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Holomorphic Motions, Fatou Linearization, and Quasiconformal Rigidity for Parabolic Germs

Yunping Jiang

1. Introduction

One of the fundamental theorems in complex dynamical systems is a theorem called the Fatou linearization theorem. This theorem provides a topological and dynamical structure of a parabolic germ. A parabolic germ f is an analytic function defined in a neighborhood of a point z_0 in the complex plane $\mathbb{C}$ such that (a) it fixes z_0 and (b) some power $(f'(z_0))^q$ of the derivative $f'(z_0)$ of f at z_0 is 1. Thus we can write $f(z)$ in the following form:

$$f(z) = z_0 + \lambda(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots, \quad z \in U,$$

where U is a neighborhood of z_0 and $\lambda = e^{2\pi pi/q}$ for p and q two relatively prime integers. The number λ is called the *multiplier* of f. Two parabolic germs f and g at two points z_0 and z_1 are said to be *topologically conjugate* if there is a homeomorphism h from a neighborhood of z_0 onto a neighborhood of z_1 such that

$$h \circ f = g \circ h.$$

If h is a K-quasiconformal homeomorphism, then we say that f and g are *K-quasiconformally conjugate*.

By a linear conjugacy $\phi(z) = z - z_0$, we may assume that $z_0 = 0$. So we only consider parabolic germs at 0,

$$f(z) = \lambda z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad z \in U.$$

Assume we are given a parabolic germ f at 0 whose multiplier $\lambda = e^{2\pi ip/q}$ with $(p, q) = 1$. Then

$$f^q(z) = z + az^{n+1} + o(z^{n+1}), \quad n \geq 1.$$

If $a \neq 0$, then $n + 1$ is called the *multiplicity* of f. Here $n = kq$ is a multiplier of q. The Leau–Fatou flower theorem states that the local topological and dynamical picture of f around 0 can be described as follows. There are n petals pairwise tangential at 0 such that each petal is mapped into the (kp)th petal counting counterclockwise from this petal. These petals are called *attracting* petals. At

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the same time, there are n *repelling* petals—that is, n other petals also pairwise tangential at 0 and for which the inverse f^{-1} maps each petal into the (kp)th petal counting counterclockwise from this petal. Thus f^q maps every attracting petal into itself and f^{-q} maps every repelling petal into itself. Furthermore, the Fatou linearization theorem states that the map

$$f^q(z)\colon \mathcal{P} \to \mathcal{P}$$

from any attracting petal $\mathcal{P}$ into itself is conjugate to $G(w) = w + 1$ from a right half-plane into itself by a conformal map.

The union of all attracting petals and repelling petals forms a neighborhood of 0. If two parabolic germs are topologically conjugate, then they have the same Leau–Fatou flowers in any neighborhood of 0. A parabolic germ is quasiconformal rigidity as follows.

THEOREM 1. *Suppose f and g are two parabolic germs at* 0 *and suppose f and g are topologically conjugate. Then, for every $\varepsilon > 0$, there are neighborhoods U_ε and V_ε about* 0 *such that $f|_{U_\varepsilon}$ and $g|_{V_\varepsilon}$ are $(1+\varepsilon)$-quasiconformally conjugate.*

The method in our proof of this theorem involves holomorphic motions. In the proof, the reader could find a beautiful application of holomorphic motions to the study of parabolic germs. A special case of this theorem has been proved by McMullen [16, Thm. 8.1]. His proof used the Ahlfors–Weill extension theorem, which states that a conformal mapping of the unit disk can be extended to a quasiconformal homeomorphism as long as the hyperbolic norm of the Schwarzian derivative of this conformal mapping is less than 2.

The idea used in this paper first appeared in [11], where we incorporated holomorphic motions into some new proofs of König's theorem and Böttcher's theorem, which provide normal forms of attracting and super-attracting germs. Furthermore, we used the same idea to prove normal-form theorems for integrable asymptotically conformal attracting germs and for integrable asymptotically conformal super-attracting germs in [12]. Here we continue this idea for parabolic germs, so this is a sequel paper in our research for applications of holomorphic motions to complex dynamical systems. However, from the technical point of view, the parabolic case is much more delicate because it is structurally unstable. Thus we need more carefully to construct quasiconformal conjugacies from holomorphic motions in order to have a sequence of quasiconformal conjugacies contained in a compact subset in the space of all quasiconformal mappings.

As an interesting by-product of our proofs of Theorems 1 and 5, we prove a theorem stating that a quasiconformal homeomorphism can be used to glue an arbitrary finite number of parabolic germs in the Riemann sphere at different points. This theorem may be viewed as a generalization of the Ahlfors–Weill extension theorem that basically considers one germ.

THEOREM 2. *Let $\{f_i\}_{i=1}^k$ denote a finite number of parabolic germs at distinct points $\{z_i\}_{i=1}^k$ in the complex plane $\mathbb{C}$. Then, for every $\varepsilon > 0$, there exist a number $r > 0$ and a $(1+\varepsilon)$-quasiconformal homeomorphism f of $\hat{\mathbb{C}}$ such that*

$$f|_{\Delta_r(z_i)} = f_i|_{\Delta_r(z_i)}, \quad i = 1, \dots, k.$$

Corollary 1. *Let $\{f_i\}_{i=1}^k$ denote a finite number of germs at distinct points $\{z_i\}_{i=1}^k$ such that $\lambda_i = f_i'(z_i) \neq 0$ for $1 \leq i \leq k$. Then, for every $\varepsilon > 0$, there exist a number $s > 0$ and a $(1+\varepsilon)$-quasiconformal homeomorphism f of $\hat{\mathbb{C}}$ such that*

$$f|_{\Delta_s(z_i)} = f_i|_{\Delta_s(z_i)}, \quad i = 1, \dots, k.$$

We prove these two results by again applying holomorphic motions. In the study of complex dynamical systems, an important method for constructing a new dynamical system from an old one is the *surgery* method, which cuts certain parts from the old one and glues some desired dynamical phenomena by quasiconformal mappings to yield a new complex dynamical system. However, it is difficult to control the Teichmüller distance between the new and old systems. Theorem 2 and Corollary 1, together with the ideas in their proofs, suggest some ways to control this Teichmüller distance.

