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Proper Holomorphic Mappings between Reinhardt Domains in $\mathbb{C}^2$

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1. Introduction

Given $\alpha \in \mathbb{R}^n$ and $z \in \mathbb{C}_*^n$, we put $|z^\alpha| := |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}$ whenever it makes sense. Let $\mathbb{A}_{r^-,r^+} = \{z \in \mathbb{C} \mid r^- < |z| < r^+\}$ for $-\infty < r^- < r^+ < \infty, r^+ > 0$. By $\mathbb{D}$ we always denote the unit disc in $\mathbb{C}$. For a domain $D \subset \mathbb{C}^n$, $D \setminus \{0\}$ is denoted by D_*.

Following [Z2], for $A = (A_k^j)_{j=1,\dots,m,\, k=1,\dots,n} \in \mathbb{Z}^{m\times n}$ and $b = (b_1,\dots,b_m) \in \mathbb{C}_*^m$ we define:

$$\varphi_A(z) := z^A := (z^{A^1},\dots,z^{A^m}), \quad z \in \mathbb{C}_*^n,$$

$$\varphi_{A,b}(z) := (b_1 z^{A^1},\dots,b_m z^{A^m}), \quad z \in \mathbb{C}_*^n.$$

Such maps are called *elementary algebraic* (or briefly *elementary* maps).

The aim of this paper is to describe nonelementary proper holomorphic maps between nonhyperbolic Reinhardt domains in $\mathbb{C}^2$ as well as the corresponding pairs of domains.

Recall that if D, G are Reinhardt domains and $f: D \to G$ is a biholomorphic mapping, then f can be represented as composition of automorphism of D and G and an elementary mapping between these domains (see [K] and [S2]). Thus, the description of nonelementary biholomorphic mappings between Reinhardt domains reduces to the investigation of their group of automorphisms. It is a general problem of complex geometry of Reinhardt domains considered in many papers. In [S1] the author used group-theoretic methods to investigate the holomorphic equivalence of bounded Reinhardt domains in $\mathbb{C}^n$ not containing the origin and thereby determined automorphisms of a certain class of Reinhardt domains. Similar results were obtained by Barrett in [Ba], although his approach was analytic. The groups of automorphisms of all bounded Reinhardt domains containing the origin were determined in [Su]. This work has been extended in [K] by dropping the assumption that the origin is included in the domain. The situation when domains D and G may be unbounded were considered for example in [S3] and [EZ].

Obviously, the problem of describing proper holomorphic mappings is harder to deal with. Proper maps between nonhyperbolic, pseudoconvex Reinhardt domains have been considered in [EZ] and [Ko]. In the bounded case, partial results

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were obtained in [BP], [LSp], and [DSe]. The final result for bounded domains in $\mathbb{C}^2$, which may be viewed as the completion of this research, was at last obtained in the paper of Isaev and Kruzhilin [IK]. The authors explicitly described all possibilities of existence of nonelementary proper holomorphic mappings between bounded Reinhardt domains in $\mathbb{C}^2$.

We generalize Isaev and Kruzhilin's results to unbounded domains (except for the case of domains whose logarithmic images of envelopes of holomorphy are equal to $\mathbb{R}^2$). Additionally, we obtain some partial results for proper maps between domains of the form $\mathbb{C}_*^2$, $\mathbb{C}^2$, and $\mathbb{C} \times \mathbb{C}_*$, and we also give some more general results related to proper holomorphic mappings.

2. Preliminaries and Statement of Results

It is well known that, for any pseudoconvex Reinhardt domain D in $\mathbb{C}^n$, its logarithmic image $\log D$ is convex. Moreover, any proper holomorphic mapping between domains D_1, D_2 in $\mathbb{C}^n$ can be extended to a proper map between the envelopes of holomorphy $\hat{D}_1, \hat{D}_2$ of D_1, D_2, respectively (see e.g. [Ke]).

Let us introduce some notation. First we define

$$\begin{aligned} V_\iota &:= \mathbb{C}^{\iota-1} \times \{0\} \times \mathbb{C}^{n-\iota} \subset \mathbb{C}^n, \quad \iota = 1, \dots, n, \\ M &:= \bigcup_{\iota=1}^{n} V_\iota. \end{aligned} \tag{1}$$

With a given Reinhardt domain D we associate the following constants:

$d(D) :=$ the maximal possible dimension of a linear subspace contained in the logarithmic image of the envelope of holomorphy of D;

$t(D) :=$ the number of j such that $\hat{D} \cap V_j \neq \emptyset$.

Moreover, in the case $D \subset \mathbb{C}^2$ we put

$s(D) :=$ the number of $j = 1, 2$ such that $V_j \cap \hat{D}$ is equal to $\mathbb{C}$;

$s_*(D) :=$ the number of $j = 1, 2$ such that $V_j \cap \hat{D}$ is equal to $\mathbb{C}_*$.

It turns out that the objects we have just introduced are invariant under proper holomorphic mappings $f : D \to G$, where D, G are Reinhardt domains in $\mathbb{C}^2$, except for the case when $\alpha\mathbb{R} + \beta \subset \log D$ for some $\alpha \in \mathbb{Q}^2$ and $\beta \in \mathbb{R}^2$. In particular, we shall obtain the following result.

