Michigan Math. J. 58 (2009)

L^2-Betti Numbers of Plane Algebraic Curves

Stefan Friedl, Constance Leidy,
& Laurentiu Maxim

1. Introduction

Let X be any topological space and let $\varphi\colon \pi_1(X) \to \Gamma$ be a homomorphism to a group (all groups are assumed to be countable). Then for $p \in \mathbb{N} \cup \{0\}$ we can consider the L^2-Betti number $b_p^{(2)}(X, \varphi) \in [0, \infty]$. We recall the definition and some of the most important properties of L^2-Betti numbers in Section 2.

Let $\mathcal{C} \subset \mathbb{C}^2$ be a reduced plane algebraic curve with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$. We write $X(\mathcal{C}) := \mathbb{C}^2 \setminus \nu\mathcal{C}$ for $\nu\mathcal{C}$ a regular neighborhood of $\mathcal{C}$ inside $\mathbb{C}^2$. We denote the meridians about the nonsingular parts of $\mathcal{C}_1, \dots, \mathcal{C}_r$ by $\mu_1, \dots, \mu_r$. Note that these meridians come with a preferred orientation because the nonsingular parts of the irreducible components $\mathcal{C}_i$ are complex submanifolds of $\mathbb{C}^2$.

It is well known (cf. Theorem 3.1) that $H_1(X(\mathcal{C}); \mathbb{Z})$ is the free abelian group generated by the meridians $\mu_1, \dots, \mu_r$. Throughout this paper we denote by ϕ the map $\pi_1(X(\mathcal{C}); \mathbb{Z}) \to \mathbb{Z}$ given by sending each meridian μ_i to 1. We also refer to ϕ as the total linking homomorphism. We henceforth call a homomorphism $\alpha\colon \pi_1(X(\mathcal{C})) \to \Gamma$ to a group *admissible* if the total linking homomorphism ϕ factors through α.

Our first result is the following.

Theorem 1.1. *Let $\mathcal{C} \subset \mathbb{C}^2$ be a reduced algebraic curve $\mathcal{C}$ whose projective completion intersects the line at infinity transversely. Let $\alpha\colon \pi_1(X(\mathcal{C})) \to \Gamma$ be an admissible homomorphism. Then*

$$b_p^{(2)}(X(\mathcal{C}), \alpha) = \begin{cases} 0 & \textit{for } p \neq 2, \\ \chi(X(\mathcal{C})) & \textit{for } p = 2. \end{cases}$$

In [DaJLe] it was shown that if $\mathcal{A}$ is an affine hyperplane arrangement in $\mathbb{C}^n$ then at most one of the L^2-Betti numbers $b_p^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \mathrm{id})$ is nonzero. Theorem 1.1 can be seen as an analogous statement for the complement of an algebraic curve in $\mathbb{C}^2$ that is in general position at infinity. Note that if Γ is a polytorsion-free abelian (PTFA) group then this theorem, together with Proposition 2.4, recovers [LMa1, Cor. 4.2].

Received December 12, 2007. Revision received June 12, 2008.
S.F. was supported by a CRM-ISM Fellowship and by CIRGET. L.M. was supported by a PSC-CUNY Research Award.

Given an algebraic curve $\mathcal{C}$, we denote by $\widetilde{X(\mathcal{C})}$ the infinite cyclic cover of $X(\mathcal{C})$ corresponding to ϕ. Given an admissible homomorphism $\alpha: \pi_1(X(\mathcal{C})) \to \Gamma$, we let $\tilde{\Gamma} := \operatorname{Im}\{\pi_1(\widetilde{X(\mathcal{C})}) \to \pi_1(X(\mathcal{C})) \xrightarrow{\alpha} \Gamma\}$ and denote the induced map $\pi_1(\widetilde{X(\mathcal{C})}) \to \tilde{\Gamma}$ by $\tilde{\alpha}$. We will now study the invariant

$$b_1^{(2)}(\widetilde{X(\mathcal{C})}, \tilde{\alpha}: \pi_1(\widetilde{X(\mathcal{C})}) \to \tilde{\Gamma}).$$

The idea of looking for invariants of the fundamental group of the complement that capture information about the topology of the curve goes back to the early work of Zariski, and it was further developed by Libgober by analogy with classical knot theory (cf. [Lib1; Lib2; Lib4; Lib6]). In particular, Libgober studied the ordinary one-variable Alexander polynomial corresponding to $X(\mathcal{C})$ whose degree is given by the ordinary Betti number of $\widetilde{X(\mathcal{C})}$ (cf. e.g. [C, p. 368]). In that sense the study of the L^2-Betti numbers of $\widetilde{X(\mathcal{C})}$ can be seen as a noncommutative generalization of the approach of Libgober.

Following work of Cochran and Harvey, the second and third author consider in [LMa1] the homomorphism

$$\pi_n: \pi_1(X(\mathcal{C})) \to \pi_1(X(\mathcal{C}))/\pi_1(X(\mathcal{C}))_r^{(n+1)} =: \Gamma_n;$$

here, given a group G, we denote by $G_r^{(n)}$ the nth term in the rational derived series (cf. [Ha1]). The group Γ_n is a PTFA group, and the authors define an invariant $\delta_n(\mathcal{C})$ as the dimension of the first homology of $\widetilde{X(\mathcal{C})}$ with coefficients in the skew field associated to $\tilde{\Gamma}_n$. Some of these invariants are computed in [LMa1; LMa2]. The main result of [LMa1] gives upper bounds on $\delta_n(\mathcal{C})$ in terms of information coming from the singularities of $\mathcal{C}$.