The paper is organized as follows. Because our proof involves holomorphic motions, we introduce this interesting topic in Section 2. In Section 3, we continue the idea in [11; 12] of giving a conceptual proof of the Fatou linearization theorem (Theorem 5) by incorporating holomorphic motions. In Section 4 we give a proof of Theorem 1; we prove Theorem 2 and Corollary 1 in Section 5.

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2. Holomorphic Motions and Quasiconformal Maps

In the study of complex analysis, the measurable Riemann mapping theorem plays an important role. A measurable function μ on $\hat{\mathbb{C}}$ is called a *Beltrami coefficient* if its L^∞-norm

$$k = \|\mu\|_\infty < 1.$$

The corresponding equation

$$H_{\bar{z}} = \mu H_z$$

is called the *Beltrami equation*. The measurable Riemann mapping theorem states that the Beltrami equation has a solution H, which is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, whose quasiconformal dilatation is less than or equal to $K = (1+k)/(1-k)$. It is called a K-quasiconformal homeomorphism.

The study of the measurable Riemann mapping theorem has a long history; in the 1820s, Gauss considered its connection with the problem of finding isothermal coordinates for a given surface. As early as 1938, Morrey [18] systematically studied homeomorphic L^2-solutions of the Beltrami equation. But it took almost twenty years until, in 1957, Bers [4] observed that these solutions are quasiconformal (see [13, p. 24]). Finally, the existence of a solution to the Beltrami equation under the most general possible circumstance (i.e., for measurable μ with

$\|\mu\|_\infty < 1$) was shown by Bojarski [6]. In this generality the existence theorem is sometimes called the *measurable Riemann mapping theorem.*

If one considers only a normalized solution to the Beltrami equation (a solution that fixes 0, 1, and ∞), then H is unique and the solution is denoted by H^μ. The solution H^μ is expressed as a power series made up of compositions of singular integral operators applied to the Beltrami equation on the Riemann sphere. In this expression, if one considers μ as a variable then the solution H^μ depends on μ analytically. This analytic dependence was emphasized by Ahlfors and Bers in their 1960 paper [2] and is essential in determining a complex structure for Teichmüller space (see [1; 13; 14; 19]). Note that when $\mu \equiv 0$, H^0 is the identity map. A 1-quasiconformal homeomorphism is conformal.

In light of the subsequent development of complex dynamics, this analytic dependence presents the even more interesting phenomenon known as *holomorphic motions.* Let

$$\Delta_r = \{c \in \mathbb{C} \mid |c| < r\}$$

denote the disk of radius $0 < r < 1$ and centered at 0. We use Δ to mean the unit disk. Given a measurable function μ on $\hat{\mathbb{C}}$ with $\|\mu\|_\infty = 1$, we have a family of Beltrami coefficients $c\mu$ for $c \in \Delta$ as well as a family of normalized solutions $H^{c\mu}$. Note that $H^{c\mu}$ is a $(1+|c|)/(1-|c|)$-quasiconformal homeomorphism. Moreover, $H^{c\mu}$ is a family that is holomorphic on c. Consider a subset E of $\hat{\mathbb{C}}$ and its image $E_c = H^{c\mu}(E)$. One can see that E_c moves holomorphically in $\hat{\mathbb{C}}$ when c moves in Δ. That is, for any point $z \in E$, $z(c) = H^{c\mu}(z)$ traces a holomorphic path starting from z as c moves in the unit disk. Surprisingly, the converse of this fact is also true, which follows from the famous λ-lemma of Mañé, Sad, and Sullivan [15] in complex dynamical systems. Let us begin to understand this fact by first defining holomorphic motions.

DEFINITION 1 (Holomorphic motions). Let E be a subset of $\hat{\mathbb{C}}$. Let

$$h(c,z)\colon \Delta \times E \to \hat{\mathbb{C}}$$

be a map. Then h is called a *holomorphic motion* of E parameterized by Δ and with base point 0 if:

(1) $h(0,z) = z$ for $z \in E$;
(2) for any fixed $c \in \Delta$, $h(c,\cdot)\colon E \to \hat{\mathbb{C}}$ is injective;
(3) for any fixed z, $h(\cdot,z)\colon \Delta \to \hat{\mathbb{C}}$ is holomorphic.

For example, for a measurable function μ on $\hat{\mathbb{C}}$ with $\|\mu\|_\infty = 1$,

$$H(c,z) = H^{c\mu}(z)\colon \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

is a holomorphic motion of $\hat{\mathbb{C}}$ parameterized by Δ and with base point 0.

Observe that continuity does not directly enter into the definition; the only restriction is in the c direction. However, continuity is a consequence of the hypotheses from the proof of the λ-lemma of [15, Thm. 2], where the following lemma was also proved.

LEMMA 1 (λ-lemma). *A holomorphic motion of a set $E \subset \hat{\mathbb{C}}$ parameterized by Δ and with base point* 0 *can be extended to a holomorphic motion of the closure of E parameterized by the same Δ and with base point* 0.

Furthermore, Mañé, Sad, and Sullivan showed in [15] that $f(c,\cdot)$ satisfies the Pesin property. In particular, when the closure of E is a domain, this property can be described as the quasiconformal property. A further study of this quasiconformal property was given by Sullivan and Thurston [21] and by Bers and Royden [5]. In [21] it is proved that there exists a universal constant $a > 0$ such that any holomorphic motion of any set $E \subset \hat{\mathbb{C}}$ parameterized by Δ and with base point 0 can be extended to a holomorphic motion of $\hat{\mathbb{C}}$ parameterized by Δ_a and with base point 0. In [5], classical Teichmüller theory is used to show that this constant actually can be taken to be $1/3$. In the same paper it is shown that, in any holomorphic motion $H(c,z)\colon \Delta\times\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $H(c,\cdot)\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a $(1+|c|)/(1-|c|)$-quasiconformal homeomorphism for any fixed $c \in \Delta$. The expectation in both [5] and [21] was that $a = 1$. This was eventually proved by Slodkowski in [20]. Several different proofs have been given for Slodkowski's theorem (cf. [3; 8; 9]).

