot h)$. This is the graph of the action $w\colon \nu^{-1}(h) \to \nu^{-1}(w \cdot h)$ obtained by restricting the action $w\colon \tilde{\mathfrak{g}}^{\mathrm{sr}} \to \tilde{\mathfrak{g}}^{\mathrm{sr}}$. By the definition of Λ_w^h, we have

$$\Lambda_w^h = \{(x, \mathfrak{b}, x, \mathrm{Ad}w(\mathfrak{b})) \in \tilde{\mathfrak{g}}^{\mathrm{sr}} \times \tilde{\mathfrak{g}}^{\mathrm{sr}} \mid x \in \mathfrak{b} \cap \mathrm{Ad}w(\mathfrak{b}),$$
$$\nu(x, \mathfrak{b}) = h,\ \nu(x, \mathrm{Ad}w(\mathfrak{b})) = \mathrm{Ad}w(h)\}.$$

The following lemma is due to Ginzburg.

Lemma 3.1. (1) *The inverse image $\nu^{-1}(h)$ is a smooth variety.*

(2) *The graph variety Λ_w^h is a smooth variety.*

Proof. See [G]. □

Next, let us consider the isomorphisms

$$\nu^{-1}(h) \xrightarrow{w} \nu^{-1}(w \cdot h) \xrightarrow{y} \nu^{-1}(yw \cdot h)$$

for elements $w, y \in W$. These isomorphisms yield a set-theoretic composition $\Lambda_w^h \circ \Lambda_y^{w \cdot h}$ defined in (2.3), and we can see that $\Lambda_w^h \circ \Lambda_y^{w \cdot h} = \Lambda_{yw}^h$. Thus we obtain the convolution product

$$*\colon K(\Lambda_w^h) \otimes K(\Lambda_y^{w \cdot h}) \to K(\Lambda_{yw}^h). \tag{3.2}$$

LEMMA 3.2. *The map $p\colon \Lambda_w^h \to \mathcal{B}$ given by $(x, \mathfrak{b}, x, \mathrm{Adw}(\mathfrak{b})) \mapsto \mathfrak{b}$ is a vector bundle over $\mathcal{B}$ with fiber $h + \mathfrak{n}$, and p is a homotopy equivalence.*

Proof. We first notice that Λ_w^h is isomorphic to $\nu^{-1}(h)$, which in turn is isomorphic to $G \times_B (h + \mathfrak{n})$ as G-equivariant vector bundles over $\mathcal{B}$. Moreover, $\mathfrak{n}$ is contractible because it is nilpotent. The lemma now follows. $\square$

ACTION OF THE WEYL GROUP W ON $H_*(\mathcal{B})$ AND $K_{\mathbb{C}}(\mathcal{B})$. Let B be a Borel subgroup containing a maximal torus T, and let W be the Weyl group $W = N_G(T)/T$. We also let $p\colon G/T \to G/B = \mathcal{B}$ be the natural projection. Then p is a homotopy equivalence and induces an isomorphism $p^*\colon H^i(\mathcal{B}) \to H^i(G/T)$. We notice that the right action of W on G/T given by $w\colon gT \to gwT$ yields a W-action on $H^i(G/T)$, and this W-action on $H^i(G/T)$ can be transferred to $H^i(\mathcal{B})$ through the isomorphism p^*. Thus we obtain a W-module structure on $H^i(\mathcal{B})$ that is called the *classical action of* W on $H^i(\mathcal{B})$; see [S] for details. Furthermore, if we apply the Poincaré duality $H_*(\mathcal{B}) \simeq H^*(\mathcal{B})$, then the graded Borel–Moore homology $H_*(\mathcal{B})$ also acquires a W-module structure by means of the classical action of W on $H^*(\mathcal{B})$.

Similarly, the homotopy equivalence $p\colon G/T \to G/B = \mathcal{B}$ yields the natural W-module structure on $K_{\mathbb{C}}(\mathcal{B})$ through the right W-action on G/T.