We will see in Theorem 2.5 that

$$\delta_n(\mathcal{C}) = b_1^{(2)}(\widetilde{X(\mathcal{C})}, \tilde{\pi}_n: \pi_1(\widetilde{X(\mathcal{C})}) \to \tilde{\Gamma}_n).$$

The following theorem can therefore be viewed as a generalization of [LMa1, Thm. 4.1]. Note that for the invariants $\delta_n(\mathcal{C})$ this new result gives a slightly better bound.

Theorem 1.2. *Let $\mathcal{C} \subset \mathbb{C}^2$ be a reduced plane algebraic curve of degree d whose projective completion intersects the line at infinity transversely. Denote the set of singular points by $P_1, \dots, P_s$ and, for a singular point P_i, denote by $\mu(\mathcal{C}, P_i)$ the associated Milnor number of the singularity germ at P_i. Let $\alpha: \pi_1(X(\mathcal{C})) \to \Gamma$ be an admissible homomorphism. Then*

$$b_1^{(2)}(\widetilde{X(\mathcal{C})}, \tilde{\alpha}: \pi_1(\widetilde{X(\mathcal{C})}) \to \tilde{\Gamma}) \leq \sum_{i=1}^{s} (\mu(\mathcal{C}, P_i) + n_i - 1) + 2g + d,$$

where n_i denotes the number of branches through P_i and g is the genus of the normalization of the projective completion of $\mathcal{C}$.

This theorem shows that the topology of the singularities imposes restrictions on the L^2-Betti numbers of the curve complement. In this sense Theorem 1.2 is in the same vein as the results of Libgober [Lib1] and Cogolludo and Florens [CoF], but see also [DiMa; Lib5; Ma] for similar results in the higher-dimensional case.

2. L^2-Betti Numbers

2.1. The von Neumann Algebra and Its Localizations

Let Γ be a countable group. Define $l^2(\Gamma) := \{f : \Gamma \to \mathbb{C} \mid \sum_{g\in\Gamma}|f(g)|^2 < \infty\}$; this is a Hilbert space. Then Γ acts on $l^2(\Gamma)$ by right multiplication—that is, $(g \cdot f)(h) = f(hg)$. This defines an injective map $\mathbb{C}[\Gamma] \to \mathcal{B}(l^2(\Gamma))$, where $\mathcal{B}(l^2(\Gamma))$ is the set of bounded operators on $l^2(\Gamma)$. We henceforth view $\mathbb{C}[\Gamma]$ as a subset of $\mathcal{B}(l^2(\Gamma))$.

Now define the *von Neumann algebra* $\mathcal{N}(\Gamma)$ to be the closure of $\mathbb{C}[\Gamma] \subset \mathcal{B}(l^2(\Gamma))$ with respect to pointwise convergence in $\mathcal{B}(l^2(\Gamma))$. Note that any $\mathcal{N}(\Gamma)$-module M has a dimension $\dim_{\mathcal{N}(\Gamma)}(M) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. We refer to [Lü, Def. 6.20] for details.

2.2. The Definition of L^2-Betti Numbers

Let X be a topological space (not necessarily compact) and let $\varphi : \pi_1(X) \to \Gamma$ be a homomorphism to a group. Denote the covering of X corresponding to φ by $\tilde{X}$. Then we can study the $\mathcal{N}(\Gamma)$-chain complex

$$C_*^{\text{sing}}(\tilde{X}) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma),$$

where $C_*^{\text{sing}}(\tilde{X})$ is the singular chain complex of $\tilde{X}$ with right Γ-action given by covering translation. Furthermore, Γ acts canonically on $\mathcal{N}(\Gamma)$ on the left. The pth L^2-Betti number is now defined as

$$b_p^{(2)}(X, \varphi) := \dim_{\mathcal{N}(\Gamma)}(H_p(C_*^{\text{sing}}(\tilde{X}) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma))) \in [0, \infty].$$

See [Lü, Def. 6.50] for more details.

In the following lemma we summarize some of the properties of L^2-Betti numbers. We refer to [Lü, Thm. 6.54, Lemma 6.53, Thm. 1.35] for the proofs.

LEMMA 2.1. *Let X be a topological space and let $\varphi : \pi_1(X) \to \Gamma$ be a homomorphism to a group.*

(1) *$b_p^{(2)}(X, \varphi)$ is a homotopy invariant of the pair (X, φ).*

(2) *$b_0^{(2)}(X, \varphi) = 0$ if $\operatorname{Im}(\varphi)$ is infinite and $b_0^{(2)}(X, \varphi) = 1/|\operatorname{Im}(\varphi)|$ if $\operatorname{Im}(\varphi)$ is finite.*

(3) *If X is a finite CW-complex, then*

$$\sum_p (-1)^p b_p^{(2)}(X, \varphi) = \chi(X),$$

where $\chi(X)$ denotes the Euler characteristic of X.

(4) *If $\operatorname{Im}(\varphi) \subset \tilde{\Gamma} \subset \Gamma$, then $b_p^{(2)}(X, \varphi : \pi_1(X) \to \tilde{\Gamma}) = b_p^{(2)}(X, \varphi : \pi_1(X) \to \Gamma)$.*

We will also make use of the following lemma.