THEOREM 1. *Let D, G be Reinhardt domains in $\mathbb{C}^2$ such that the set of proper holomorphic mappings from D onto G is nonempty. Then*

$$d(D) = d(G). \tag{2}$$

If, moreover, $d(D) = d(G) = 0$, then

$$(s(D), s_*(D), t(D)) = (s(G), s_*(G), t(G)). \tag{3}$$

Recall here that pseudoconvex Reinhardt domains that are algebraically biholomorphic to bounded Reinhardt domains have been described in [Z1]. This result is of key importance for our considerations, so we quote it next.

THEOREM 2. *Assume that D is a pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then the following conditions are equivalent*:

(i) *D is Brody hyperbolic—that is, any holomorphic mapping from $\mathbb{C}$ to D is constant*;

(ii) (a) $\log D$ *contains no affine lines and*
 (b) *$D \cap V_j$ is either empty or c-hyperbolic, $j = 1, \dots, n$ (viewed as a domain in $\mathbb{C}^{n-1}$)*;

(iii) *D is algebraically biholomorphic to a bounded Reinhardt domain—in other words, there is an $A \in \mathbb{Z}^{n\times n}$ with $|\det A| = 1$ such that $\varphi_A(D)$ is bounded and $(\varphi_A)|_D$ is a biholomorphism onto the image.*

We say that a Reinhardt domain D is *hyperbolic* if its envelope of holomorphy $\hat{D}$ is algebraically equivalent to a bounded Reinhardt domain. Note that a Reinhardt domain D in $\mathbb{C}^2$ satisfies condition (ii) of Theorem 2 if and only if $s(D) = s_*(D) = d(D) = 0$.

Let D_1, D_2 be Reinhardt domains in $\mathbb{C}^2$ and let $f : D_1 \to D_2$ be a proper holomorphic mapping. Assume that f is nonelementary. Our aim is to derive the explicit formulas for the mapping f as well as for the domains D_1, D_2.

In view of Theorem 2, we see that the case $d(D_i) = s(D_i) = s_*(D_i) = 0$, $i = 1, 2$, has been described in [IK]. Moreover, in [EZ] and [Ko] the authors gave explicit formulas for all proper holomorphic mappings $f : D_1 \to D_2$ between pseudoconvex Reinhardt domains D_1 and D_2 such that $d(D_1) = d(D_2) = 1$—that is, domains whose logarithmic image is equal to a strip or a half-plane. One may apply direct and tedious calculations to determine all possibilities of the form of Reinhardt domains D_1' and D_2' whose envelopes of holomorphy are equal to D_1 and D_2, respectively, and such that the restriction $f|_{D_1'} : D_1' \to D_2'$ is proper.

On the other hand, there is no proper holomorphic mapping between hyperbolic and nonhyperbolic domains (see Lemma 6), so we shall focus our considerations on proper holomorphic mappings between nonhyperbolic domains.

Summing up, to obtain a desired description of the set of nonelementary proper holomorphic mappings between nonhyperbolic Reinhardt domains D_1, D_2 in $\mathbb{C}^2$ whose envelopes of holomorphy do not contain $\mathbb{C}_*^2$, it suffices to confine ourselves to the cases when $d(D_1) = d(D_2) = 0$ and $s(D_1) = s(D_2) \neq 0$ or $s_*(D_1) = s_*(D_2) \neq 0$.

We are now in position to formulate the main result of this paper.

THEOREM 3. *Let D_1, D_2 be nonhyperbolic Reinhardt domains in $\mathbb{C}^2$ such that $d(D_1) = d(D_2) = 0$ and $s(D_i) \neq 0$ or $s_*(D_i) \neq 0$, $i = 1, 2$. Assume that there is a proper, nonelementary holomorphic mapping $f : D_1 \to D_2$. Then one of the following two scenarios obtains.*

(i) *Up to a permutation of the components of f and the variables, the map f has the form*

$$f(z,w) = (\mu_1 z^k B(C_1 z^{p_1} w^{q_1}), \mu_2 w^l), \tag{4}$$

where $k, l \in \mathbb{N}$, $p_1, q_1 > 0$ are relatively prime integers, B is a nonconstant finite Blaschke product nonvanishing at 0, $C_1 > 0$, and $\mu_1, \mu_2 \in \mathbb{C}_*$. In this case, the domains D_1 and D_2 have the form

$$D_i = \{(z,w) \in \mathbb{C}^2 : C_i |z|^{p_i} |w|^{q_i} < 1,\ |w| < E_i\} \setminus (P_i \times \{0\}), \quad i = 1,2, \tag{5}$$

where $E_1, E_2 > 0$, $p_2, q_2 > 0$ are relatively prime integers satisfying the equation $q_2/p_2 = kq_1/lp_1$, and P_1 is any closed proper Reinhardt subset of $\mathbb{C}$. (Then, obviously, P_2 is of the form $\{\mu_1 \zeta^k B(0) : \zeta \in P_1\}$.)