According to [CG, Prop. 6.4.19], there is a W-module isomorphism

$$\phi\colon \mathbb{C}[T]/\mathcal{I} \to K_{\mathbb{C}}(\mathcal{B}). \tag{3.3}$$

If we combine ϕ with the cohomological Chern character isomorphism

$$\mathrm{ch}^*\colon K_{\mathbb{C}}(\mathcal{B}) \to H^*(\mathcal{B}),$$

then we obtain a W-equivariant algebra isomorphism

$$\psi\colon \mathbb{C}[T]/\mathcal{I} \to H^*(\mathcal{B}), \tag{3.4}$$

which is nothing but the Borel isomorphism introduced in Section 3 (see [CG, Sec. 6.4] for details).

In the following proposition we show that, for a given smooth quasi-projective variety X, the graded Borel–Moore homology $H_*(X)$ is isomorphic to the cohomology ring $H^*(X)$ as $\mathbb{C}$-algebras.

PROPOSITION 3.3. *Let $H_*(X)$ and $H^*(X)$ be equipped with $\mathbb{C}$-algebra structures via the convolution product and the cup product, respectively. Then the Poincaré duality $H_*(X) \simeq H^*(X)$ yields a $\mathbb{C}$-algebra isomorphism.*

Proof. We first recall from Section 2 that the convolution $*\colon H_*(X) \otimes H_*(X) \to H_*(X)$ reduces to the intersection pairing $\cap\colon H_*(X) \otimes H_*(X) \to H_*(X)$. However, by definition the intersection pairing $\cap\colon H_i(X) \otimes H_j(X) \to H_{i+j-\dim_{\mathbb{R}} X}(X)$ is induced from the standard $\cup$-product $\cup\colon H^{\dim_{\mathbb{R}} X - i}(X) \otimes H^{\dim_{\mathbb{R}} X - j}(X) \to H^{2\dim_{\mathbb{R}} X - i - j}(X)$ through the Poincaré duality . The result now follows. $\square$

COROLLARY 3.4. *Let X be a smooth quasi-projective variety. Then the homological Chern character map* $\mathrm{ch}_*\colon K_{\mathbb{C}}(X) \to H_*(X)$ *is a $\mathbb{C}$-algebra homomorphism.*

Proof. We recall that the cohomological Chern character map $\mathrm{ch}^*\colon K_{\mathbb{C}}(X) \to H^*(X)$ is a ring homomorphism. Then the result follows from Proposition 3.3 and the construction of ch_*. □

COROLLARY 3.5. *Let $\rho\colon H^*(\mathcal{B}) \to H_*(\mathcal{B})$ be the Poincaré duality isomorphism for the flag variety $\mathcal{B}$. Then the composition map $\rho \circ \psi\colon \mathbb{C}[T]/\mathcal{I} \to H_*(\mathcal{B})$ is a W-equivariant algebra isomorphism, where ψ is the isomorphism in* (3.4).

Proof. Observe that $\rho\colon H^*(\mathcal{B}) \to H_*(\mathcal{B})$ is a W-module isomorphism by the construction of the W-module structure on $H_*(\mathcal{B})$. The result is now immediate from Proposition 3.3 and the W-equivariant algebra isomorphism ψ. □

In addition to the classical action of W on $H_*(\mathcal{B})$, we have another Weyl group action on $H_*(\mathcal{B})$ via the convolution product. In more detail, we first recall from the work of Ginzburg that the group algebra $\mathbb{C}[W]$ is isomorphic to $H_{\dim_{\mathbb{R}} Z}(Z)$ as $\mathbb{C}$-algebras when $H_{\dim_{\mathbb{R}} Z}(Z)$ is equipped with the convolution product. We also have the convolution product

$$*\colon H_{\dim_{\mathbb{R}} Z}(Z) \otimes H_*(\mathcal{B}) \to H_*(\mathcal{B}) \tag{3.5}$$

under the circumstances $M_1 = M_2 = \widetilde{\mathcal{N}}$, $M_3 = \{\text{point}\}$, $Z = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} \subset M_1 \times M_2$, and $\mathcal{B} \simeq \mu^{-1}(0) \subset M_2 \times M_3$, where $\mu\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$ is the first projection map.

