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Special Loci in Moduli of Marked Curves

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1. Introduction

This paper concerns "special loci" in the moduli space $\mathfrak{M}_{g,[n]}$ parameterizing the smooth projective curves of genus g with n unordered marked points. Classically, one fixes a finite-order diffeomorphism φ of a compact orientable topological surface S of genus g with n marked points. The *special locus* associated to φ corresponds to the set of complex structures that can be put on S such that φ is an automorphism of the associated marked algebraic curve. The main theorem of this paper (Theorem 2.18) uses scheme theory to reformulate the notion of special locus in purely algebraic terms. A related result (Corollary 2.23) shows how the notion can often be further reformulated combinatorially in many cases. As a consequence of these results, the notion of special locus can be extended to curves over more general algebraically closed fields, including the characteristic-p case. In the last section, we consider some examples both in characteristic 0 and characteristic p. It turns out that special loci in characteristic p behave differently than the analogous special loci in characteristic 0 because of differences in the corresponding Riemann–Hurwitz formulas.

This paper builds upon previous work by González-Díez, Harvey, and Schneps. González-Díez and Harvey [GoH] considered curves of genus $g \geq 2$ over the complex numbers without marked points, and they studied the loci of those with a given automorphism group acting in a specified topological way. Cornalba [C] gave a complete classification of the irreducible subvarieties corresponding to the curves whose automorphism group contains a fixed cyclic subgroup of prime order in the case $g \geq 1$, $n = 0$, over the complex numbers (but without specifying the topological action). Later, Schneps [Sc1] considered the cyclic case for genus 0 with n marked points and for genus 1 with $n = 1$ or 2 marked points, which correspond to specifying a finite-order diffeomorphism of the underlying real 2-manifold. Related work has also been done by Magaard, Shaska, Shpectorov, and Völklein [MSSV], where the group but not the topological behavior is specified.

The main result in Section 2 provides a purely algebraic definition of special locus using scheme theory and without reference to differential or topological notions. In order to do this, we rely on the fact that the classical ("differential") special locus is irreducible. What we do is define two automorphisms α, α' of

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marked curves X, X' as being *algebraically equivalent* if there is a flat family $\mathfrak{X}$ of marked curves containing X, X' as fibres, together with an automorphism θ of the curve X_η corresponding to the generic member of the family, such that θ specializes to both α and α'. The *algebraic special locus* of α is then defined as the set of points in the moduli space that have an automorphism algebraically equivalent to α (see Definition 2.14). Our main theorem (Theorem 2.18) then uses topology and algebraic geometry to show the equivalence of the classical special locus with our algebraic special locus over the complex numbers. This allows the notion of special locus to be carried over to curves defined over other fields. When there are enough marked points, a more concrete characterization of special loci in terms of permutations is also given (Corollary 2.23). Some explicit examples are given in Section 3 in characteristic p, including a complete description of special loci for marked curves of genus 0 in characteristic p.

2. Special Loci in Moduli

2.1. Differential Special Loci

In this paper, we adopt several definitions from [Sc1] with some adjustments. In particular, we use the term "differential special locus" for the original term "special locus" in order to distinguish this from other related notions.

We fix S once and for all to be an orientable topological surface of genus g equipped with n distinct ordered marked points $s_1, \dots, s_n$. We say that S is of *type* (g, n).

Throughout this section, we only work over the complex numbers $\mathbb{C}$. In order to give the definition of special locus, we first need to give the following definitions.

DEFINITION 2.1. An n-unordered (resp. n-ordered) *marked* Riemann surface is a Riemann surface X of genus g together with n unordered (resp. ordered) distinct marked points $x_1, \dots, x_n$. We denote X with unordered marked points $x_1, \dots, x_n$ by $(X; x_1, \dots, x_n)$.

DEFINITION 2.2 (cf. [Sc1, Sec. 2.1]). A *parameterized* (ordered) marked Riemann surface of genus g consists of the following data:

(1) an n-ordered marked Riemann surface X of genus g with ordered marked points $x_1, \dots, x_n$;
(2) a *parameterization*—that is, a diffeomorphism $\Phi\colon S \to X$ such that $\Phi(s_i) = x_i$ for $1 \le i \le n$.

