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First-order Univalence Criteria, Interior Chord-arc Conditions, and Quasidisks

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Dedicated to Professor Christian Pommerenke on the occasion of his seventy-fifth birthday

1. Introduction

In all that follows, G and R will denote domains in the complex plane $\mathbb{C}$; G will always be simply connected, and $0 \notin R$. A first-order univalence criterion for G is a condition of the form

$$f'(G) \subset R, \tag{1.1}$$

or, somewhat more generally (and for present considerations more conveniently) of the form

$$\log f'(G) \subset R', \tag{1.2}$$

which implies that f is univalent on G. By (1.2) we mean of course that $f'(z) = e^{g(z)}$, where $g(G) \subset R'$. We will be concerned in large measure with the particular case in which $R' = \alpha S_0$, where S_0 is the infinite strip

$$S_0 = \{z : -1 < \Re\{z\} < 1\}.$$

The third author has studied the problem of determining criteria of the form (1.1) for smoothly bounded Jordan domains G that are sharp in the sense that there is no R_1 properly containing R for which the condition $f'(G) \subset R_1$ implies univalence (see [Ge] and the references therein). In this paper we examine two further aspects of first-order univalence criteria. First of all, in Section 2 we prove a theorem from which it follows immediately that there will be a criterion of either of these forms if and only if G satisfies an interior chord-arc condition—that is, if and only if there is an L such that

$$l(z_1, z_2) = \inf\left\{\int_\gamma |dz| : \gamma \subset G,\, z_1, z_2 \in \gamma\right\} \le L|z_1 - z_2| \tag{1.3}$$

for all $z_1, z_2 \in G$, where the γ are arcs.

To put our other results in context, we briefly discuss univalence criteria on quasidisks involving the pre-Schwarzian derivative $P_f(z) = f''(z)/f'(z)$ and the Schwarzian derivative $S_f(z) = (P_f(z))' - \frac{1}{2}(P_f(z))^2$. There is a classical result of Ahlfors [A] to the effect that, if G is a quasidisk, then there is some $\beta_S > 0$ such that

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$$|S_f(z)| \le \frac{\beta_S}{\operatorname{dist}(z,\partial G)^2} \tag{1.4}$$

is a univalence criterion for G, where here (and in what follows) $\operatorname{dist}(a,X) = \inf\{|a-z| : z \in X\}$. From this it follows easily that if G is a quasidisk then there is also some $\beta_P > 0$ such that

$$|P_f(z)| \le \frac{\beta_P}{\operatorname{dist}(z,\partial G)} \tag{1.5}$$

is a univalence criterion in G. It is well known and easily proved that if g maps the unit disk into S_0 then

$$|g'(z)| \le \frac{4}{\pi(1-|z|^2)}. \tag{1.6}$$

Applying (1.5) and (1.6) for fixed $w \in G$ to $g(z) = \frac{1}{\alpha}\log f'(w+rz)$ with $|z|<1$ and $r = \operatorname{dist}(w,\partial G)$, one sees that if G is a quasidisk then

$$\log f'(G) \subset \alpha S_0 \tag{1.7}$$

is a univalence criterion in G for any α for which $|\alpha| \le \frac{\pi}{4}\beta_P$. Conversely, Gehring [G1] showed that any G that has a univalence criterion (1.4) is necessarily a quasidisk, and subsequently Astala and Gehring [AG] established the stronger fact that any G having a univalence criterion of the form (1.5) must be a quasidisk. In light of these results, it makes sense to ask whether there might be an even stronger implication to the effect that all G having a univalence criterion of the form (1.7) are quasidisks. We show in Section 3 that if G has a univalence criterion (1.7) for some $\alpha > 0$, then G must indeed be a quasidisk; the corresponding results of [G1] and [AG] just mentioned follow immediately from this. Moreover, this result yields a new characterization of quasidisks—namely, that $G \subset \mathbb{C}$ is a quasidisk if and only if it has a univalence criterion (1.1) in which R is an annulus centered at 0. In Section 4 we show, on the other hand, that for any $\alpha \notin \mathbb{R}$ there is a G that is not a quasidisk but that nevertheless has a univalence criterion $\log f'(G) \subset \varepsilon\alpha S_0$ for some $\varepsilon > 0$.

In what follows, $\Delta(a,\rho) = \{z : |z-a| < \rho\}$, $\Delta = \Delta(0,1)$. The diameter of a set X is denoted by $\operatorname{diam}(X)$.

2. Interior Chord-arc Conditions and First-order Univalence Criteria

We shall prove the following theorem.

THEOREM 1. *Let $G \ne \mathbb{C}$ be a simply connected domain. Then the following statements are equivalent.*

(a) *G satisfies an interior chord-arc condition; that is, there is a constant L_1 such that $l(z_1,z_2) \le L_1|z_1-z_2|$ for all $z_1,z_2 \in G$.*
(b) *There is an $\varepsilon > 0$ such that $f'(G) \subset \Delta(1,\varepsilon)$ is a univalence criterion for G.*

(c) *There is a constant* L_2 *such that* $g'(G) \subset \Delta$ *implies* $|g(z_1) - g(z_2)| \le L_2|z_1 - z_2|$ *for all* $z_1, z_2 \in G$.

REMARKS. (i) We do not consider domains that are not simply connected, although investigation of that case would be of interest.

(ii) Condition (c) of this theorem is the Hardy–Littlewood property of order 1, as explained in [KW].