Then, through the identification $\mathbb{C}[W] \simeq H_{\dim_{\mathbb{R}} Z}(Z)$, we can define a W-action on $H_*(\mathcal{B})$ via the convolution product in (3.5). Thus we have two W-actions on $H_*(\mathcal{B})$: one is obtained from the classical W-action on $H_*(\mathcal{B})$, and the other comes from the convolution action in (3.5). However, it is known that these two W-actions on $H_*(\mathcal{B})$ coincide.

LEMMA 3.6. *The W-action on $H_*(\mathcal{B})$ arising from the convolution product in* (3.5) *is the same as the one obtained from the classical W-action on $H_*(\mathcal{B})$.*

Proof. See [CG, Claim 3.6.17]. □

For given $w_1, w_2 \in W$, let

$$\Phi\colon (\mathbb{C}[T]/\mathcal{I})w_1 \otimes (\mathbb{C}[T]/\mathcal{I})w_2 \to K_{\mathbb{C}}(\mathcal{B}) \otimes K_{\mathbb{C}}(\mathcal{B})$$

be a map defined by $\Phi(e^{\lambda}w_1 \otimes e^{\mu}w_2) = \phi(e^{\lambda}) \otimes w_1 \cdot \phi(e^{\mu})$, where ϕ is the W-module isomorphism in (3.3). Then Φ becomes a vector space isomorphism. We also let $\Psi\colon (\mathbb{C}[T]/\mathcal{I})w \to K_{\mathbb{C}}(\mathcal{B})$ be a W-module isomorphism given by $\Psi(e^{\lambda}w) = \phi(e^{\lambda})$, and we write $\rho\colon H^*(\mathcal{B}) \to H_*(\mathcal{B})$ for the Poincaré duality isomorphism.

Now we present the main result of this paper.

THEOREM 3.7. *The following diagram commutes for any w_1, w_2 in W:*

$$\begin{tikzcd}
(\mathbb{C}[T]/\mathcal{I})w_1 \otimes (\mathbb{C}[T]/\mathcal{I})w_2 \arrow[r, "\textit{multiplication}"] \arrow[d, "\Phi"', "\wr"] & (\mathbb{C}[T]/\mathcal{I})w_1w_2 \arrow[d, "\Psi", "\wr"'] \\
K_{\mathbb{C}}(\Lambda^h_{w_1}) \otimes K_{\mathbb{C}}(\Lambda^{w_1(h)}_{w_2}) \arrow[r, "*"] \arrow[d, "\textit{Lemma 3.2}"', "\|"] & K_{\mathbb{C}}(\Lambda^h_{w_1w_2}) \arrow[d, "\textit{Lemma 3.2}", "\|"'] \\
K_{\mathbb{C}}(\mathcal{B}) \otimes K_{\mathbb{C}}(\mathcal{B}) \arrow[r, "*=\otimes"] \arrow[d, "\mathrm{ch}^*\otimes\mathrm{ch}^*"', "\wr"] & K_{\mathbb{C}}(\mathcal{B}) \arrow[d, "\mathrm{ch}^*", "\wr"'] \\
H^*(\mathcal{B}) \otimes H^*(\mathcal{B}) \arrow[r, "\cup"] \arrow[d, "\rho\otimes\rho"', "\wr"] & H^*(\mathcal{B}) \arrow[d, "\rho", "\wr"'] \\
H_*(\mathcal{B}) \otimes H_*(\mathcal{B}) \arrow[r, "*=\cap"] & H_*(\mathcal{B}),
\end{tikzcd}$$

where the first row is induced by multiplication in $\bigoplus_{w\in W}(\mathbb{C}[T]/\mathcal{I})w$ and the second row is obtained from (3.2).

Proof. We first recall that the induced K-group homomorphisms from homotopy equivalences commute with the convolution products. From this we obtain that the second rectangular diagram in Theorem 3.7 commutes.