LEMMA 2.2. *Let $f : Y \to Z$ be a map of topological spaces such that $\pi_1(Y) \to \pi_1(Z)$ is surjective. Assume that we are given a homomorphism $\beta : \pi_1(Z) \to \Gamma$. Then*

$$b_1^{(2)}(Y, \pi_1(Y) \xrightarrow{f_*} \pi_1(Z) \xrightarrow{\beta} \Gamma) \geq b_1^{(2)}(Z, \beta).$$

Proof. We denote the homomorphism $\pi_1(Y) \xrightarrow{f_*} \pi_1(Z) \xrightarrow{\beta} \Gamma$ by β as well. Note that an Eilenberg–Maclane space K for $\pi_1(Z)$ is given by adding handles of degree > 2 to Z. In particular $b_1^{(2)}(Z, \beta) = b_1^{(2)}(K, \beta)$. By the homotopy invariance of the L^2-Betti numbers we know that, for any other Eilenberg–Maclane space, we get the same invariant for $\pi_1(Z)$.

Because $f_* : \pi_1(Y) \to \pi_1(Z)$ is surjective, we can also build an Eilenberg–Maclane space K' for $\pi_1(Z)$ by adding handles of degree ≥ 2 to Y. By the preceding discussion we therefore get

$$b_1^{(2)}(Z, \beta) = b_1^{(2)}(K, \beta) = b_1^{(2)}(K', \beta).$$

It now remains to show that $b_1^{(2)}(Y, \beta) \geq b_1^{(2)}(K', \beta)$. Since K' is given by adding handles of degree ≥ 2 to Y, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
C_2(Y; \mathcal{N}(\Gamma)) & \longrightarrow & C_1(Y; \mathcal{N}(\Gamma)) & \longrightarrow & C_0(Y; \mathcal{N}(\Gamma)) & \longrightarrow & 0 \\
\downarrow & & \downarrow = & & \downarrow = & & \\
P \oplus C_2(Y; \mathcal{N}(\Gamma)) & \longrightarrow & C_1(Y; \mathcal{N}(\Gamma)) & \longrightarrow & C_0(Y; \mathcal{N}(\Gamma)) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
C_2(K'; \mathcal{N}(\Gamma)) & \longrightarrow & C_1(K'; \mathcal{N}(\Gamma)) & \longrightarrow & C_0(K'; \mathcal{N}(\Gamma)) & \longrightarrow & 0,
\end{array}$$

where P is the free $\mathcal{N}(\Gamma)$-module generated by the extra 2-handles of K'. This shows that the map $H_1(Y; \mathcal{N}(\Gamma)) \to H_1(K'; \mathcal{N}(\Gamma))$ is surjective. But then the claim on L^2-Betti numbers follows immediately from [Lü, Thm. 6.7]. □

2.3. The L^2-Betti Numbers and the Cochran–Harvey Invariants

Recall that a group Γ is called *locally indicable* if for every finitely generated nontrivial subgroup $H \subset \Gamma$ there exists an epimorphism $H \to \mathbb{Z}$. We will also need the notion of an *amenable* group. We refer to [Lü, p. 256] for the definition of an amenable group, but note that any solvable group is amenable and that groups containing the free group on two generators are not amenable. In the following we refer to a locally indicable torsion-free amenable group as a LITFA group.

Denote by S the set of nonzero divisors of the ring $\mathcal{N}(\Gamma)$. By [R, Prop. 2.8] (see also [Lü, Thm. 8.22]), the pair $(\mathcal{N}(\Gamma), S)$ satisfies the right Ore condition. We now let $\mathcal{U}(\Gamma) := \mathcal{N}(\Gamma)S^{-1}$; this ring is called the *algebra of operators affiliated to* $\mathcal{N}(\Gamma)$. For any $\mathcal{U}(\Gamma)$-module M we also have a dimension $\dim_{\mathcal{U}(\Gamma)}(M)$. By [Lü, Thm. 8.31],

$$b_p^{(2)}(X, \varphi) = \dim_{\mathcal{U}(\Gamma)}(H_p(C_*^{\text{sing}}(\tilde{X}) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{U}(\Gamma))).$$

We collect some properties of LITFA groups in the following well-known theorem.

THEOREM 2.3. *Let Γ be a LITFA group.*

(1) *All nonzero elements in $\mathbb{Z}[\Gamma]$ are nonzero divisors in $\mathcal{N}(\Gamma)$.*
(2) *$\mathbb{Z}[\Gamma]$ is an Ore domain and embeds in its classical right ring of quotients $\mathbb{K}(\Gamma)$.*
(3) *$\mathbb{K}(\Gamma)$ is flat over $\mathbb{Z}[\Gamma]$.*
(4) *There exists a monomorphism $\mathbb{K}(\Gamma) \to \mathcal{U}(\Gamma)$ that makes the following diagram commute:*

$$\begin{array}{ccc} \mathbb{Z}[\Gamma] & \longrightarrow & \mathbb{K}(\Gamma) \\ & \searrow & \downarrow \\ & & \mathcal{U}(\Gamma). \end{array}$$