THEOREM 3 (Holomorphic motion theorem). *Suppose*

$$h(c,z)\colon \Delta \times E \to \hat{\mathbb{C}}$$

is a holomorphic motion of a set $E \subset \hat{\mathbb{C}}$ parameterized by Δ and with base point 0. *Then h can be extended to a holomorphic motion*

$$H(c,z)\colon \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

of $\hat{\mathbb{C}}$ that is also parameterized by Δ and with base point 0. *Moreover, for every $c \in \Delta$,*

$$H(c,\cdot)\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

is a $(1+|c|)/(1-|c|)$-quasiconformal homeomorphism of $\hat{\mathbb{C}}$. The Beltrami coefficient of $H(c,\cdot)$ given by

$$\mu(c,z) = \frac{\partial H(c,z)}{\partial \bar{z}} \Bigg/ \frac{\partial H(c,z)}{\partial z}$$

is a holomorphic function from Δ into the unit ball of the Banach space $\mathcal{L}^\infty(\mathbb{C})$ of all essentially bounded measurable functions on $\mathbb{C}$.

The reader can read our recent expository paper [10] for a complete proof of this theorem and related topics.

3. Leau–Fatou Flowers and Linearization

Since the idea in [11] and [12] plays an important role in the proof of Theorem 1, we begin by using it to give a conceptual proof of the Fatou linearization theorem. This proof is again an application of holomorphic motions.

Suppose $f(z)$ is a parabolic germ at 0. Then there is a constant $0 < r_0 < 1/2$ such that $f(z)$ is conformal with the Taylor expansion

$$f(z) = e^{2\pi pi/q}z + \text{higher-order terms}, \quad (p,q) = 1,\ |z| < r_0.$$

Suppose $f^m \not\equiv \mathrm{id}$ for all $m > 0$. Then, for appropriate r_0, we have

$$f^q(z) = z(1 + az^n + \varepsilon(z)), \quad a \neq 0,\ |z| < r_0,$$

where n is a multiple of q and $\varepsilon(z)$ is given by a convergent power series of the form

$$\varepsilon(z) = a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots, \quad |z| < r_0.$$

Suppose $N \subset \Delta_{r_0}$ is a neighborhood of 0. A simply connected open set $\mathcal{P} \subset N \cap f^q(N)$ with $f^q(\mathcal{P}) \subset \mathcal{P}$ and $0 \in \bar{\mathcal{P}}$ is called an *attracting petal* if $f^m(z)$ for $z \in \mathcal{P}$ converges uniformly to 0 as $m \to \infty$. An attracting petal $\mathcal{P}'$ for f^{-1} is called a *repelling petal* at 0.

THEOREM 4 (Leau–Fatou flower). *There exist n attracting petals $\{\mathcal{P}_i\}_{i=0}^{n-1}$ and n repelling petals $\{\mathcal{P}_j'\}_{j=0}^{n-1}$ such that*

$$N_0 = \bigcup_{i=0}^{n-1} \mathcal{P}_i \cup \bigcup_{j=0}^{n-1} \mathcal{P}_j'$$

is a neighborhood of 0.

See [17] for a proof of this theorem.

For each attracting petal $\mathcal{P} = \mathcal{P}_i$, consider the change of coordinate

$$w = \phi(z) = \frac{d}{z^n}, \quad d = -\frac{1}{na},$$

on $\mathcal{P}$. Suppose the image of $\mathcal{P}$ under $\phi(z)$ is a right half-plane

$$R_\tau = \{w \in \mathbb{C} \mid \Re w > \tau\}.$$

Then

$$z = \phi^{-1}(w) = \sqrt[n]{\frac{d}{w}} : R_\tau \to \mathcal{P}$$

is a conformal map. The form of f^q in the w-plane is

$$F(w) = \phi \circ f \circ \phi^{-1}(w) = w + 1 + \eta\left(\frac{1}{\sqrt[n]{w}}\right), \quad \Re w > \tau,$$

where $\eta(\xi)$ is an analytic function in a neighborhood of 0. Suppose

$$\eta(\xi) = b_1\xi + b_2\xi^2 + \cdots, \quad |\xi| < r_1,$$

is a convergent power series for some $0 < r_1 \leq r_0$. Take $0 < r < r_1$ such that

$$|\eta(\xi)| \leq \frac{1}{2} \quad \forall |\xi| \leq r.$$

Then $F(R_\tau) \subset R_\tau$ for any $\tau \geq 1/r^n$ because

$$\Re F(w) = \Re w + 1 + \Re\eta\left(\frac{1}{\sqrt[n]{w}}\right) \geq \Re w + \frac{1}{2} \quad \text{for all } \Re w \geq \tau.$$

THEOREM 5 (Fatou linearization theorem). *Suppose $\tau > 1/r^n + 1$ is a real number. Then there is a conformal map $\Psi(w)\colon R_\tau \to \Omega$ such that*

$$F(\Psi(w)) = \Psi(w+1) \quad \forall w \in R_\tau.$$

Here we give a new proof by incorporating holomorphic motions. For other proofs, see [7;17].

3.1. Construction of a Holomorphic Motion

For any $x \geq \tau$, let

$$E_{0,x} = \{w \in \mathbb{C} \mid \Re w = x\}$$

and

$$E_{1,x} = \{w \in \mathbb{C} \mid \Re w = x+1\}$$

and let

$$E_x = E_{0,x} \cup E_{1,x}.$$

Then E_x is a subset of $\hat{\mathbb{C}}$. Define

$$H_x(w) = \begin{cases} w, & w \in E_{0,x}; \\ \Phi(w) = w + \eta\big(1/\sqrt[n]{w-1}\big), & w \in E_{1,x}. \end{cases}$$

