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The Möbius Geometry of Hypersurfaces

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1. Introduction

Suppose r is a defining function for a twice differentiable hypersurface $M^{2n-1} \subset \mathbb{C}^n$ near $p \in M$. In complex form, the Taylor expansion for r is given by

$$r(p+t) = r(p) + 2\,\text{Real} \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p) t_j + L_{r,p}(t, \bar{t}) + \text{Real}\, Q_{r,p}(t,t) + o(|t|^2),$$

where $t = (t_1, \dots, t_n)$,

$$L_{r,p}(s, \bar{t}) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) s_j \bar{t}_k,$$

and

$$Q_{r,p}(s,t) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(p) s_j t_k.$$

It is a familiar fact in several complex variables that the hermitian quadratic form $L_{r,p}$ is invariant under biholomorphism. (Restricted to the complex tangent space, this is exactly the Levi form.) It is less familiar that the non-hermitian form $Q_{r,p}$ is invariant under Möbius transformation when restricted to the complex tangent space. This is established in Section 2.

Our main result is the following.

Theorem 1. *Suppose that $M^{2n-1} \subset \mathbb{C}^n$ is a non–Levi-flat, three times differentiable hypersurface and that, for all $p \in M$,*

$$Q_{r,p}(s,s) = 0 \quad \text{for } s = (s_1, \dots, s_n) \text{ with } \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p) s_j = 0. \tag{1}$$

Then M is contained in a hermitian quadric surface in $\mathbb{C}^n$.

Condition (1) is independent of the choice of defining function.

The proof of Theorem 1 uses the structural equations for a hypersurface and is similar to a proof the author used for an earlier characterization of the Bochner–Martinelli kernel [2]. An earlier analytic proof of Theorem 1 that requires the

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hypersurface to be eight times differentiable was given by Detraz and Trépreau [4]. They also characterized the situation for Levi-flat hypersurfaces as follows.

PROPOSITION 1 [4]. *Suppose that $M^{2n-1} \subset \mathbb{C}^n$ is twice differentiable and Levi-flat and that, for all $p \in M$,*

$$Q_{r,p}(s,s) = 0 \quad \text{for } s = (s_1, \dots, s_n) \text{ with } \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p) s_j = 0.$$

Then M is foliated by (the germs of) complex hyperplanes.

As applications of Theorem 1, we will prove the following local versions of results obtained by the author in [3]. The first application also extends a result proved by Boas [1] and Wegner [8] to the case of a weighted measure. In our usage, the weights will be positive, twice differentiable functions.

THEOREM 2. *Let $M^{2n-1} \subset \mathbb{C}^n$ be a non–Levi-flat, three times differentiable hypersurface. Then there is a positive measure on M for which the Bochner–Martinelli transform is self-adjoint if and only if M is contained in a hermitian quadric surface.*

For the Levi-flat case, only two derivatives are needed, and M must be foliated by complex hyperplanes as follows from the Detraz and Trépreau result.

THEOREM 3. *Let $M^{2n-1} \subset \mathbb{C}^n$ be a three times differentiable, lineally convex hypersurface. Then there is a positive measure on M for which the Leray–Aĭzenberg transform is self-adjoint if and only if M is contained in the Möbius image of a sphere.*

(The Leray–Aĭzenberg transform is the integral operator whose kernel is constructed using the supporting hyperplanes.)

It would be an interesting problem to estimate the norms of the Bochner–Martinelli transform and the Leray–Aĭzenberg transform in terms of invariant quantities derived from the quadratic forms L and Q.

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2. Möbius Invariance of the Second Fundamental Form

In this section we establish the biholomorphic invariance of L and the additional Möbius invariance of Q when restricted to the complex tangent space. We also demonstrate that the vanishing of Q on the complex tangent space is independent of the choice of defining function.

By a Möbius transformation on $\mathbb{C}^n$ we mean that, after embedding $\mathbb{C}^n$ in $\mathbb{CP}^n$ in the usual way, the transformation acts linearly in the homogeneous coordinates. Alternatively, a Möbius transformation is a fractional linear transformation.

DEFINITION. A Möbius transformation is a function $F = (f_1, \ldots, f_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ where $f_j = g_j/g_{n+1}$,

$$g_j(z) = a_{j,1}z_1 + \cdots + a_{j,n}z_n + a_{j,n+1},$$

and $\det(a_{j,k})_{j,k=1,\ldots,n+1} = 1$.

The condition $\det(a_{j,k}) = 1$ acts only as a normalization. Indeed, if $\det(a_{j,k}) \neq 0$ then one can divide the rows of $(a_{j,k})$ by an appropriate constant in order to make $\det(a_{j,k}) = 1$. This has no affect on the transformation itself. Under composition, Möbius transformations form a group that acts on $\mathbb{C}^n$ and is isomorphic to $\mathrm{SL}_{n+1}(\mathbb{C})$.

The complex tangent space at p consists of those vectors $s = (s_1, \ldots, s_n)$ for which $\sum_j (\partial r/\partial z_j)(p)s_j = 0$. This subspace of the tangent space is independent of the choice of defining function.

PROPOSITION 2. *Suppose that $M^{2n-1} \subset \mathbb{C}^n$ is twice differentiable near $p \in M$ and that $w = F(z)$ is biholomorphic in a neighborhood U of p. Let $r \in C^2(U)$ be a defining function for M near p. Then $M' = F(M \cap U)$ is twice differentiable and has defining function $r \circ F^{-1}$, and*

$$L_{r,p}(s, \bar{t}) = L_{r\circ F^{-1}, F(p)}(F'(p)s, \overline{F'(p)t}) \tag{2}$$

for all $s, t \in \mathbb{C}^n$. In addition, if F is a Möbius transformation and if s and t are in the complex tangent space of M, then

$$Q_{r,p}(s, t) = Q_{r\circ F^{-1}, F(p)}(F'(p)s, F'(p)t). \tag{3}$$

Proof. Suppose that $F = (f_1, \ldots, f_n)$. Then a direct computation shows that (2) is valid:

$$L_{r,p}(s, \bar{t}) = \sum_{j,k} \frac{\partial^2((r \circ F^{-1}) \circ F)}{\partial z_j \partial \bar{z}_k} s_j \bar{t}_k = \sum_{j,k,l,m} \frac{\partial^2(r \circ F^{-1})}{\partial w_l \partial \bar{w}_m} \left(\frac{\partial f_l}{\partial z_j} s_j\right)\left(\overline{\frac{\partial f_m}{\partial z_k}} \bar{t}_k\right),$$

where the partial derivatives are evaluated at p or $F(p)$ as appropriate. The right-hand side of this equation is exactly $L_{r\circ F^{-1}, F(p)}(F'(p)s, \overline{F'(p)t})$.

Likewise, working with the left-hand side of (3) yields

$$\begin{aligned} Q_{r,p}(s, t) &= \sum_{j,k} \frac{\partial^2((r \circ F^{-1}) \circ F)}{\partial z_j \partial z_k} s_j t_k \\ &= \sum_{j,k,l,m} \frac{\partial^2(r \circ F^{-1})}{\partial w_l \partial w_m} \left(\frac{\partial f_l}{\partial z_j} s_j\right)\left(\frac{\partial f_m}{\partial z_k} t_k\right) + \sum_{j,k,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \frac{\partial^2 f_l}{\partial z_j \partial z_k} s_j t_k, \end{aligned}$$

where the partial derivatives are evaluated at p or $F(p)$ as appropriate. The first summation on the right-hand side of this equation is $Q_{r\circ F^{-1}, F(p)}(F'(p)s, F'(p)t)$. Hence we need only check that, for a Möbius transformation F and for vectors s and t in the complex tangent space,

$$\sum_{j,k,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \frac{\partial^2 f_l}{\partial z_j \partial z_k} s_j t_k = 0.$$

