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Convergence of the Kähler–Ricci Flow and Multiplier Ideal Sheaves on del Pezzo Surfaces

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1. Introduction

Let X be an n-dimensional compact complex manifold with positive first Chern class $c_1(X)$. Such manifolds are called *Fano* manifolds. The Kähler–Ricci flow on X is defined by the equation

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}}, \tag{1}$$

where $R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det g_{\alpha\bar{\beta}}$ is the Ricci curvature tensor of the hermitian metric $\sum_{i,j} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$. If the class of the Kähler form $\hat{\omega} = \frac{i}{2\pi} \sum_{i,j} \hat{g}_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is $c_1(X)$, then the Kähler–Ricci flow preserves the class of $i \sum_{i,j} \hat{g}_{i\bar{j}} dz_i \wedge d\bar{z}_j$, so we can write

$$g_{i\bar{j}} = \hat{g}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \phi$$

for the solution to the Kähler–Ricci flow with initial condition

$$g_{i\bar{j}}(0) = \hat{g}_{i\bar{j}}.$$

Equation (1) can be reformulated as

$$\frac{\partial}{\partial t}\phi = \log \frac{\det g_{\alpha\bar{\beta}}}{\det \hat{g}_{\alpha\bar{\beta}}} + \phi - \hat{f}, \quad \phi(0) = c_0 \in \mathbb{R}, \tag{2}$$

where $\hat{f}$ is the Ricci potential; that is, for $\hat{R}_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det \hat{g}_{\alpha\bar{\beta}}$ we have $\hat{R}_{i\bar{j}} - \hat{g}_{i\bar{j}} = \partial_i \partial_{\bar{j}} \hat{f}$. It was proven in [Ca] that the solution to (1) exists for all $t > 0$. This paper investigates the issue of convergence based on the following theorem, which first appeared in [PSeS]. The version given here, which is stronger than the one in [PSeS], is based on [PS].

Theorem 1.1 [PSeS; PS]. *Let X be a Fano manifold. Consider the Ricci flow in the form of* (2) *with the initial value c_0 specified by* [PSeS, (2.10)]. *Then the following two statements are equivalent.*

(i) *There exists a $p > 1$ such that*

$$\sup_{t \geq 0} \int_X e^{-p\phi} \hat{\omega}^n < \infty.$$

(ii) *The family of metrics $g_{i\bar{j}}(t)$ converges in C^∞-norm exponentially fast to a Kähler–Einstein metric.*

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The preceding theorem will allow us to formulate the sufficient Criterion 1.6 for statement (ii) to hold, in analogy to Nadel's criterion for the existence of Kähler–Einstein metrics (see [N1; DKo; He]). It is well known that some Fano manifolds do not possess a Kähler–Einstein metric (e.g., $\mathbb{P}^2$ blown up in one or two points; see Section 5), so we cannot expect (ii) to hold in general on a Fano manifold. In this paper, we mention no necessary condition for (ii) other than the existence of a Kähler–Einstein metric.

First, we quickly recall the basics of multiplier ideal sheaves. The following is the standard definition of the multiplier ideal sheaf pertaining to a plurisubharmonic function on a complex manifold.

THEOREM 1.2 [N1]. *Let φ be a plurisubharmonic function on the complex manifold X. Then the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the subsheaf of $\mathcal{O}_X$ defined by*

$$\mathcal{I}(\varphi)(U) = \{f \in \mathcal{O}_X(U) : |f|^2 e^{-\varphi} \in L^1_{\mathrm{loc}}(U)\}$$

for every open set $U \subseteq X$. It is a coherent subsheaf.

Multiplier ideal sheaves have turned out to be very useful in algebraic geometry, mainly because of the following vanishing theorem. They are usually defined using the notion of a singular hermitian metric on a line bundle, which in general is a metric h that is given on a small open set U by $h = e^{-\varphi}$, where φ is $L^1(U)$. If φ is plurisubharmonic for every U, then the multiplier ideal sheaf $\mathcal{I}(h)$ attached to h is defined by $\mathcal{I}(h)(U) = \mathcal{I}(\varphi)(U)$ provided $h = e^{-\varphi}$ on U.

THEOREM 1.3 (Nadel's vanishing theorem). *Let X be a compact complex Kähler manifold. Let L be a line bundle on X equipped with a singular hermitian metric such that the curvature current $-\frac{i}{2\pi}\partial\bar{\partial}\log h$ is positive definite in the sense of currents—in other words, there exist a smooth positive definite $(1,1)$-form ω and an $\varepsilon > 0$ such that $-\frac{i}{2\pi}\partial\bar{\partial}\log h \geq \varepsilon\omega$. Then*

$$H^q(X, (K_X + L) \otimes \mathcal{I}(h)) = 0 \quad \textit{for all } q \geq 1.$$

We now develop a Nadel-type criterion for Theorem 1.1(ii) to hold. If we assume that Theorem 1.1(ii) does not hold then, according to the theorem, for all $p > 1$ there exists a sequence of times $t_i \to \infty$ with

$$\lim_{i\to\infty} \int_X e^{-p\phi(t_i)}\hat{\omega}^n = \infty.$$

In fact, also

$$\lim_{i\to\infty} \int_X \exp\biggl\{-p\biggl(\phi(t_i) - \frac{1}{V}\int_X \phi(t_i)\hat{\omega}^n\biggr)\biggr\}\hat{\omega}^n = \infty,$$

where $V = \int_X \hat{\omega}^n$. Let ψ be an L^1 limit of the sequence $\phi(t_i) - \frac{1}{V}\int_X \phi(t_i)\hat{\omega}^n$. By semicontinuity [ASi; DKo], $\|e^{-\psi}\|_{L^p(X)} = \infty$. If $G \subseteq \mathrm{Aut}(X)$ is a compact subgroup and if $\hat{\omega}$ is G-invariant, then we can assume ψ and $\mathcal{I}(p\psi)$ to be G-invariant as well.

We have

$$\hat{\omega} + \frac{i}{2\pi}\partial\bar{\partial}\psi \geq 0.$$

Let $\hat{h}$ be a smooth G-invariant hermitian metric for the anticanonical line bundle $-K_X$ with $\frac{1}{2\pi i}\partial\bar{\partial}\log\hat{h} = \hat{\omega} \in c_1(X)$. The singular G-invariant hermitian metric $\hat{h}^{1+\lfloor p\rfloor}\cdot e^{-p\psi}$ is a singular metric for $-(1+\lfloor p\rfloor)K_X$ with positive curvature in the sense of currents:

$$-\frac{i}{2\pi}\partial\bar{\partial}\log(\hat{h}^{1+\lfloor p\rfloor}\cdot e^{-p\psi}) = -\frac{i}{2\pi}\partial\bar{\partial}\log(\hat{h}^{1+\lfloor p\rfloor-p}\cdot\hat{h}^{p}\cdot e^{-p\psi})$$
$$\geq (1+\lfloor p\rfloor - p)\hat{\omega}.$$

Note that for all $p > 1$ we have $1+\lfloor p\rfloor - p > 0$.