(ii) *Up to a permutation of the components of f and the variables, the map f has the form*

$$f(z,w) = ((e^{it_1} z^{a_1} + s)^{a_2}, e^{it_2} \exp(2\bar{s} e^{it_1} z^{a_1} + |s|^2)^{-c_2} w^{c_1 c_2}), \tag{6}$$

where $a_1, a_2, c_1, c_2 \in \mathbb{N}$, $s \in \mathbb{C}_*$, and $t_1, t_2 \in \mathbb{R}$. In this case, domains have the forms

$$D_i = \{(z,w) \in \mathbb{C}^2 : |w| < C_i \exp(-E_i |z|^{k_i})\}, \quad i = 1,2, \tag{7}$$

where $k_1 = 2a_1$, $k_2 = 2/a_2$, and $C_1, C_2, E_1, E_2 > 0$.

As mentioned before, in Section 4 we shall also obtain some results related to proper mappings $f\colon D \to G$ in the case when $d(D) = d(G) = 2$. It is clear that, for any pseudoconvex domain D in $\mathbb{C}^2$, $d(D) = 2$ if and only if $\log D = \mathbb{R}^2$.

3. Proofs

We start with some preliminary results.

LEMMA 4. *Let $\varphi\colon D_1 \to D_2$ be a proper holomorphic mapping, where $D_1, D_2 \subset \mathbb{C}^n$ are pseudoconvex Reinhardt domains.*

(i) *Assume that $d(D_2) = 0$ and suppose that there is a nonconstant holomorphic mapping $\psi\colon \mathbb{C} \to D_1$. Then $\varphi(\psi(\mathbb{C})) \subset M$.*

(ii) *If $\tilde{\psi}\colon \mathbb{C} \to D_2$ is a nonconstant holomorphic mapping and if $d(D_1) = 0$, then $\varphi^{-1}(\tilde{\psi}(\mathbb{C})) \subset M$.*

Proof. (i) By Lemma 6 in [JPf], there exist a nonempty open set $U \subset \mathbb{R}^n$ and a positive R such that, for any $v \in U$, the set $\log D_2$ is contained in $\{x \in \mathbb{R}^n : x_1 v_1 + \cdots + x_n v_n < R\}$. Thus, there are linearly independent $\alpha^1, \ldots, \alpha^n \in \mathbb{R}^n$, $\alpha^\iota = (\alpha_1^\iota, \ldots, \alpha_n^\iota)$, such that D_2 is contained in $\{z \in \mathbb{C}^n : |z^{\alpha^\iota}| < e^R\}$, $\iota = 1, \ldots, n$. Put

$$u_\iota(z) = |\varphi(\psi(z))^{\alpha^\iota}|, \quad z \in \mathbb{C},\ \iota = 1, \ldots, n. \tag{8}$$

The u_ι are obviously bounded and subharmonic functions on $\mathbb{C}$, so they are constant; say, $u_\iota = \rho_\iota$ with $\iota = 1, \ldots, n$. It suffices to observe that $\rho_\iota = 0$ for some ι. Indeed, if $\rho_\iota \neq 0$ for every $\iota = 1, \ldots, n$, then clearly $\sum_{j=1}^n \alpha_j^\iota \log|\varphi_j(\psi(z))| = \log \rho_\iota$. By applying Cramer rules we would find the mapping $\varphi \circ \psi$ to be constant (recall that $\alpha^1, \dots, \alpha^n$ are linearly independent). However, this would be in contradiction with the properness of the mapping φ (since the mapping ψ is unbounded).

(ii) Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_*$ and $R > 1$ be such that $\log D_1$ is contained in $\{x \in \mathbb{R}^n : x_1\alpha_1 + \dots + x_n\alpha_n < R\}$ and, for any $t \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n : x_1\alpha_1 + \dots + x_n\alpha_n = t\} \cap \log D_1$ is bounded. Put $u(z) = |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}$ for $z \in D_1$. It is well known that the function

$$v(z) = v_\alpha(z) = \max u(\varphi^{-1}(\tilde{\psi}(z))), \quad z \in \mathbb{C}, \tag{9}$$

is subharmonic. Because it is bounded, we find that v is constant. Let ρ be such that $v = \rho$. Similarly as in the previous part of the proof, it is sufficient to show that ρ is equal to 0.

Suppose not. One could then see that there is a sequence $\{w_\mu\}_{\mu=1}^\infty \subset D_1$ such that $|w_\mu^\alpha| = \rho$ for any $\mu \in \mathbb{N}$ and $w_\mu \to w_0 \in \partial D_1$, $\mu \to \infty$. Moreover, $|w^\alpha| \le \rho$ for every w such that $\varphi(w) \in \tilde{\psi}(\mathbb{C})$. Take the supporting hyperplane H of $\log D_1$ at the point $\log w_0$, and let $\beta \in \mathbb{R}^n$ be such that $H = \{x \in \mathbb{R}^n : x_1\beta_1 + \dots + x_n\beta_n = \hat{\rho}\}$ for some $\hat{\rho} \in \mathbb{R}$. Repeating the same reasoning as before (here the assumption of the boundedness of $H \cap \log D_1$ is unnecessary) but now applied to a function $v = v_\beta$ (see (9)), we find that there is a $\tilde{\rho} < e^{\hat{\rho}}$ such that $|w^\beta| \le \tilde{\rho}$ for any $w \in \varphi^{-1}(\tilde{\psi}(\mathbb{C}))$. However, $|w_\mu^\beta| \to e^{\hat{\rho}}$ $(\mu \to \infty)$, which immediately gives the desired contradiction. □