Proof of Theorem 1. (a) ⇒ (b). Suppose that (a) holds and that $f' = 1 + \varepsilon h$, where $h(G) \subset \Delta$. Then for $z_1 \ne z_2$ we have $f(z_1) - f(z_2) = z_1 - z_2 + \varepsilon \int_\gamma h(z)\,dz$, where γ is an arc of length λ with

$$l(z_1, z_2) < \lambda < (L_1 + 1)|z_1 - z_2|$$

joining z_2 to z_1 in G. It follows that

$$|f(z_1) - f(z_2) - (z_1 - z_2)| < \varepsilon(L_1 + 1)|z_1 - z_2|.$$

Hence $f(z_1) \ne f(z_2)$ if, for example,

$$\varepsilon = \frac{1}{L_1 + 1}. \tag{2.1}$$

(b) ⇒ (c). Suppose that (b) holds and $g'(G) \subset \Delta$ but that, for some distinct $z_1, z_2 \in G$,

$$|g(z_1) - g(z_2)| > \frac{|z_1 - z_2|}{\varepsilon}. \tag{2.2}$$

Let $\delta_g = g(z_1) - g(z_2)$ and $\delta_z = z_1 - z_2$, and consider $f(z) = z - (\delta_z/\delta_g)g(z)$. Then by (2.2), $f'(G) \subset \Delta(1, \varepsilon)$, so that f is univalent in G. But

$$f(z_1) - f(z_2) = \delta_z - \frac{\delta_z}{\delta_g}\delta_g = 0,$$

which is a contradiction. Thus we have that (c) holds with

$$L_2 = \frac{1}{\varepsilon}. \tag{2.3}$$

(c) ⇒ (a). This follows directly from Corollary 1 of [KW] in the limiting case $k = 1$; for the convenience of the reader, we repeat the proof given there.

LEMMA 1. *There is a function* $c(\varepsilon) > 0$ *defined for all* $\varepsilon > 0$ *with the following property. Let h be a complex-valued function on* $(0, \infty)$ *for which*

$$|h(x)| \le 1 \quad \textit{and} \quad |h'(x)| \le \frac{1}{x} \quad \textit{for all } x \in (0, \infty).$$

Then there is an F analytic in the right half-plane $\mathbb{H} = \{z : \Re\{z\} > 0\}$ *such that*

$$|F(x) - h(x)| < \varepsilon \quad \textit{for } x \in (0, \infty)$$

and $\|F\|_\infty < c(\varepsilon)$.

Proof. Let h be as in the hypothesis. The function $w = e^z$ maps the horizontal strip $|\Im\{z\}| < \frac{\pi}{2}$ conformally onto $\mathbb{H}$, and clearly the function $H(u) = h(e^u)$ satisfies

$$|H(u)| \le 1 \tag{2.4}$$

and

$$|H'(u)| = |h'(e^u)|e^u \le 1 \tag{2.5}$$

for all $u \in \mathbb{R}$. If we can find a function f defined in this strip with $\|f\|_\infty < c(\varepsilon)$ there and with $|f(u) - H(u)| < \varepsilon$ for all $u \in \mathbb{R}$, then our required function is $F(z) = f(\log z)$ with $\log 1 = 0$. Thus we work with functions H on $\mathbb{R}$ that satisfy (2.4) and (2.5).

Let $\psi(z) = (\sin^2 z)/z^2$, and for $n \ge 1$ set

$$f_n(w) = \frac{n}{\pi}\int_{-\infty}^{\infty} \psi(n(w-t))H(t)\,dt.$$

Since $\int_{-\infty}^{\infty}(\sin^2 x)/x^2\,dx = \pi$, a standard argument shows that $f_n(u) \to H(u)$ uniformly on $\mathbb{R}$ as $n \to \infty$ and that the rate of convergence is independent of H under conditions (2.4) and (2.5). With regard to the bound $\|f\|_\infty < c(\varepsilon)$, for $w = x + iy$ with $|y| \le \frac{\pi}{2}$ we have

$$\begin{aligned}
|f_n(w)| &\le \frac{n}{\pi}\int_{-\infty}^{\infty}|\psi(n(w-t))|\,dt = \frac{n}{\pi}\int_{-\infty}^{\infty}|\psi(n(iy-t))|\,dt \\
&= \frac{n}{\pi}\int_{-\infty}^{\infty}|\psi(niy+nt)|\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty}|\psi(niy+t)|\,dt \\
&\le B_n + \frac{1}{\pi}\int_{|t|\ge 1}\frac{A_n}{t^2}\,dt,
\end{aligned}$$

where

$$\begin{aligned}
B_n &= \frac{2}{\pi}\sup\left\{|\psi(niy+t)| : |t| \le 1,\ |y| \le \frac{\pi}{2}\right\}, \\
A_n &= \sup\left\{|\sin(niy+t)|^2 : t \in \mathbb{R},\ |y| \le \frac{\pi}{2}\right\}.
\end{aligned}$$

Since A_n and B_n are finite, the proof is complete. □

Returning to the proof of Theorem 1, we now assume that condition (c) holds and show that G satisfies an interior chord-arc condition. Let ϕ map $\mathbb{H}$ one-to-one onto G in such a way that z_1 and z_2 are images of points on $\mathbb{R}^+$. It is easy to see that this is possible by observing, for example, that if ξ maps Δ onto G with $\xi(0) = z_1$ then, for an appropriate α with $|\alpha| = 1$, $\xi(\alpha z)$ maps points of $(-1, 1)$ onto z_1 and z_2. Then to obtain ϕ one just composes ξ with a mapping of $\mathbb{H}$ onto Δ that takes $\mathbb{R}^+$ onto $(-1, 1)$. Now consider $h(t) = e^{-i\arg\{\phi'(t)\}}$. Since $h'(t)/h(t) = -i\Im\{\phi''(t)/\phi'(t)\}$, it follows that $|h'(t)| \le |\phi''(t)/\phi'(t)|$. Using the fact that the function $\tau(z) = \phi\left(t\frac{1+z}{1-z}\right)$, $t > 0$, is univalent on Δ, a straightforward calculation—together with the classical coefficient bound $|a_2| \le 2$ for normalized univalent functions on Δ—shows that $|\phi''(t)/\phi'(t)| \le 3/t$, so that $|h'(t)| \le 3/t$. Thus, on applying Lemma 1 we find that there is a positive C and an analytic function F on $\mathbb{H}$ such that

$$|F(x) - h(x)| \le \tfrac{1}{2} \quad \text{for all } \ x \in (0, \infty) \ \text{ and } \ \|F\|_\infty < C.$$