Letting $p = \frac{3}{2}$ (or any other number in the interval $]1,2[$) yields the following.

THEOREM 1.4 (Nadel-type criterion). *Let X be a Fano manifold. Assume that Theorem 1.1(ii) does not hold. Then the G-invariant singular hermitian metric $h = \hat{h}^2\cdot e^{-(3/2)\psi}$ on the line bundle $-2K_X$ is such that*

(i) *the curvature of h is positive definite in the sense of currents,*
(ii) $0 \neq \mathcal{I}\big(\frac{3}{2}\psi\big) \neq \mathcal{O}_X$.

The multiplier ideal sheaf $\mathcal{I}\big(\frac{3}{2}\psi\big)$ is also G-invariant. In particular, every element of G maps the zero set $V\big(\mathcal{I}\big(\frac{3}{2}\psi\big)\big)$ to itself.

Note that we can apply Nadel's vanishing theorem with $\tilde{h} = hh_E = \hat{h}^2\cdot e^{-(3/2)\psi}h_E$ and $L = -2K_X + E$, where E is an arbitrary line bundle with semipositive metric h_E, to obtain

$$H^q(X,(K_X+L)\otimes\mathcal{I}(h)) = H^q\big(X,(-K_X+E)\otimes\mathcal{I}\big(\tfrac{3}{2}\psi\big)\big) = 0 \quad \text{for all } q \geq 1. \tag{3}$$

DEFINITION 1.5. An ideal subsheaf $\mathcal{I} \subseteq \mathcal{O}_X$ is said to satisfy *Property (Van)* if, for every semipositive line bundle E,

$$H^q(X,(-K_X+E)\otimes\mathcal{I}) = 0 \quad \text{for all } q \geq 1.$$

The discussion so far can be summed up in the following sufficient criterion.

CRITERION 1.6. *Let X be a Fano manifold, and let G be a compact subgroup of* $\mathrm{Aut}(X)$. *Let there be no nontrivial G-invariant subsheaf $\mathcal{I} \subseteq \mathcal{O}_X$ that satisfies Property (Van). Then Theorem 1.1(ii) holds.*

A criterion of this kind is the essence of Nadel's technique, which can be applied under similar circumstances based on the continuity method for the Monge–Ampère equation to show the existence of Kähler–Einstein metrics on certain Fano manifolds [N1; DKo; He]). However, it is easier to handle Kähler–Einstein metrics instead of the Ricci flow with Nadel's method because one can work with a G-invariant singular hermitian metric for $-K_X$ instead of $-2K_X$, resulting in a cohomology vanishing statement for a G-invariant multiplier ideal sheaf $\mathcal{I}$ of the form

$$H^q(X,(K_X-K_X)\otimes\mathcal{I}) = H^q(X,\mathcal{I}) = 0 \quad \forall q \geq 1. \tag{4}$$

Note that (4) yields more information on the zero set of $\mathcal{I}$ than (3). In fact, the information in (4) is strong enough to prove the existence of Kähler–Einstein metrics on all del Pezzo surfaces of degree 4, 5, and 6 [He]. In this paper we show that the information in (3) can be used to establish Theorem 1.1(ii) for certain nongeneric del Pezzo surfaces with large automorphism group. In particular, we will prove the following result.

MAIN THEOREM 1.7. *Let X be one of the following del Pezzo surfaces*:

(i) $\mathbb{P}^2$ *blown up in four points in general position*;
(ii) $\mathbb{P}^2$ *blown up in five points in general position, where* $\operatorname{Aut}(X)$ *equals* $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_2^4 \rtimes (\mathbb{Z}_3 \rtimes \mathbb{Z}_2)$, *or* $\mathbb{Z}_2^4 \rtimes (\mathbb{Z}_5 \rtimes \mathbb{Z}_2)$.

Then Theorem 1.1(ii) holds.

The method of proof also applies to certain del Pezzo surfaces of low degree if their automorphism group is large enough, as in the case of the Fermat cubic hypersurface in $\mathbb{P}^3$ (see the remarks in Section 5).

REMARK 1.8. It is a result of Perelman that, given the existence of a Kähler–Einstein metric, the Kähler–Ricci flow will converge to it in the sense of Cheeger–Gromov. It should be noted that Theorem 1.7 does not assume the existence of a Kähler–Einstein metric. In fact, it proves the existence as an obvious corollary to the convergence statement (ii) of Theorem 1.1. Moreover, note again that the convergence in Theorem 1.1(ii) is very strong—namely, *exponentially fast in the* C^∞*-norm*.

Finally, let us remark that the nonexistence statements on multiplier ideal sheaves that are established to prove Theorem 1.7 also yield a new result on Tian's holomorphic invariant $\alpha_G(X)$ (see [T1]). In fact, the following theorem is immediate from our considerations.

THEOREM 1.9. *Let X be one of the del Pezzo surfaces described in Theorem 1.7. Then* $\alpha_{\operatorname{Aut}(X)}(X) \geq 2$.

According to the comment immediately following Theorem 1.7, our methods also give $\alpha_{\operatorname{Aut}(X)}(X) \geq 2$ for certain del Pezzo surfaces X of low degree. When X is the Fermat cubic hypersurface in $\mathbb{P}^3$, Tian already found this lower bound in [T1, Thm. 4.3].

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2. Classification and Basic Properties of del Pezzo Surfaces

DEFINITION 2.1. A *del Pezzo surface* is a two-dimensional compact complex manifold X whose anticanonical line bundle $-K_X$ is ample. We call the self-intersection number $(-K_X)^2 = K_X^2$ the *degree* of X. We will denote the degree also by δ.

We now gather some important facts about del Pezzo surfaces that result in the standard classification (see [MHaz; De; Ha]).

FACTS 2.2. For every del Pezzo surface X, the Picard group $\operatorname{Pic} X$ satisfies

$$\operatorname{rank} \operatorname{Pic} X + \delta = 10.$$

In particular, $\delta \leq 9$.

If $\delta = 9$, then X is isomorphic to $\mathbb{P}^2$. If $\delta = 8$, then X is isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\tilde{\mathbb{P}}^2$ (i.e., $\mathbb{P}^2$ blown up at one point).

If $7 \geq \delta \geq 1$, then X is isomorphic to $\mathbb{P}^2$ blown up at $r = 9 - \delta$ points that have the following properties:

(i) no three points lie on a line;
(ii) no six points lie on a conic;
(iii) no seven points lie on a cubic such that the eighth is a double point of the cubic.