We say that X is of *type* (g, n).

DEFINITION 2.3 [Sc1, Sec. 2.1]. Two parameterized marked Riemann surfaces X (with marked points $x_1, \dots, x_n$ and parameterization Φ) and X' (with marked points $x'_1, \dots, x'_n$ and parameterization Φ') are said to be *isomorphic* if there exist (a) an isomorphism $\alpha\colon X \to X'$ of Riemann surfaces with $\alpha(x_i) = x'_i$ for $1 \le i \le n$ and (b) a diffeomorphism $h\colon S \to S$, with $h(s_i) = s_i$ for $1 \le i \le n$, that is isotopic to the identity via a family of diffeomorphisms $h_t\colon S \to S$ with $h_t(s_i) = s_i$ for $t \in [0,1]$ and each i and such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\Phi"] \arrow[d, "h"'] & X \arrow[d, "\alpha"] \\ S \arrow[r, "\Phi'"] & X' \end{tikzcd}$$

REMARK 2.4. The *Teichmüller space* $\mathcal{T}_{g,n}$ is the set of isomorphism classes of parameterized marked Riemann surfaces of type (g,n). In fact, it is well known that the Teichmüller space forms a simply connected complex analytic space of dimension $3g-3+n$ (cf. [Na, Thm. 3.2.3]).

DEFINITION 2.5 [Sc1, Sec. 2.1]. (a) We define the *full mapping class group* $\Gamma_{g,[n]}$ by setting

$$\Gamma_{g,[n]} = \mathrm{Diff}^{+}([S])/\mathrm{Diff}^{0}(S),$$

where $\mathrm{Diff}^{+}([S])$ denotes the group of orientation-preserving diffeomorphisms of S that fixes $\{s_1,\dots,s_n\}$ as a set and $\mathrm{Diff}^{0}(S)$ is the subgroup of those that are isotopic to the identity.

(b) We define the *pure mapping class group* (or *pure subgroup of the full mapping class group*) $\Gamma_{g,n}$ by setting

$$\Gamma_{g,n} = \mathrm{Diff}^{+}(S)/\mathrm{Diff}^{0}(S),$$

where $\mathrm{Diff}^{+}(S)$ is the subgroup of $\mathrm{Diff}^{+}([S])$ consisting of diffeomorphisms that fix each marked point s_i.

REMARK 2.6. For the definition of mapping class group, Schneps [Sc1] and Hain and Looijenga [HaL] use diffeomorphisms of a compact orientable surface of genus g, whereas González-Díez, Harvey, and Maclachlan [GoH; MaH] use homeomorphisms of a compact orientable surface of genus g. But these definitions of the mapping class group are equivalent because every homeomorphism of a compact orientable surface S of genus g can be approximated by a diffeomorphism of S up to homotopy (cf. [Hi, Sec. 5, Lemma 1.5]). So we can use all the results about the mapping class group from the papers just cited.

REMARK 2.7. The mapping class group $\Gamma_{g,[n]}$ acts on the Teichmüller space $\mathcal{T}_{g,n}$ as follows. If $\psi \in \Gamma_{g,[n]}$, let ψ' denote a lifting of ψ to a diffeomorphism of S; then ψ' maps the marked Riemann surface (Φ, X) to $(\Phi \circ \psi', X)$ [Sc1, Sec. 2.1]. The *unordered moduli space* $\mathfrak{M}_{g,[n]}$, parameterizing smooth curves of genus g together with an unordered set of n-distinct marked points, is realized as the quotient of the Teichmüller space $\mathcal{T}_{g,n}$ by the action of the mapping class group $\Gamma_{g,[n]}$. Similarly, the *ordered moduli space* $\mathfrak{M}_{g,n}$, parameterizing smooth curves of genus g together with an ordered set of n-distinct marked points, is the quotient of $\mathcal{T}_{g,n}$ by the pure subgroup $\Gamma_{g,n}$ of $\Gamma_{g,[n]}$.