Since $H_x(w)$ is injective on both $E_{0,x}$ and $E_{1,x}$ and since

$$\Re(H_x(w)) \geq \Re(w) - \frac{1}{2} = x + 1 - \frac{1}{2} = x + \frac{1}{2}, \quad w \in E_{1,x},$$

the images of $E_{0,x}$ and $E_{1,x}$ under $H_x(w)$ do not intersect. Hence $H_x(w)$ is injective. Moreover, $H_x(w)$ conjugates $F(w)$ to the linear map $w \mapsto w+1$ on $E_{0,x}$; that is,

$$F(H_x(w)) = H_x(w+1) \quad \forall w \in E_{0,x}.$$

We introduce a complex parameter $c \in \Delta$ into $\eta(\xi)$ as follows. Define

$$\eta(c,\xi) = \eta\big(cr\xi\sqrt[n]{x-1}\big) = b_1\big(cr\xi\sqrt[n]{x-1}\big) + b_2\big(cr\xi\sqrt[n]{x-1}\big)^2 + \cdots$$

for $|c| < 1$ and $|\xi| \leq 1/\sqrt[n]{x-1}$. Since $\left|cr\xi\sqrt[n]{x-1}\right| \leq r$, it follows that $\eta(c,\xi)$ is a convergent power series and that $|\eta(c,\xi)| \leq 1/2$ for $|c| < 1$ and $|\xi| \leq 1/\sqrt[n]{x-1}$. We are thus led to introduce a complex parameter $c \in \Delta$ for $H_x(w)$ by defining

$$H_x(c,w) = \begin{cases} w, & (c,w) \in \Delta \times E_{0,x}; \\ \Phi(w) = w + \eta\big(c, 1/\sqrt[n]{w-1}\big), & (c,w) \in \Delta \times E_{1,x}. \end{cases}$$

LEMMA 2. *The map $H_x(c,w)\colon \Delta \times E_x \to \hat{\mathbb{C}}$ is a holomorphic motion.*

Proof. (1) It is clear that $H_x(0,w) = w$ for $w \in E_x$.

(2) By Rouché's theorem it follows that, for any fixed $c \in \Delta$, $H_x(c,\cdot)$ is injective on $E_{0,x}$ and on $E_{1,x}$. Since

$$\Re H_x(c,w) = \Re w + \Re\eta\left(c, \frac{1}{\sqrt[n]{w-1}}\right) \geq \Re w - \frac{1}{2} \quad \forall w \in E_{1,x},$$

the images of $E_{0,x}$ and $E_{1,x}$ under $H_x(c,\cdot)$ do not intersect. Therefore, $H_x(c,\cdot)$ is injective on E_x.

(3) For any fixed $w \in E_{0,x}$ we have $H_x(c,w) = w$, so $H(\cdot, w)$ is a holomorphic function of c. For any fixed $w \in E_{1,x}$,

$$H_x(c,w) = w + \eta\left(\frac{cr\sqrt[n]{x-1}}{\sqrt[n]{w-1}}\right),$$

which is a convergent power series of c and so is holomorphic. We have proved the lemma. □

By Theorem 3,

$$H_x(c,w)\colon \Delta \times E_x \to \hat{\mathbb{C}}$$

can be extended to a holomorphic motion

$$\tilde{H}_x(c,w)\colon \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$

For each $c \in \Delta$,

$$h_c(w) = \tilde{H}_x(c,w)\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

is a $(1+|c|)/(1-|c|)$-quasiconformal homeomorphism. If $c(x) = 1/\big(r\sqrt[n]{x-1}\big)$, then $h_{c(x)}$ is a quasiconformal extension of $H_x(w)$ to $\hat{\mathbb{C}}$ whose quasiconformal dilatation is less than or equal to

$$K(x) = \frac{1+1/r\sqrt[n]{x-1}}{1-1/r\sqrt[n]{x-1}}.$$

Note that $K(x) \to 1$ as $x \to \infty$.

3.2. Construction of Quasiconformal Conjugacies

Suppose

$$S_x = \{w \in \mathbb{C} \mid x \leq \Re w \leq x+1\}$$

is the strip bounded by two lines $\Re w = x$ and $\Re w = x+1$. Consider the restriction of $h_{c(x)}(w)$ on S_x, which we still denote as $h_{c(x)}(w)$.

For any $w_0 \in R_\tau \cup E_{0,\tau}$, let $w_m = F^m(w_0)$. Since $w_m - w_{m+1}$ tends to 1 as m goes to ∞ uniformly on $R_\tau \cup E_{0,\tau}$, it follows that

$$\frac{w_n - w_0}{m} = \frac{1}{m}\sum_{k=1}^{m}(w_k - w_{k-1}) \to 1$$

uniformly on $R_\tau \cup E_{0,\tau}$ as m goes to ∞. So w_m is asymptotic to m as m goes to ∞ uniformly in any bounded set of $R_\tau \cup E_{0,\tau}$.

Let $x_0 = \tau$ and $x_m = \Re(F^m(x_0))$. Then x_m is asymptotic to m as m goes to ∞. For each $m > 0$, let

$$\Upsilon_m = F^{-m}(E_{0,x_m} \cup \{\infty\})$$

be a curve passing through $x_0 = \tau$ and ∞; let

$$\Omega_m = F^{-m}(R_{x_m})$$

be a domain with boundary Υ_m.

Let

$$S_{i,x_m} = F^{-i}(S_{x_m}), \quad i = m, m+1, \ldots, 1, 0, -1, \ldots, -m+1, -m, \ldots.$$

Then

$$\Omega_m = \bigcup_{-\infty}^{i=m} S_{i,x_m}.$$

Let

$$A_m = \{w \in \mathbb{C} \mid \tau + m \le \Re w \le \tau + m + 1\}$$

and let $A_{i,m} = A_m - i$ for $i = m, m+1, \ldots, 1, 0, -1, \ldots, -m+1, -m, \ldots$. Let

$$\beta_m(w) = w + x_m - \tau - m \colon \mathbb{C} \to \mathbb{C}.$$

Then $\beta_m(w)$ is a conformal map and

$$\beta_m(A_m) = S_{x_m}.$$

Define

$$\psi_m(w) = h_{c(x_m)} \circ \beta_m(w).$$

Then $\psi_m(w)$ is a $K(x_m)$-quasiconformal homeomorphism on A_m. Moreover,

$$F(\psi_m(w)) = \psi_m(w+1) \quad \text{for all } \Re w = m + \tau.$$

Now define

$$\psi_m(w) = F^{-i}(\psi_m(w+i)) \quad \forall w \in A_{i,m}$$

for $i = m, m-1, \ldots, 1, 0, -1, \ldots, -m+1, -m, \ldots$. Then $\psi_m(w)$ is a $K(x_m)$-quasiconformal homeomorphism from R_τ to Ω_m and

$$F(\psi_m(w)) = \psi_m(w+1) \quad \forall w \in R_\tau.$$

3.3. Improvement to Conformal Conjugacy

Let $w_0 = \tau$ and $w_m = F^m(w_0)$ for $m = 1, 2, \ldots$. Recall that

$$R_{x_m} = \{w \in \mathbb{C} \mid \Re w > x_m\},$$

where $x_m = \Re w_m$.