Here, $C = 3c\left(\frac{1}{6}\right)$ is a universal constant. Now we define f on G by setting $f'(\phi(z)) = F(z)$ so that, obviously, $f'(G) \subset C\Delta$. Consider the curve $\gamma = \phi([a_1, a_2])$, where $\phi(a_k) = z_k$ for $k = 1, 2$. Then

$$\begin{aligned} f(z_2) - f(z_1) &= \int_\gamma f'(\zeta)\, d\zeta = \int_{a_1}^{a_2} f'(\phi(x))\phi'(x)\, dx = \int_{a_1}^{a_2} F(x)\phi'(x)\, dx \\ &= \int_{a_1}^{a_2} h(x)\phi'(x)\, dx + \int_{a_1}^{a_2} (F(x) - h(x))\phi'(x)\, dx. \end{aligned}$$

Thus, since $h(x)\phi'(x) = |\phi'(x)|$,

$$\begin{aligned} |f(z_2) - f(z_1)| &= \left| \int_\gamma f'(\zeta)\, d\zeta \right| \\ &\ge \int_{a_1}^{a_2} |\phi'(x)|\, dx - \frac{1}{2}\int_{a_1}^{a_2} |\phi'(x)|\, dx = \frac{1}{2}\int_\gamma |dz|. \end{aligned}$$

Therefore, γ is a curve joining z_1 to z_2 in G for which $\int_\gamma |dz| \le 2C|g(z_2) - g(z_1)|$, where $g = f/C$. But since $f'(G) \subset C\Delta$ we have $g'(G) \subset \Delta$ and so, by our assumption that (c) holds, $\int_\gamma |dz| \le 2CL_2|z_1 - z_2|$. Passing to the infimum we see that (a) holds with

$$L_1 = 2CL_2. \tag{2.6}$$

□

REMARK. The proof shows that L_1, ε, and L_2 depend on each other in accordance with (2.1), (2.3), and (2.6).

It is clear that (1.1) can be a univalence criterion only if $0 \notin R$, so that G has a univalence criterion of this form if and only if it has one of the form (1.2). We have the following result.

COROLLARY. *Let $G \subset \mathbb{C}$ be a simply connected domain. Then there is some domain R for which $f'(G) \subset R$ is a univalence criterion if and only if G satisfies an interior chord-arc condition.*

Proof. That (a) of the theorem implies that there is such an R follows from (b). On the other hand, if there is such an R and $\Delta(w, \delta) \subset R$ then it is clear that $f'(G) \subset \frac{1}{w}\Delta(w, \delta) = \Delta\left(1, \frac{\delta}{|w|}\right)$ is a univalence criterion for G, so that (a) holds. □

REMARK. Since the disk just considered cannot contain 0, it must be that $\delta < |w|$. From this one concludes that, for any domain R for which (1.1) is a univalence criterion, $\operatorname{dist}(w, \partial R) < |w|$ for all $w \in R$.

3. Characterization of Quasidisks in Terms of First-order Univalence Criteria

The literature contains numerous characterizations of quasidisks (see e.g. [G3; L]). Theorem 4.5 and Section 4.6 of [NV] show that a simply connected domain $G \subset \mathbb{C}$, $G \neq \mathbb{C}$, is a quasidisk if and only if G satisfies an interior chord-arc condition (1.3) and G is a John disk; that is, there is some constant L' such that, for any rectilinear crosscut (a, b) of G, at least one of the two components of $G \backslash (a, b)$ has diameter at most $L'|b - a|$. (See also [P, Thm. 5.9] for this characterization of quasidisks in the case of bounded domains.)

Theorem 2. *Let $G \neq \mathbb{C}$ be a simply connected domain. Then G has a univalence criterion of the form*

$$\log f'(G) \subset \alpha S_0 \tag{3.1}$$

for some $\alpha > 0$ if and only if G is a quasidisk.

Proof. That quasidisks have univalence criteria of this form follows, as indicated in the Introduction, from the result of Ahlfors [A]. Conversely, let G be a domain for which (3.1) is a univalence criterion. Since there is an $\varepsilon = \varepsilon(\alpha)$ such that $|\Re\{\log(1 + \varepsilon w)\}| \le \alpha$ for all $w \in \Delta$, we see that $f'(G) \subset \Delta(1, \varepsilon)$ implies (3.1). Thus G satisfies condition (b) of Theorem 1; hence it follows from that theorem and the Remark thereafter that G necessarily has the interior chord-arc property with a constant L, as in (1.3), which depends only on α. We therefore need only show that G is a John disk—in other words, that there is a constant $L' = L'(\alpha)$ such that, if (a, b) is any rectilinear crosscut of G, then one of the two components G_1, G_2 of $G \backslash (a, b)$ has diameter at most $L'|b - a|$. For notational simplicity we may assume without loss of generality that $a = -1$ and $b = 1$. We need to obtain an upper bound for $\min\{\operatorname{diam}(G_1), \operatorname{diam}(G_2)\}$ that depends only on α. For definiteness we assume that G_1 lies to the left of $(-1, 1)$ as this interval is traversed from -1 to 1.

Remark. A mapping $q : G \to \mathbb{C}$ is a local Q-quasi-isometry if it is a local homeomorphism for which the upper and lower limits of $|q(w) - q(z)|/|w - z|$ as $w \to z$ are in $\left[\frac{1}{Q}, Q\right]$ for all z in G; analytic functions satisfying (3.1) are obviously local e^{α}-quasi-isometries. Loosely speaking, to prove the local quasi-isometry counterpart of Theorem 2, Gehring [G2] constructed a local Q-quasi-isometry q of G for which $q(z) = ze^{i\phi_Q(|z|)}$ on $G_1 \backslash 2\Delta$ and $q(z) = z$ on $G_2 \backslash 2\Delta$, where $\phi_Q(r)$ is a real-valued function (that depends on Q) tending to ∞ as $r \to \infty$. The proof we are giving essentially amounts to an adaptation of this construction to the much more restrictive class of analytic functions satisfying a condition of the form (3.1).