DEFINITION 2.8 (cf. [Sc1, Sec. 2.1]). If φ is an element of finite order in the full or pure mapping class group, then we consider the set of points in Teichmüller space fixed by φ. The image of this set in the quotient moduli space $\mathfrak{M}_{g,n}$ or $\mathfrak{M}_{g,[n]}$ is called the *differential special locus of* φ, denoted respectively by $\mathfrak{M}_{\varphi}(S)$ or $\mathfrak{M}_{\varphi}[S]$.

REMARK 2.9. A finite-order mapping class can be represented by a self-homeomorphism of that same order. This was proved algebraically by Nielsen [N] in the $n = 0$ case; it may well be true for more general n, too.

REMARK 2.10. For all genus g, if $n > n_0(g) = 2g + 2$ then special loci in the ordered moduli space $\mathfrak{M}_{g,n}$ are trivial because automorphisms of a marked curve X can't fix more than $2g + 2$ points except for the identity (see Remark 2.20). So in this paper we restrict the special loci in the unordered moduli space $\mathfrak{M}_{g,[n]}$.

In Corollary 2.13 we give a characterization of the differential special locus that does not refer to Teichmüller space or mapping class group; this will make it possible to carry the concept over to characteristic p. First we introduce the notion of differential equivalence as follows.

DEFINITION 2.11. Suppose $(X; x_1, \dots, x_n)$ and $(X'; x'_1, \dots, x'_n)$ are two unordered marked Riemann surfaces with genus g in the unordered moduli space $\mathfrak{M}_{g,[n]}$. Let α be a finite-order automorphism of X and let α' be a finite-order automorphism of X'. Then α and α', which fix the marked points as a set, are said to be *differentially equivalent* if there exists a diffeomorphism $\Psi\colon X \to X'$ that maps the set of marked points of X to the set of marked points of X' such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

PROPOSITION 2.12. *Let $(X; x_1, \dots, x_n)$ be an unordered marked Riemann surface in $\mathfrak{M}_{g,[n]}$ and let α be a finite-order automorphism of X. Pick a parameterization Φ of X such that (X, Φ) is a point in Teichmüller space. Let $\psi\colon S \to S$ be the diffeomorphism induced by α and Φ. Let φ be the equivalence class of ψ in the full mapping class group.*

(a) *Then (X, Φ) is fixed by φ under the action of the mapping class group on Teichmüller space.*
(b) *Let X' (with unordered marked points $x'_1, \dots, x'_n$) be a marked Riemann surface in $\mathfrak{M}_{g,[n]}$. Then there exists a parameterization Φ' of X' such that the point (X', Φ') in Teichmüller space is also fixed by φ if and only if there exists an automorphism α' of X' that is differentially equivalent to α.*

Proof. (a) By assumption of the proposition, we have the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & X \\ \psi\big\downarrow & & \big\downarrow\alpha \\ S & \xrightarrow{\Phi} & X \end{array}$$

which is equivalent to the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & X \\ \mathrm{id}\big\downarrow & & \big\downarrow\alpha \\ S & \xrightarrow{\Phi\psi} & X \end{array}$$

where id: $S \to S$ is the identity. So (X, Φ) is fixed by φ in Teichmüller space.

(b) Now suppose that there exists a parameterized marked Riemann surface (X', Φ') that is also fixed by φ. Then there exist an automorphism $\alpha'': X' \to X'$ and a diffeomorphism $h: S \to S$ with $h(s_i) = s_i$ for $1 \leq i \leq n$ that is isotopic to the identity and such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'} & X' \\ h\big\downarrow & & \big\downarrow\alpha'' \\ S & \xrightarrow{\Phi'\psi} & X' \end{array}$$

By assumption of the proposition, we have the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & X \\ \psi\big\downarrow & & \big\downarrow\alpha \\ S & \xrightarrow{\Phi} & X \end{array}$$