So suppose $f_j = g_j/g_{n+1}$ $(1 \le j \le n)$, $g_j(z) = a_{j,1}z_1 + \cdots + a_{j,n}z_n + a_{j,n+1}$ $(1 \le j \le n+1)$, and $\det(a_{j,k}) = 1$. A straightforward computation then shows that

$$\sum_{j,k,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \frac{\partial^2 f_l}{\partial z_j \partial z_k} s_j t_k$$
$$= \sum_{j,k,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \left(-a_{l,j} \frac{a_{n+1,k}}{g_{n+1}^2} - a_{l,k} \frac{a_{n+1,j}}{g_{n+1}^2} + 2g_l \frac{a_{n+1,j} a_{n+1,k}}{g_{n+1}^3} \right) s_j t_k. \quad (4)$$

Moreover, since s and t are in the complex tangent space, we have

$$\sum_j \frac{\partial r}{\partial z_j} s_j = \sum_{j,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \frac{\partial f_l}{\partial z_j} s_j = \sum_{j,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \left(\frac{a_{l,j}}{g_{n+1}} - g_l \frac{a_{n+1,j}}{g_{n+1}^2} \right) s_j = 0$$

and

$$\sum_k \frac{\partial r}{\partial z_k} t_k = \sum_{k,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \frac{\partial f_l}{\partial z_k} t_k = \sum_{k,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \left(\frac{a_{l,k}}{g_{n+1}} - g_l \frac{a_{n+1,k}}{g_{n+1}^2} \right) t_k = 0.$$

Using these last identities along with (4), we conclude that

$$\sum_{j,k,l} \frac{\partial(r \circ F^{-1})}{\partial w_l} \frac{\partial^2 f_l}{\partial z_j \partial z_k} s_j t_k$$
$$= -\left(\sum_j \frac{\partial r}{\partial z_j} s_j \right) \sum_k \frac{a_{n+1,k}}{g_{n+1}} t_k - \sum_j \frac{a_{n+1,j}}{g_{n+1}} s_j \left(\sum_k \frac{\partial r}{\partial z_k} t_k \right) = 0.$$

Thus the proposition is proved. □

PROPOSITION 3. *Let r and $\tilde{r}$ be defining functions for a twice differentiable hypersurface $M^{2n-1} \subset \mathbb{C}^n$ with $\tilde{r} = h \cdot r$ for a twice differentiable function $h > 0$. Then $Q_{\tilde{r},p}(s,t) = h \cdot Q_{r,p}(s,t)$ for vectors s and t in the complex tangent space. In particular, if $Q_{r,p}(s,t) = 0$ then also $Q_{\tilde{r},p}(s,t) = 0$.*

Proof. This follows readily from the calculation

$$Q_{h \cdot r,p}(s,t) = \sum_{j,k=1}^{n} \frac{\partial^2 (h \cdot r)}{\partial z_j \partial z_k} s_j t_k$$
$$= \sum_{j,k=1}^{n} \left(\frac{\partial^2 h}{\partial z_j \partial z_k} r + \frac{\partial h}{\partial z_j} \frac{\partial r}{\partial z_k} + \frac{\partial h}{\partial z_k} \frac{\partial r}{\partial z_j} + h \frac{\partial^2 r}{\partial z_j \partial z_k} \right) s_j t_k.$$

The first terms in the sum vanish because $r = 0$ on M, and the second and third terms vanish because, respectively, t and s are in the complex tangent space. The remaining terms are exactly $h \cdot Q_{r,p}(s,t)$. □

3. Normalization

PROPOSITION 4. *Suppose that a twice differentiable hypersurface $M^{2n-1} \subset \mathbb{C}^n$ has defining function r with nonzero gradient and that, at a fixed $p \in M$,*

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(p) s_j s_k = 0 \quad \textit{for } s = (s_1, \ldots, s_n) \textit{ with } \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p) s_j = 0.$$

Then there are a Möbius transformation F with $F(p) = 0$ and $\varepsilon_j \in \{-1, 0, +1\}$ such that $M' = F(M)$ can be defined by

$$r'(z) = \frac{1}{2}(z_n + \bar{z}_n) + \sum_{j=1}^{n-1} \varepsilon_j |z_j|^2 + o(|z|^2). \tag{5}$$

Proof. To begin, we use a preliminary Möbius transformation (composed of a translation and rotation); this allows us to assume that $p = 0$ and that M can be defined by

$$r(z) = \frac{1}{2}(z_n + \bar{z}_n) + \sum_{j,k=1}^{n} b_{j,k} z_j \bar{z}_k + \text{Real} \sum_{j=1}^{n} c_j z_j z_n + o(|z|^2).$$

Here $b_{j,k} \in \mathbb{C}$ with $b_{j,k} = \bar{b}_{k,j}$ for $1 \le j, k \le n$, and $c_j \in \mathbb{C}$ for $1 \le j \le n$. In this situation, the hypothesis of the proposition is that $r_{jk}(0) = 0$ for $1 \le j, k < n$, so there are no $z_j z_k$ terms that appear in r for $1 \le j, k < n$. (We reiterate that the hypothesis of the theorem is preserved by Möbius transformation and is independent of the choice of defining function.)

We need to identify a further Möbius transformation F so that $F(0) = 0$ and the surface $M' = F(M)$ can be defined as in (5). We mention that subscripts on defining functions will always refer to partial derivatives.

Before doing so, we multiply the defining function r by the positive function

$$h(z) = 1 - 2\,\text{Real}(2b_{1,n} z_1 + \cdots + 2b_{n-1,n} z_{n-1} + b_{n,n} z_n) + o(|z|)$$

and continue to call the new defining function r. This has the effect of eliminating the $z_j \bar{z}_n$ and $\bar{z}_j z_n$ terms while introducing possibly new constants c_j. The same surface M is then defined more economically by

$$r(z) = \frac{1}{2}(z_n + \bar{z}_n) + \sum_{j,k=1}^{n-1} b_{j,k} z_j \bar{z}_k + \text{Real} \sum_{j=1}^{n} c_j z_j z_n + o(|z|^2). \tag{6}$$

We now proceed to identify the transformation F so that (5) holds for $r' = r \circ F^{-1}$.

To do this, suppose that $F^{-1} = (f_1, \ldots, f_n)$ where $f_j = g_j / g_{n+1}$ $(1 \le j \le n)$ and $g_j(z) = a_{j,1} z_1 + \cdots + a_{j,n} z_n + a_{j,n+1}$ $(1 \le j \le n+1)$. To make $F(0) = 0$, it is necessary that $a_{j,n+1} = 0$ for $j \le n$. We choose the normalization $a_{n+1,n+1} = 1$ rather than the usual normalization $\det(a_{j,k}) = 1$. This does not affect the set of transformations, but it does simplify the subsequent computations. Then, evaluated at 0, we find

$$\frac{\partial f_l}{\partial z_j} = \frac{\partial g_l}{\partial z_j} \frac{1}{g_{n+1}} - \frac{\partial g_{n+1}}{\partial z_j} \frac{g_l}{g_{n+1}^2} = a_{l,j} - a_{n+1,j} a_{l,n+1} = a_{l,j},$$

where in the last step we used $a_{l,n+1} = 0$. It follows that, when evaluated at 0,

$$\frac{\partial (r \circ F^{-1})}{\partial z_j} = \sum_{l=1}^{n} \frac{\partial r}{\partial z_l} \frac{\partial f_l}{\partial z_j} = \frac{1}{2} \frac{\partial f_n}{\partial z_j} = \frac{a_{n,j}}{2}.$$

So for F (and therefore F^{-1}) to preserve the tangent plane at 0, it is necessary that $a_{n,j} = 0$ for $j < n$. Again, for simplicity we specify that $a_{n,n} = 1$. We have then specified the $(n+1)$th column and the nth row of the $(n+1) \times (n+1)$ matrix $(a_{j,k})$. In particular,

$$\begin{aligned} a_{j,n+1} &= 0 \quad \text{if } j \le n, \\ a_{n+1,n+1} &= 1, \\ a_{n,j} &= 0 \quad \text{if } j < n, \\ a_{n,n} &= 1. \end{aligned}$$

With these choices, the first-order expansion of $r \circ F^{-1}$ is still as claimed in (5).