Any set of $r = 9 - \delta$ points satisfying these three properties will be said to be in *general* position; conversely, the result of blowing up $1 \leq r \leq 8$ points in general position in $\mathbb{P}^2$ is a del Pezzo surface. For $1 \leq r \leq 4$ general points blown up, there results in each case a unique del Pezzo surface. The reason is that, for any two sets of points $P_1, \dots, P_r$ and $Q_1, \dots, Q_r$ $(r \leq 4)$ with each set in general position, there is an element $A \in \operatorname{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3, \mathbb{C})$ with $A(P_i) = Q_i$ $(1 \leq i \leq r)$.

For our understanding of del Pezzo surfaces, the following facts about the anticanonical line bundle are also important.

FACTS 2.3. Let $1 \leq r \leq 8$. Let X be obtained by blowing up general points P_i, $i = 1, \dots, r$. Let E_i denote the exceptional (-1)-curve that is the preimage of P_i. Let $\pi : X \to \mathbb{P}^2$ denote the blowup map. Then

$$K_X = \pi^* K_{\mathbb{P}^2} + \sum_{i=1}^{r} E_i.$$

This yields

$$\dim H^0(X, -K_X) = 10 - r.$$

For $1 \leq r \leq 6$, the complete linear system $|-K_X|$ gives an embedding into $\mathbb{P}^{9-r} = \mathbb{P}^{\delta}$. For $r = 7$, it gives a double cover of $\mathbb{P}^2$. The complete linear system $|-2K_X|$ gives an embedding into $\mathbb{P}^6$. For $r = 8$, $|-K_X|$ has a unique base point, $|-2K_X|$ gives a double cover of a singular quadric surface in $\mathbb{P}^3$, and $|-3K_X|$ gives an embedding into $\mathbb{P}^6$.

Finally, it turns out that the number of (-1)-curves exceeds r on every del Pezzo surface of degree at most 7. The reason is that, when blowing up two points in $\mathbb{P}^2$, the proper transform of the unique line through the two points becomes a (-1)-curve as well. When blowing up five points, the unique conic through the five points also becomes a (-1)-curve. It is easy to count these (-1)-curves: for $r = 1, \dots, 8$ their numbers are, respectively, $1, 3, 6, 10, 16, 27, 56, 240$. It is interesting that for $r=1,\dots,6$ under the map given by $|-K_X|$, all (-1)-curves become lines in projective space. Hence they are often referred to as *lines* on X.

3. The Case of Four Points Blown Up

Let X be a del Pezzo surface obtained from blowing up four points. We may, and do, assume these to be $P_1=[1,0,0]$, $P_2=[0,1,0]$, $P_3=[0,0,1]$, and $P_4=[1,1,1]$. It is known that $\mathrm{Aut}(X)$ is the Weyl group of the root system of Dynkin type D_5, which is S_5 (see [K]; see also [Wi]). However, we would like to understand $\mathrm{Aut}(X)$ more concretely.

First all of, there is a subgroup S_4 of projectivities in $\mathrm{PGL}(3,\mathbb{C})=\mathrm{Aut}(\mathbb{P}^2)$ that preserve the set $\{P_1,P_2,P_3,P_4\}$. These projectivities lift to X, and we can write $S_4\subset\mathrm{Aut}(X)$.

In addition, for every $i=1,\dots,4$ there exists a quadratic Cremona transformation Cr_i that leaves P_i fixed and has the three remaining points as indeterminacy locus. (Note that such a Cr_i is defined only up to the action of the $S_3\subset S_4$ consisting of automorphisms fixing P_i; for our purposes, it does not matter which Cr_i we choose.) All Cr_i extend to automorphisms of X. In light of this, we can write $\mathrm{Aut}(X)$ set-theoretically as a disjoint union

$$\mathrm{Aut}(X)=S_4\uplus\left(\biguplus_{i=1}^{4}\mathrm{Cr}_i\circ S_4\right).$$

In Sections 3.1 and 3.2 we will prove Theorem 1.7(i) by means of Criterion 1.6.

3.1. Zero-dimensional Multiplier Ideal Sheaves

Let $\mathcal{I}\subseteq\mathcal{O}_X$ be an $\mathrm{Aut}(X)$-invariant ideal sheaf satisfying Property (Van). In particular, for E the trivial line bundle,

$$H^q(X,(-K_X)\otimes\mathcal{I})=0\quad\text{for all } q\geq 1. \tag{5}$$

Let $V=V(\mathcal{I})$. Let $\dim V=0$; that is, V consists of a finite number of points. Consider the short exact sequence

$$0\to\mathcal{I}(-K_X)\to\mathcal{O}_X(-K_X)\to\mathcal{O}_V(-K_X)\to 0.$$

Taking the corresponding long exact sequence yields

$$\begin{aligned}0\to H^0(X,\mathcal{I}(-K_X))\to H^0(X,\mathcal{O}_X(-K_X))\to H^0(V,\mathcal{O}_V(-K_X))\\ \to H^1(X,\mathcal{I}(-K_X)).\end{aligned}$$

From (5) it follows that $H^1(X,\mathcal{I}(-K_X))=0$. Therefore, the map

$$H^0(X,\mathcal{O}_X(-K_X))\to H^0(V,\mathcal{O}_V(-K_X))$$

is surjective. We saw in Section 2 that $\dim H^0(X,\mathcal{O}_X(-K_X))=10-4=6$. As a result, V consists of at most six points.

PROPOSITION 3.1. *There is no* $\mathrm{Aut}(X)$*-invariant ideal sheaf* $\mathcal{I}\subseteq\mathcal{O}_X$ *satisfying Property (Van) with* $\dim V(\mathcal{I})=0$.

Proof. We assume that $\mathcal{I}$ exists and then derive a contradiction. Because $V = V(\mathcal{I})$ has at most six points, the contradiction arises from the claim that all orbits of $\mathrm{Aut}(X)$ have cardinality at least 8. In the sequel, we prove this claim.

If the cardinality of an orbit were 7 or less then it would actually be 6 or less, because the order of $\mathrm{Aut}(X)$ is not divisible by 7. The stabilizer subgroup of a point in an orbit of cardinality 6 or less would be of order $\frac{120}{6} = 20$ or more. However, the only subgroups of S_5 of order 20 or more are S_5, A_5, S_4, and the Frobenius group $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, none of which has a faithful two-dimensional complex representation (on the tangent space to any point in the orbit). This yields the contradiction we sought.