Combining the preceding two commutative diagrams, we have the following commutative diagram:

$$\begin{array}{ccccc} & & S & \xrightarrow{\Phi'} & X' \\ & & h\big\downarrow & & \big\downarrow\alpha'' \\ X & \xrightarrow{\Phi^{-1}} & S & \xrightarrow{\Phi'\psi} & X' \\ \alpha\big\downarrow & & \psi\big\downarrow & & \big\| \\ X & \xrightarrow{\Phi^{-1}} & S & \xrightarrow{\Phi'} & X' \end{array}$$

Let $\Psi = \Phi'\Phi^{-1}: X \to X'$ and $\alpha' = \Phi'\psi\Phi'^{-1}: X' \to X'$; then we get the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

So α and α' are differentially equivalent.

Conversely, suppose there exists an automorphism α' of X' that is differentially equivalent to α. Then there exists a diffeomorphism $\Psi\colon X \to X'$ that maps the set of marked points to the set of marked points and such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc} S & \xrightarrow{\Phi} & X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\psi}\downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{\Phi} & X & \xrightarrow{\Psi} & X' \end{array}$$

Let $\Phi' = \Phi\Psi$; then the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'} & X' \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{\Phi'} & X' \end{array}$$

This is equivalent to the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'} & X' \\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{\Phi'\psi} & X' \end{array}$$

where id: $S \to S$ is the identity. So (X', Φ') is fixed by φ. □

COROLLARY 2.13. *Let $(X; x_1, \dots, x_n)$ be an unordered marked Riemann surface in $\mathfrak{M}_{g,[n]}$ and let α be a finite-order automorphism of X. The differential special locus of α is the set of points on the moduli space $\mathfrak{M}_{g,[n]}$ that have an automorphism differentially equivalent to α.*

Proof. By Definition 2.8, consider the set of points in Teichmüller space fixed by α. Then the differential special locus of α is the image of this set in the quotient moduli space. By Proposition 2.12, let $(X'; x'_1, \dots, x'_n)$ be any unordered marked Riemann surface in $\mathfrak{M}_{g,[n]}$. Then there exists a parameterized marked Riemann surface

(X', Φ') (where Φ' is a parameterization) in Teichmüller space that is also fixed by α if and only if there exists an automorphism α' of X' that is differentially equivalent to α. Therefore, the differential special locus of α is the set of points on the moduli space $\mathfrak{M}_{g,[n]}$ that have an automorphism differentially equivalent to α. □

2.2. *Algebraic Special Loci*

Let K be an algebraically closed field. Unlike in the previous section, where we only worked over the complex numbers $\mathbb{C}$, now we generalize the definition of special loci to any algebraically closed field K. To do this, we first give the following definition of algebraic special loci.

Definition 2.14. Suppose $(X; x_1, \dots, x_n)$ and $(X'; x'_1, \dots, x'_n)$ are two unordered marked curves with genus g in $\mathfrak{M}_{g,[n]}$.

(a) Let α and α' be finite-order automorphisms of X and X' (respectively) that fix the marked points as a set. We say that α and α' are *algebraically equivalent* if there exists (i) a connected flat family $\mathfrak{X} \to T$ of curves of genus g with n marked points such that (X, α) and (X', α') are two fibers and (ii) an automorphism θ of the curve A that corresponds to the generic point ξ in T such that θ specializes to both α and α'.

(b) The *algebraic special locus* of α is the set of points in the moduli space $\mathfrak{M}_{g,[n]}$ that have an automorphism algebraically equivalent to α; we denote this locus by $\mathfrak{M}^{\mathrm{alg}}_{g,[n]}(\alpha)$.

Remark 2.15. Because $\mathfrak{M}_{g,[n]}$ is a coarse moduli space, for the flat family $\mathfrak{X} \to T$ (in Definition 2.14) of curves of genus g with n marked points there is a morphism $h\colon T \to \mathfrak{M}_{g,[n]}$ such that, for each closed point $t \in T$, the curve X_t together with its marked points is in the isomorphism class of marked curves determined by the point $h(t) \in \mathfrak{M}_{g,[n]}$.