To normalize the hermitian quadratic terms, notice first that

$$\frac{\partial^2 (r \circ F^{-1})}{\partial z_j \partial \bar{z}_k} = \sum_{l,m=1}^{n} \frac{\partial^2 r}{\partial z_l \partial \bar{z}_m} \frac{\partial f_l}{\partial z_j} \overline{\frac{\partial f_m}{\partial z_k}} = \sum_{l,m=1}^{n} \frac{\partial^2 r}{\partial z_l \partial \bar{z}_m} a_{l,j} \overline{a_{m,k}}. \tag{7}$$

In particular, since already $a_{n,j} = 0$ for $j < n$, it follows that if $1 \le j, k < n$ then

$$\frac{\partial^2 (r \circ F^{-1})}{\partial z_j \partial \bar{z}_k} = \sum_{l,m=1}^{n-1} \frac{\partial^2 r}{\partial z_l \partial \bar{z}_m} a_{l,j} \overline{a_{m,k}}.$$

We can then choose an invertible submatrix $(a_{j,k})_{j,k=1,\dots,n-1}$ such that $(r \circ F^{-1})_{j\bar{j}} = \varepsilon_j$ for $1 \le j < n$ (where $\varepsilon_j \in \{-1, 0, +1\}$) and $(r \circ F^{-1})_{j\bar{k}} = 0$ otherwise. In fact, the submatrix is the composition of a unitary transformation and an invertible diagonal matrix.

We next determine conditions on the constants $a_{j,n}$, $1 \le j < n$, so that on the right-hand side of (5) there will still be no terms $z_j \bar{z}_n$, $1 \le j < n$. From (7) this means

$$\frac{\partial^2 (r \circ F^{-1})}{\partial z_j \partial \bar{z}_n} = \sum_{l=1}^{n-1} a_{l,j} \left(\sum_{m=1}^{n} \frac{\partial^2 r}{\partial z_l \partial \bar{z}_m} \overline{a_{m,n}} \right) = 0 \quad \text{for } 1 \le j < n \tag{8}$$

because $a_{n,j} = 0$ for $1 \le j < n$. If $(a^{j,k})_{j,k=1,\dots,n-1}$ is the inverse of the submatrix $(a_{j,k})_{j,k=1,\dots,n-1}$ then, after multiplying (8) by $a^{j,k}$ and summing on $1 \le j < n$, we find the equivalent condition

$$\sum_{m=1}^{n} \frac{\partial^2 r}{\partial z_k \partial \bar{z}_m} \overline{a_{m,n}} = \sum_{m=1}^{n-1} \frac{\partial^2 r}{\partial z_k \partial \bar{z}_m} \overline{a_{m,n}} = 0 \quad \text{for } 1 \le k < n. \tag{9}$$

(The first equality in (9) uses (6).) We choose $a_{m,n} = 0$ for $m < n$. Notice also—from (7) and (6) and from our existing choices—that

$$\frac{\partial^2 (r \circ F^{-1})}{\partial z_n \partial \bar{z}_n} = \sum_{l,m=1}^{n} \frac{\partial^2 r}{\partial z_l \partial \bar{z}_m} a_{l,n} \overline{a_{m,n}} = \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n} = 0.$$

So far, then, we have chosen constants so that the hermitian terms on the right-hand side of (5) are as claimed.

We still have to determine constants $a_{n+1,j}$ for $1 \le j \le n$ such that $(r \circ F^{-1})_{nj} = 0$ when these partials are evaluated at 0. To do this, notice that the second-order partial derivatives of the g_j are identically zero. So evaluated at 0, we find

$$\begin{aligned}\frac{\partial^2 f_l}{\partial z_j \partial z_k} &= -\frac{\partial g_l}{\partial z_j}\frac{\partial g_{n+1}}{\partial z_k}\frac{1}{g_{n+1}^2} - \frac{\partial g_{n+1}}{\partial z_j}\frac{\partial g_l}{\partial z_k}\frac{1}{g_{n+1}^2} + \frac{\partial g_{n+1}}{\partial z_j}\frac{\partial g_{n+1}}{\partial z_k}\frac{2g_l}{g_{n+1}^3}\\ &= -a_{l,j}a_{n+1,k} - a_{n+1,j}a_{l,k}.\end{aligned}$$

Because we have just chosen $a_{l,n} = 0$ for $l < n$, we also find that

$$\begin{aligned}\frac{\partial^2 (r \circ F^{-1})}{\partial z_n \partial z_j} &= \sum_{l,m=1}^{n} \frac{\partial^2 r}{\partial z_l \partial z_m}\frac{\partial f_l}{\partial z_n}\frac{\partial f_m}{\partial z_j} + \sum_{m=1}^{n}\frac{\partial r}{\partial z_m}\frac{\partial^2 f_m}{\partial z_n \partial z_j}\\ &= \sum_{m=1}^{n} \frac{\partial^2 r}{\partial z_n \partial z_m}\frac{\partial f_n}{\partial z_n}\frac{\partial f_m}{\partial z_j} + \frac{1}{2}\frac{\partial^2 f_n}{\partial z_n \partial z_j}\\ &= \sum_{m=1}^{n-1}\frac{c_m}{2}a_{m,j} + c_n a_{n,j} - \frac{1}{2}(a_{n+1,j} + a_{n+1,n}a_{n,j}).\end{aligned}$$

In particular, when $j = n$,

$$\frac{\partial^2 (r \circ F^{-1})}{\partial z_n \partial z_n} = \sum_{m=1}^{n-1}\frac{c_m}{2}a_{m,n} + c_n a_{n,n} - \frac{1}{2}(a_{n+1,n} + a_{n+1,n}a_{n,n}) = c_n - a_{n+1,n}.$$

So to make $(r \circ F^{-1})_{nn} = 0$, we choose $a_{n+1,n} = c_n$. Furthermore, if $1 \le j < n$ then

$$\frac{\partial^2 (r \circ F^{-1})}{\partial z_n \partial z_j} = \sum_{m=1}^{n-1}\frac{c_m}{2}a_{m,j} - \frac{1}{2}a_{n+1,j},$$

where we have used $a_{n,j} = 0$. Hence, to make $(r \circ F^{-1})_{nj} = 0$ we also choose

$$a_{n+1,j} = \sum_{m=1}^{n-1} c_m a_{m,j}.$$

The constants on the right-hand side of this equation are already determined. □

4. Restatement of the Vanishing Condition

Our proof of Theorem 1 uses classical differential geometry. We use the following notation, much of which is used in the book by Hicks [6], for instance.

The coordinates $(z_1, \dots, z_n) \in \mathbb{C}^n$ correspond with coordinates $(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$ according to $z_j = x_j + ix_{j+n}$. Under this identification, the real Euclidean space inherits a complex structure $J \colon T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ that corresponds with multiplication by $i = \sqrt{-1}$ and is given by $J(\partial_{x_j}) = \partial_{x_{j+n}}$, $J(\partial_{x_{j+n}}) = -\partial_{x_j}$. This structure preserves the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $T\mathbb{R}^{2n}$. In fact, $J^* = -J$ and $J^2 = -I$. The real tangent space of M is denoted by TM. Then the complex tangent space is the subspace $HM = TM \cap J(TM)$.

Let N be a unit normal vector on M. The direction orthogonal to HM in TM is then JN. For $X \in TM$, let $d = d_X$ be the Riemannian connection that M inherits as a submanifold of $\mathbb{R}^n$. This connection is naturally symmetric and metric, so $[X, Y] = d_X Y - d_Y X$ for $X, Y \in TM$ and $X\langle Y, Z\rangle = \langle d_X Y, Z\rangle + \langle Y, d_X Z\rangle$ for $X, Y, Z \in TM$. The complex structure and the Riemannian connection commute with one another.