By definition, the composition map $\mathrm{ch}^* \circ \rho \colon K_{\mathbb{C}}(\mathcal{B}) \to H_*(\mathcal{B})$ yields the homological Chern character map ch_*. Then, by Theorem 2.2, the following diagram commutes:

$$\begin{tikzcd}
K_{\mathbb{C}}(\mathcal{B}) \otimes K_{\mathbb{C}}(\mathcal{B}) \arrow[r, "*=\otimes"] \arrow[d, "(\rho\otimes\rho)\circ(\mathrm{ch}^*\otimes\mathrm{ch}^*)=\mathrm{ch}_*\otimes\mathrm{ch}_*"', "\wr"] & K_{\mathbb{C}}(\mathcal{B}) \arrow[d, "\rho\circ\mathrm{ch}^*=\mathrm{ch}_*", "\wr"'] \\
H_*(\mathcal{B}) \otimes H_*(\mathcal{B}) \arrow[r, "*=\cap"] & H_*(\mathcal{B}).
\end{tikzcd} \tag{3.6}$$

Moreover, the definition of the intersection pairing in Section 2 implies that the last rectangular diagram in Theorem 3.7 also commutes. Hence, we obtain the following commutative diagram:

$$\begin{tikzcd}
K_{\mathbb{C}}(\mathcal{B}) \otimes K_{\mathbb{C}}(\mathcal{B}) \arrow[r, "*=\otimes"] \arrow[d, "\mathrm{ch}^*\otimes\mathrm{ch}^*"', "\wr"] & K_{\mathbb{C}}(\mathcal{B}) \arrow[d, "\mathrm{ch}^*", "\wr"'] \\
H^*(\mathcal{B}) \otimes H^*(\mathcal{B}) \arrow[r, "\cup"] \arrow[d, "\rho\otimes\rho"', "\wr"] & H^*(\mathcal{B}) \arrow[d, "\rho", "\wr"'] \\
H_*(\mathcal{B}) \otimes H_*(\mathcal{B}) \arrow[r, "*=\cap"] & H_*(\mathcal{B}).
\end{tikzcd} \tag{3.7}$$

Now we shall show that the following diagram commutes, which is the main part of this proof:

$$
\begin{array}{ccc}
(\mathbb{C}[T]/\mathcal{I})w_1 \otimes (\mathbb{C}[T]/\mathcal{I})w_2 & \xrightarrow{\text{multiplication}} & (\mathbb{C}[T]/\mathcal{I})w_1 w_2 \\
{\scriptstyle (\rho\otimes\rho)\circ(\mathrm{ch}^*\otimes\mathrm{ch}^*)\circ\Phi}\Big\downarrow \wr & & \wr\Big\downarrow{\scriptstyle \rho\circ\mathrm{ch}^*\circ\Psi} \\
H_*(\mathcal{B}) \otimes H_*(\mathcal{B}) & \xrightarrow{\quad *=\cap \quad} & H_*(\mathcal{B}).
\end{array}
\tag{3.8}
$$

Let $e^\lambda w_1 \in (\mathbb{C}[T]/\mathcal{I})w_1$ and $e^\mu w_2 \in (\mathbb{C}[T]/\mathcal{I})w_2$. Then

$$
\begin{aligned}
&((\rho \otimes \rho) \circ (\mathrm{ch}^* \otimes \mathrm{ch}^*) \circ \Phi)(e^\lambda w_1 \otimes e^\mu w_2) \\
&\qquad\qquad\qquad = (\rho \circ \mathrm{ch}^* \circ \phi)(e^\lambda) \otimes w_1 \cdot (\rho \circ \mathrm{ch}^* \circ \phi)(e^\mu)
\end{aligned}
$$

because ρ, ch^*, and ϕ are W-module isomorphisms (see Corollary 3.5 and [CG, Sec. 6.4]). Notice that by Lemma 3.6 we obtain