Proof. The first claim follows from results of Linnell [Li] and Burns and Hale [BH]. Note that it implies in particular that all nonzero elements in $\mathbb{Z}[\Gamma]$ are nonzero divisors in $\mathbb{Z}[\Gamma]$. The second part now follows from [DLiMSY, Cor. 6.3]. The third part is a well-known property of Ore localizations (cf. e.g. [R, p. 99]). Finally, the last statement follows from the definitions of $\mathbb{K}(\Gamma)$ and $\mathcal{U}(\Gamma)$ as Ore localizations and the fact that $\mathbb{Z}[\Gamma] \setminus \{0\} \subset S$. □

We recall that a group Γ is called polytorsion-free abelian (PTFA) if there exists a normal series

$$1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{n-1} \subset \Gamma_n = \Gamma$$

such that Γ_i/Γ_{i-1} is torsion-free abelian. PTFA groups have played an important role in several recent papers (e.g. [C; COT; Ha1; LMa1]).

It is easy to see that PTFA groups are LITFA. Note that the quotients $\pi/\pi_r^{(n)}$ of a group by terms in the rational derived series are PTFA (cf. [Ha1]). The following proposition relates L^2-Betti numbers to ranks of modules over skew fields. It seems to be well known (see e.g. [Ha2, p. 8]), but for the sake of completeness we quickly outline the proof.

PROPOSITION 2.4. *Let $\varphi\colon \pi_1(X) \to \Gamma$ be a homomorphism to a LITFA group Γ. Then*

$$b_p^{(2)}(X, \varphi) = \dim_{\mathbb{K}(\Gamma)}(H_p(X; \mathbb{K}(\Gamma))).$$

Proof. By Theorem 2.3 we have an inclusion $\mathbb{K}(\Gamma) \to \mathcal{U}(\Gamma)$. Since $\mathbb{K}(\Gamma)$ is a skew field, it follows that any $\mathbb{K}(\Gamma)$-module is free. We deduce that $\mathcal{U}(\Gamma)$ is flat as a $\mathbb{K}(\Gamma)$-module. In particular, if $d = \dim_{\mathbb{K}(\Gamma)}(H_p(X; \mathbb{K}(\Gamma))) < \infty$ then

$$\begin{aligned} \dim_{\mathcal{U}(\Gamma)}(H_p(X; \mathcal{U}(\Gamma))) &= \dim_{\mathcal{U}(\Gamma)}(H_p(X; \mathbb{K}(\Gamma)) \otimes_{\mathbb{K}(\Gamma)} \mathcal{U}(\Gamma)) \\ &= \dim_{\mathcal{U}(\Gamma)}(\mathbb{K}(\Gamma)^d \otimes_{\mathbb{K}(\Gamma)} \mathcal{U}(\Gamma)) \\ &= \dim_{\mathcal{U}(\Gamma)}(\mathcal{U}(\Gamma)^d) = d. \end{aligned}$$

The case of $d = \dim_{\mathbb{K}(\Gamma)}(H_p(X; \mathbb{K}(\Gamma))) = \infty$ follows similarly. □

We now recall the definition of the Cochran–Harvey invariants (which in this context were first studied in [LMa1]). Let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}^2$. Furthermore, let $\alpha\colon \pi_1(X(\mathcal{C})) \to \Gamma$ be an admissible homomorphism to a LITFA group. Recall that "admissible" means that there exists a map $\phi\colon \Gamma \to \mathbb{Z}$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(\mathbb{C}^2 \setminus \mathcal{C}) & \xrightarrow{\quad\alpha\quad} & \Gamma \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & \mathbb{Z}. & \end{array}$$

Also recall that we denote by $\tilde{\Gamma}$ the kernel of $\phi\colon \Gamma \to \mathbb{Z}$ and by $\tilde{\alpha}$ the induced homomorphism $\pi_1(\widetilde{X(\mathcal{C})}) \to \tilde{\Gamma}$.

Now consider the homomorphism $\pi_1(X(\mathcal{C})) \to \pi_1(X(\mathcal{C}))/\pi_1(X(\mathcal{C}))_r^{(n+1)} = \Gamma_n$. It is easy to see that this homomorphism is admissible. As in [LMa1], we now define

$$\delta_n(\mathcal{C}) = \dim_{\mathbb{K}(\tilde{\Gamma}_n)}(H_1(\widetilde{X(\mathcal{C})}; \mathbb{K}(\tilde{\Gamma}_n)).$$

The following theorem, which is an immediate corollary to Proposition 2.4, now shows that the L^2-Betti numbers considered in this paper can be viewed as a generalization of the Cochran–Harvey invariants of plane algebraic curves.

THEOREM 2.5. *Let $\mathcal{C} \subset \mathbb{C}^2$ be an algebraic curve $\mathcal{C}$ and let $\alpha\colon \pi_1(X(\mathcal{C})) \to \Gamma$ be an admissible homomorphism to a LITFA group. Then*

$$\dim_{\mathbb{K}(\tilde{\Gamma})}(H_1(\widetilde{X(\mathcal{C})}; \mathbb{K}(\tilde{\Gamma})) = b_1^{(2)}(\widetilde{X(\mathcal{C})}, \tilde{\alpha}\colon \pi_1(\widetilde{X(\mathcal{C})}) \to \tilde{\Gamma}).$$

3. Proof of Theorem 1.1 and Theorem 1.2

3.1. Plane Algebraic Curves and Their Topology

From now on let $\mathcal{C} \subset \mathbb{C}^2$ be an algebraic curve with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$. Recall that we write $X(\mathcal{C}) = \mathbb{C}^2 \setminus \nu\mathcal{C}$. We now also write $Y(\mathcal{C}) = \partial(\overline{\nu\mathcal{C}}) = \partial(\overline{X(\mathcal{C})})$. Note that $Y(\mathcal{C}) \subset X(\mathcal{C})$. The following theorem summarizes some well-known results on the topology of $X(\mathcal{C})$.