The Weingarten map is the operator $S\colon TM \to TM$ given by $X \in TM \to S(X) = d_X N$. This operator is self-adjoint. Connected to S is the second fundamental form, which is the symmetric bilinear form $b(X, Y) = \langle SX, Y\rangle = \langle d_X N, Y\rangle$. The main structural equation for a hypersurface in Euclidean space is the Codazzi equation; it says that, for $X, Y, Z \in TM$,

$$\langle d_X SY - d_Y SX, Z\rangle = \langle S[X, Y], Z\rangle.$$

These equations are the compatibility conditions between the induced metric and the second fundamental form for a surface in Euclidean space.

The following lemma shows how to express the vanishing of Q in this geometric context.

LEMMA 1. *Suppose that $M^{2n-1} \subset \mathbb{C}^n$ is twice differentiable and that, for all $p \in M$,*

$$Q_{r,p}(s, s) = 0 \quad \textit{for } s = (s_1, \ldots, s_n) \textit{ with } \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p) s_j = 0. \tag{10}$$

Then $b(X, JX) = 0$ and $b(X, X) = b(JX, JX)$ for all $X \in HM$.

Proof. We may assume the defining function is normalized so that $|\nabla r| \equiv 2$. In complex notation, $N = (r_{\bar 1}, \ldots, r_{\bar n})$. (The subscripts indicate taking antiholomorphic partial derivatives; the factor of 2 arises from $\partial_{\bar z_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{x_{j+n}})$.) Suppose also that $X = (s_1, \ldots, s_n) \in HM$. Then $JX = (is_1, \ldots, is_n)$ and, using the centered dot to represent the complex dot product, we find

$$\begin{aligned}
b(X, JX) &= \langle d_X N, JX\rangle \\
&= \mathrm{Real}[d_X N \cdot \overline{JX}] \\
&= \mathrm{Real}\Bigg(\sum_{j=1}^{n} (s_j \partial_{z_j} + \bar s_j \partial_{\bar z_j})(r_{\bar 1}, \ldots, r_{\bar n}) \cdot (-i\bar s_1, \ldots, -i\bar s_n)\Bigg) \\
&= \mathrm{Real}\Bigg(\sum_{j,k=1}^{n} -ir_{j\bar k} s_j \bar s_k - ir_{\bar j\bar k} \bar s_j \bar s_k\Bigg) \\
&= \mathrm{Imag}\Bigg(\sum_{j,k=1}^{n} -r_{jk} s_j s_k\Bigg) = 0.
\end{aligned}$$

For the second claim, replace X by $X + JX \in HM$. Then

$$\begin{aligned} 0 &= b(X + JX, J(X + JX)) \\ &= b(X + JX, JX - X) \\ &= b(X, JX) - b(X, X) + b(JX, JX) - b(JX, X) \\ &= -b(X, X) + b(JX, JX). \end{aligned}$$

This proves the lemma. □

We mention that Hermann [5] proved an analogous result for the Levi form; namely, $\mathcal{L}(X, Y) = b(X, Y) + b(JX, JY)$ for $X, Y \in HM$.

5. Proof in Dimension Two

In complex dimension two, the vanishing condition says that the second fundamental form for $M^3 \subset \mathbb{C}^2$ can be given by the 3×3 matrix of real functions

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \lambda & 0 \\ \gamma & 0 & \lambda \end{pmatrix}.$$

The rows and columns correspond to the vectors JN, X, and JX, respectively, where $X \in HM$. These vectors can be assumed to have unit length. The Weingarten map is then given by

$$\begin{aligned} S(JN) &= \alpha JN + \beta X + \gamma JX, \\ S(X) &= \beta JN + \lambda X, \\ S(JX) &= \gamma JN + \lambda JX. \end{aligned}$$

We choose X (and therefore JX) as in the following lemma. Then the connection on M can be described quite simply in terms of the second fundamental form.

Lemma 2. *Suppose $M^3 \subset \mathbb{C}^2$ is defined by $r = r(z_1, z_2)$, which is normalized so that $|\nabla r| \equiv 2$. In complex notation, $N = (r_{\bar{1}}, r_{\bar{2}})$ and $JN = (ir_{\bar{1}}, ir_{\bar{2}})$. The complex tangent space is spanned by $X = (r_2, -r_1)$ and $JX = (ir_2, -ir_1)$. Furthermore, if $Y \in TM$ then $\langle d_Y X, JX\rangle = -\langle JN, d_Y N\rangle$. In particular:*

$$\begin{aligned} \langle d_{JN} X, JX\rangle &= -\alpha, \\ \langle d_X X, JX\rangle &= -\beta, \\ \langle d_{JX} X, JX\rangle &= -\gamma; \end{aligned}$$

and $d_X JX = -\lambda JN + \beta X$ and $d_{JX} X = \lambda JN - \gamma JX$.

Proof. Again using the dot to represent the complex dot product, we find

$$\begin{aligned} \langle d_Y X, JX\rangle = \mathrm{Real}[Y(X) \cdot \overline{JX}] &= \mathrm{Real}[Y(r_2, -r_1) \cdot (-ir_{\bar{2}}, ir_{\bar{1}})] \\ &= -\mathrm{Real}[Y(r_2, r_1) \cdot (ir_{\bar{2}}, ir_{\bar{1}})] \\ &= -\mathrm{Real}[(ir_{\bar{1}}, ir_{\bar{2}}) \cdot Y(r_1, r_2)] \\ &= -\mathrm{Real}[JN \cdot \overline{Y(N)}] = -\langle JN, d_Y N\rangle. \end{aligned}$$

The remaining claims are special cases of this fact. □

Using Lemma 2, the Codazzi equation, and the symmetry of the connection, we now prove the following lemma.

LEMMA 3. *Suppose $M^3 \subset \mathbb{C}^2$ is three times differentiable and has second fundamental form as described previously. If $\lambda \neq 0$, then*

$$\begin{aligned} X(\beta) &= -3\beta\gamma, \\ JX(\gamma) &= +3\beta\gamma, \\ X(\gamma) &= -\alpha\lambda + \lambda^2 + 2\beta^2 - \gamma^2, \\ JX(\beta) &= +\alpha\lambda - \lambda^2 + \beta^2 - 2\gamma^2, \\ X(\alpha) &= -\alpha\gamma + 2\gamma\lambda, \\ JX(\alpha) &= +\alpha\beta - 2\beta\lambda, \\ X(\lambda) &= -3\gamma\lambda, \\ JX(\lambda) &= +3\beta\lambda. \end{aligned}$$

Before giving the proof, we make a few extra remarks about simplifying inner products. For instance, in the proof we use repeatedly the derivation property of d and the orthonormality of X, JX, JN. As an example,

$$\begin{aligned} \langle d_X(\alpha JN), X\rangle &= X(\alpha)\langle JN, X\rangle + \alpha\langle d_X JN, X\rangle \\ &= X(\alpha)\cdot 0 + \alpha\langle d_X JN, X\rangle = \alpha\langle d_X JN, X\rangle. \end{aligned}$$

We also use the fact that J commutes with d and is antisymmetric. So

$$\langle d_X JN, X\rangle = \langle J d_X N, X\rangle = -\langle d_X N, JX\rangle = -\langle S(X), JX\rangle = 0$$

and, in particular, $\langle d_X(\alpha JN), X\rangle = \alpha\cdot 0 = 0$, as is used in part (a) of the proof.

Some expressions are simplified by combining the metric property of d with the antisymmetry of J. For instance, in part (c) we use

$$\langle d_{JN} X, JN\rangle = \langle X, -d_{JN} JN\rangle = \langle X, -J d_{JN} N\rangle = \langle JX, d_{JN} N\rangle = \gamma,$$

the first identity coming from $JN\langle X, JN\rangle = 0 = \langle d_{JN} X, JN\rangle + \langle X, d_{JN} JN\rangle$. One more kind of simplification uses the fact that J preserves the inner product. (This also follows from $J^* = -J$ and $J^2 = -I$.) For instance, in part (c) we use

$$\langle d_X JN, JX\rangle = \langle J d_X N, JX\rangle = \langle d_X N, X\rangle = \lambda.$$

We are now set for the proof.