COROLLARY 5. *Let $D, G \subset \mathbb{C}^n$ be pseudoconvex Reinhardt domains such that $d(D) = 0$ and $d(G) \ge 1$. Then the sets* $\operatorname{Prop}(D, G)$ *and* $\operatorname{Prop}(G, D)$ *are empty.*

Proof. Take $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ such that $\alpha\mathbb{R} + \beta \subset \log G$. Note that for any $z \in G$ the set $\psi_z(\mathbb{C})$ is contained in G, where ψ_z is given by

$$\psi_z(\zeta) = (z_1 e^{\alpha_1\zeta}, \dots, z_n e^{\alpha_n\zeta}), \quad \zeta \in \mathbb{C}. \tag{10}$$

Thus, if $f : D \to G$ (or $g : G \to D$) is a proper holomorphic mapping then, by Lemma 4, $f^{-1}(G) \subset M$ (respectively, $g(G) \subset M$). This immediately yields a contradiction. □

LEMMA 6. *Let $D, G \subset \mathbb{C}^n$ be domains. Assume that D is bounded and that G is not Brody-hyperbolic. Then there is no proper holomorphic mapping from D onto G.*

Proof. Suppose that $\varphi : D \to G$ is a proper holomorphic mapping. Put $A = \{z \in D : \det \varphi'(z) = 0\}$. The set A is a variety in D and, by the properness of φ, $A \ne D$. Moreover, there is an integer m such that $\#\varphi^{-1}(w) = m$ for any $w \in G \setminus \varphi(A)$.

Put

$$\pi_k(\lambda) = \sum_{1 \le i_1 < \dots < i_k \le m} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m,\ k = 1, \dots, m$$

and $\pi = (\pi_1, \dots, \pi_m)$. Moreover, for $z^j = (z_1^j, \dots, z_n^j) \in \mathbb{C}^n$, $j = 1, \dots, m$, define

$$\sigma(z^1, \dots, z^m) := (\pi(z_1^1, \dots, z_1^m), \dots, \pi(z_n^1, \dots, z_n^m)). \tag{11}$$

Obviously $\sigma\colon (\mathbb{C}^n)^m \to \mathbb{C}^{nm}$ is a proper holomorphic mapping with multiplicity equal to $(m!)^n$.

Let $\varphi^{-1}(w) = \{\zeta_1(w), \dots, \zeta_m(w)\}$ for $w \in G \setminus \varphi(A)$. Since φ is locally biholomorphic near any $\zeta_i(w)$, $i = 1, \dots, m$, and since the mapping σ given by (11) is symmetric, it follows that the mapping $\psi = \sigma \circ (\zeta_1, \dots, \zeta_m)$ is holomorphic in $G \setminus \varphi(A)$. Since $\varphi(A)$ is analytic in G and since ψ is bounded, we may extend ψ to a bounded holomorphic mapping on the whole G. Let $\tilde{\psi}$ be such an extension. Take any $\gamma\colon \mathbb{C} \to G$ nonconstant and holomorphic. Then $\tilde{\psi} \circ \gamma$ is bounded and holomorphic on $\mathbb{C}$; in particular, $\tilde{\psi} \circ \gamma$ is constant.

Take any $z' \in \mathbb{C}$. If $\gamma(z')$ belongs to $G \setminus \varphi(A)$, then obviously $\tilde{\psi}(\gamma(z')) = \sigma(\zeta_1(\gamma(z)), \dots, \zeta_m(\gamma(z)))$. Suppose now that $\gamma(z')$ is a critical value of φ. Let $x = (x_1, \dots, x_m) \in (\mathbb{C}^n)^m$ be such that $\tilde{\psi}(\gamma(z)) = \sigma(x)$, $z \in \mathbb{C}$.

Take any ζ such that $\varphi(\zeta) = \gamma(z')$, and let $(\zeta_n) \subset D \setminus A$ be such that $\zeta_n \to \zeta$. Observe that $\sigma(\varphi^{-1}(\varphi(\zeta_n))) = \tilde{\psi}(\varphi(\zeta_n)) \to \sigma(x)$. In particular, using properness of σ, we find that $\zeta \in \sigma^{-1}(\sigma(x))$; thus we have shown that $\varphi^{-1}(\gamma(z')) \subset \sigma^{-1}(\sigma(x_1, \dots, x_1))$.

It follows that, for any $w \in \gamma(\mathbb{C})$, $\varphi^{-1}(w)$ is contained in the finite set $\sigma^{-1}(\sigma(x))$. Because the mapping γ is unbounded, we immediately get a contradiction with the properness of φ. □

REMARK 7. Since the mapping $\mathbb{C} \setminus \{0, 1\} \ni z \to \frac{1}{z(z-1)} \in \mathbb{C}$ is proper, Lemma 6 does not hold if we assume only that the domain D is Brody-hyperbolic (instead of bounded). On the other hand, in the class of pseudoconvex Reinhardt domains we know that the property of Brody-hyperbolicity implies boundedness up to algebraic mappings, so we easily see that there is no proper holomorphic mapping between hyperbolic and nonhyperbolic pseudoconvex Reinhardt domains.