THEOREM 3.1. (1) *$\pi_1(X(\mathcal{C}))$ is normally generated by the meridians about the nonsingular parts of the irreducible components, and $H_1(X(\mathcal{C}); \mathbb{Z})$ is a free abelian group of rank r with basis given by these meridians.*

(2) *$X(\mathcal{C})$ is homotopy equivalent to a 2-complex.*

(3) *If $\mathcal{C}$ intersects the line at infinity transversely, then $\pi_1(Y(\mathcal{C})) \to \pi_1(X(\mathcal{C}))$ is surjective.*

Proof. The first statement follows from the fact that by gluing in disks at the meridians we kill the fundamental group; the claim about the first homology group follows from Lefschetz duality (cf. [Lib1, p. 835] or [Di, p. 103]). The second statement follows because $X(\mathcal{C})$ has the homotopy type of a 2-dimensional complex affine variety (cf. also [Di, Thm. 1.6.8] or [Mi, Thm. 7.2]). The third statement follows from applying the Lefschetz hyperplane theorem (cf. e.g. [Di, p. 25]) and by an argument similar to the proof of Theorem 4.1 in [LMa1]. □

3.2. Proof of Theorem 1.1

From now on we assume that the algebraic curve $\mathcal{C}$ intersects the line at infinity transversely. Let $\alpha\colon \pi_1(X(\mathcal{C})) \to \Gamma$ be an admissible homomorphism.

Since Γ is infinite and since $X(\mathcal{C})$ is homotopy equivalent to a 2-complex, it follows from Lemma 2.1 that $b_p^{(2)}(X(\mathcal{C}),\alpha) = 0$ for $p = 0$ and $p > 2$. It therefore remains to show that $b_1^{(2)}(X(\mathcal{C}),\alpha) = 0$, from which the statement on $b_2^{(2)}(X(\mathcal{C}),\alpha)$ will then follow immediately by Lemma 2.1(3). We denote the homomorphism $\pi_1(Y(\mathcal{C})) \to \pi_1(X(\mathcal{C})) \xrightarrow{\alpha} \Gamma$ by α as well. By Theorem 3.1(3) and Lemma 2.2, it is enough to prove that $b_1^{(2)}(Y(\mathcal{C}),\alpha) = 0$.

Let $\mathbb{B}^4 \subset \mathbb{C}^2$ be a sufficiently large closed ball in the sense that $\operatorname{int}(\mathbb{B}^4) \setminus (\mathcal{C} \cap \operatorname{int}(\mathbb{B}^4))$ is diffeomorphic to $\mathbb{C}^2 \setminus \mathcal{C}$. Such a ball exists by [Di, Thm. 1.6.9]. Note in particular that all singularities of $\mathcal{C}$ lie in the interior of $\mathbb{B}^4$. By the homotopy invariance of the L^2-Betti numbers, we can abuse notation and thus also denote $\mathbb{B}^4 \cap X(\mathcal{C})$ and $\mathbb{B}^4 \cap Y(\mathcal{C})$ by $X(\mathcal{C})$ and $Y(\mathcal{C})$, respectively.

Given a point $P = (x_P, y_P) \in \mathbb{C}^2$ and $\varepsilon > 0$, we write $\mathbb{B}^4(P,\varepsilon) = \{(x,y) \in \mathbb{C}^2 \mid |x - x_P|^2 + |y - y_P|^2 \le \varepsilon^2\}$ and $S^3(P,\varepsilon) = \partial\mathbb{B}^4(P,\varepsilon)$. Now let $\operatorname{Sing}(\mathcal{C}) := \{P_1,\dots,P_s\}$ denote the set of singularities of $\mathcal{C}$. Then there exist $\varepsilon_1,\dots,\varepsilon_s > 0$ such that

(1) $\mathbb{B}^4(P_i,\varepsilon_i)$ are pairwise disjoint,
(2) $\mathbb{B}^4(P_i,\varepsilon_i) \subset \operatorname{int}(\mathbb{B}^4)$, and
(3) $\mathbb{B}^4(P_i,\varepsilon_i) \setminus (\mathcal{C} \cap \mathbb{B}^4(P_i,\varepsilon_i))$ is the cone on $S^3(P_i,\varepsilon_i) \setminus (\mathcal{C} \cap S^3(P_i,\varepsilon_i))$.

Such ε_i exist by Thom's first isotopy lemma (see [Di, Sec. 5] for details). For $i = 1,\dots,s$ we write $S_i^3 = \partial(\mathbb{B}^4(P_i,\varepsilon_i))$, $L_i := S_i^3 \cap \mathcal{C}$, and $X(L_i) := S_i^3 \setminus \nu L_i$.