Proof of Lemma 3. We begin by applying the Codazzi equation to all combinations of tangent vectors. In particular, we apply the identity

$$\langle d_X SY - d_Y SX, Z\rangle = \langle d_X Y - d_Y X, SZ\rangle$$

to combinations of vectors X, Y, Z taken from among the special tangent vectors X, JX, JN for the surface M^3. (This reformulated statement of the Codazzi equation follows from the statement in the previous section after using the symmetry of the second fundamental form and the symmetry of the connection.)

(a) X, JN, X:

$$\begin{aligned}\langle d_X S(JN) - d_{JN} S(X), X\rangle &\\ = \langle d_X(\alpha JN + \beta X + \gamma JX) - d_{JN}(\beta JN + \lambda X), X\rangle &\\ = \alpha\langle d_X JN, X\rangle + X(\beta) + \gamma\langle d_X JX, X\rangle - \beta\langle d_{JN} JN, X\rangle - JN(\lambda) &\\ = \alpha\cdot 0 + X(\beta) + \gamma\beta + \beta\gamma - JN(\lambda) &\end{aligned}$$

and

$$\begin{aligned}\langle d_X JN - d_{JN} X, S(X)\rangle &= \langle d_X JN - d_{JN} X, \beta JN + \lambda X\rangle\\ &= \beta\cdot 0 + \lambda\langle d_X JN, X\rangle - \beta\langle d_{JN} X, JN\rangle - \lambda\cdot 0\\ &= \lambda\cdot 0 - \beta\gamma\end{aligned}$$

so that $X(\beta) = JN(\lambda) - 3\beta\gamma$.

(b) JX, JN, JX:

$$\begin{aligned}\langle d_{JX} S(JN) - d_{JN} S(JX), JX\rangle &\\ = \langle d_{JX}(\alpha JN + \beta X + \gamma JX) - d_{JN}(\gamma JN + \lambda JX), JX\rangle &\\ = \alpha\langle d_{JX} N, X\rangle + \beta\langle d_{JX} X, JX\rangle + JX(\gamma) - \gamma\langle d_{JN} N, X\rangle - JN(\lambda) &\\ = \alpha\cdot 0 - \beta\gamma + JX(\gamma) - \gamma\beta - JN(\lambda) &\end{aligned}$$

and

$$\begin{aligned}\langle d_{JX} JN - d_{JN} JX, S(JX)\rangle &= \langle d_{JX} JN - d_{JN} JX, \gamma JN + \lambda JX\rangle\\ &= \gamma\cdot 0 + \lambda\langle d_{JX} N, X\rangle - \gamma\langle d_{JN} X, N\rangle - \lambda\cdot 0\\ &= \lambda\cdot 0 + \gamma\beta\end{aligned}$$

so that $JX(\gamma) = JN(\lambda) + 3\beta\gamma$.

(c) X, JN, JX:

$$\begin{aligned}\langle d_X S(JN) - d_{JN} S(X), JX\rangle &\\ = \langle d_X(\alpha JN + \beta X + \gamma JX) - d_{JN}(\beta JN + \lambda X), JX\rangle &\\ = \alpha\langle d_X N, X\rangle + \beta\langle d_X X, JX\rangle + X(\gamma) - \beta\langle d_{JN} N, X\rangle - \lambda\langle d_{JN} X, JX\rangle &\\ = \alpha\lambda - \beta^2 + X(\gamma) - \beta^2 + \alpha\lambda &\end{aligned}$$

and

$$\begin{aligned}\langle d_X JN - d_{JN} X, S(JX)\rangle &\\ = \langle d_X JN - d_{JN} X, \gamma JN + \lambda JX\rangle &\\ = \gamma\cdot 0 + \lambda\langle d_X N, X\rangle - \gamma\langle d_{JN} X, JN\rangle - \lambda\langle d_{JN} X, JX\rangle = \lambda^2 - \gamma^2 + \alpha\lambda &\end{aligned}$$

so that $X(\gamma) = -\alpha\lambda + \lambda^2 + 2\beta^2 - \gamma^2$.

(d) JX, JN, X:

$$\begin{aligned}\langle d_{JX}S(JN) - d_{JN}S(JX), X\rangle &= \langle d_{JX}(\alpha JN + \beta X + \gamma JX) - d_{JN}(\gamma JN + \lambda JX), X\rangle\\ &= \alpha\langle d_{JX}JN, X\rangle + JX(\beta) + \gamma\langle d_{JX}JX, X\rangle\\ &\quad - \gamma\langle d_{JN}JN, X\rangle - \lambda\langle d_{JN}JX, X\rangle\\ &= -\alpha\lambda + JX(\beta) + \gamma^2 + \gamma^2 - \alpha\lambda\end{aligned}$$

and

$$\begin{aligned}\langle d_{JX}JN - d_{JN}JX, S(X)\rangle &= \langle d_{JX}JN - d_{JN}JX, \beta JN + \lambda X\rangle\\ &= \beta\cdot 0 + \lambda\langle d_{JX}JN, X\rangle - \beta\langle d_{JN}X, N\rangle - \lambda\langle d_{JN}JX, X\rangle\\ &= -\lambda^2 + \beta^2 - \alpha\lambda\end{aligned}$$

so that $JX(\beta) = \alpha\lambda - \lambda^2 + \beta^2 - 2\gamma^2$.

(e) X, JN, JN:

$$\begin{aligned}\langle d_X S(JN) - d_{JN}S(X), JN\rangle &= \langle d_X(\alpha JN + \beta X + \gamma JX) - d_{JN}(\beta JN + \lambda X), JN\rangle\\ &= X(\alpha) + \beta\langle d_X X, JN\rangle + \gamma\langle d_X X, N\rangle - JN(\beta) - \lambda\langle d_{JN}X, JN\rangle\\ &= X(\alpha) + \beta\cdot 0 - \gamma\lambda - JN(\beta) - \lambda\gamma\end{aligned}$$

and

$$\begin{aligned}\langle d_X JN - d_{JN}X, S(JN)\rangle &= \langle d_X JN - d_{JN}X, \alpha JN + \beta X + \gamma JX\rangle\\ &= \alpha\cdot 0 + \beta\langle d_X JN, X\rangle + \gamma\langle d_X N, X\rangle - \alpha\langle d_{JN}X, JN\rangle\\ &\quad - \beta\cdot 0 - \gamma\langle d_{JN}X, JX\rangle\\ &= \beta\cdot 0 + \gamma\lambda - \alpha\gamma + \gamma\alpha\end{aligned}$$

so that $X(\alpha) = JN(\beta) + 3\gamma\lambda$.

(f) JX, JN, JN:

$$\begin{aligned}\langle d_{JX}S(JN) - d_{JN}S(JX), JN\rangle &= \langle d_{JX}(\alpha JN + \beta X + \gamma JX) - d_{JN}(\gamma JN + \lambda JX), JN\rangle\\ &= JX(\alpha) + \beta\langle d_{JX}X, JN\rangle + \gamma\langle d_{JX}X, N\rangle - JN(\gamma) - \lambda\langle d_{JN}X, N\rangle\\ &= JX(\alpha) + \beta\lambda + \gamma\cdot 0 - JN(\gamma) + \beta\lambda\end{aligned}$$

and

$$\begin{aligned}\langle d_{JX}JN - d_{JN}JX, S(JN)\rangle &= \langle d_{JX}JN - d_{JN}JX, \alpha JN + \beta X + \gamma JX\rangle\\ &= \alpha\cdot 0 + \beta\langle d_{JX}JN, X\rangle + \gamma\langle d_{JX}N, X\rangle - \alpha\langle d_{JN}X, N\rangle\\ &\quad - \beta\langle d_{JN}JX, X\rangle - \gamma\cdot 0\\ &= -\beta\lambda + \gamma\cdot 0 + \alpha\beta - \alpha\beta\end{aligned}$$

so that $JX(\alpha) = JN(\gamma) - 3\beta\lambda$.