For any $\tilde{w}_0 \in R_{x_{m+1}}$, let $\tilde{w}_m = F^m(\tilde{w}_0)$ for $m = 1, 2, \ldots$. Since

$$F'(w) = 1 + O\left(\frac{1}{|w|^{1+1/n}}\right), \quad w \in R_\tau,$$

and since $\tilde{w}_m/m \to 1$ as $m \to \infty$ uniformly on any compact set, there is a constant $C > 0$ such that

$$C^{-1} \le \frac{|\tilde{w}_m - w_m|}{|\tilde{w}_1 - w_1|} = \prod_{k=1}^{m} \frac{|\tilde{w}_{k+1} - w_{k+1}|}{|\tilde{w}_k - w_k|} = \prod_{k=1}^{m}\left(1 + O\left(\frac{1}{k^{1+1/n}}\right)\right) \le C$$

as long as w_1 and $\tilde{w}_1$ remain in the same compact set. Since

$$w_{m+1} = w_m + 1 + \eta\left(\frac{1}{\sqrt[n]{w_m}}\right) \quad \text{and} \quad \left|\eta\left(\frac{1}{\sqrt[n]{w_m}}\right)\right| \le \frac{1}{2},$$

the distance between w_{m+1} and R_{x_m} is greater than or equal to $1/2$. Hence the disk $\Delta_{1/2}(w_{m+1})$ is contained in R_{x_m}, which implies that the disk $\Delta_{1/(2C)}(w_1)$ is contained in Ω_m for every $m = 0, 1, \ldots$. Thus the sequence

$$\psi_m(w)\colon R_\tau \to \Omega_m, \quad m = 1, 2, \ldots,$$

is contained in a weakly compact subset of the space of quasiconformal mappings. Let

$$\Psi(w)\colon R_\tau \to \Omega$$

be a limiting mapping of a subsequence. Then Ψ is 1-quasiconformal and thus conformal and satisfies

$$F(\Psi(w)) = \Psi(w+1) \quad \forall w \in R_\tau.$$

This completes the proof of Theorem 5.

4. Quasiconformal Rigidity, Proof of Theorem 1

In this section, we prove Theorem 1 by using an idea similar to that used in the proof of Theorem 5.

4.1. Conformal Conjugacies on Attracting Petals

Suppose f and g are two topologically conjugate parabolic germs, and suppose that $f^m, g^m \not\equiv \mathrm{id}$ for all $m > 0$. (If some $f^m \equiv$ identity then $g^m \equiv$ identity, too.) Suppose λ and $n+1$ are their common multiplier and multiplicity. Suppose $0 < r_0 < 1/2$ such that both f and g are conformal in Δ_{r_0}. Without loss of generality, we assume that $\lambda = 1$ and that f and g have the respective forms

$$f(z) = z(1 + z^n + o(z^n)) \quad \text{and} \quad g(z) = z(1 + z^n + o(z^n)), \quad |z| < r_0.$$

By Theorem 4, for any small neighborhood $N \subset \Delta_{r_0}$ there exist n attracting petals $\{P_{i,f}\}_{i=0}^{n-1}$ and n repelling petals $\{P'_{i,f}\}_{i=0}^{n-1}$ for f in N. Let us assume that every $P_{i,f}$ is the maximal attracting petal in N. Similarly, we have the same pattern of attracting petals $\{P_{i,g}\}_{i=0}^{n-1}$ and the repelling petals $\{P'_{i,g}\}_{i=0}^{n-1}$ for g.

From Theorem 5 and also [17, p. 107], for every $0 \le i \le n-1$ there is a conformal map

$$\psi_i\colon P_{i,g} \to P_{i,f}$$

such that

$$f(\psi_i(z)) = \psi_i(g(z)), \quad z \in P_{i,g}.$$

4.2. Repelling Petals

For each $0 \le i \le n-1$, let

$$w = \phi(z) = -\frac{1}{nz^n}$$

be the change of coordinate. Then f and g in the w-coordinate system have the respective forms

$$F(w) = w + 1 + \eta_f\left(\frac{1}{\sqrt[n]{w}}\right) \quad \text{and} \quad G(w) = w + 1 + \eta_g\left(\frac{1}{\sqrt[n]{w}}\right),$$

where both

$$\eta_f(\xi) = a_1\xi + a_2\xi + \cdots \quad \text{and} \quad \eta_g(\xi) = b_1\xi + b_2\xi + \cdots, \quad |\xi| < r_1,$$

are convergent power series for some number $0 < r_1 < r_0$. Take a number $0 < r < r_1$ such that

$$|\eta_f(\xi)|, |\eta_g(\xi)| \le \frac{1}{4}, \quad |\xi| \le r.$$

Without loss of generality, we assume that $\eta_g(w) \equiv 0$; that is, $G(w) = w + 1$.

Suppose that both repelling petals $P'_{i,f}$ and $P'_{i,g}$ are changed to a left half-plane

$$L_{-r^n} = \{w \in \mathbb{C} \mid \Re w < -r^n\}.$$

4.3. Construction of a Holomorphic Motion

Take $\tau_0 = r^n$. Let

$$U_{\tau_0} = \{w \in \mathbb{C} \mid \Im w > \tau_0\}$$

be an upper half-plane and let

$$D_{-\tau_0} = \{w \in \mathbb{C} \mid \Im w < -\tau_0\}$$

be a lower half-plane. Define

$$\Psi(w) = \begin{cases} \phi \circ \psi_i \circ \phi^{-1}(w), & w \in U_{\tau_0}; \\ \phi \circ \psi_{i+1} \circ \phi^{-1}(w), & w \in D_{\tau_0}. \end{cases}$$

(If $i + 1 = n$, we consider it as 0.) Then

$$F(\Psi(w)) = \Psi(G(w)), \quad w \in U_{\tau_0} \cup D_{\tau_0}.$$

We have the property that $\Psi(w)/w \to 1$ as $w \to \infty$ (see [17, p. 109]).