Before proceeding with the proof of Theorem 2, we briefly remind the reader of the mapping properties of $\sin z$ and its inverse. For the remainder of this section and in distinction to the notation of Section 2, $\mathbb{H}^+$ and $\mathbb{H}^-$ will denote the upper and lower half-planes $\{z : \Im\{z\} > 0\}$ and $\{z : \Im\{z\} < 0\}$, respectively. Because $\sin z$ maps the half-strip $\left\{z : \Im\{z\} > 0,\ |\Re\{z\}| < \frac{\pi}{2}\right\}$ one-to-one onto $\mathbb{H}^+$, with $-\frac{\pi}{2}$, $\frac{\pi}{2}$, and ∞i corresponding (respectively) to -1, 1, and ∞, it follows immediately from the reflection principle that $\sin z$ effects a one-to-one mapping of $\mathbb{H}^+$ onto the universal covering surface $\mathbb{U}_1$ of $\mathbb{C}\backslash[-1,1]$. An additional application of the reflection principle shows that $\sin z$ maps the domain $\mathbb{E} = \mathbb{H}^+ \cup \mathbb{H}^- \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ one-to-one onto the simply connected surface $\mathbb{U}$ consisting of $\mathbb{U}_1$, its reflection $\mathbb{U}_2$ across $(-1,1)$, and this interval. The monodromy principle implies that $\sin z$ has a uniquely defined inverse on any simply connected domain, such as G, for which $(-1,1)$ is a crosscut. Henceforth we denote this inverse on G by γ. We have that

$$\gamma(G_1) \subset \mathbb{H}^+ \quad \text{and} \quad \gamma(G_2) \subset \mathbb{H}^-. \tag{3.2}$$

Let $\delta > 0$ be such that

$$\sin\left(\Delta\left(\tfrac{\pi}{2}, \tfrac{1}{2}\right) \cap \mathbb{H}^+\right) \supset \Delta(1, 2\delta)\backslash[-1,1].$$

Let $z \in G_1$ be such that $|z-1| < \frac{\delta}{L}$. From the fact that G satisfies an interior chord-arc condition with constant L it easily follows that G_1 and G_2 also have the same property, from which it follows in turn that for any $t \in \left(1 - \frac{\delta}{L}, 1\right)$ there is an arc C of length less than 2δ that joins t to z in G_1. From this we conclude that $C\backslash\{t\} \subset \Delta(1,2\delta)\backslash[-1,1]$, so that $\gamma(z) \in \Delta\left(\frac{\pi}{2}, \frac{1}{2}\right) \cap \mathbb{H}^+$; that is, $\gamma\left(G_1 \cap \Delta\left(1, \frac{\delta}{L}\right)\right) \subset \Delta\left(\frac{\pi}{2}, \frac{1}{2}\right) \cap \mathbb{H}^+$. In the same manner one sees that $\gamma\left(G_2 \cap \Delta\left(1, \frac{\delta}{L}\right)\right) \subset \Delta\left(\frac{\pi}{2}, \frac{1}{2}\right) \cap \mathbb{H}^-$ and that analogous statements hold for $\Delta\left(-1, \frac{\delta}{L}\right)$. In light of the mapping properties of $\sin z$ in neighborhoods of $\pm\frac{\pi}{2}$, this means that

$$\gamma(z) = \gamma_1\left(\sqrt{z-1}\right) + \tfrac{\pi}{2} \quad \text{and} \quad \gamma(z) = \gamma_2\left(\sqrt{z+1}\right) - \tfrac{\pi}{2} \tag{3.3}$$

for $z \in \Delta\left(1, \frac{\delta}{L}\right)$ and $z \in \Delta\left(-1, \frac{\delta}{L}\right)$, respectively, where γ_1 and γ_2 are analytic in $\Delta\left(0, \sqrt{\delta/L}\right)$ with simple zeros at 0. Because

$$\gamma(z) = -i \log\left(iz + \sqrt{1-z^2}\right) = -i\left(\log iz + \log\left(1 + \sqrt{1 - 1/z^2}\right)\right),$$

with appropriate values of the logarithm and root, there is clearly an absolute constant A for which

$$|\gamma(z)| \geq A \log|z| \quad \text{for } z \in G\backslash 2\Delta. \tag{3.4}$$

Let $u(z) = (z^2+z+1)^4 = (z-e^{2\pi i/3})^4(z-e^{4\pi i/3})^4$. Clearly, $|u(z)| < u(1) = 81$ for $z \in \bar{\Delta}\backslash\{1\}$, so that

$$|\arg\{81 - u(z)\}| < \tfrac{\pi}{2} \quad \text{for } z \in \Delta. \tag{3.5}$$

Let g map $\mathbb{E}$ one-to-one onto Δ with $g\left(-\frac{\pi}{2}\right) = e^{2\pi i/3}$ and $g\left(\frac{\pi}{2}\right) = e^{4\pi i/3}$ and with the point at infinity on the boundary of $\mathbb{H}^+$ corresponding to 1. Clearly, g is analytic everywhere on $\partial\mathbb{E}$ with the exception of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, and in neighborhoods of these points g is given by

$$g(z) = g_1\left(\sqrt{z + \tfrac{\pi}{2}}\right) + e^{2\pi i/3} \quad \text{and} \quad g(z) = g_2\left(\sqrt{z - \tfrac{\pi}{2}}\right) + e^{4\pi i/3}, \tag{3.6}$$