Remark 3.2. The nonexistence of faithful two-dimensional complex representations of S_5, A_5, S_4, and $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ is a known fact in the representation theory of finite groups. For the reader merely looking for a reference, we suggest [Dor, Sec. 26], where a complete proof is given for $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$. Alternatively, character theory can be used to find, in an elementary way, all possible complex representations of the groups in question (see e.g. [FH, Lect. 2]). With the lists of all possible representations at hand, it is easy to check that there are no faithful two-dimensional ones among them.

3.2. One-dimensional Multiplier Ideal Sheaves

Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an $\mathrm{Aut}(X)$-invariant ideal sheaf satisfying Property (Van). Let $V = V(\mathcal{I})$, and let $\dim_x V = 1$ for all $x \in V$. We assume that the scheme defined by $\mathcal{I} \subseteq \mathcal{O}_X$ has no embedded points; that is, $\mathcal{I} = \mathcal{O}_X(-D)$ for some effective divisor D. Then there exist an effective divisor D' on $\mathbb{P}^2$ and a sequence $m_1, \dots, m_4 \in \mathbb{Z}$ such that

$$\mathcal{O}_X(-D) = \pi^*\mathcal{O}_{\mathbb{P}^2}(-D') \otimes \mathcal{O}_X\left(\sum_{i=1}^{4} m_i E_i\right).$$

Note that the support of D' is $\pi(V)$.

It follows from Property (Van) and [N2, Thm. 2.1] or [L] that

$$R^i\pi_*(\mathcal{I}(-K_X)) = 0 \quad \text{for } i > 0$$

and

$$H^i(\mathbb{P}^2, \pi_*(\mathcal{I}(-K_X))) = 0 \quad \text{for } i > 0.$$

In particular,

$$H^2(\mathbb{P}^2, \pi_*(\mathcal{I}(-K_X)) = 0.$$

By the standard projection formula [F, p. 281] applied twice, for some $k_1, \dots, k_4 \in \mathbb{N}$ we have

$$\begin{aligned}
\pi_*(\mathcal{I}(-K_X)) &= \pi_*\left(\pi^*(-K_{\mathbb{P}^2}) \otimes \mathcal{O}_X\left(-\sum_{i=1}^{4} E_i\right) \otimes \mathcal{I}\right) \\
&= \mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}) \otimes \pi_*\left(\pi^*\mathcal{O}_{\mathbb{P}^2}(-D') \otimes \mathcal{O}_X\left(\sum_{i=1}^{4}(m_i - 1)E_i\right)\right) \\
&= \mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(-D') \otimes \mathfrak{m}_{P_1}^{k_1} \otimes \cdots \otimes \mathfrak{m}_{P_4}^{k_4}.
\end{aligned}$$

Here we have used that, for all $k \in \mathbb{N}$ and $i = 1, \ldots, 4$, $\pi_*(\mathcal{O}_X(kE_i)) = \mathcal{O}_{\mathbb{P}^2}$ and $\pi_*(\mathcal{O}_X(-kE_i)) = \mathfrak{m}_{P_i}^k$.

Let $\mathcal{O}_{\mathbb{P}^2}(-D') \cong \mathcal{O}_{\mathbb{P}^2}(-d)$ with $d \geq 1$. Then, by Serre duality,

$$\begin{aligned}
0 &= H^2(\mathbb{P}^2, \pi_*(\mathcal{I}(-K_X)))\\
&= H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(-D') \otimes \mathfrak{m}_{P_1}^{k_1} \otimes \cdots \otimes \mathfrak{m}_{P_4}^{k_4})\\
&= H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(-D'))\\
&= H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(-d))\\
&= H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2K_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(d))\\
&= H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-6)).
\end{aligned}$$

Note that $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-6)) = 0$ if and only if $d \leq 5$. From this information, we will now extract information on V and show that no such V can exist. First we record two lemmas.

LEMMA 3.3. $\mathrm{Aut}(X)$ *acts effectively on any* $\mathrm{Aut}(X)$*-invariant irreducible curve.*

Proof. For any given element of $\mathrm{Aut}(X)$, it is easy to list the irreducible curves that are left pointwise fixed by the given element (if any exist). However, none of these curves are $\mathrm{Aut}(X)$-invariant. □

LEMMA 3.4 [H, Exer. 20.18]. *The number of singular points of an irreducible plane curve of degree d is no more than its arithmetic genus, which is* $g_a = \frac{1}{2}(d-1)(d-2)$.

PROPOSITION 3.5. *There is no* $\mathrm{Aut}(X)$*-invariant ideal sheaf* $\mathcal{I} \subseteq \mathcal{O}_X$ *satisfying Property (Van) with* $\dim_x V(\mathcal{I}) = 1$ *for all* $x \in V(\mathcal{I})$ *and such that the scheme defined by* $\mathcal{I} \subseteq \mathcal{O}_X$ *has no embedded points.*

Proof. We assume that $\mathcal{I}$ exists and derive a contradiction. We can assume without loss of generality (w.l.o.g.) that no (-1)-curve is contained in $V = V(\mathcal{I})$, because otherwise the $\mathrm{Aut}(X)$-invariance would imply that all ten (-1)-curves are contained in V and that $\pi(V)$ has at least six irreducible one-dimensional components, in violation of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-6)) = 0$.

All elements f of $\mathrm{Aut}(X)$ are induced by an automorphism or birational map of $\mathbb{P}^2$ (again denoted by f). For an irreducible curve C not contained in the exceptional set of π we thus have $\pi(f(C)) = f(\pi(C))$, where $f(\pi(C))$ is defined to be the closure of $f(\pi(C) \cap \mathrm{Dom}(f))$.

First, let us assume that V is irreducible. If $d = 1, 2$, then $V = \mathbb{P}^1$. This is a contradiction, because $S_5 = \mathrm{Aut}(X)$ acts effectively on V by Lemma 3.3 but $S_5 \not\subseteq \mathrm{PGL}(2, \mathbb{C})$ (see [GB, Chap. 2]).

If $d = 3$, then $g_a = 1$. Since $\mathrm{Aut}(X)$ acts on V with orbits of length at least 8, V is smooth by Lemma 3.4. We saw in Lemma 3.3 that S_5 acts effectively on all S_5-invariant curves. However, it is not a subgroup of the automorphism group of any elliptic curve, because it is well known (see [Mi, p. 64]) that the possible automorphism groups of an elliptic curve are

$$\mathbb{Z}_2 \ltimes \mathbb{C}/\Gamma, \quad \mathbb{Z}_4 \ltimes \mathbb{C}/\Gamma, \quad \mathbb{Z}_6 \ltimes \mathbb{C}/\Gamma, \tag{6}$$

none of which can contain S_5. We have obtained a contradiction.