Let $a = e^{-2\pi\tau_0}$. Consider the covering map

$$\xi = \beta(w) = e^{2\pi i w} : \mathbb{C} \to \mathbb{C} \setminus \{0\},$$

which maps U_{τ_0} to $\Delta_a \setminus \{0\}$ and $D_{-\tau_0}$ to $\mathbb{C} \setminus \bar{\Delta}_{1/a}$.

The inverse of $w = \beta^{-1}(\xi)$ is a multivalued analytic function on $\mathbb{C} \setminus \{0\}$. We take one branch as β^{-1}. Since $\Psi(w)$ is asymptotic to w as $w \to \infty$, the map

$$\theta(\xi) = \beta \circ \Psi \circ \beta^{-1}(\xi)$$

is analytic in Δ_a and in $\bar{\Delta}^c_{1/a} = \hat{\mathbb{C}} \setminus \bar{\Delta}_{1/a}$. Suppose that

$$\theta(\xi) = \xi + a_2\xi^2 + \cdots, \quad |\xi| < a,$$

and

$$\theta(\xi) = \xi + \frac{b_1}{\xi} + \cdots, \quad |\xi| > \frac{1}{a},$$

are two convergent power series.

For any $\tau > \tau_0$, let $\varepsilon = e^{-2\pi\tau}$. Suppose $\bar{\Delta}^c_{1/\varepsilon} = (\hat{\mathbb{C}} \setminus \bar{\Delta}_{1/\varepsilon})$, and let

$$E = \Delta_\varepsilon \cup \bar{\Delta}^c_{1/\varepsilon};$$

E is a subset of $\hat{\mathbb{C}}$. We now introduce a complex parameter $c \in \Delta$ into $\theta(\xi)$ such that it is a holomorphic motion of E parameterized by Δ and with base point 0.

Define

$$\theta(c,\xi) = \frac{\varepsilon}{ca}\theta\left(\frac{ca\xi}{\varepsilon}\right) = \xi + a_2\left(\frac{ca}{\varepsilon}\right)\xi^2 + \cdots, \quad |c| < 1,\ |\xi| \le \varepsilon,$$

and

$$\theta(c,\xi) = \frac{ca}{\varepsilon}\theta\left(\frac{\varepsilon\xi}{ca}\right) = \xi + \frac{b_1}{\xi}\left(\frac{ca}{\varepsilon}\right)^2 + \cdots, \quad |c| < 1,\ |\xi| \ge \frac{1}{a}.$$

We claim that

$$\theta(c,\xi)\colon \Delta \times E \mapsto \hat{\mathbb{C}}$$

is a holomorphic motion. Here is a proof.

(1) It is clear that $\theta(0,\xi) = \xi$ for all $\xi \in E$.

(2) For any fixed $c \neq 0 \in \Delta$, $\theta(c,\xi)$ on Δ_ε is a conjugation map of $\theta(\xi)$ by the linear map $\xi \mapsto (ca/\varepsilon)\xi$, and $\theta(c,\xi)$ on $\bar{\Delta}^c_{1/\varepsilon}$ is a conjugation map of $\theta(\xi)$ by the linear map $\xi \mapsto (\varepsilon/(ca))\xi$. Hence they are injective. Because the image $\theta(c,\Delta_\varepsilon)$ is contained in Δ_a and the image $\theta(c,\bar{\Delta}^c_{1/\varepsilon})$ is contained in $\bar{\Delta}^c_{1/a}$, they do not intersect. So $\theta(c,\cdot)$ on E is injective.

(3) For any fixed $\xi \in \Delta_\varepsilon$, since $|ca\xi/\varepsilon| < a$ for $|c| < 1$, $\theta(\cdot,\xi)$ is a convergent power series of c. So $\theta(\cdot,\xi)$ is holomorphic on c. For any fixed $\xi \in \bar{\Delta}^c_{1/\varepsilon}$, since $|\varepsilon\xi/(ca)| > 1/a$ for $|c| < 1$, $\theta(\cdot,\xi)$ is a convergent power series of c. So $\theta(\cdot,\xi)$ is holomorphic on c. We have proved the claim.

Let $E_\tau = U_\tau \cup D_{-\tau}$. Then $\beta(E_\tau) = E$. Thus we can lift the holomorphic motion

$$\theta(c,\xi)\colon \Delta \times E \to \hat{\mathbb{C}}$$

to obtain a holomorphic motion

$$h_0(c,w)\colon \Delta \times E_\tau \to \hat{\mathbb{C}}.$$

When $c(\tau) = \varepsilon/a$, $h(c(\tau),w) = \Psi(w)$.

Let $w_1 = -\tau + i\tau$ and $w_2 = -\tau - i\tau$. Consider the vertical segment connecting them,

$$s_\tau = \{tw_1 + (1-t)w_2 \mid 0 \le t \le 1\}.$$

Let

$$s'_\tau = s_\tau + 1 = \{tw_1 + (1-t)w_2 + 1 \mid 0 \le t \le 1\}.$$

Define

$$h_1(c, tw_1 + (1-t)w_2) = th_0(c,w_1) + (1-t)h_0(c,w_2)\colon \Delta \times s_\tau \to \hat{\mathbb{C}}$$

and

$$h_2(c, tw_1 + (1-t)w_2 + 1) = F(h_1(c, tw_1 + (1-t)w_2)\colon \Delta \times s'_\tau \to \hat{\mathbb{C}}.$$

Both of these maps are holomorphic motions. Since

$$\begin{aligned}&h_2(c, tw_1 + (1-t)w_2 + 1)\\&\quad = th_0(c, w_1) + (1-t)h_0(c, w_2) + 1 + \eta(th_0(c, w_1) + (1-t)h_0(c, w_2))\end{aligned}$$

and since

$$|\eta(w)| \le 1/4 \quad \text{for all } |w| \ge \tau,$$

the images of these two holomorphic motions do not intersect. Therefore, we define a holomorphic motion

$$h(c, w) = \begin{cases} h_0(c, w), & (c, w) \in \Delta \times E_\tau, \\ h_1(c, w), & (c, w) \in \Delta \times s_\tau, \\ h_2(c, w), & (c, w) \in \Delta \times s'_\tau, \end{cases}$$

of $\Sigma = E_\tau \cup s_\tau \cup s'_\tau$ parameterized by Δ and with base point 0.