respectively, with appropriate branches of the root, where g_1 and g_2 are analytic at 0 with simple zeros there. In addition, $g(1/z)$ is analytic in neighborhoods of the point at infinity in both $\mathbb{H}^+$ and $\mathbb{H}^-$, so that $g'(z)$ is bounded outside of any neighborhood of $\{-\frac{\pi}{2}, \frac{\pi}{2}\}$. Since $|\gamma'(z)| = |1-z^2|^{-1/2}$ is bounded outside of any neighborhood of $\{-1,1\}$, it follows that the derivative of $g(\gamma(z))$ is bounded in the complement of the $\left(\frac{1}{3}\sqrt{\delta/L}\,\right)$-neighborhood of $\{-1,1\}$ by a constant that depends only on α. In addition, a straightforward calculation together with (3.3) and (3.6) shows that

$$g(\gamma(z)) = h_1((z+1)^{1/4}) + e^{2\pi i/3} \quad \text{and} \quad g(\gamma(z)) = h_2((z-1)^{1/4}) + e^{4\pi i/3},$$

where h_1 and h_2 are analytic in $\Delta\left(0, \sqrt{\delta/L}\,\right)$ with simple zeros at 0, whence the derivative of $u_1(z) = u(g(\gamma(z)))$ is bounded in the $\left(\frac{1}{2}\sqrt{\delta/L}\,\right)$-neighborhood of $\{-1,1\}$. Thus, there is an $A_0 = A_0(\alpha)$ such that

$$|u_1'(z)| \le A_0 \quad \text{for } z \in G. \tag{3.7}$$

From (3.2) and (3.4) we have

$$|u_1(z) - 81| \le \frac{A_1}{\log|z|} \quad \text{for } z \in G_1 \backslash 2\Delta, \tag{3.8}$$

and since (as observed previously) $|u(z)| < 81$ for $z \in \bar{\Delta}\backslash\{1\}$, it follows that

$$|u_1(z) - 81| \ge A_2, \quad z \in G \backslash (G_1 \backslash 2\Delta) = G_2 \cup (G \cap 2\Delta), \tag{3.9}$$

where A_1 and A_2 are positive universal constants. Relation (3.5) implies that

$$|\Re\{i \log(81 - u_1(z))\}| < \tfrac{\pi}{2} \quad \text{for } z \in G.$$

We define

$$H(z) = \frac{-2i}{\pi} \log(81 - u_1(z)),$$

so that

$$H(G) \subset S_0 \tag{3.10}$$

and, because of (3.7) and (3.9),

$$|H'(z)| \le A_3 \quad \text{for } z \in G \cap 2\Delta, \tag{3.11}$$

where $A_3 = A_3(\alpha)$.

For each $z_0 \in \mathbb{U}$ with $|z_0| \ge 2$, we have $\Delta(z_0, |z_0| - 1) \cap [-1,1] = \emptyset$. This means that such a z_0 has a neighborhood U such that H coincides in U with a function H_1 (manufactured using an appropriate branch of the inverse of $\sin z$) that is analytic in $\Delta(z_0, |z_0| - 1)$ and for which $H_1(\Delta(z_0, |z_0| - 1)) \subset S_0$. It follows from (1.6) that

$$|H'(z)| \le \frac{4}{\pi(|z|-1)} \quad \text{for } z \in G \backslash 2\Delta.$$

Taking (3.11) into account, we see that

$$|H'(z)| \le \frac{A_4}{|z|+1} \quad \text{for } z \in G \tag{3.12}$$

for some $A_4 = A_4(\alpha)$.

Now consider the function

$$F_\varepsilon(z) = z e^{\varepsilon H(z)}$$

on G. We have

$$\log F'_\varepsilon(z) = \log\big(e^{\varepsilon H(z)}(1 + \varepsilon z H'(z))\big) = \varepsilon H(z) + \log(1 + \varepsilon z H'(z)),$$

so that, in light of (3.12),

$$|\log F'_\varepsilon(z) - \varepsilon H(z)| \leq 2\varepsilon A_4 \ \text{ on } G$$

provided that $\varepsilon \in [0, 1/2A_4]$. Consequently, taking (3.10) into account shows us that

$$\log F'_\varepsilon(G) \subset (\min\{\alpha/2, 1\}) S_0 \quad \text{for } 0 \leq \varepsilon \leq \varepsilon_0, \tag{3.13}$$

where

$$\varepsilon_0 = \varepsilon_0(\alpha) = \frac{\min\{\alpha/2, 1, 1/2A_4(\alpha)\}}{1 + 2A_4(\alpha)}.$$

For all $\varepsilon \in [0, \varepsilon_0]$, the function F_ε maps G univalently onto $G_\varepsilon = F_\varepsilon(G)$; moreover, by considering $f \circ F_\varepsilon$, one sees that the condition

$$\log f'(G_\varepsilon) \subset (\alpha/2) S_0$$

is a univalence criterion for G_ε. Thus, there is some $\lambda_0 = \lambda_0(\alpha) \geq 1$ such that, for all $\varepsilon \in [0, \varepsilon_0]$, G_ε has the interior chord-arc property with constant λ_0. That is, if $p, q \in G_\varepsilon$ then there is an arc in G_ε joining p to q whose length is at most $\lambda_0 |q - p|$. From the definition of F_ε and the fact that, by (3.9), H is bounded on G_2 it follows that there is an $\varepsilon_1 = \varepsilon_1(\alpha) > 0$ such that

$$|F_\varepsilon(z) - z| \leq \frac{1}{4\lambda_0}|z| \quad \text{for all } z \in G_2 \text{ and all } 0 \leq \varepsilon \leq \varepsilon_1. \tag{3.14}$$

On the other hand, it follows from (3.8) that given $\varepsilon > 0$ there is some $K_1 = K_1(\varepsilon) \geq 4$ depending only on ε and such that, for all $z \in G_1$ for which $|z| \geq K_1$, we have $\Im\{\varepsilon H(z)\} \geq 2\pi$. Thus, if $\operatorname{diam}(G_1) \geq K_1$ then there is a point

$$z_1 \in G_1 \quad \text{with } |z_1| = K_1 \text{ and } \Im\{\varepsilon H(z_1)\} \geq 2\pi. \tag{3.15}$$