Remark 2.16. In a flat family of tamely ramified Galois covers of smooth curves, the number of branch points must remain constant, with no coalescing of branch points. Namely, if branch points of ramification indices a and b coalesce to a branch point of ramification index e in a family of covers of degree d, then by the Riemann–Hurwitz formula the contribution of these points to the genus is $\frac{d}{a}(a-1) + \frac{d}{b}(b-1) \geq d$ on the general fibre and $\frac{d}{e}(e-1) < d$ on the special fibre, which contradicts the fact that the genus must be constant in the family. As a consequence, if X and X' are in $\mathfrak{M}^{\mathrm{alg}}_{g,[n]}(\alpha)$, where the order of α is prime to the characteristic and where α' is the corresponding automorphism of X', then the cyclic covers $X \to Y := X/\langle\alpha\rangle$ and $X' \to Y' := X'/\langle\alpha'\rangle$ have the same numerical data. In other words, the number of marked points, the number of ramification points of any given index e, and the number of marked points that are ramification points of index e are all the same.

Lemma 2.17. *Let U be a contractible space and let Y be a Riemann surface of genus g. Let $\sigma_1, \sigma_2, \dots, \sigma_r$ be r disjoint continuous sections of the projection map*

$\pi : Y \times U \to U$. Let $u_0 \in U$. Then there is a diffeomorphism $\Psi : Y \times U \to Y \times U$ preserving π such that these r disjoint continuous sections all map to constant sections under Ψ (possibly after shrinking U around u_0).

Proof. By Remark 2.6, it suffices to show that there exists a homeomorphism $\Psi : Y \times U \to Y \times U$ preserving π such that these r disjoint continuous sections all map to constant sections under Ψ (possibly after shrinking U around u_0).

For $i = 1, \dots, r$, let P_i be the points of Y such that $\sigma_i(u_0) = (P_i, u_0)$. Since the P_i are distinct, we may choose disjoint open neighborhoods D_i of P_i in Y such that D_i is homeomorphic to an open disk. We can shrink U to a contractible space $U' \subset U$ such that, for any $u' \in U'$, $\sigma_i(u') \subset D_i \times \{u'\}$. So replacing U by U', we may assume that U has this property.

Let $D = \bigcup D_i \subset Y$. It suffices to construct a homeomorphism $\Psi : Y \times U \to Y \times U$ that restricts to the identity from $(Y - D) \times U$ to $(Y - D) \times U$ and such that, for any i, $\Psi(\sigma_i(u)) = (P_i, u)$ for every $u \in U$.

For any i, we first construct a homeomorphism $\Psi_i : \bar{D}_i \times U \to \bar{D}_i \times U$ such that, for any $u \in U$, $\Psi_i(\sigma_i(u)) = (P_i, u)$ and restricts the identity on the boundary of $\bar{D}_i$. For $i = 1, \dots, r$, define $\xi_i(u) \in D_i$ by $\sigma_i(u) = (\xi_i(u), u)$. In particular, $P_i = \xi_i(u_0)$. Also identify $\bar{D}_i$ with the closed unit disk and identify P_i with the origin.

Now we construct Ψ_i. For any point $Q \neq \xi_i(u)$ in the disk D_i and for any $u \in U$, the ray from $\xi_i(u)$ to Q intersects the boundary of $\bar{D}_i$ at a unique point, say R_i. There is a unique point $g_{i,u}(Q)$ that is on the line segment connecting P_i and R_i and that satisfies the equation

$$\frac{d(\xi_i(u), Q)}{d(\xi_i(u), R_i)} = \frac{d(P_i, g_{i,u}(Q))}{d(P_i, R_i)},$$

where $d(\xi_i(u), Q)$ denotes the distance from $\xi_i(u)$ to Q. Also let $g_{i,u}(\xi_i(u)) = P_i$; this defines a map $g_{i,u} : \bar{D}_i \to \bar{D}_i$. Note that $g_{i,u} = \mathrm{id}$ on the boundary of $\bar{D}_i$. By construction, $g_{i,u} : \bar{D}_i \to \bar{D}_i$ is a homeomorphism for any i, u.