$$w_1 \cdot (\rho \circ \mathrm{ch}^* \circ \phi)(e^\mu) = [\Lambda_{w_1}] * (\rho \circ \mathrm{ch}^* \circ \phi)(e^\mu),$$

where by $[\Lambda_{w_1}]$ we denote the fundamental class corresponding to w_1 through the identification $\mathbb{C}[W] \simeq H_{\dim_{\mathbb{R}} Z}(Z)$. Hence $(\rho\circ\mathrm{ch}^*\circ\phi)(e^\lambda)\otimes w_1\cdot(\rho\circ\mathrm{ch}^*\circ\phi)(e^\mu)$ goes to $(\rho \circ \mathrm{ch}^* \circ \phi)(e^\lambda) * [\Lambda_{w_1}] * (\rho \circ \mathrm{ch}^* \circ \phi)(e^\mu)$ under the convolution product $*\colon H_*(\mathcal{B}) \otimes H_*(\mathcal{B}) \to H_*(\mathcal{B})$.

On the other hand, $e^\lambda w_1 \otimes e^\mu w_2$ goes to $e^\lambda(w_1 \cdot e^\mu)w_1w_2$ under the multiplication map. It now suffices to see the image $(\rho \circ \mathrm{ch}^* \circ \phi)(e^\lambda(w_1 \cdot e^\mu))$. We recall from (3.4) that $\psi = \mathrm{ch}^* \circ \phi\colon \mathbb{C}[T]/\mathcal{I} \to H^*(\mathcal{B})$ and that ψ is a W-equivariant algebra isomorphism. Therefore,

$$
\begin{aligned}
(\rho \circ \mathrm{ch}^* \circ \phi)(e^\lambda(w_1 \cdot e^\mu)) &= \rho(\psi(e^\lambda) \cup w_1 \cdot \psi(e^\mu)) \\
&= (\rho \circ \psi)(e^\lambda) * w_1 \cdot (\rho \circ \psi)(e^\mu) \\
&= (\rho \circ \mathrm{ch}^* \circ \phi)(e^\lambda) * [\Lambda_{w_1}] * (\rho \circ \mathrm{ch}^* \circ \phi)(e^\mu).
\end{aligned}
$$

Hence, diagram (3.8) commutes.

Since all vertical maps in the diagrams (3.6), (3.7), and (3.8) are isomorphisms, the theorem is now immediate if we combine those commutative diagrams. □

For given $w \in W$ and $h \in \mathfrak{g}^{\mathrm{sr}}$, take the disjoint union set $Z \amalg \left(\coprod_{\lambda\in\mathbb{C}^\times} \Lambda_w^{\lambda h}\right)$. We now consider a map $\pi\colon Z \amalg \left(\coprod_{\lambda\in\mathbb{C}^\times} \Lambda_w^{\lambda h}\right) \to \mathbb{C} \cdot h = \{\lambda h \mid \lambda \in \mathbb{C}\}$ defined by $\pi(Z) = 0$ and $\pi(\Lambda_w^{\lambda h}) = \lambda h$ for each $\lambda \in \mathbb{C}^\times$. Then, the map π induces a locally trivial fibration $\pi\colon \coprod_{\lambda\in\mathbb{C}^\times} \Lambda_w^{\lambda h} \to \mathbb{C}^\times \cdot h$. Thus we have a specialization homomorphism

$$\lim\nolimits_{\mathbb{C}h\to 0}\colon H_*\left(\textstyle\coprod_{\lambda\in\mathbb{C}^\times} \Lambda_w^{\lambda h}\right) \to H_*(Z). \tag{3.9}$$

On the other hand, from the homomorphism $\rho\circ\mathrm{ch}^*\circ\Psi\colon (\mathbb{C}[T]/\mathcal{I})w \to H_*(\mathcal{B})$ in Theorem 3.7 we obtain a homomorphism

$$\theta_w^h\colon (\mathbb{C}[T]/\mathcal{I})w \to H_*\left(\textstyle\coprod_{\lambda\in\mathbb{C}^\times} \Lambda_w^{\lambda h}\right). \tag{3.10}$$