From this it follows that as t varies from 0 to 1, $F_{\varepsilon t}(z_1)$ describes a spiral such that the segment $[0, F_{\varepsilon t}(z_1)]$ turns through an angle of at least 2π radians. Since $H(G) \subset S_0$, this spiral lies in the (thin) annulus

$$A = \{z : |z_1| e^{-\varepsilon} \leq |z| \leq |z_1| e^{\varepsilon}\}.$$

Let $\varepsilon_2 = \varepsilon_2(\alpha) > 0$ be so small that

$$e^{\varepsilon_2} - e^{-\varepsilon_2} < \frac{1}{4\lambda_0}.$$

Then

$$e^{\varepsilon_2} - e^{-\varepsilon_2} + \frac{1}{4\lambda_0} < \frac{1}{2\lambda_0}.$$

Finally, let $\varepsilon = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, so that ε depends only on α. If both $\operatorname{diam}(G_1)$ and $\operatorname{diam}(G_2)$ are greater than $K_1(\varepsilon)$, then there is a point z_1 satisfying (3.15) and a point

$$z_2 \in \partial\Delta(0, |z_1|) \cap G_2.$$

Since, by (3.14),

$$|F_{\varepsilon t}(z_2) - z_2| \le \frac{1}{4\lambda_0}|z_2|$$

and since, for some $t \in [0,1]$, $\arg\{F_{\varepsilon t}(z_1)\} = \arg\{z_2\}$, we have

$$\begin{aligned} |F_{\varepsilon t}(z_2) - F_{\varepsilon t}(z_1)| &\le |F_{\varepsilon t}(z_2) - z_2| + |z_2 - F_{\varepsilon t}(z_1)| \\ &< \frac{1}{4\lambda_0}|z_2| + (e^{\varepsilon_2} - e^{-\varepsilon_2})|z_2| < \frac{|z_2|}{2\lambda_0} = \frac{|z_1|}{2\lambda_0}. \end{aligned}$$

But by (3.13), any arc in $F_{\varepsilon t}(G)$ that joins $F_{\varepsilon t}(z_2)$ and $F_{\varepsilon t}(z_1)$ has length at least e^{-1} times the length of its inverse image. However, the length of any arc joining z_1 to z_2 in G is at least $2(|z_1| - 1)$, so that

$$\frac{2}{e}(|z_1| - 1) \le \lambda_0|F_{\varepsilon t}(z_2) - F_{\varepsilon t}(z_1)| \le \frac{|z_1|}{2};$$

that is,

$$\frac{4}{e} \le \frac{|z_1|}{|z_1| - 1} \le \frac{4}{3},$$

since $|z_1| \ge K_1 \ge 4$. This is a contradiction, so at least one of $\operatorname{diam}(G_1)$ and $\operatorname{diam}(G_2)$ must be at most $K_1(\varepsilon)$. Since ε depends only on α and given our normalizing assumption $|b - a| = 2$, we have thus shown that G is a John disk with a constant $L' = K_1(\varepsilon)/2$ that depends only on α. □

4. The Case of Nonvertical Strips

The well-known Noshiro–Warschawski theorem [No; W] states that if G is convex and $f'(G)$ is contained in any half-plane whose boundary contains 0, then f is univalent, so that $\log f'(G) \subset \frac{\pi}{2}iS_0$ is a univalence criterion for any such G. Because S_0 is a convex domain that is not a John disk, the case $G = S_0$ itself shows that, for pure imaginary α, the existence of a univalence criterion of the form (1.7) does not imply that G is a quasidisk.

THEOREM 3. *For any $\alpha \in \partial\Delta\backslash\{1,-1\}$, there is a simply connected domain $G \ne \mathbb{C}$ that is not a John disk but for which*

$$\log f'(G) \subset \xi\alpha S_0 \tag{4.1}$$

is a univalence criterion for G for some $\xi > 0$.

Proof. The cases of $\alpha = \pm i$ have already been dealt with. For simplicity we assume that $-\frac{\pi}{2} < \arg\{\alpha\} < 0$; the case $0 < \arg\{\alpha\} < \frac{\pi}{2}$ requires only minor notational changes. Let $\eta > 0$ be such that the intersection of $\eta\alpha S_0$ with any vertical line has length at most $\frac{\pi}{2}$. (The largest possible value of η is $\frac{\pi}{4}|\Im\{\alpha\}|$, but this is immaterial.) Then e^z is one-to-one on $\eta\alpha S_0 \cup (i\pi + \eta\alpha S_0)$, so that $e^{\eta\alpha S_0} \cap -e^{\eta\alpha S_0} = \emptyset$. We set

$$P_\delta = \bigcup\{e^{it\alpha}\Delta(1,\delta) : t \in \mathbb{R}\} = \bigcup\{\Delta(e^{it\alpha}, |e^{it\alpha}|\delta) : t \in \mathbb{R}\}.$$

It is easy to see that there is some $\delta_0 \in (0, \eta)$ such that

$$e^{(\delta/2)\alpha S_0} \subset P_\delta \subset e^{\eta\alpha S_0} \quad \text{for } 0 < \delta \le \delta_0. \tag{4.2}$$

Henceforth we consider only $\delta \in (0, \delta_0]$. We define

$$G_\delta = P_\delta \cup \Delta \cup -P_\delta. \tag{4.3}$$

From (4.2) it follows that $P_\delta \cap -P_\delta = \emptyset$ for $\delta \in (0, \delta_0]$. The domains P_δ and $-P_\delta$ spiral outward from the origin to infinity. We will show that there is a $\delta \in (0, \delta_0]$ such that for G_δ, which is clearly not a John disk, (4.1) is a univalence criterion for all sufficiently small ξ. Briefly, this is so because (a) for all sufficiently small ξ, (4.1) with $G = P_\delta \cup \Delta$ or $G = -P_\delta \cup \Delta$ implies univalence, since each of these domains is a quasidisk; and (b) for all sufficiently small ξ, any function satisfying (4.1) and $f(0) = 0$ maps P_δ into a domain very close (in the appropriate sense) to P_δ, so that it stays away from $-P_\delta$ (which obviously has the analogous property).