For $c(\tau) = \varepsilon/a$, $h(c(\tau), w)$ is a conjugacy from F to G on $E_\tau \cup s_\tau$; that is,

$$F(h(c(\tau), w)) = h(c(\tau), G(w)), \quad w \in E_\tau \cup s_\tau.$$

By Theorem 3, there is an extended holomorphic motion of $h(c, w)$,

$$H(c, w)\colon \Delta \times \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}},$$

such that for each $c \in \Delta$, $H(c, \cdot)$ is a $(1 + |c|)/(1 - |c|)$-quasiconformal homeomorphism of $\hat{\mathbb{C}}$.

Let $H(w) = H(c(\tau), w)$ and

$$K(\tau) = \frac{1 + c(\tau)}{1 - c(\tau)}.$$

Note that $K(\tau) \to 1$ as $\tau \to \infty$. Then $H(w)$ is a $K(\tau)$-quasiconformal homeomorphism of $\hat{\mathbb{C}}$ such that

$$H(w) = h(c(\tau), w) \quad \forall w \in \Sigma.$$

Let

$$A_0 = \{w \in \mathbb{C} \mid -\tau \le \Re w \le -\tau + 1\}$$

and $A_{-m} = A_0 - m$ for $m = 1, 2, \ldots$. Define

$$\Psi(w) = F^{-m}(H(w + m)), \quad w \in A_{-m},\ m = 0, 1, \ldots.$$

Then $\Psi(w)$ is a $K(\tau)$-quasiconformal homeomorphism defined on the left half-plane

$$L_{-\tau+1} = \{w \in \mathbb{C} \mid \Re w \le -\tau + 1\}$$

and extends $\Psi(w)$ on $U_\tau \cup D_{-\tau}$.

Now let

$$\psi(z) = \phi^{-1} \circ \Psi \circ \phi(z).$$

It extends

$$\psi_i\colon P_{i,g} \to P_{i,f} \quad \text{and} \quad \psi_{i+1}\colon P_{i+1,g} \to P_{i+1,f}$$

in a small neighborhood N to a $K(\tau)$-quasiconformal homeomorphism

$$\psi(z)\colon P_{i,g} \cup P'_{i,g} \cup P_{i+1,g} \to P_{i,f} \cup P'_{i,f} \cup P_{i+1,f}$$

and

$$f \circ \psi(z) = \psi \circ g(z) \quad \forall z \in P_{i,g} \cup P'_{i,g} \cup P_{i+1,g}.$$

If we work out these expressions for every $0 \le i \le n-1$, we find that for any $\varepsilon > 0$ there is a neighborhood U_ε of 0 and a $(1+\varepsilon)/(1-\varepsilon)$-quasiconformal homeomorphism

$$\psi(z)\colon U_\varepsilon \to V_\varepsilon = \psi(U_\varepsilon)$$

that extends every $\psi_i\colon P_{i,g} \to P_{i,f}$ in U_ε and such that

$$f \circ \psi(z) = \psi \circ g(z) \quad \forall z \in U_\varepsilon.$$

This completes the proof of Theorem 1.

5. Gluing Germs in the Riemann Sphere, Proofs of Theorem 2 and Corollary 1

Suppose $\Delta_r(z)$ is the disk of radius $r > 0$ centered at z.

Proof of Theorem 2. Denote

$$B_i(r) = f_i(\Delta_r(z_i)).$$

Let $r_0 > 0$ be a number such that

$$B_i(r) \cap B_j(r) = \emptyset, \quad 1 \le i \ne j \le k,\ 0 < r \le r_0.$$

Let

$$E_r = \bigcup_{i=1}^{n} \bar{\Delta}_r(z_i)$$

be a closed subset of $\hat{\mathbb{C}}$.

For any $0 < r \le r_0$, write

$$f_i(z) = z + a_{i,2}(z-z_i)^2 + \cdots + a_{i,n}(z-z_i)^n + \cdots, \quad |z-z_i| \le r.$$

Let

$$\eta_i(\xi) = a_{i,2}\xi^2 + \cdots + a_{i,n}\xi^n + \cdots.$$

Then

$$f_i(z) = z + \eta_i(z-z_i), \quad |z-z_i| \le r.$$

Let $\phi(z)$ be defined on E_r as

$$\phi(z) = f_i(z) = z + \eta_i(z-z_i) \quad \text{for } |z-z_i| \le r,\ i = 1, \ldots, k.$$

We introduce a complex parameter $c \in \Delta$ into $\phi(z)$ as follows. Define

$$\phi(c,z) = z + \frac{r}{cr_0}\eta_i\left(\frac{cr_0}{r}(z-z_i)\right), \quad |z-z_i| \le r,\ i = 1, \ldots, k.$$

Then

$$\phi(c,z)\colon \Delta \times E_r \to \hat{\mathbb{C}}$$

is a map. We will check that it is a holomorphic motion.

For any fixed $c \in \Delta$, we have

$$\phi'_z(c, z) = 1 + \eta'_i\left(\frac{cr_0}{r}(z - z_i)\right), \quad |z - z_i| \le r,\ i = 1, \ldots, k.$$

By picking $r_0 > 0$ small enough, we can assume that

$$|f'_i(z)| = |1 + \eta'_i(z - z_i)| \ge 1 - |\eta'_i(z - z_i)| > 0, \quad |z - z_i| < r_0,\ i = 1, \ldots, k.$$

Thus

$$\phi'_z(c, z) \ne 0 \quad \text{for all } |z - z_i| \le r,\ i = 1, \ldots, k.$$

We get that $\phi(c, z)$ on each $\Delta_r(z_i)$ is injective. But images of $\Delta_r(z_i)$ and $\Delta_r(z_j)$, $1 \le i \ne j \le k$, are pairwise disjoint under $\phi(c, z)$. Hence $\phi(c, z)$ is injective on E_r.