For a Reinhardt domain D in $\mathbb{C}^n$, let $I(D)$ denote the set of $i = 1, \dots, n$ for which the intersection $V_i \cap D$ is not c-hyperbolic (viewed as a domain in $\mathbb{C}^{n-1}$). Put

$$D^{\mathrm{hyp}} = D \setminus \Big(\bigcup_{i \in I(D)} V_i \Big). \tag{12}$$

It is clear that $D^{\mathrm{hyp}} = D$ if D is c-hyperbolic or $D \subset \mathbb{C}^n_*$. In the sequel, we use $\hat{D}^{\mathrm{hyp}}$ to denote the set $(\hat{D})^{\mathrm{hyp}}$.

We are now in position to formulate the following statement.

THEOREM 8. *Let D_1, D_2 be pseudoconvex Reinhardt domains in $\mathbb{C}^2$ such that $\log D_i$ contains no affine lines, $i = 1, 2$ (i.e., $d(D_1) = d(D_2) = 0$). If $\varphi\colon D_1 \to D_2$ is a proper holomorphic mapping, then $\varphi(D_1^{\mathrm{hyp}}) \subset D_2^{\mathrm{hyp}}$ and the restriction $\varphi|_{D_1^{\mathrm{hyp}}}\colon D_1^{\mathrm{hyp}} \to D_2^{\mathrm{hyp}}$ is proper.*

Proof. It clearly suffices to prove the following statements.

(a) If $D_1 \cap V_1 \in \{\mathbb{C}, \mathbb{C}_*\}$, then $\varphi(D_1 \cap V_1)$ is contained either in V_1 or in V_2. In particular, if $D_2 \cap V_2$ is, moreover, c-hyperbolic or empty, then $\varphi(D_1 \cap V_1) \subset V_1$.

(b) If $D_2 \cap V_1 \in \{\mathbb{C}, \mathbb{C}_*\}$ and $D_1 \cap V_2$ is neither $\mathbb{C}$ nor $\mathbb{C}_*$, then $D_1 \cap V_1 \in \{\mathbb{C}, \mathbb{C}_*\}$ and $\varphi^{-1}(D_2 \cap V_1) \subset V_1$.

(a) From Lemma 4(i) applied to the mapping $\psi(z) = (0, e^z)$, $z \in \mathbb{C}$, we get that $\varphi(D_1 \cap V_1) \subset M$. It follows that $\varphi_1(0,z)\varphi_2(0,z) = 0$ for any $z \in \mathbb{C}_*$, so $\varphi_1(0,\cdot) \equiv 0$ or $\varphi_2(0,\cdot) \equiv 0$. The second statement is clear.

(b) If $D_1 \cap V_1$ were neither $\mathbb{C}$ nor $\mathbb{C}_*$, then D_1 would be biholomorphic to a bounded domain, which clearly contradicts Lemma 6. Thus $D_1 \cap V_1 \in \{\mathbb{C}, \mathbb{C}_*\}$.

Suppose that $D_1 \cap V_2$ is nonempty (otherwise, Lemma 4(ii) finishes the proof). Pseudoconvexity implies that $\pi_1(D_1)$ is a bounded subset of $\mathbb{C}$, where $\pi_1 \colon \mathbb{C}^2 \to \mathbb{C}$ denotes a projection onto the first variable. Thus, the function given by

$$v(z) := \max|\pi_1(\varphi^{-1}(0,z))|, \quad z \in \mathbb{C}_*, \tag{13}$$

is constant (because it is bounded and subharmonic). Moreover, from Lemma 4 we have that $\varphi^{-1}(D_2 \cap V_1) \subset M$.

Now, one may easily verify that $v = 0$. □

Corollary 9. *Let* $D_1, D_2 \subset \mathbb{C}^2$ *be Reinhardt domains such that* $d(D_1) = d(D_2) = 0$. *Assume that* $\operatorname{Prop}(D_1, D_2)$ *is nonempty. Then*

$$(s(D_1), s_*(D_1), t(D_1)) = (s(D_2), s_*(D_2), t(D_2)). \tag{14}$$

Proof. Any proper holomorphic map between domains D_1, D_2 may be extended to the proper map between their envelopes of holomorphy $\hat{D}_1, \hat{D}_2$, respectively. Moreover, it is well known (see [IK, Cor. 0.3]) that, if there exists a proper holomorphic mapping between two given bounded domains, then there also exists an elementary algebraic proper holomorphic mapping between these domains.

Thus, our result is a direct consequence of Theorems 8 and 2 and properties of algebraic mappings. □

By applying the methods used in previous theorems, we may easily show the following result.

Proposition 10. *There are no proper holomorphic mappings between domains* D, G *and* G, D *in the following cases.*

(i) *There exist a nonconstant, negative, plurisubharmonic function on* D *and a* $G = \mathbb{C}^n \setminus E$ *for some pluripolar set* E *in* $\mathbb{C}^n$.

(ii) D *is hyperconvex* (*i.e., there is a negative plurisubharmonic exhaustion function for* D) *and* G *is not Brody-hyperbolic.*

Proof. (i) Obviously $\operatorname{Prop}(\mathbb{C}^n \setminus E, D) = \emptyset$.

Suppose that $\varphi \colon D \to \mathbb{C}^n \setminus E$ is proper and holomorphic. Let $u \in \mathcal{PSH}(D)$ be nonconstant and negative. Put $v(z) = \max u(\varphi^{-1}(z))$, $z \in \mathbb{C}^n \setminus E$. It is seen that the function v is constant. In particular, there is a $\rho < 0$ such that $u \le \rho$ and $u(w_0) = \rho$ for some $w_0 \in D$; a contradiction.