We set

$$L_0 = e^{i\alpha\mathbb{R}} = \{e^{i\alpha t} : t \in \mathbb{R}\}.$$

Clearly, for any $t \in \mathbb{R}$, P_δ and L_0 are mapped onto themselves by the function $e^{i\alpha t}z$. Let

$$\mathcal{F}(\varepsilon, \delta) = \{f : \log f'(P_\delta) \subset \varepsilon\alpha S_0\}.$$

Since P_δ is a quasidisk, it follows (cf. the explanation of (1.7)) that there is an $\varepsilon(\delta) > 0$ such that all $f \in \mathcal{F}(\varepsilon(\delta), \delta)$ are univalent. Suppose $\varepsilon \in (0, \varepsilon(\delta))$ and $f \in \mathcal{F}(\varepsilon, \delta)$, and set

$$g(z) = \frac{f(z)}{f'(1)}.$$

By (4.2), $e^{(\delta/2)\alpha S_0} \subset P_\delta$ and the appropriate branch of $\log g'$ maps $e^{(\delta/2)\alpha S_0}$ into $2\varepsilon\alpha S_0$ with $\log g'(1) = 0$. The function

$$h(z) = \frac{1}{2\varepsilon\alpha} \log g'(e^{(\delta/2)\alpha z})$$

maps S_0 into itself with $h(0) = 0$. Because the function $H(z) = -\frac{2i}{\pi} \log \frac{1+z}{1-z}$ maps Δ one-to-one onto S_0, we have

$$h \circ H(z) = H \circ \omega(z),$$

where $\omega\colon \Delta \to \Delta$ with $\omega(0) = 0$. If we express it as $H(x)$, $x \in (-1, 1)$, then

$$\begin{aligned} |h(it)| &= \left| h\left(-\frac{2i}{\pi} \log \frac{1+x}{1-x} \right) \right| = \left| -\frac{2i}{\pi} \log \frac{1+\omega(x)}{1-\omega(x)} \right| \\ &\le 1 + \frac{2}{\pi} \log \frac{1+|x|}{1-|x|} = 1 + |H(x)| = 1 + |t|, \end{aligned}$$

where the inequality follows from the Schwarz lemma together with the fact that $\left|\arg\left\{\frac{1+\omega(z)}{1-\omega(z)}\right\}\right| < \frac{\pi}{2}$. Thus $|h(it)| \le 1 + |t|$ for all $t \in \mathbb{R}$, so that

$$|\log g'(e^{i\alpha t})| \le 2\varepsilon\left(1 + \tfrac{2}{\delta}|t|\right)$$

and therefore

$$|f'(e^{i\alpha t})| \le |f'(1)|e^{4\varepsilon|t|/\delta}e^{2\varepsilon}.$$

We set $w(t) = f(e^{i\alpha t})$, so that $w'(t) = i\alpha f'(e^{i\alpha t})e^{i\alpha t}$; consequently, by the foregoing bound we have

$$|w'(t)| \le |f'(1)|e^{4\varepsilon|t|/\delta + t5\tau_0}e^{2\varepsilon},$$

where $5\tau_0 = -\Im\{\alpha\} > 0$. Thus $\int_{-\infty}^{0}|w'(t)|\,dt$ converges as long as $4\varepsilon/\delta < 5\tau_0$, so that $\lim_{t\to-\infty} f(e^{i\alpha t})$ exists for $0 < \varepsilon \le \tau_0\delta$. We call this limit $f(0)$. Now let $\varepsilon_1(\delta) = \min\{\varepsilon(\delta), \tau_0\delta\}$. To normalize, we assume without loss of generality that $f \in \mathcal{F}(\varepsilon_1(\delta),\delta)$ has $f(0) = 0$. With this convention, for any $M, r > 0$ the family

$$\{f \in \mathcal{F}(\varepsilon_1(\delta),\delta) : |f'(1)| \le M\}$$

is compact with respect to uniform convergence on the curve $(L_0 \cup \{0\}) \cap r\Delta$ and, in addition, $0 \in \partial f(P_\delta)$ for all $f \in \mathcal{F}(\varepsilon_1(\delta),\delta)$.

LEMMA 2. *With notation as before, for each* $\delta \in (0,\delta_0]$ *and each* $\tau > 0$ *there is an* $\varepsilon_2(\delta,\tau) \le \varepsilon_1(\delta)$ *such that, for all* $f \in \mathcal{F}(\varepsilon_2(\delta,\tau),\delta)$,

$$|f(e^{it\alpha}) - e^{is\alpha}| < \tau|e^{is\alpha}| \quad \textit{for all } t \in \mathbb{R}, \tag{4.4}$$

where $s = s(t) \in \mathbb{R}$ *is defined by* $|f(e^{it\alpha})| = |e^{is\alpha}|$.

Proof. We use a compactness argument. Suppose, to the contrary, that there are real sequences $\{\varepsilon_n\} \subset (0,\varepsilon_1(\delta))$, $\varepsilon_n \to 0$, and $\{t_n\}$ as well as functions $f_n \in \mathcal{F}(\varepsilon_n,\delta)$ for which

$$|f_n(e^{it_n\alpha}) - e^{is_n\alpha}| \ge \tau|e^{is_n\alpha}|,$$

where $s_n = s(t_n)$. The functions

$$g_n(z) = f_n(e^{it_n\alpha}z)/e^{is_n\alpha}$$

are univalent in P_δ with $|g_n(1)| = 1$ and $g_n(0) = 0$. Since $0 \in \partial g_n(P_\delta)$, it follows from the fact that $\operatorname{dist}(1,\partial P_\delta) = \delta$ and the $\frac{1}{4}$-theorem that $|g_n(1)/\delta g_n'(1)| \ge \frac{1}{4}$, so that $|g_n'(1)| \le \frac{4}{\delta}$. Thus $\{g_n\}$ is a normal family of functions with

$$g_n \in \mathcal{F}(\varepsilon_n,\delta) \quad \text{and} \quad |g_n(1) - 1| \ge \tau.$$