Let T_i, $i = 1,\dots,s$, be the boundaries of $S_i^3 \setminus \nu L_i$. These are unions of tori, and we denote the connected components of T_i by $T_i^1,\dots,T_i^{n_i}$. Let $F_j := \mathcal{C}_j \setminus \bigl(\bigcup \operatorname{int}(\mathbb{B}_i^4) \cap \mathcal{C}_j\bigr)$ for $j = 1,\dots,r$. Then $F_1,\dots,F_r$ are the connected components of $F := \mathcal{C} \cap \bigl(\mathbb{C}^2 \setminus \bigcup_{i=1}^s \operatorname{int}(\mathbb{B}_i^4)\bigr)$. We write $Y(F) = Y(\mathcal{C}) \cap \bigl(\mathbb{C}^2 \setminus \bigcup_{i=1}^s \operatorname{int}(\mathbb{B}_i^4)\bigr)$ and denote the connected components of $Y(F)$ by $Y(F_1),\dots,Y(F_r)$. We can therefore decompose

$$Y(\mathcal{C}) = \bigcup_{i=1,\dots,r} Y(F_i) \cup_{T_1\cup\dots\cup T_s} \bigcup_{i=1,\dots,s} X(L_i).$$

We need the following definition.

Definition. Let M be a 3-manifold and let $\psi \in H^1(M;\mathbb{Z})$. We say that (M,ψ) *fibers over* S^1 if the homotopy class of maps $M \to S^1$ determined by $\psi \in H^1(M;\mathbb{Z}) = [M,S^1]$ contains a representative that is a fiber bundle over S^1.

Milnor [Mi, Thm. 4.8] showed that for $i = 1,\dots,s$ the pair $(X(L_i),\phi_i)$ fibers over S^1, where $\phi_i\colon H_1(X(L_i);\mathbb{Z}) \to \mathbb{Z}$ is induced by the (local) total linking number homomorphism—that is, by sending all meridians (with the induced orientation) about the components of L_i to 1 (see e.g. [Di, pp. 76–77]). Note that ϕ_i is precisely the homomorphism given by homomorphism

$$\pi_1(X(L_i)) \to \pi_1(Y(\mathcal{C})) \to \pi_1(X(\mathcal{C})) \xrightarrow{\phi} \mathbb{Z}.$$

For $i = 1, \dots, r$ we now consider $Y(F_i)$. Picking a trivialization of the normal bundle of F_i allows us to identify $Y(F_i)$ with $F_i \times S^1$. Consider the homomorphism

$$\psi_i : \pi_1(F_i \times S^1) \to \pi_1(Y(\mathcal{C})) \to \pi_1(X(\mathcal{C})) \xrightarrow{\phi} \mathbb{Z}.$$

Since the homomorphism $\pi_1(S^1) \to \pi_1(F_i \times S^1) \xrightarrow{\psi_i} \mathbb{Z}$ is surjective, it is well known that $(F_i \times S^1, \psi_i)$ fibers over S^1 and that the fiber is diffeomorphic to F_i. It follows from the preceding discussion that the fibrations $F_i \times S^1 \to S^1$ and $X(L_i) \to S^1$, when restricted to the tori T_i^j, correspond to the same classes in $H^1(T_i^j; \mathbb{Z})$. Since fibrations of a torus that lie in the same cohomology class are isotopic, it follows that we can glue the fibrations $F_i \times S^1 \to S^1$ and $X(L_i) \to S^1$ to get a fibration $\pi : Y(\mathcal{C}) \to S^1$ such that $\pi_* : \pi_1(Y(\mathcal{C})) \to \pi_1(S^1) = \mathbb{Z}$ equals $\pi_1(Y(\mathcal{C})) \to \pi_1(X(\mathcal{C})) \xrightarrow{\phi} \mathbb{Z}$.

We now recall the following theorem [Lü, Thm. 1.39].

THEOREM 3.2. *Let M be a compact 3-manifold and let $\psi \in H^1(M; \mathbb{Z})$ be such that (M, ψ) fibers over S^1. If $\beta : \pi_1(M) \to G$ is a homomorphism to a group G such that ψ factors through β, then $b_p^{(2)}(M, \beta) = 0$ for all p.*

Since α is admissible it follows now that $b_p^{(2)}(Y(\mathcal{C}), \alpha) = 0$. This concludes the proof of Theorem 1.1.

3.3. Proof of Theorem 1.2

Let $\mathcal{C} \subset \mathbb{C}^2$ be a reduced algebraic curve in general position at infinity. We pick $\mathbb{B}^4$ as in the previous section, and again we abuse notation by using $X(\mathcal{C})$ and $Y(\mathcal{C})$ to denote $\mathbb{B}^4 \cap X(\mathcal{C})$ and $\mathbb{B}^4 \cap Y(\mathcal{C})$, respectively.