If $d = 4$ then $g_a = 3$. Again, V is smooth. We will obtain a contradiction by analyzing the Riemann–Hurwitz formula of the cyclic branched covering $V \to V/S_5$. Since the only nontrivial cyclic subgroups of S_5 are $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$, the cardinality of a given fiber of the covering is in the set $\{120, 60, 40, 30, 24\}$. Thus, the Riemann–Hurwitz formula reads

$$\begin{aligned} 2g_a - 2 = 4 = 120(2g_{V/S_5} - 2) + 60j_1(2-1) + 40j_2(3-1) \\ + 30j_3(4-1) + 24j_4(5-1) \end{aligned}$$

for some $j_1, j_2, j_3, j_4 \in \mathbb{N}$. Obviously, g_{V/S_5} must be zero, so the formula can be simplified to

$$122 = 30j_1 + 40j_2 + 45j_3 + 48j_4.$$

It is easy to check that this relation cannot be satisfied by any combination of non-negative integers $j_1, j_2, j_3, j_4 \in \mathbb{N}$, so we have reached a contradiction.

If $d = 5$ then $g_a = 6$. Again V is smooth, and the Riemann–Hurwitz relation becomes

$$125 = 30j_1 + 40j_2 + 45j_3 + 48j_4.$$

This relation is satisfied precisely for the values $j_1 = j_4 = 0$, $j_2 = 2$, and $j_3 = 1$. We will derive a contradiction from the fact $j_2 = 2$, which means that there are two branch points in $V/S_5 = \mathbb{P}^1$ over each of which are 40 points that each have a stabilizer group isomorphic to $\mathbb{Z}_3$.

Let g be the element of $\mathrm{Aut}(X)$ induced by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

On $\mathbb{P}^2$, this matrix has three fixed points:

$$P_4 = [1,1,1], \quad Q_1 := [1, e^{2\pi i/3}, e^{-2\pi i/3}], \quad Q_2 := [1, e^{4\pi i/3}, e^{-4\pi i/3}].$$

Accordingly, g has four fixed points on X: the two above Q_1, Q_2 and two above P_4. The latter correspond to the intersection of the lifts of the two lines $\overline{P_4 Q_1}$ and $\overline{P_4 Q_2}$ with the exceptional curve E_4 over P_4.

It is trivial to check that g generates a cyclic subgroup $\mathbb{Z}_3$ in $\mathrm{Aut}(X) = S_5$. According to the Sylow theorems, there are ten (see [W]) such subgroups in S_5, all of which are conjugate to each other. Because they are all conjugate, the union of all their respective fixed points cannot consist of more than $10 \cdot 4 = 40$ points. However, $j_2 = 2$ in the Riemann–Hurwitz relation requires that there be 80 such points—a contradiction.

Next we treat the case where the number of irreducible components of V is 2. For $d = 2$, the two components of $\pi(V)$ are lines. If the two lines intersected outside of $\{P_1, P_2, P_3, P_4\}$, then the components of V would intersect in precisely one point as well. Yet this is impossible because the minimum orbit length of the action of $\mathrm{Aut}(X)$ is 8. Therefore, the two lines must each go through the same P_i. It is now obvious that they are not invariant under $S_4 \subset S_5$.

For $d = 3$, the components of $\pi(V)$ are a line and a conic. They must meet in one (with multiplicity 2) or two of the $\{P_1, P_2, P_3, P_4\}$. We again obtain a contradiction.

For $d = 4$, there are two cases. First, the components of V could be a rational curve and an elliptic curve. They each must be invariant under S_5, in which case we obtained a contradiction earlier. Second, the components of $\pi(V)$ could be two conics. Since the length of any orbit of the action of S_5 is at least 8, these conics must intersect precisely in the points $P_1 = [1,0,0]$, $P_2 = [0,1,0]$, $P_3 = [0,0,1]$, and $P_4 = [1,1,1]$ so that they become separated under the blowing up. Note that any conic through these four points is of the form

$$a_0X_1X_2 + a_1X_0X_2 + (-a_0 - a_1)X_0X_1 = 0,$$

yet any such conic is mapped to a line under Cr_4, a contradiction.

For $d = 5$, the components of V are a rational curve and an elliptic curve or a rational curve and a curve of genus 3. From the rational curve, which is preserved, we get a contradiction.

Now we treat the case of three irreducible components. For $d = 3$, the components of $\pi(V)$ are lines that meet in at most three of the points P_1, P_2, P_3, P_4. Thus they cannot be invariant under the S_4 action.

For $d = 4$, the three components of $\pi(V)$ are two lines and one conic. All points of intersection must be contained in $\{P_1, P_2, P_3, P_4\}$, but the multiplicity will not be the same at all P_i, $i = 1, \ldots, 4$. This is again impossible by the S_4-symmetry.

For $d = 5$, $\pi(V)$ consists of either two lines and an elliptic curve or one line and two conics. In either case, we can argue as before to obtain a contradiction.

Now for the case of four irreducible components. If $d = 4$, then the components of $\pi(V)$ are lines whose six (or fewer) points of intersection must be $\{P_1, P_2, P_3, P_4\}$ (counted with multiplicity). However, the multiplicity cannot be distributed symmetrically, and we derive a contradiction from the action of S_4.

If $d = 5$ then the components of $\pi(V)$ are three lines and a conic. The eight points of intersection (counted with multiplicity) must either be contained entirely in $\{P_1, P_2, P_3, P_4\}$ or be disjoint from it. Containment is impossible because of the S_4-symmetry. Now, again by the S_4-symmetry, all three lines must be disjoint from $\{P_1, P_2, P_3, P_4\}$. As a consequence, the equations of the three lines are of the form

$$a_0X_0 + a_1X_1 + a_2X_2 = 0$$

with $a_0, a_1, a_2 \neq 0$. Under Cr_4 such lines are mapped to conics of the form

$$a_0X_1X_2 + a_1X_0X_2 + a_2X_0X_1 = 0,$$

which is a contradiction.

Finally, we treat the case of five components. The only possibility is $d = 5$ and $\pi(V)$ consisting of five lines. Since lines that are disjoint from $\{P_1, P_2, P_3, P_4\}$ are mapped to conics by Cr_4, all five lines must have nonempty intersection with $\{P_1, P_2, P_3, P_4\}$. Clearly, one point must be contained in two lines. By the S_4-symmetry, all points must be contained in two lines. Arguing again from the minimum orbit length of 8, we see that it is impossible for six or fewer points of intersection to be outside $\{P_1, P_2, P_3, P_4\}$, so all ten must be in this set. However, it is impossible to distribute the ten points onto the four points with equal multiplicity. Therefore, the S_4-action gives our final contradiction. $\square$

Propositions 3.1 and 3.5 prove Theorem 1.7(i) for pure-dimensional $V(\mathcal{I})$ in the absence of embedded points. It is clear that the argument used to handle multiplier ideal sheaves with zero-dimensional zero set goes through when one-dimensional components (with or without embedded points) are present. So in order to prove Theorem 1.7(i) completely, it remains only to indicate how to handle a purely one-dimensional zero set that has embedded points as a scheme.