There is a subsequence of $\{g_n\}$ that tends locally uniformly in P_δ and uniformly on $(L_0 \cup \{0\}) \cap 2\Delta$ to a function g_0 with $|g_0(1)| = 1$, $g_0(0) = 0$, and $|g_0(1) - 1| \ge \tau$. But since $\log g_n'(P_\delta) \subset \varepsilon_n\alpha S_0$ and $\varepsilon_n \to 0$ as $n \to \infty$, it follows that g_0' must have a constant value $c \in (L_0 \cup \{0\}) \cap \frac{4}{\delta}\Delta$. Since $|g_0(1)| = 1$ and $g_0(0) = 0$, we have $|c| = 1$. Since we are assuming that $\alpha \notin \mathbb{R}$, we have $L_0 \cap \partial\Delta = \{1\}$, so that, in fact, $c = 1$. But then $g_0(z) = z$, which is a contradiction because $|g_0(1) - 1| \ge \tau$. □

We return now to the proof of Theorem 3. An almost identical compactness argument shows that for each $\delta \in (0,\delta_0]$ and each $\tau > 0$ there is an $\varepsilon_3(\delta,\tau) \le \varepsilon_2(\delta,\tau)$ such that, for all $f \in \mathcal{F}(\varepsilon_3(\delta,\tau),\delta)$,

$$\left|\frac{e^{it\alpha}f'(e^{it\alpha})}{e^{is\alpha}} - 1\right| < \tau \quad \text{for all } t \in \mathbb{R}, \tag{4.5}$$

where $s = s(t)$ is, as before, the number such that $|f(e^{it\alpha})| = |e^{is\alpha}|$.

Now (1.6) shows that if h is analytic in $\Delta(a,\rho)$ and if $h(\Delta(a,\rho)) \subset \varepsilon\alpha S_0$, then $|h'(a+z)| \le \frac{K\varepsilon}{\rho-|z|}$ for $|z|<\rho$. Here, $K=\frac{4}{\pi}$. Thus, if

$$\log f'(\Delta(a,\rho)) \subset \varepsilon\alpha S_0$$

then

$$\left|\frac{f''}{f'}(a+z)\right| \le \frac{K\varepsilon}{\rho-|z|}.$$

From this we obtain

$$\left|\log\frac{f'(a+z)}{f'(a)}\right| \le -K\varepsilon\log\frac{\rho-|z|}{\rho} = \log\left(1-\frac{|z|}{\rho}\right)^{-K\varepsilon},$$

so that

$$|f'(z+a)| \le |f'(a)|\left(1-\frac{|z|}{\rho}\right)^{-K\varepsilon},$$

from which in turn it follows on integrating that

$$|f(a+z)-f(a)| \le \frac{\rho|f'(a)|}{1-K\varepsilon} \tag{4.6}$$

for $|z|\le\rho$.

Let $\delta\in(0,\delta_0]$, $f\in\mathcal{F}(\varepsilon_3(\delta,\tau),\delta)$, $\varepsilon\le\varepsilon_3(\delta,\tau)$, and $z\in\Delta(e^{it\alpha},|e^{it\alpha}|\delta)$. Then by (4.6) and Lemma 2 we have that

$$\begin{aligned}|f(z)-e^{is\alpha}| &\le |f(z)-f(e^{it\alpha})|+|f(e^{it\alpha})-e^{is\alpha}|\\ &\le \frac{|e^{it\alpha}||f'(e^{it\alpha})|\delta}{1-K\varepsilon}+\tau|e^{is\alpha}|.\end{aligned} \tag{4.7}$$

But from (4.5) it follows that $|e^{it\alpha}f'(e^{it\alpha})-e^{is\alpha}|<\tau|e^{is\alpha}|$, so that

$$|e^{it\alpha}f'(e^{it\alpha})|<(\tau+1)|e^{is\alpha}|.$$

Therefore, in light of (4.7), we have

$$|f(z)-e^{is\alpha}| \le \left(\frac{(\tau+1)\delta}{1-K\varepsilon}+\tau\right)|e^{is\alpha}|. \tag{4.8}$$

Let $2\delta<\delta_0$. Then, for $\tau>0$ sufficiently close to 0 and $\varepsilon\in(0,\varepsilon_3(\delta,\tau))$ also sufficiently small, it follows that $\frac{(\tau+1)\delta}{1-K\varepsilon}+\tau<\delta_0$, so that (4.8) and (4.2) imply $f(z)\in e^{\eta\alpha S_0}$. In other words, for these values of δ, τ, and ε,

$$f(P_\delta)\subset e^{\eta\alpha S_0}. \tag{4.9}$$

Obviously, an analogous statement holds for $-P_\delta$. But since $\Delta\cup P_\delta$ and $-P_\delta\cup\Delta$ are quasidisks, there is some $\xi\in(0,\varepsilon]$ for which (4.1) with $G=P_\delta\cup\Delta$ or $G=-P_\delta\cup\Delta$ implies univalence. If $\log f'(G_\delta)\subset\xi\alpha S_0$, then the only way such an f could fail to be univalent would be for $f(p_1)=f(p_2)$, where $p_1\in P_\delta\backslash\Delta$ and $p_2\in(-P_\delta)\backslash\Delta$. But this is impossible by (4.9) and its analogue for $-P_\delta$, since $e^{\eta\alpha S_0}\cap -e^{\eta\alpha S_0}=\emptyset$. Thus, indeed, $\log f'(G_\delta)\subset\xi\alpha S_0$ is a univalence criterion for $G=G_\delta$. Since G_δ is obviously not a John disk, the proof of Theorem 3 is complete. □

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