(ii) It is clear that the set $\operatorname{Prop}(G, D)$ is empty. Suppose that $\varphi \colon D \to G$ is proper and holomorphic. Let u be a negative plurisubharmonic exhaustion function for D and let $\psi \colon \mathbb{C} \to G$ be a nonconstant holomorphic mapping. Put $v(\zeta) = \max u(\varphi^{-1}(\psi(\zeta)))$, $\zeta \in \mathbb{C}$. The function v is subharmonic on $\mathbb{C}$. Since $v < 0$, it is constant. Hence we find that $\varphi^{-1}(\psi(\mathbb{C}))$ is a relatively compact subset of D; a contradiction. □

PROPOSITION 11. *Let $D, G \subset \mathbb{C}^2$ be pseudoconvex Reinhardt domains. If $d(D) \neq d(G)$, then there is no proper holomorphic mapping between D and G.*

Proof. In view of Corollary 5, it suffices to consider the case when $d(D) = 2$ or $d(G) = 2$. The condition $d(D) < 2$ means that the logarithmic image of the envelope of holomorphy of domain D is contained in a half-space. More precisely, $D \subset \{z \in \mathbb{C}^2 : |z_1|^{\alpha_1}|z_2|^{\alpha_2} < r\}$ for some $\alpha \in \mathbb{R}^2_*$, $0 < r < \infty$. Therefore, the assertion follows immediately from Proposition 10. □

Proof of Theorem 1. This is a direct consequence of Corollary 9 and Proposition 11. □

Proof of Theorem 3. Let $f : \hat{D}_1 \to \hat{D}_2$ also denote the extension of the mapping f to a proper mapping between the envelopes of holomorphy of D_1 and D_2. By Theorem 8, the restriction $f|_{\hat{D}_1^{\text{hyp}}} : \hat{D}_1^{\text{hyp}} \to \hat{D}_2^{\text{hyp}}$ is proper.

If $s(D_1) = 2$ or $s_*(D_1) = t(D_1) = 1$, then $\hat{D}_1^{\text{hyp}}$ would be contained in $\mathbb{C}^2_*$ and, by Theorem 2 and the description in [IK], we find that $f|_{\hat{D}_1^{\text{hyp}}}$ would be elementary algebraic. It is clear that the identity principle gives a contradiction.

Therefore, we may assume that $t(D_1) = t(D_2) = 2$, $s(D_1) = s(D_2) = 1$, and $s_*(D_1) = s_*(D_2) = 0$. Up to a permutation of components, we may suppose that $\hat{D}_i \cap V_2 = V_2$ and that $\hat{D}_i \cap V_1$ is bounded, $i = 1, 2$. Hence there are $k_1, k_2 \in \mathbb{N}$ such that $\hat{D}_i$ is contained in $\{(z, w) \in \mathbb{C}^2 : |z||w|^{k_i} < c_i\}$ for some positive constants c_i, $i = 1, 2$. It follows that Φ_{A_i}, where $A_i = \left(\begin{smallmatrix} 1 & k_i \\ 0 & 1 \end{smallmatrix}\right)$, is a biholomorphic mapping from $\hat{D}_i^{\text{hyp}}$ onto the bounded set $\Phi_{A_i}(\hat{D}_i^{\text{hyp}})$, $i = 1, 2$. In particular,

$$g := \Phi_{A_2} \circ f \circ \Phi_{A_1^{-1}} : \Phi_{A_1}(\hat{D}_1^{\text{hyp}}) \to \Phi_{A_2}(\hat{D}_2^{\text{hyp}}) \tag{15}$$

is a proper holomorphic mapping between two bounded domains in $\mathbb{C}^2$. Now, using the description in [IK], it is straightforward to observe that one of two possibilities may hold:

(i) $\hat{D}_i^{\text{hyp}} = \{(z, w) \in \mathbb{C}^2 : C_i|z|^{p_i}|w|^{p_i k_i + q_i} < 1, 0 < |w| < C_i'\}$, where p_i, q_i are relatively prime integers such that $p_i k_i + q_i > 0$, $p_i > 0$, $q_i \leq 0$, and $C_i, C_i' > 0$, $i = 1, 2$;

(ii) $\hat{D}_1^{\text{hyp}} = \{(z, w) \in \mathbb{C}^2 : 0 < |w| < C_1 \exp(-E_1|z|^{2a_1}|w|^{2k_1 a_1 - 2b_1})\}$ and $\hat{D}_2^{\text{hyp}} = \{(z, w) \in \mathbb{C}^2 : 0 < |w| < C_2 \exp(-E_2|z|^{2/a_2}|w|^{2k_2/a_2 - 2b_2/a_2 c_2})\}$, where $a_i, b_i, c_i \in \mathbb{N}$ and $C_i, E_i > 0$ for $i = 1, 2$.