Toward this end, let $\mathcal{I} = \bigcap_{j\in J} \mathfrak{p}_j$ be a primary decomposition of $\mathcal{I}$. Let $\tilde{\mathcal{I}} = \bigcap_{j\in \tilde{J}} \mathfrak{p}_j$, where $\tilde{J} \subseteq J$ is such that there are no embedded points, but set-theoretically $V(\mathcal{I}) = V(\tilde{\mathcal{I}})$. Then there is a short exact sequence

$$0 \to \mathcal{I}(-K_X) \to \tilde{\mathcal{I}}(-K_X) \to (\tilde{\mathcal{I}}/\mathcal{I})(-K_X) \to 0.$$

Since $H^1(X, \mathcal{I}(-K_X)) = 0$, the map

$$H^0(X, \tilde{\mathcal{I}}(-K_X)) \to H^0(X, (\tilde{\mathcal{I}}/\mathcal{I})(-K_X))$$

is surjective. Clearly, $\dim H^0(X, (\tilde{\mathcal{I}}/\mathcal{I})(-K_X))$ is at least as large as the cardinality of the locus of embedded points, and

$$\dim H^0(X, \tilde{\mathcal{I}}(-K_X)) \le \dim H^0(X, \mathcal{O}_X(-K_X)) = 6.$$

Thus, the cardinality of the locus of embedded points is at most 6. Since this locus is preserved by $\mathrm{Aut}(X)$, we obtain a contradiction in exactly the same way as we did in the proof of Proposition 3.1.

4. Five Points Blown Up

Let X be a del Pezzo surface obtained by blowing up five points. We can find an automorphism of $\mathbb{P}^2$ that takes the five points to $P_1 = [1,0,0]$, $P_2 = [0,1,0]$, $P_3 = [0,0,1]$, $P_4 = [1,1,1]$, and $P_5 = [a,b,c]$ with $(a,b,c) \in (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)\setminus \{(1,1,1)\}$. (The reason for $a,b,c \neq 0$ is that no three of these points lie on a line.)

The structure of $\mathrm{Aut}(X)$ is described in [Ho1], for example (see also [Wi]). It turns out that this structure is always of the form

$$\mathrm{Aut}(X) = \mathbb{Z}_2^4 \rtimes G_{P_5},$$

where G_{P_5} is a subgroup of S_5 depending on the point P_5. The possibilities for G_{P_5} are: (i) {id}; (ii) $\mathbb{Z}_2$; (iii) $\mathbb{Z}_4$; (iv) $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$; and (v) $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$. The elements of G_{P_5} are lifts of those elements of $\mathrm{PGL}(3,\mathbb{C})$ that map the set $\{P_1, P_2, P_3, P_4, P_5\}$ to itself. For a generic point P_5, we have $G_{P_5} = \{\mathrm{id}\}$. More precisely, we have the following statement.

Proposition 4.1. *One has $G_{P_5} \neq \{\mathrm{id}\}$ if and only if $P_5 = [1, \xi, 1+\xi]$ with $\xi \in \mathbb{C}\setminus\{0,1,-1\}$. Moreover, $G_{P_5} = \mathbb{Z}_2$ holds precisely when $\xi^2 + 1 \neq 0$ and $\xi^2 \pm \xi \pm 1 \neq 0$.*

When $G_{P_5} \neq \{\mathrm{id}\}$, [Bl, Prop. 8.1.11] gives explicitly the elements of $G_{P_5} \subset \mathrm{PGL}(3,\mathbb{C})$ and their action on the set $\{P_1, P_2, P_3, P_4, P_5\}$. Based on Proposition 4.1, it is clear that there exists one element of $\mathrm{PGL}(3,\mathbb{C})$ that is contained in G_{P_5} whenever it is not the trivial group—namely,

$$[X_0, X_1, X_2] \mapsto [X_2 - X_1, X_2 - X_0, X_2].$$

Next, we take a closer look at the elements of $\mathbb{Z}_2^4 \subseteq \mathrm{Aut}(X)$. The following two birational involutions of $\mathbb{P}^2$ lift to elements of $\mathrm{Aut}(X)$. We define Cr_{45} to be

$$\mathrm{Cr}_{45}([X_0, X_1, X_2]) = [aX_1X_2, bX_0X_2, cX_0X_1].$$

This is a quadratic Cremona transformation that exchanges P_4 and P_5 and has $\{P_1, P_2, P_3\}$ as indeterminacy locus. We abuse notation and use Cr_{45} to signify both the birational involution and the corresponding element of $\mathrm{Aut}(X)$.

Moreover, we let σ_1 be the following cubic birational involution of $\mathbb{P}^2$:

$$\begin{aligned}\sigma_1([X_0, X_1, X_2]) = [&-aX_1X_2((c-b)X_0 + (a-c)X_1 + (b-a)X_2),\\ &X_1(a(c-b)X_1X_2 + b(a-c)X_0X_2 + c(b-a)X_0X_1),\\ &X_2(a(c-b)X_1X_2 + b(a-c)X_0X_2 + c(b-a)X_0X_1)].\end{aligned}$$

We use σ_1 to denote both the birational involution and the corresponding element of $\mathrm{Aut}(X)$. It is easy to see that the strict transform of the cubic curve C given by

$$\begin{aligned}b(a-c)X_0^2X_2 + c(b-a)X_0^2X_1 + a(a-c)X_1^2X_2&\\ + a(b-a)X_1X_2^2 + 2a(c-b)X_0X_1X_2 &= 0\end{aligned}$$

is precisely the set of points fixed pointwise by the lift of σ_1. In addition, the following lemma tells us that C is invariant under every element of $\mathbb{Z}_2^4$.

LEMMA 4.2. *Let A be an abelian group acting on a set M. For $g \in A$, let*

$$M^g = \{x \in M : gx = x\}.$$

Then M^g is invariant under every element of A.

Proof. For $x \in M^g$, for any $h \in A$ we have

$$g(hx) = h(gx) = hx;$$

that is, $hx \in M^g$ also. □

Closer inspection (cf. [K; Ho1; Wi]) shows that $\mathbb{Z}_2^4 \subseteq \mathrm{Aut}(X)$ contains the lifts of ten quadratic Cremona involutions Cr_{ij} $(1 \leq i < j \leq 5)$ that exchange P_i and P_j and have the remaining three points as indeterminacy locus. Again, we abuse notation and denote the maps before and after the lift by the same symbols Cr_{ij}.