Now define $\Psi_i : \bar{D}_i \times U \to \bar{D}_i \times U$ by $\Psi_i(Q, u) = (g_{i,u}(Q), u)$. This is a homeomorphism from $\bar{D}_i \times U$ to $\bar{D}_i \times U$ such that $\xi_i(u)$ maps to the center P_i of D_i for any u. Moreover, these homeomorphisms Ψ_i together extend to a homeomorphism $\Psi : Y \times U \to Y \times U$, which is the identity on $(Y - D) \times U$. This mapping then has the desired properties. □

THEOREM 2.18. *Over the complex numbers $\mathbb{C}$, let $(X; x_1, \dots, x_n)$ be an unordered marked curve of genus g in $\mathfrak{M}_{g,[n]}$, where $n \geq 3$ if $g = 0$ and $n \geq 1$ if $g = 1$. Let α be a finite-order automorphism of X that fixes the marked points as a set. Then the algebraic special locus of α is the same as the differential special locus of α.*

Proof. Suppose X' is a marked curve of genus g with unordered marked points $x'_1, \dots, x'_n$ and suppose α' is a finite-order automorphism of X' that preserves the marked points as a set. Suppose α' and α are algebraically equivalent. Let $Y = X/\langle\alpha\rangle$ and $Y' = X'/\langle\alpha'\rangle$. Then, by Remark 2.16, Y and Y' each have the same genus g' and the same number of n' marked points.

Now the subvariety $\mathfrak{M}^{\mathrm{alg}}_{g,[n]}(\alpha)$ maps to a subvariety $\mathfrak{M}'_\alpha$ in the moduli space $\mathfrak{M}_{g',[n']}$ by mapping the parameterizing point of X to the parameterizing point of Y. Let $\mathfrak{X}$ be the family of curves corresponding to the subvariety $\mathfrak{M}^{\mathrm{alg}}_{g,[n]}(\alpha)$. Consider the family $\mathfrak{Y}$ of curves corresponding to the subvariety $\mathfrak{M}'_\alpha$; we claim that this family is locally trivial in the metric topology.

We need to show that $\mathfrak{Y}$ is locally trivial. Let $u_0 \in \mathfrak{M}'_\alpha$ be the point corresponding to Y. Pick a contractible neighborhood U of u_0 in $\mathfrak{M}'_\alpha$ and consider the family $\mathfrak{Y}_U$ of curves corresponding to U. We know that $\mathfrak{Y}_U$ is a subspace of $\mathfrak{Y}$. Since all of the curves in $\mathfrak{Y}_U$ have the same genus g', they are diffeomorphic. (A classical proof may be found in Ahlfors and Sario [ASa] and a modern Morse-theoretic proof in Wallace [W].) Moreover, by the corollary to the isotopy lemma of [GP, p. 142], there exists a diffeomorphism from Y to Y, isotopic to the identity, that maps the set of n' marked points to another set of n' marked points. Hence all of the curves in $\mathfrak{Y}_U$ are diffeomorphic to Y and map the marked points to the marked points of Y. Since U is contractible, there is a diffeomorphism from $\mathfrak{Y}_U$ to $Y \times U$ that preserves the projections to U. Let b be the number of branch points of the covering map from X to Y and let $r = n' + b$; then there are r disjoint continuous sections of the projection map $\pi\colon Y \times U \to U$. In fact, the projection map $\pi\colon Y \times U \to U$ is a trivial cover and each component is trivial, so it consists of sections. Hence, by Lemma 2.17 (possibly after shrinking U), there is a diffeomorphism $\Psi\colon Y \times U \to Y \times U$ such that these r disjoint continuous sections all map to constant sections under Ψ. So $\mathfrak{Y}$ is a locally trivial family.