(g) X, JX, JX:

$$\begin{aligned}
&\langle d_X S(JX) - d_{JX} S(X), JX\rangle \\
&\quad = \langle d_X(\gamma JN + \lambda JX) - d_{JX}(\beta JN + \lambda X), JX\rangle \\
&\quad = \gamma\langle d_X N, X\rangle + X(\lambda) - \beta\langle d_{JX} N, X\rangle - \lambda\langle d_{JX} X, JX\rangle \\
&\quad = \gamma\lambda + X(\lambda) - \beta\cdot 0 + \lambda\gamma
\end{aligned}$$

and

$$\begin{aligned}
&\langle d_X JX - d_{JX} X, S(JX)\rangle \\
&\quad = \langle d_X JX - d_{JX} X, \gamma JN + \lambda JX\rangle \\
&\quad = \gamma\langle d_X X, N\rangle + \lambda\cdot 0 - \gamma\langle d_{JX} X, JN\rangle - \lambda\langle d_{JX} X, JX\rangle \\
&\quad = -\gamma\lambda - \gamma\lambda + \lambda\gamma
\end{aligned}$$

so that $X(\lambda) = -3\gamma\lambda$.

(h) X, JX, X:

$$\begin{aligned}
&\langle d_X S(JX) - d_{JX} S(X), X\rangle \\
&\quad = \langle d_X(\gamma JN + \lambda JX) - d_{JX}(\beta JN + \lambda X), X\rangle \\
&\quad = \gamma\langle d_X JN, X\rangle + \lambda\langle d_X JX, X\rangle - \beta\langle d_{JX} JN, X\rangle - JX(\lambda) \\
&\quad = \gamma\cdot 0 + \beta\lambda + \beta\lambda - JX(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
\langle d_X JX - d_{JX} X, S(X)\rangle &= \langle d_X JX - d_{JX} X, \beta JN + \lambda X\rangle \\
&= \beta\langle d_X X, N\rangle + \lambda\langle d_X JX, X\rangle - \beta\langle d_{JX} X, JN\rangle - \lambda\cdot 0 \\
&= -\beta\lambda + \beta\lambda - \beta\lambda
\end{aligned}$$

so that $JX(\lambda) = 3\beta\lambda$.

So far we have established the following eight equalities:

$$\begin{aligned}
X(\beta) &= JN(\lambda) - 3\beta\gamma, \\
JX(\gamma) &= JN(\lambda) + 3\beta\gamma, \\
X(\gamma) &= -\alpha\lambda + \lambda^2 + 2\beta^2 - \gamma^2, \\
JX(\beta) &= +\alpha\lambda - \lambda^2 + \beta^2 - 2\gamma^2, \\
X(\alpha) &= JN(\beta) + 3\gamma\lambda, \\
JX(\alpha) &= JN(\gamma) - 3\beta\lambda, \\
X(\lambda) &= -3\gamma\lambda, \\
JX(\lambda) &= +3\beta\lambda.
\end{aligned}$$

The lemma will be proved as soon as we verify

(i) $JN(\lambda) = 0$,
(ii) $X(\alpha) = -\alpha\gamma + 2\gamma\lambda$, and
(iii) $JX(\alpha) = \alpha\beta - 2\beta\lambda$.

For this we use the symmetry of the connection. In particular, we apply the identity $d_X JX - d_{JX} X = [X, JX]$ to each of λ, β, and γ. Alternatively, Lemma 2 says $-2\lambda JN + \beta X + \gamma JX = [X, JX]$. We also use the identities that have been proved already.

Proof of (i). Applying the identity to λ, we find

$$(-2\lambda JN + \beta X + \gamma JX)(\lambda) = -2\lambda JN(\lambda) + \beta(-3\gamma\lambda) + \gamma(3\beta\lambda) = -2\lambda JN(\lambda)$$

and

$$\begin{aligned} &X(JX(\lambda)) - JX(X(\lambda)) \\ &\qquad = X(3\beta\lambda) - JX(-3\gamma\lambda) \\ &\qquad = 3[(JN(\lambda) - 3\beta\gamma)\lambda + \beta(-3\gamma\lambda)] + 3[(JN(\lambda) + 3\beta\gamma)\lambda + \gamma(3\beta\lambda)] \\ &\qquad = 6\lambda JN(\lambda), \end{aligned}$$

so $8\lambda JN(\lambda) = 0$. Since $\lambda \neq 0$, this proves (i).

Proof of (ii). Applying the identity to β, we find

$$\begin{aligned} &(-2\lambda JN + \beta X + \gamma JX)(\beta) \\ &\qquad = -2\lambda(X(\alpha) - 3\gamma\lambda) + \beta(-3\beta\gamma) + \gamma(\alpha\lambda - \lambda^2 + \beta^2 - 2\gamma^2) \\ &\qquad = -2\lambda X(\alpha) + \gamma\alpha\lambda + 5\gamma\lambda^2 - 2\beta^2\gamma - 2\gamma^3 \end{aligned}$$

and

$$\begin{aligned} &X(JX(\beta)) - JX(X(\beta)) \\ &\quad = X(\alpha\lambda - \lambda^2 + \beta^2 - 2\gamma^2) - JX(-3\beta\gamma) \\ &\quad = \lambda X(\alpha) + (\alpha - 2\lambda)X(\lambda) + 2\beta X(\beta) - 4\gamma X(\gamma) + 3\beta JX(\gamma) + 3\gamma JX(\beta) \\ &\quad = \lambda X(\alpha) + (\alpha - 2\lambda)(-3\gamma\lambda) + 2\beta(-3\beta\gamma) - 4\gamma(-\alpha\lambda + \lambda^2 + 2\beta^2 - \gamma^2) \\ &\qquad + 3\beta(3\beta\gamma) + 3\gamma(\alpha\lambda - \lambda^2 + \beta^2 - 2\gamma^2) \\ &\quad = \lambda X(\alpha) + 4\alpha\gamma\lambda - \gamma\lambda^2 - 2\gamma^3 - 2\beta^2\gamma, \end{aligned}$$

so $3\lambda X(\alpha) = -3\alpha\gamma\lambda + 6\gamma\lambda^2$ and $X(\alpha) = -\alpha\gamma + 2\gamma\lambda$. This proves (ii).

Proof of (iii). Finally, applying the identity to γ, we find

$$\begin{aligned} &(-2\lambda JN + \beta X + \gamma JX)(\gamma) \\ &\qquad = -2\lambda(JX(\alpha) + 3\beta\lambda) + \beta(-\alpha\lambda + \lambda^2 + 2\beta^2 - \gamma^2) + \gamma(3\beta\gamma) \\ &\qquad = -2\lambda JX(\alpha) - \beta\alpha\lambda - 5\beta\lambda^2 + 2\beta^3 + 2\beta\gamma^2 \end{aligned}$$

and

$$
\begin{aligned}
&X(JX(\gamma)) - JX(X(\gamma)) \\
&\quad = X(3\beta\gamma) - JX(-\alpha\lambda + \lambda^2 + 2\beta^2 - \gamma^2) \\
&\quad = 3\beta X(\gamma) + 3\gamma X(\beta) + \lambda JX(\alpha) + (\alpha - 2\lambda)JX(\lambda) \\
&\qquad - 4\beta JX(\beta) + 2\gamma JX(\gamma) \\
&\quad = 3\beta(-\alpha\lambda + \lambda^2 + 2\beta^2 - \gamma^2) + 3\gamma(-3\beta\gamma) \\
&\qquad + \lambda JX(\alpha) + (\alpha - 2\lambda)(3\beta\lambda) - 4\beta(\alpha\lambda - \lambda^2 + \beta^2 - 2\gamma^2) + 2\gamma(3\beta\gamma) \\
&\quad = \lambda JX(\alpha) - 4\alpha\beta\lambda + \beta\lambda^2 + 2\beta^3 + 2\beta\gamma^2
\end{aligned}
$$

so that $3\lambda JX(\alpha) = 3\alpha\beta\lambda - 6\beta\lambda^2$ and $JX(\alpha) = \alpha\beta - 2\beta\lambda$. This proves (iii). □

LEMMA 4. *Let $M^3 \subset \mathbb{C}^2$ be as previously described. If $\lambda \neq 0$ and $\alpha + \lambda \neq 0$, then*

$$\Lambda = \frac{\beta^2 + \gamma^2 - \alpha\lambda}{(\alpha + \lambda)^2}$$

is constant on M.