Let $\alpha : \pi_1(X(\mathcal{C})) \to \Gamma$ be an admissible homomorphism. Denote the induced map $\pi_1(Y(\mathcal{C})) \to \pi_1(X(\mathcal{C})) \xrightarrow{\phi} \mathbb{Z}$ by ϕ'. Note that, by Theorem 3.1(3), the map ϕ' is surjective as well. Now denote by $\widetilde{Y(\mathcal{C})}$ and $\widetilde{X(\mathcal{C})}$ the infinite cyclic covers corresponding to ϕ' and ϕ, respectively. It follows easily that the induced map

$$\pi_1(\widetilde{Y(\mathcal{C})} \to \pi_1(\widetilde{X(\mathcal{C})})$$

is still surjective. But by Lemma 2.2 we then also have

$$b_1^{(2)}(\widetilde{Y(\mathcal{C})}, \tilde{\alpha}) \geq b_1^{(2)}(\widetilde{X(\mathcal{C})}, \tilde{\alpha}).$$

As we saw before, $(Y(\mathcal{C}), \phi)$ fibers over S^1. It follows that $\widetilde{Y(\mathcal{C})}$ is homotopy equivalent to the fiber Σ of the fibration, and we see that $b_1^{(2)}(\widetilde{Y(\mathcal{C})}, \tilde{\alpha}) = b_1^{(2)}(\Sigma, \tilde{\alpha})$. Since Σ is a compact surface with boundary, it follows immediately from Lemma 2.1 that

$$b_1^{(2)}(\Sigma, \tilde{\alpha}) = -\chi(\Sigma) + b_0^{(2)}(\Sigma, \tilde{\alpha}) \leq -\chi(\Sigma) + 1.$$

It therefore remains to compute $\chi(\Sigma)$.

We denote the fibers of the fibrations $X(L_i) \to S^1$ by Σ_i, and we denote the fibers of the fibrations $X(F_i) \to S^1$ by F_i'. Recall that F_i' is diffeomorphic to F_i. Note that Σ is the result of gluing the set of fibers $\{\Sigma_i\}$ and the surfaces $\{F_i'\}$ along the longitudes of the links L_i. Because the Euler characteristic of the longitudes is 0, we obtain

$$\chi(\Sigma) = \sum_{i=1}^{s} \chi(\Sigma_i) + \sum_{i=1}^{r} \chi(F_i).$$

By [Di, p. 78] we have $\chi(\Sigma_i) = 1 - \mu(\mathcal{C}, P_i)$, where $\mu(\mathcal{C}, P_i)$ denotes the Milnor number of the singularity P_i.

Now let $\mathcal{D}$ be the projective completion of $\mathcal{C}$. Topologically, $\mathcal{D}$ is given by adding disks to the boundary components of $\mathcal{C}$ at "infinity". Since $\mathcal{C}$ has degree d and is in general position at infinity, there are exactly d such components. Since gluing in a disk increases the Euler characteristic by 1, we have that

$$\chi(\mathcal{D}) = \chi(\mathcal{C}) + d.$$

Recall that the normalization of $\mathcal{D}$ is defined to be the curve $\hat{\mathcal{D}}$ without singularities obtained from $\mathcal{D}$ by blow-ups. Note that $\chi(\hat{\mathcal{D}})$ can be computed as follows.

Let $\mathcal{D}'$ be the result of first removing balls around the singularities, and let $\mathcal{D}''$ be the result of gluing in disks to all the boundary components of $\mathcal{D}'$. Then $\mathcal{D}''$ is topologically equivalent to $\mathcal{D}$ blown up at the singularities; in particular,

$$\chi(\hat{\mathcal{D}}) = \chi(\mathcal{D}'').$$

Since gluing in a disk increases the Euler characteristic by 1, we also get that

$$\chi(\hat{\mathcal{D}}) = \chi(\mathcal{D}') + b_0(\partial\mathcal{D}').$$

Hence in our situation it follows that

$$\chi(\hat{\mathcal{D}}) = \sum_{i=1}^{r} \chi(F_i) + \sum_{i=1}^{s} n_i + d.$$

In summary, we therefore see that

$$\begin{aligned} b_1^{(2)}(\widetilde{X(\mathcal{C})}) &\le b_1^{(2)}(\widetilde{Y(\mathcal{C})}) \\ &\le -\chi(\Sigma) + 1 \\ &= -\sum_{i=1}^{s} \chi(\Sigma_i) - \sum_{i=1}^{r} \chi(F_i) + 1 \\ &= \sum_{i=1}^{s} (\mu(\mathcal{C}, P_i) - 1) - \chi(\hat{\mathcal{D}}) + \sum_{i=1}^{s} n_i + d + 1 \\ &\le \sum_{i=1}^{s} (\mu(\mathcal{C}, P_i) + n_i - 1) + 2g(\hat{\mathcal{D}}) + d. \end{aligned}$$

This completes the proof of Theorem 1.2. We conclude with three remarks.

Remark. (1) For Γ a LITFA group we saw in Proposition 2.4 that the L^2-Betti numbers are determined by ranks of homology modules over skew fields. In that case, the flatness of certain rings involved shows that the statement of Theorem 1.2 is an immediate consequence of Theorem 1.2 (we refer to [LMa1] for details). This approach does not seem to work if Γ is not a LITFA group.

(2) Our methods carry over to prove generalizations of Theorem 4.5, Theorem 4.7, and Corollary 4.8 in [LMa1]. We leave the task of formulating and proving the precise statements to the interested reader.