First suppose that (i) holds. From [IK] it follows that g must be of the form $g(z, w) = (\lambda_1 z^a w^b B(C_1 z^{p_1} w^{q_1}), \lambda_2 w^c)$, $(z, w) \in \Phi(\hat{D}_1^{\text{hyp}})$, where $a, b, c \in \mathbb{Z}$, $a, c > 0$, $aq_1 - bp_1 < 0$, $q_2/p_2 = (aq_1 - bp_1)/cp_1$, B is a Blaschke product nonvanishing at 0, and $\lambda_1, \lambda_2 \in \mathbb{C}_*$. Put $\tilde{q}_i = p_i k_i + q_i$. It is obvious that p_i and $\tilde{q}_i$ are relatively prime. Moreover, from the form of $\hat{D}_i$ we get that $\tilde{q}_i > 0$. An easy computation gives

$$f(z, w) = (\mu_1 z^a w^{ak_1 - ck_2 + b} B(C_1 z^{p_1} w^{\tilde{q}_1}), \mu_2 w^c), \quad (z, w) \in \hat{D}_1^{\text{hyp}},$$

for some constants μ_1, μ_2. Because f may be extended properly on $\hat{D}_1$, we have $ak_1 - ck_2 + b = 0$. Moreover, it is clear that $\tilde{q}_2/p_2 = a\tilde{q}_1/cp_1$.

It is straightforward to see that any Reinhardt subdomain of $\hat{D}_1$ mapped properly by f onto a Reinhardt domain and whose envelopes of holomorphy coincides with $\hat{D}_1$ is equal to $\hat{D}_1 \setminus P_1 \times \{0\}$, where P_1 is any closed Reinhardt subset of $\mathbb{C}$.

Now suppose that (ii) holds. Denote $m_1 := k_1 a_1 - b_1$ and $m_2 := k_2 c_2 - b_2$. Similarly as before, by taking into account the form of $\hat{D}_1$ and $\hat{D}_2$ we can see that $m_1, m_2 \geq 0$.

For $s \in \mathbb{C}_*$ and $t_1, t_2 \in \mathbb{R}$, put

$$h_1(z) := e^{it_1} z + s \quad \text{and} \quad h_2(z) = e^{it_2} \exp(2\bar{s}e^{it_1 z} + |s|^2), \quad z \in \mathbb{C}.$$

An easy calculation and formula for the mapping g (see [IK]) then yield

$$f(z,w) = (h_1(z^{a_1} w^{m_1})^{a_2} h_2(z^{a_1} w^{m_1})^{m_2} w^{-m_2 c_2}, h_2(z^{a_1} w^{m_2})^{-c_2} w^{c_1 c_2}), \quad (z,w) \in \hat{D}_1^{\text{hyp}}. \tag{16}$$

Since f may be extended through V_1, it follows that $m_1 = m_2 = 0$.

Finally, one may easily verify that any Reinhardt subdomain of $\hat{D}_1$ whose envelope of holomorphy coincides with $\hat{D}_1$ and that is mapped properly by f onto a Reinhardt domain is equal to $\hat{D}_1$. □

4. Remarks on the Proper Holomorphic Mappings $f\colon D \to G$ between Reinhardt Domains When $d(D) = d(G) = 2$

It is widely recognized that the structures of $\operatorname{Aut}(\mathbb{C}^2)$, $\operatorname{Aut}(\mathbb{C}_*^2)$, and $\operatorname{Aut}(\mathbb{C} \times \mathbb{C}_*)$ are complicated and that a full description of these groups seems to be not known. Proper maps are harder to deal with, so describing the set of proper holomorphic mappings between pseudoconvex Reinhardt domains D_1 and D_2 in the case when $\log D_i = \mathbb{R}^2$, $i = 1, 2$, is more difficult.

In this section we present some partial results related to these problems.

Proposition 12. *The sets* $\operatorname{Prop}(\mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}_*)$, $\operatorname{Prop}(\mathbb{C} \times \mathbb{C}, \mathbb{C}_* \times \mathbb{C}_*)$, *and* $\operatorname{Prop}(\mathbb{C} \times \mathbb{C}_*, \mathbb{C}_* \times \mathbb{C}_*)$ *are empty.*

Proof. First suppose that $f\colon \mathbb{C}^2 \to \mathbb{C} \times \mathbb{C}_*$ is proper and holomorphic. Obviously there exists a holomorphic mapping $\psi\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $f = (\psi_1, e^{\psi_2})$. One can easily verify that the mapping ψ is proper; in particular, ψ is surjective. Thus there is a discrete sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}^2$ such that $\psi(z_n) = (0, 2n\pi i)$ for $n \in \mathbb{N}$. It follows that $f(z_n) = (0,1)$ for $n \in \mathbb{N}$. From this we immediately obtain a contradiction.

To show that $\operatorname{Prop}(\mathbb{C}^2, \mathbb{C}_*^2) = \emptyset$ we proceed similarly.

Now suppose that $g\colon \mathbb{C} \times \mathbb{C}_* \to \mathbb{C}_*^2$ is holomorphic and proper. It can be seen that there exists a holomorphic mapping $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $g(z, e^w) = (e^{\varphi_1(z,w)}, e^{\varphi_2(z,w)})$ for $z, w \in \mathbb{C}$.