Proof. Using Lemma 2, $[X, JX] = -2\lambda JN + \beta X + \gamma JX$. So to prove that Λ is constant on M, it is enough to show that $X(\Lambda) = 0$ and $JX(\Lambda) = 0$.

To show that $X(\Lambda) = 0$, first notice that

$$
\begin{aligned}
&X(\beta^2 + \gamma^2 - \alpha\lambda) \\
&\quad = 2\beta X(\beta) + 2\gamma X(\gamma) - \lambda X(\alpha) - \alpha X(\lambda) \\
&\quad = 2\beta(-3\beta\gamma) + 2\gamma(-\alpha\lambda + \lambda^2 + 2\beta^2 - \gamma^2) - \lambda(-\alpha\gamma + 2\gamma\lambda) - \alpha(-3\gamma\lambda) \\
&\quad = -2\gamma(\beta^2 + \gamma^2 - \alpha\lambda).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
X(\Lambda) &= \frac{-2\gamma(\beta^2 + \gamma^2 - \alpha\lambda)}{(\alpha + \lambda)^2} - 2(\beta^2 + \gamma^2 - \alpha\lambda)\frac{X(\alpha) + X(\lambda)}{(\alpha + \lambda)^3} \\
&= \frac{-2(\beta^2 + \gamma^2 - \alpha\lambda)}{(\alpha + \lambda)^3}(\gamma(\alpha + \lambda) + (-\alpha\gamma + 2\gamma\lambda) + (-3\gamma\lambda)) = 0.
\end{aligned}
$$

Likewise, to show that $JX(\Lambda) = 0$, observe that

$$
\begin{aligned}
&JX(\beta^2 + \gamma^2 - \alpha\lambda) \\
&\quad = 2\beta JX(\beta) + 2\gamma JX(\gamma) - \lambda JX(\alpha) - \alpha JX(\lambda) \\
&\quad = 2\beta(\alpha\lambda - \lambda^2 + \beta^2 - 2\gamma^2) + 2\gamma(3\beta\gamma) - \lambda(\alpha\beta - 2\beta\lambda) - \alpha(3\beta\lambda) \\
&\quad = 2\beta(\beta^2 + \gamma^2 - \alpha\lambda).
\end{aligned}
$$

It follows that

$$\begin{aligned}JX(\Lambda) &= \frac{2\beta(\beta^2+\gamma^2-\alpha\lambda)}{(\alpha+\lambda)^2} - 2(\beta^2+\gamma^2-\alpha\lambda)\frac{JX(\alpha)+JX(\lambda)}{(\alpha+\lambda)^3}\\ &= \frac{2(\beta^2+\gamma^2-\alpha\lambda)}{(\alpha+\lambda)^3}(\beta(\alpha+\lambda)-(\alpha\beta-2\beta\lambda)-(3\beta\lambda)) = 0.\end{aligned}$$

Thus the lemma is proved. □

Completion of the proof in dimension two. Here we prove Theorem 1 in the case $n = 2$. To do this, normalize the surface so that it can be defined near $p = 0$ by $r(z) = \frac{1}{2}(z_2+\bar{z}_2) + \varepsilon z_1\bar{z}_1 + o(|z|^2)$ for $\varepsilon = \pm 1$. Then perform a further Möbius transformation

$$F(z_1, z_2) = \left(\frac{z_1}{z_2-\varepsilon}, \frac{\varepsilon z_2+1}{z_2-\varepsilon}\right)$$

so that $M' = F^{-1}(M)$ is defined by $r'(z) = \varepsilon|z_1|^2 + \varepsilon|z_2|^2 - \varepsilon + o(|(z_1, z_2+\varepsilon)|^2)$ and is therefore osculated to second order by the unit sphere at $p' = (0, -\varepsilon)$. This means that $\alpha = \lambda = 1$ and $\beta = \gamma = 0$ at p', so $\Lambda = -\frac{1}{4}$ at p'. By Lemma 4, $\Lambda = -\frac{1}{4}$ on all of M'. We have

$$\begin{aligned}\frac{\beta^2+\gamma^2-\alpha\lambda}{(\alpha+\lambda)^2} = -\frac{1}{4} &\iff \beta^2+\gamma^2 = \alpha\lambda - \frac{1}{4}(\alpha+\lambda)^2\\ &\iff \beta^2+\gamma^2 = -\frac{1}{4}(\alpha-\lambda)^2.\end{aligned}$$

The left-hand side of the last identity is nonnegative and the right-hand side is nonpositive, so both sides must be zero on M'. In particular, $\alpha \equiv \lambda$ and $\beta \equiv \gamma \equiv 0$ on M', so M' is everywhere umbilic. It then follows that M' is spherical (see e.g. [6, p. 36]). Since the Möbius image of a sphere is a hermitian quadric, the lemma is proved. □

6. Proof in Higher Dimensions

The proof in higher dimensions involves slices of complex dimension two. We first show that if $M^{2n-1} \subset \mathbb{C}^n$ is a hypersurface that satisfies condition (1), then a nontrivial intersection of M^{2n-1} with a two-dimensional vector space is a surface that also satisfies condition (1).

Suppose the vector space is spanned by $\zeta, \eta \in \mathbb{C}^n$. If M is defined by $r(z)$, then the surface of intersection $M^{\zeta,\eta} \subset \mathbb{C}^2$ can be defined by $r^{\zeta,\eta}(w) = r(w_1\zeta + w_2\eta)$. The complex tangent space is spanned by $(s_1, s_2) = \left(-\sum_k r_k\eta_k, \sum_k r_k\zeta_k\right)$. To verify condition (1) for $M^{\zeta,\eta}$, one shows first the identity

$$\sum_{j,k=1}^{2} \frac{\partial^2 r^{\zeta,\eta}}{\partial w_j \partial w_k} s_j s_k = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k} t_j t_k \tag{11}$$

for $t_j = -\zeta_j \sum_l r_l\eta_l + \eta_j \sum_l r_l\zeta_l$. (We omit the details.) One sees readily that $t = (t_1, \ldots, t_n)$ is in the complex tangent space of M, since

$$\sum_j r_j t_j = \sum_j r_j \left(-\zeta_j \sum_k r_k \eta_k + \eta_j \sum_k r_k \zeta_k \right) = 0.$$

The right-hand side of (11) is zero because condition (1) holds on M. It follows that condition (1) holds on $M^{\zeta,\eta}$ as well.

We finish the proof of Theorem 1 as follows. By Proposition 4, one can use a Möbius transformation to normalize $M^{2n-1} \subset \mathbb{C}^n$ so that it has defining function

$$r(z) = \frac{1}{2}(z_n + \bar{z}_n) + \sum_{j=1}^{n-1} \varepsilon_j |z_j|^2 + o(|z|^2)$$

for z near $0 \in M$ and $\varepsilon_j \in \{-1, 0, +1\}$. (Notice that r is not uniquely determined, but the second-order information does identify a unique quadric.) Under condition (1), we must show that the $o(|z|^2)$ terms can be taken to be zero. If $e_n = (0, \dots, 0, 1)$ then it will be enough to check that, for a dense set of $\zeta = (\zeta_1, \dots, \zeta_{n-1}, 0) \in S^{2n-3} \times \{0\}$, the surface M^{ζ,e_n} is hermitian quadric. In particular, a dense subset of M is then contained in the quadric obtained by truncating the $o(|z|^2)$ terms from $r(z)$.