(3) Given a knot K, we denote by $X(K) = S^3 \setminus \nu K$ its exterior and by $\widetilde{X(K)}$ the infinite cyclic cover of $X(K)$. When K is a nontrivial fibered knot it follows from the preceding that $b_1^{(2)}(\widetilde{X(K)}, \mathrm{id}) = 2\,\mathrm{genus}(K) - 1$. Given any nontrivial knot K, we write $\tilde{\pi} = \pi_1(\widetilde{X(K)})$. By Proposition 2.4, the sequence of L^2-Betti numbers $b_1^{(2)}(\widetilde{X(K)}, \tilde{\pi} \to \tilde{\pi}/\tilde{\pi}^{(n)})$, $n \geq 1$, equals the sequence of Cochran invariants $\delta_n(K)$, which was shown in [C] to be a never-decreasing sequence of invariants that all give lower bounds on $2\,\mathrm{genus}(K) - 1$. Cochran's result can be interpreted as stating that the L^2-Betti number corresponding to "bigger" (PTFA-) quotients of $\tilde{\pi}$ give better bounds on $2\,\mathrm{genus}(K) - 1$. It therefore seems natural to conjecture that "in the limit" we have equality—in other words, that $b_1^{(2)}(\widetilde{X(K)}, \mathrm{id}) = 2\,\mathrm{genus}(K) - 1$.

References

[BH] R. G. Burns and V. W. Hale, *A note on group rings of certain torsion-free groups,* Canad. Math. Bull. 15 (1972), 441–445.

[C] T. Cochran, *Noncommutative knot theory,* Algebr. Geom. Topol. 4 (2004), 347–398.

[COT] T. Cochran, K. Orr, and P. Teichner, *Knot concordance, Whitney towers and L^2-signatures,* Ann. of Math. (2) 157 (2003), 433–519.

[CoF] J. I. Cogolludo and V. Florens, *Twisted Alexander polynomials of plane algebraic curves,* J. London Math. Soc. (2) 76 (2007), 105–121.

[DaJLe] M. W. Davis, T. Januszkiewicz, and I. J. Leary, *The l^2-cohomology of hyperplane complements,* Groups, Geom. Dyn. 1 (2007), 301–309.

[Di] A. Dimca, *Singularities and topology of hypersurfaces,* Universitext, Springer-Verlag, New York, 1992.

[DiMa] A. Dimca and L. Maxim, *Multivariable Alexander invariants of hypersurface complements,* Trans. Amer. Math. Soc. 359 (2007), 3505–3528.

[DLiMSY] J. Dodziuk, P. Linnell, V. Mathai, T. Schick, and S. Yates, *Approximating L^2-invariants, and the Atiyah conjecture,* Comm. Pure Appl. Math. 56 (2003), 839–873.

[Ha1] S. Harvey, *Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm,* Topology 44 (2005), 895–945.

[Ha2] ———, *Homology cobordism invariants and the Cochran–Orr–Teichner filtration of the link concordance group,* Geom. Topol. 12 (2008), 387–430.

[LMa1] C. Leidy and L. Maxim, *Higher-order Alexander invariants of plane algebraic curves,* Internat. Math. Res. Notices 2006 (2006), article ID 12976.

[LMa2] ———, *Obstructions on fundamental groups of plane curve complements,* Proceedings of the IX workshop on real and complex singularities (Sao Carlos, 2006) (to appear).

[Lib1] A. Libgober, *Alexander polynomial of plane algebraic curves and cyclic multiple planes,* Duke Math. J. 49 (1982), 833–851.

[Lib2] ———, *Alexander invariants of plane algebraic curves,* Singularities, part 2 (Arcata, 1981), Proc. Sympos. Pure Math., 40, pp. 135–143, Amer. Math. Soc., Providence, RI, 1983.

[Lib3] ———, *On the homotopy type of the complement to plane algebraic curves,* J. Reine Angew. Math. 367 (1986), 103–114.

[Lib4] ———, *On the homology of finite abelian covers,* Topology Appl. 43 (1992), 157–166.

[Lib5] ———, *Homotopy groups of the complements to singular hypersurfaces, II,* Ann. of Math. (2) 139 (1994), 117–144.

[Lib6] ———, *Characteristic varieties of algebraic curves,* Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), NATO Sci. Ser. II Math. Phys. Chem., 36, pp. 215–254, Kluwer, Dordrecht, 2001.

[Li] P. Linnell, *Zero divisors and* $L^2(G)$, C. R. Acad. Sci. Paris Sér. I Math. 315 (1992), 49–53.

[Lü] W. Lück, *L^2-invariants: Theory and applications to geometry and K-theory,* Ergeb. Math. Grenzgeb. (3), 44, Springer-Verlag, Berlin, 2002.

[Ma] L. Maxim, *Intersection homology and Alexander modules of hypersurface complements,* Comment. Math. Helv. 81 (2006), 123–155.

[Mi1] J. Milnor, *Morse theory,* Ann. of Math. Stud., 51, Princeton Univ. Press, Princeton, NJ, 1963.

[Mi2] ———, *Singular points of complex hypersurfaces,* Ann. of Math. Stud., 61, Princeton Univ. Press, Princeton, NJ, 1968.

[R] H. Reich, *Group von Neumann algebras and related algebras,* Thesis, Univ. of Göttingen, 1998.

S. Friedl
Université du Québec à Montréal
Montréal, Québec
Canada

and

University of Warwick
Coventry
United Kingdom

sfriedl@gmail.com

C. Leidy
Department of Mathematics
and Computer Science
Wesleyan University
Middletown, CT 06459

cleidy@wesleyan.edu

L. Maxim
Institute of Mathematics of the Romanian Academy
P.O. Box 1-764
70700 Bucharest
Romania

and

Department of Mathematics & Computer Science
CUNY – Lehman College
Bronx, NY 10468

laurentiu.maxim@lehman.cuny.edu