Fix $z \in \mathbb{C}$. Put $\tilde{g}_i = g_i(z,\cdot)$ and $\tilde{\varphi}_i = \varphi_i(z,\cdot)$, $i = 1, 2$. Since $\tilde{g}_i(e^w) = e^{\tilde{\varphi}_i(w)}$, we find that $\tilde{\varphi}_i'(w) = \zeta_i(e^w)$, $w \in \mathbb{C}$, where ζ_i is a holomorphic function given by the formula $\zeta_i(\lambda) = \lambda \tilde{g}_i'(\lambda)/\tilde{g}_i(\lambda)$, $\lambda \in \mathbb{C}_*$. Expanding ζ_i to the Laurent series gives $\tilde{\varphi}_i(w) = a_i w + \sum_{n \in \mathbb{Z}_*} a_{in} e^{nw}$ for some $a_i = a_i(z) \in \mathbb{C}$ and $a_{in} = a_{in}(z) \in \mathbb{C}$.

Thus, there is a holomorphic mapping $\hat{\varphi}_i(\cdot) = \hat{\varphi}_i(z, \cdot)$ on $\mathbb{C}_*$ such that $\tilde{\varphi}_i(w) = a_i w + \hat{\varphi}_i(e^w)$, $w \in \mathbb{C}$. Since $e^{a_i w} = \tilde{g}_i(e^w)/e^{\hat{\varphi}_i(e^w)}$, we immediately find that $a_i \in \mathbb{Z}$, $i = 1, 2$.

Therefore, $\varphi_i(z, w) = a_i(z) w + \hat{\varphi}_i(z, e^w)$ for $z, w \in \mathbb{C}$ and $i = 1, 2$. In particular,

$$g(z, w) = (w^{a_1(z)} e^{\hat{\varphi}_1(z,w)}, w^{a_2(z)} e^{\hat{\varphi}_2(z,w)}), \quad (z, w) \in \mathbb{C} \times \mathbb{C}_*. \tag{17}$$

It is straightforward to verify that

$$a_i(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\frac{\partial g_i}{\partial \lambda}(z, \lambda)}{g_i(z, \lambda)} \, d\lambda, \quad z \in \mathbb{C},$$

whence a_i is constant (recall that $a_i(z) \in \mathbb{Z}$) and therefore $\hat{\varphi}_i$ is holomorphic on $\mathbb{C} \times \mathbb{C}_*$, $i = 1, 2$.

Note that we may assume that $a_2 = 0$ (if $a_1 a_2 \neq 0$, then one may compose g with a proper holomorphic mapping $F\colon \mathbb{C}_*^2 \to \mathbb{C}_*^2$ given by the formula $F(z, w) = (z^{a_2}, w^{a_1}/z^{a_2})$).

Put

$$h(z, w) = (w^{a_1} e^{\hat{\varphi}_1(z,w)}, \hat{\varphi}_2(z, w)), \quad (z, w) \in \mathbb{C} \times \mathbb{C}_*,$$

and observe that the mapping $h\colon \mathbb{C} \times \mathbb{C}_* \to \mathbb{C}_* \times \mathbb{C}$ is proper.

Now we may obtain a contradiction by proceeding exactly as in the case of $\operatorname{Prop}(\mathbb{C}^2, \mathbb{C} \times \mathbb{C}_*)$. □

COROLLARY 13. $\operatorname{Prop}(A \times \mathbb{C}, A \times \mathbb{C}_*)$ *is empty for any domain* $A \subset \mathbb{C}$.

Proof. If $\#(\mathbb{C} \setminus A) \leq 1$ then the result follows directly from Proposition 12. So assume that $\#(\mathbb{C} \setminus A) > 1$, and let $f\colon A \times \mathbb{C} \to A \times \mathbb{C}_*$ be proper and holomorphic. By the uniformization theorem there exist a universal covering $\pi : \mathbb{D} \to A$ and a $\psi \in \mathcal{O}(\mathbb{D} \times \mathbb{C}, \mathbb{D})$ such that

$$f(\pi(\lambda), w) = (\pi(\psi(\lambda, w)), f_2(\pi(\lambda), w)) \quad \text{for any } (\lambda, w) \in \mathbb{D} \times \mathbb{C}.$$

Fix any $\lambda \in \mathbb{D}$ and note that the mapping $\psi(\lambda, \cdot)$ is constant. From the properness of f it easily follows that the mapping $f_2(\pi(\lambda), \cdot)\colon \mathbb{C} \to \mathbb{C}_*$ is proper; a contradiction. □

REMARK 14. Because $\phi\colon \mathbb{C}_* \ni z \to z + 1/z \in \mathbb{C}$ is proper, there exist proper holomorphic maps from $\mathbb{C}_*^2$ onto $\mathbb{C}^2$, from $\mathbb{C}_*^2$ onto $\mathbb{C} \times \mathbb{C}_*$, and from $\mathbb{C} \times \mathbb{C}_*$ onto $\mathbb{C}^2$. Clearly such maps cannot be elementary.

On the other hand, the preceding results and those obtained in [IK] and [Ko] imply that, if there exists a proper holomorphic mapping between two Reinhardt domains $D_1, D_2 \subset \mathbb{C}^2$ such that $\alpha\mathbb{R} + \beta$ is not contained in $\log D_1$ for any $\alpha \in \mathbb{Q}^2$ and $\beta \in \mathbb{R}^2$ (hence also in $\log D_2$; see [Ko]), then there also exists an elementary algebraic mapping between these domains.

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