Moreover, $\mathbb{Z}_2^4 \subseteq \mathrm{Aut}(X)$ contains the lifts of five cubic involutions σ_i $(1 \leq i \leq 5)$. By comparing the respective action on the set of the sixteen (-1)-curves (which determines any automorphism uniquely; see [K] or [Ho1]), it is easily verified that

$$\mathrm{Cr}_{ij} \circ \mathrm{Cr}_{kl} = \mathrm{Cr}_{kl} \circ \mathrm{Cr}_{ij},$$
$$\mathrm{Cr}_{ij} \circ \mathrm{Cr}_{jk} = \mathrm{Cr}_{ik}.$$

Moreover, for i, j, k, l, m all distinct, we have

$$\sigma_m = \mathrm{Cr}_{ij} \circ \mathrm{Cr}_{kl}.$$

In particular,

$$\sigma_j = \mathrm{Cr}_{1j} \circ \sigma_1 \quad (2 \leq j \leq 5). \tag{7}$$

The Cr_{ij} have precisely four fixed points each (both before and after the lifting). Thus, the only $\mathbb{Z}_2^4$-invariant curves on which $\mathbb{Z}_2^4$ does not act effectively are the lifts of the curves C pertaining to the cubic involutions. For $a, b, c \neq 0$, however, the curves C are smooth elliptic curves (easy exercise). Therefore, the group $\mathbb{Z}_2^4 \subset \mathrm{Aut}(X)$ acts effectively on any $\mathrm{Aut}(X)$-invariant irreducible curve that is not an elliptic curve.

In Sections 4.1 and 4.2 we will prove Theorem 1.7(ii) by means of Criterion 1.6.

4.1. Zero-dimensional Multiplier Ideal Sheaves

Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an $\mathrm{Aut}(X)$-invariant ideal sheaf satisfying Property (Van). Let $V = V(\mathcal{I})$ and let $\dim V = 0$. We saw in the previous section that the map

$$H^0(X, \mathcal{O}_X(-K_X)) \to H^0(V, \mathcal{O}_V(-K_X))$$

is surjective. Since $\dim H^0(X, \mathcal{O}_X(-K_X)) = 10 - 5 = 5$, it follows that V consists of at most five points.

Proposition 4.3. *If $G_{P_5} = \mathbb{Z}_4, \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, or $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$, then there is no* $\mathrm{Aut}(X)$-*invariant ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ satisfying Property (Van) with* $\dim V(\mathcal{I}) = 0$.

Proof. We assume that $\mathcal{I}$ exists and then derive a contradiction. Since $V = V(\mathcal{I})$ has at most five points, the statement follows from the claim that all orbits of $\mathrm{Aut}(X)$ have cardinality at least 8. In the sequel, we prove this claim.

We first consider the action of $\mathbb{Z}_2^4 \subset \mathrm{Aut}(X)$ on X. There are no orbits of length less than 4, which can be shown as follows. If we assume that there is such an orbit, then the stabilizer subgroup $(\mathbb{Z}_2^4)_P$ of a point P in the orbit would be a subgroup of order 8 or 16 in $\mathbb{Z}_2^4$. The only such groups are $\mathbb{Z}_2^3$ and $\mathbb{Z}_2^4$, but by Schur's lemma these groups do not permit faithful two-dimensional complex representations (on the tangent space to any point in the orbit)—a contradiction.

Now let $P \in X$ be such that the cardinality of the orbit of P under $\mathbb{Z}_2^4$ is $\#\mathbb{Z}_2^4 P = 4$. The list in [Bl, Prop. 8.1.11] gives the action of G_{P_5} on the elements of the set $\{P_1, P_2, P_3, P_4, P_5\}$. This data determines the action on the sixteen (-1)-curves and therefore determines the automorphism uniquely.

When $G_{P_5} = \mathbb{Z}_4, \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, or $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$, the claim is implied by $\#(\mathbb{Z}_2^4 \rtimes G_{P_5})_P \leq 8$. A case-by-case analysis shows that if $\#(\mathbb{Z}_2^4 \rtimes G_{P_5})_P > 8$ then

$$\mathbb{Z}_2^3 \subseteq (\mathbb{Z}_2^4 \rtimes G_{P_5})_P \cap \mathbb{Z}_2^4,$$

which yields a contradiction by Schur's lemma. Note that the strict inequality $\#(\mathbb{Z}_2^4 \rtimes G_{P_5})_P > 8$ is necessary, because otherwise $(\mathbb{Z}_2^4 \rtimes G_{P_5})_P = \mathbb{Z}_2^2 \rtimes \mathbb{Z}_2 \ncong \mathbb{Z}_2^3$ might (and does) occur. □

4.2. One-dimensional Multiplier Ideal Sheaves

In this section, we prove the following result.

PROPOSITION 4.4. *If $G_{P_5} = \mathbb{Z}_4, \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, or $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$, then there is no* $\mathrm{Aut}(X)$-*invariant ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ satisfying Property (Van) with* $\dim_x V(\mathcal{I}) = 1$ *for all $x \in V(\mathcal{I})$ and such that the scheme defined by $\mathcal{I} \subseteq \mathcal{O}_X$ has no embedded points.*

Proof. We assume that $\mathcal{I}$ exists and derive a contradiction. We can again assume w.l.o.g. that no (-1)-curve is contained in $V = V(\mathcal{I})$.

First, let us assume that V is irreducible. If $d = 1$ or 2, then $V = \mathbb{P}^1$. This is a contradiction, because $\mathbb{Z}_2^4 \subset \mathrm{Aut}(X)$ acts effectively on V yet $\mathbb{Z}_2^4 \not\subseteq \mathrm{PGL}(2, \mathbb{C})$ (see [GB, Chap. 2]).

If $d = 3$, then $g_a = 1$. Since $\mathrm{Aut}(X)$ acts on V with orbits of length at least 8, it follows from Lemma 3.4 that V is smooth. We know that $\mathbb{Z}_2^3 \rtimes G_{P_5}$ acts effectively on V. However, it is not a subgroup of any of the groups listed in (6). We have obtained a contradiction.

If $d = 4$ then $g_a = 3$. Again, V is smooth. Since $\mathbb{Z}_2^4 \subset \mathrm{Aut}(X)$ acts effectively on V, we can get a contradiction by analyzing the Riemann–Hurwitz formula for the cyclic branched covering $V \to V/\mathbb{Z}_2^4$. Clearly, all nontrivial cyclic subgroups of $\mathbb{Z}_2^4$ are isomorphic to $\mathbb{Z}_2$. Therefore, all fibers of the branched covering have cardinality either 8 or 16, and the Riemann–Hurwitz formula reads

$$2g_a - 2 = 4 = 16(2g_{V/\mathbb{Z}_2^4} - 2) + 8j$$

for some $j \in \mathbb{N}$. Since 4 is not divisible by 8, we have obtained a contradiction.