Given the result of Section 5, we now need only check that there is a dense set of $\zeta \in S^{2n-3} \times \{0\}$ for which M^{ζ,e_n} is non–Levi-flat. This is easy, since M^{ζ,e_n} has defining function

$$r^{\zeta,e_n}(w) = \frac{1}{2}(w_2 + \bar{w}_2) + |w_1|^2 \sum_j \varepsilon_j |\zeta_j|^2 + o(|w|^2).$$

Since M is non–Levi-flat, not all of the ε_j are zero and so $\sum_j \varepsilon_j |\zeta_j|^2 \neq 0$ except for a set of codimension one. Evidently this set is dense, so the theorem is proved.

7. Proof of Theorems 2 and 3

We remark that the Bochner–Martinelli and Leray–Aĭzenberg kernels are special cases of Cauchy–Fantappiè kernels. See Range [7] for a nice treatment of this larger topic.

We first prove the following proposition.

PROPOSITION 5. *Suppose $M^{2n-1} \subset \mathbb{C}^n$ is three times differentiable, is non–Levi-flat, and has unit normal vector N_w at $w \in M$. Then there is a twice differentiable function $h > 0$ on M with*

$$h(w) N_w \cdot (\bar{w} - \bar{z}) = h(z) \bar{N}_z \cdot (z - w) \quad \textit{for all } w, z \in M \tag{12}$$

if and only if M is contained in a hermitian quadric surface in $\mathbb{C}^n$.

(The dot product means to sum the products of the complex coordinates.)

Proof. Assuming (12), choose a defining function r such that $|\nabla r(w)| = 2h(w)$. Then $h(w)\bar{N}_w = (r_1(w), \dots, r_n(w))$, where the subscripts refer to the holomorphic partial derivatives of r (i.e., $r_j = \partial r / \partial w_j$ and $r_{\bar{j}} = \partial r / \partial \bar{w}_j$).

Then (12) can be written as

$$\sum_j r_{\bar{j}}(w)(\bar{w}_j - \bar{z}_j) = \sum_j r_j(z)(z_j - w_j) \quad \text{for all } w, z \in M. \tag{13}$$

Furthermore, using the Taylor expansions

$$r_{\bar{j}}(w) = r_{\bar{j}}(z) + \sum_k [r_{\bar{j}k}(z)(w_k - z_k) + r_{\bar{j}\bar{k}}(z)(\bar{w}_k - \bar{z}_k)] + o(|w - z|)$$

and

$$\begin{aligned} r(w) = r(z) &+ \sum_j [r_j(z)(w_j - z_j) + r_{\bar{j}}(z)(\bar{w}_j - \bar{z}_j)] \\ &+ \text{Real} \sum_{j,k} r_{jk}(z)(w_j - z_j)(w_k - z_k) \\ &+ \sum_{j,k} r_{j\bar{k}}(z)(w_j - z_j)(\bar{w}_k - \bar{z}_k) + o(|w - z|^2) \end{aligned}$$

along with $r(w) = 0 = r(z)$ for $w, z \in M$, one can replace (13) for $w, z \in M$ by

$$i \sum_{j,k} \text{Imag}[r_{jk}(z)(w_j - z_j)(w_k - z_k)] + o(|w - z|^2) = 0.$$

Considering just the quadratic terms and taking the limit as w approaches z, one sees that this implies

$$\text{Imag} \sum_{j,k} r_{jk}(z) s_j s_k = 0 \quad \text{for all } s = (s_1, \ldots, s_n) \in TM_z. \tag{14}$$

In particular, if $s \in HM_z$, then applying (14) to both $s \in TM_z$ and $\sqrt{i}s \in TM_z$ gives $\sum_{j,k} r_{jk}(z) s_j s_k = 0$. So (1) holds, and M is contained in a hermitian quadric.

The reverse direction (that M contained in a hermitian quadric implies (12)) is trivial, so we omit the proof. □

Theorems 2 and 3 follow from the proposition in a manner identical to the situation for the corresponding theorems in the global case [3]. For completeness, we outline the proofs here as well.

Proof of Theorem 2. The Bochner–Martinelli kernel is defined by

$$K(z, w) = \frac{(n-1)!}{2\pi^n} \frac{N_w \cdot (\bar{w} - \bar{z})}{|w - z|^{2n}} \quad \text{for } w \in M, \ z \neq w.$$

If $d\sigma_E$ is Euclidean surface measure, then the Bochner–Martinelli transform is the operator $f \to \mathcal{K}f$ defined for $f \in L^2(M)$ by

$$\mathcal{K}f(z) = \lim_{\varepsilon \downarrow 0} \int_{w \in M, |z-w| > \varepsilon} f(w) K(z, w) \, d\sigma_E.$$

By the Calderón–Zygmund theory of singular integrals, the limit exists for almost all $z \in M$, and $\mathcal{K}$ is bounded on $L^2(M)$. Furthermore, the $L^2(M)$ adjoint of $\mathcal{K}$ has kernel $\overline{K(w,z)}$, and $\mathcal{K}$ is self-adjoint in $L^2(M)$ if and only if $K(z,w) = \overline{K(w,z)}$ for all $z, w \in M$, $z \neq w$.

If one replaces Euclidean measure with the weighted measure $h^{-1} d\sigma_E$ for some twice differentiable function $h > 0$ on the boundary, then with respect to the new measure the transform has kernel $h(w)K(z,w)$. Furthermore, $\mathcal{K}$ is self-adjoint if and only if $h(w)K(z,w) \equiv h(z)\overline{K(w,z)}$. This holds precisely when (12) is satisfied. So Theorem 2 follows from Proposition 5. □

Proof of Theorem 3. A lineally convex hypersurface is one for which the complex tangent space never intersects the domain itself; so if $T_w^c(M) = \{w + v \in \mathbb{C}^n : \sum_j r_j(w)v_j = 0\}$, then $T_w^c(M) \subset \mathbb{C}^n \setminus M$ for all $w \in M$.

For this kind of hypersurface, the Leray–Aĭzenberg transform is the operator defined for $f \in L^2(M)$ by

$$\mathcal{C}f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{w\in M} f(w) \frac{\partial r(w) \wedge (\bar{\partial}\partial r(w))^{n-1}}{\left(\sum r_j(w)(w_j - z_j)\right)^n} \quad \text{for } z \notin M,$$

where the derivatives in the denominator refer to the holomorphic derivatives of r; that is, $r_j = \partial r/\partial w_j$. Similarly, $r_{\bar{j}} = \partial r/\partial \bar{w}_j$.

Given the convexity condition, $i^{-n}\partial r(w) \wedge (\bar{\partial}\partial r(w))^{n-1}$ is a positive multiple of Euclidean surface measure.

Furthermore, if $\left(\sum r_j(w)(w_j - z_j)\right)^n = \left(\sum r_{\bar{j}}(z)(\bar{z}_j - \bar{w}_j)\right)^n$ for all $w, z \in M$, then $\sum r_j(w)(w_j - z_j) = \sum r_{\bar{j}}(z)(\bar{z}_j - \bar{w}_j)$ for all $w, z \in M$. This can be proved via a simple argument using Taylor expansions.

From the kernel, then, we find that $\mathcal{C}$ is self-adjoint with respect to weighted measure on the boundary if and only if there is a twice differentiable function $h > 0$ such that

$$h(w)\sum_j r_j(w)(w_j - z_j) = h(z)\sum_j r_{\bar{j}}(z)(\bar{z}_j - \bar{w}_j) \quad \text{for all } w, z \in M.$$

The vector $(r_{\bar{1}}(w), \dots, r_{\bar{n}}(w))$ is a multiple of the normal vector N_w and so, after taking conjugates, this is the same condition as (12) for a possibly different function h. Hence Theorem 3 also follows from Proposition 5. □

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