It is clear that

$$\phi(0, z) = z, \quad z \in E_r.$$

For any fixed $z \in \Delta_r(z_i)$, $1 \le i \le k$,

$$\phi(c, z) = z + \frac{r}{cr_0}\eta_i\left(\frac{cr_0}{r}(z - z_i)\right).$$

Since

$$\left|\frac{cr_0}{r}(z - z_i)\right| < r_0,$$

it follows that

$$\eta_i\left(\frac{cr_0}{r}(z - z_i)\right)$$

is a convergent power series of $c \ne 0 \in \Delta$. For $c = 0$, we have $\phi(0, z) = z$ and so $\phi(c, z)$ is holomorphic with respect to $c \in \Delta$. Therefore,

$$\phi(c, z)\colon \Delta \times E(r) \to \hat{\mathbb{C}}$$

is a holomorphic motion.

Following Theorem 3, we have an extended holomorphic motion

$$\tilde{\phi}(c, z)\colon \Delta \times \hat{\mathbb{C}};$$

that is, $\tilde{\phi}(c, z)|_{\Delta \times E_r} = \phi(c, z)$. Moreover, $\tilde{\phi}(c, \cdot)$ is a $(1 + |c|)/(1 - |c|)$-quasiconformal mapping for any $c \in \Delta$.

Let

$$f(z) = \tilde{\phi}\left(\frac{r}{r_0}, z\right).$$

Then $f(z)$ is a $(1 + r/r_0)/(1 - r/r_0)$-quasiconformal homeomorphism. Furthermore,

$$f|_{\Delta_r(z_i)} = \tilde{\phi}\left(\frac{r}{r_0}, z\right)\Big|_{\Delta_r(z_i)} = \phi\left(\frac{r}{r_0}, z\right)\Big|_{\Delta_r(z_i)} = f_i|_{\Delta_r(z_i)}.$$

Thus, for any given $\varepsilon > 0$, we take $r = (2\varepsilon r_0)/(1 + \varepsilon)$; then f is a $(1 + \varepsilon)$-quasiconformal mapping and extends f_i for all $i = 1, 2, \ldots, k$. We have completed the proof. □

Proof of Corollary 1. First suppose that $r_0 > 0$ and also that

$$f_i(z) = z_i + \lambda_i(z - z_i), \quad z \in D_{r_0}(z_i),\ \lambda_i \neq 0,\ 1 \le i \le k.$$

Suppose

$$\bar{\Delta}_{r_0}(z_i) \cap \bar{\Delta}_{r_0}(z_j) = \emptyset \quad \text{for all } 0 \le i \neq j \le k.$$

Let

$$a = \max\{|\log \lambda_i| \mid 1 \le i \le k\},$$

and let

$$s = r_0 e^{-a/r}$$

for any $0 < r < r_0$.

Let $\Delta_s(z_i)$ and $E_s = \bigcup_{i=1}^k \bar{\Delta}_s(z_i)$. Define

$$\phi(c, z) = z_i + e^{(c/r)\log \lambda_i}(z - z_i), \quad c \in \Delta,\ z \in \bar{\Delta}_s(z_i).$$

We will check that

$$\phi(c, z)\colon \Delta \times E_s \to \hat{\mathbb{C}}$$

is a holomorphic motion.

For $c = 0$, we have $\phi(0, z) = z$ for all $z \in E_s$.

For each fixed $c \in \Delta$, $\phi(c, z)$ on each $\bar{\Delta}_s(z_i)$ is injective but the image of $\Delta_s(z_i)$ under $\phi(c, z)$ is contained in $\Delta_{r_0}(z_i)$. So $\phi(c, z)$ on E_s is injective.

For fixed $z \in E_s$, it is clear that $\phi(c, z)$ is holomorphic with respect to $c \in \Delta$.

Therefore,

$$\phi(c, z)\colon \Delta \times E_s \to \hat{\mathbb{C}}$$

is a holomorphic motion.

By Theorem 3, we have an extended holomorphic motion

$$\tilde{\phi}(c, z)\colon \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}};$$

that is, $\tilde{\phi}(c, z)|_{\Delta \times E_s} = \phi(c, z)$. Moreover, for any $c \in \Delta$, $\tilde{\phi}(c, \cdot)$ is a $(1 + |c|)/(1 - |c|)$-quasiconformal homeomorphism.

Let $f(z) = \tilde{\phi}(r, z)$. Then $f(z)$ is a $(1 + r)/(1 - r)$-quasiconformal homeomorphism. Furthermore,

$$f|_{\Delta_s(z_i)} = \tilde{\phi}(r, z)|_{\Delta_s(z_i)} = \phi(r, z)|_{\Delta_s(z_i)} = f_i|_{\Delta_s(z_i)}.$$

Now we consider the general situation,

$$f_i(z) = z_i + \lambda_i(z - z_i) + a_{2,i}(z - z_i)^2 + \cdots, \quad z \in \Delta_{r_0}(z_i),\ \lambda_i \neq 0,\ 1 \le i \le k.$$

Let

$$g_i(z) = z_i + \lambda_i^{-1}(z - z_i), \quad 1 \le i \le k.$$

Then

$$F_i(z) = f_i \circ g_i(z) = z + \frac{a_{2,i}}{\lambda_i^2}(z - z_i)^2 + \cdots, \quad 1 \le i \le k,$$

are all parabolic germs.

From Theorem 2 and the argument just given, for any $\varepsilon > 0$ we have $0 < s < r \le r_0$ and two $\sqrt{1 + \varepsilon}$-quasiconformal homeomorphisms $F(z)$ and $G(z)$ of $\hat{\mathbb{C}}$ such that

$$F|_{\Delta_r(z_i)} = F_i|_{\Delta_r(z_i)} \quad \text{and} \quad G|_{\Delta_s(z_i)} = g_i^{-1}|_{\Delta_s(z_i)}$$

and such that

$$G(\Delta_s(z_i)) \subset \Delta_r(z_i).$$

Then $f(z) = F \circ G(z)$ is a $(1+\varepsilon)$-quasiconformal homeomorphism of $\hat{\mathbb{C}}$ such that

$$f|_{\Delta_s(z_i)} = F \circ G|_{\Delta_s(z_i)} = f_i \circ g_i \circ g_i^{-1}|_{\Delta_s(z_i)} = f_i|_{\Delta_s(z_i)}.$$

We have completed the proof. □

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Department of Mathematics
Queens College of the City University of New York
Flushing, NY 11367

and

Department of Mathematics
Graduate School of the City University of New York
New York, NY 10016

yunping.jiang@qc.cuny.edu