Combining the homomorphisms in (3.9) and (3.10), we obtain the homomorphism

$$\theta_w\colon (\mathbb{C}[T]/\mathcal{I})w \xrightarrow{\ \theta_w^h\ } H_*\big(\textstyle\coprod_{\lambda\in\mathbb{C}^\times}\Lambda_w^{\lambda h}\big) \xrightarrow{\ \lim_{\mathbb{C}h\to 0}\ } H_*(Z), \tag{3.11}$$

which yields the homomorphism

$$\Theta\colon \bigoplus_{w\in W}(\mathbb{C}[T]/\mathcal{I})w \to H_*(Z) \tag{3.12}$$

after we assemble the homomorphisms θ_w in (3.11). It is known that θ_w does not depend on the choice of h [CG, Lemma 3.4.11].

COROLLARY 3.8. *The homomorphism* Θ *in* (3.12) *is a* $\mathbb{C}$*-algebra homomorphism if* $H_*(Z)$ *is equipped with the convolution product.*

Proof. We recall that the specialization map commutes with the convolution product in Borel–Moore homology. Now the result is immediate from Theorem 3.7. □

Finally, we construct the following isomorphism.

THEOREM 3.9. *The algebra homomorphism* $\Theta\colon \bigoplus_{w\in W}(\mathbb{C}[T]/\mathcal{I})w \to H_*(Z)$ *is a* $\mathbb{C}$*-algebra isomorphism.*

Proof. It suffices to show that Θ is bijective. We first consider the diagram

$$\begin{tikzcd}
 & \bigoplus_{w\in W}(\mathbb{C}[T]/\mathcal{I})w \arrow[dl, "\bigoplus_{w,\lambda}\Psi"'] \arrow[dr, "\bigoplus_w \theta_w^h"] & \\
\bigoplus_{w\in W} K_{\mathbb{C}}\big(\coprod_{\lambda\in\mathbb{C}^\times}\Lambda_w^{\lambda h}\big) \arrow[rr, "\bigoplus_{w,\lambda}\mathrm{ch}_*"] \arrow[d, "\bigoplus\lim^1"'] & & \bigoplus_{w\in W} H_*\big(\coprod_{\lambda\in\mathbb{C}^\times}\Lambda_w^{\lambda h}\big) \arrow[d, "\bigoplus\lim^2"] \\
K_{\mathbb{C}}(Z) \arrow[rr, "\mathrm{ch}_*"'] & & H_*(Z).
\end{tikzcd}$$

This diagram is commutative because:

(1) $\theta_w^h = \bigoplus_{\lambda\in\mathbb{C}^\times}(\rho\circ\mathrm{ch}^*\circ\Psi) = \big(\bigoplus_{\lambda\in\mathbb{C}^\times}\mathrm{ch}_*\big)\circ\big(\bigoplus_{\lambda\in\mathbb{C}^\times}\Psi\big)$; and

(2) the homological Chern character commutes with the specialization homomorphism in K-theory and with the Borel–Moore homology.

We also observe that the composition map $\big(\bigoplus_{w,\lambda}\Psi\big)\circ\big(\bigoplus\lim^1\big)$ in the diagram is a nonequivariant version of the map in [CG, Lemma 7.3.11, 7.3.13]. Thus the map $\big(\bigoplus_{w,\lambda}\Psi\big)\circ\big(\bigoplus\lim^1\big)$ is bijective by the argument in [CG, Lemma 7.3.11, 7.3.13]. The result now follows because the homological Chern character map $\mathrm{ch}_*\colon K_{\mathbb{C}}(Z)\to H_*(Z)$ is an isomorphism of abelian groups. □

COROLLARY 3.10. *There is a* $\mathbb{C}$*-algebra isomorphism*

$$\mathbb{C}[T]/\mathcal{I}\,\sharp\,\mathbb{C}[W] \simeq H_*(Z).$$

Proof. The proof is immediate from (3.1) and Theorem 3.9. □

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