If $d = 5$ then $g_a = 6$. Once again, V is smooth. In the preceding Riemann–Hurwitz formula, the left-hand side now becomes $2g_a - 2 = 10$, which is also not divisible by 8. Contradiction.

Next, we treat the case where the number of irreducible components is 2. For $d = 2, 3$, the two components are smooth rational curves. If each component (call them V_1, V_2) is invariant under $\mathbb{Z}_2^4 \subset \mathrm{Aut}(X)$, then we have a contradiction as before. So let us assume that there is a $g \in \mathbb{Z}_2^4 \subset \mathrm{Aut}(X)$ such that $g(V_1) = V_2$. Let

$$G_1 = \{g \in \mathbb{Z}_2^4 : g(V_1) = V_1\}.$$

Then

$$\mathbb{Z}_2^4 = G_1 \uplus gG_1.$$

Therefore, the index of G_1 in $\mathbb{Z}_2^4$ is 2 and so $G_1 = \mathbb{Z}_2^3$, which yields a contradiction because $\mathbb{Z}_2^3 \not\subseteq \mathrm{PGL}(2, \mathbb{C})$.

If $d = 4$, then V may be the disjoint union of a smooth elliptic curve and a rational curve or the disjoint union of two smooth rational curves. In either case, this is impossible. When $d = 5$, V may be the disjoint union of a smooth elliptic curve and a smooth rational curve or the disjoint union of a smooth curve of genus 3 and a smooth rational curve. In any case, this is impossible.

Next, let us assume that there are three irreducible components V_1, V_2, V_3. If $d = 3, 4$, then V is the union of three smooth rational curves. If the action of $\mathbb{Z}_2^4 \subset \mathrm{Aut}(X)$ on the three components leaves one component invariant, then $\mathbb{Z}_2^4 \subset \mathrm{PGL}(2, \mathbb{C})$, which is a contradiction.

If no component is invariant under $\mathbb{Z}_2^4 \subset \mathrm{Aut}(X)$, then there exist $g_2, g_3 \in \mathbb{Z}_2^4$ such that $g_2(V_1) = V_2$ and $g_3(V_1) = V_3$. However, a brief computation reveals that $(g_2 \cdot g_3)^2(V_1) \in V_2$, which means that the order of $g_2 \cdot g_3$ cannot be 2. However, all nontrivial elements of $\mathbb{Z}_2^4$ have order 2—a contradiction.

When $d = 5$, one of the components might be a smooth elliptic curve, but in any case one obtains a contradiction as before.

Finally, in the cases of four and five irreducible components, all components are rational curves. If there are two components whose union is $\mathbb{Z}_2^4$-invariant, then we obtain a contradiction as in the case of two irreducible components. If not, then there exist $g_2, g_3 \in \mathbb{Z}_2^4$ and components V_1, V_2, V_3 such that $g_2(V_1) = V_2$ and $g_3(V_1) = V_3$. Again, we obtain a contradiction. □

For the reasons given at the end of Section 3, the Propositions 4.3 and 4.4 together yield Theorem 1.7(ii).

5. Comments on the Cases of $r \neq 4, 5$

5.1. One or Two Points Blown Up. For $\mathbb{P}^2$ blown up at one or two points, one can show that the so-called Calabi–Futaki invariant does not vanish (see e.g. [T3, Exam. 3.10, 3.11] for details). The nonvanishing of this invariant is an obstruction to the existence of a Kähler–Einstein metric. If $g_{i\bar{j}}(t)$ $(t \to \infty)$ were to converge, then it would necessarily converge against a Kähler–Einstein metric. So the statement of Theorem 1.1(ii) cannot hold on $\mathbb{P}^2$ blown up at one or two points.

5.2. Three Points Blown Up. In this paper we make no statement about this case, because both zero- and one-dimensional zero sets of multiplier ideal sheaves cannot be ruled out using Property (Van).

In the zero-dimensional case, the problem is that the surjectivity of the map

$$H^0(X, \mathcal{O}_X(-K_X)) \to H^0(V, \mathcal{O}_V(-K_X))$$

limits the cardinality of the zero set to $10 - 3 = 7$. However, the six points of intersection in the "hexagon" formed by the six (-1)-curves are clearly $\mathrm{Aut}(X)$-invariant, and we are unable to rule out that they are the $\mathrm{Aut}(X)$-invariant zero set of a multiplier ideal sheaf satisfying (5).

Similarly, the union of the six (-1)-curves is clearly $\mathrm{Aut}(X)$-invariant, and it is not possible to rule out that it forms the zero set of a multiplier ideal sheaf based on Property (Van). Clearly, this case merits further investigation.

5.3. SIX OR MORE POINTS BLOWN UP. For a generic del Pezzo surface X of degree 3, $\mathrm{Aut}(X)$ is unfortunately the trivial group (see [K]). There are of course nongeneric X that have extra automorphisms, and a nice list of these can be found in [Do, Table 10.3]. Recall that del Pezzo surfaces of degree 3 are precisely the smooth cubic hypersurfaces of $\mathbb{P}^3$, and perhaps the most important example is the Fermat cubic surface in $\mathbb{P}^3$ given by

$$Z_0^3 + Z_1^3 + Z_2^3 + Z_3^3 = 0.$$

Its automorphism group is $\mathbb{Z}_3^3 \rtimes S_4$, acting in the obvious way. The order of this group is $27 \cdot 24 = 648$. In this case, the same analysis as in Section 4 does yield statement (ii) of Theorem 1.1. The arithmetic argument involving the Riemann–Hurwitz formula even becomes unnecessary because, according to the well-known Hurwitz bound, the automorphism group of a Riemann surface of genus $g \geq 2$ has order at most $84(g-1)$, which is less than 648 for $g \leq 8$. We leave the details to the reader.

At the other end of the spectrum, on a del Pezzo surface X of degree 1, the unique base point of the linear system $|-K_X|$ is fixed by all automorphisms. Therefore, unfortunately, $\mathrm{Aut}(X)$ acts with a fixed point regardless of the nature of X, and we are unable to handle this case.

On a del Pezzo surface X of degree 2, the linear system $|-K_X|$ gives a two-sheeted cover of $\mathbb{P}^2$ branched along a smooth curve C of degree 4 in $\mathbb{P}^2$. This cover defines an involutive automorphism of X called the *Geiser involution*. On a generic X, this is the only nontrivial automorphism (i.e., $\mathrm{Aut}(X) = \mathbb{Z}_2$). However, certain nongeneric X do have extra automorphisms. A list of these X and their automorphism groups, together with a lucid exposition of the topic, can be found in [Do, Table 10.4]. We do not go into any details regarding this case.

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