Now consider the family $\mathfrak{X}_U$ of curves in the family $\mathfrak{X}$ corresponding to the finite branched cover of $\mathfrak{Y}_U$. (This total space exists because U is contractible.) Since all of the curves in $\mathfrak{X}_U$ have an automorphism that is algebraically equivalent to α and since $\mathfrak{Y}$ is locally trivial, there exist diffeomorphisms among these curves in $\mathfrak{X}_U$ that commute with their automorphism. So $\mathfrak{X}$ is locally trivial.

Because $\mathfrak{M}'_\alpha$ is connected, by the foregoing argument there exists a diffeomorphism $\Psi\colon X \to X'$ that maps the set of marked points of X to the set of marked points of X' and such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

In other words, α and α' are differentially equivalent. Therefore, α' is in the differential special locus of α.

Conversely, let $\mathfrak{M}_{g,[n]}(\alpha)$ be the differential special locus of α.

First we need to show that the differential special locus $\mathfrak{M}_{g,[n]}(\alpha)$ is an irreducible subvariety of $\mathfrak{M}_{g,[n]}$ over the complex numbers $\mathbb{C}$. By [GoH, Thm. 1, p. 79] we know that the differential special locus $\mathfrak{M}_{g,[n]}(\alpha)$ is an irreducible subvariety of $\mathfrak{M}_{g,[n]}$ when $n = 0$—that is, in the case of no marked points. In the case of n marked points, the differential special locus $\mathfrak{M}_{g,[n]}(\alpha) \cong \mathfrak{M}_{g,[0]}(\alpha) \times (Y^{n_1} - \tilde{\Delta})$, where $Y = X/\langle\alpha\rangle$, $n_1 = n/\deg(\alpha)$, and $\tilde{\Delta}$ denotes the multidiagonal of points on

Y. From this we deduce that the special locus $\mathfrak{M}_{g,[n]}(\alpha)$ is an irreducible subvariety of $\mathfrak{M}_{g,[n]}$ for general n.

Now we prove that $\mathfrak{M}_{g,[n]}(\alpha)$ has a finite branched cover that is the parameter space of a family of curves of genus g with n marked points with a family of automorphism specializing to α at every complex point. We know that $\mathfrak{M}_{g,[n]}(\alpha)(\mathbb{C})$ consists of the $\mathbb{C}$-points of a closed irreducible subvariety of $\mathfrak{M}_{g,[n]}$. Let K be the function field of $\mathfrak{M}_{g,[n]}(\alpha)$ with algebraic closure $\bar{K}$. Because $\bar{K}$ has a finite transcendental degree over $\mathbb{Q}$, there exists an embedding $i\colon \bar{K} \to \mathbb{C}$. This corresponds to a $\mathbb{C}$-point of $\mathfrak{M}_{g,[n]}(\alpha)$, which corresponds to a curve X_i with an automorphism of topological type α. Since the generic point in $\mathfrak{M}_{g,[n]}$ (corresponding to a curve X_η^0) is defined over K and hence over $\bar{K}$, this automorphism is defined over $\bar{K}$ because $\bar{K}$ is algebraically closed and X has finitely many automorphisms (since $n \geq 3$ if $g = 0$ and $n \geq 1$ if $g = 1$). Hence this generic point is defined over L, where L is some finite extension of K. So the field extension L over K corresponds to irreducible finite branched cover from $\mathfrak{N}$ to $\mathfrak{M}_{g,[n]}(\alpha)$ and thus we have a family of curves of genus g with n marked points parameterizing $\mathfrak{N}$ with automorphism corresponding to α via the embedding map i. For any $\mathbb{C}$-point of $\mathfrak{N}$ (over a $\mathbb{C}$-point of $\mathfrak{M}_{g,[n]}(\alpha)$), the automorphism of the family specializes to an automorphism of the curve corresponding to the $\mathbb{C}$-point. This varies algebraically and continuously, so it is constant. But at the point corresponding to X_i, the automorphism is of type α, so all are of type α. Hence the differential special locus $\mathfrak{M}_{g,[n]}(\alpha)$ of α is also the algebraic special locus of α. □

PROPOSITION 2.19. *Let K be an algebraically closed field. Then every nontrivial automorphism of a nonsingular complete K-curve of genus g has at most $2g+2$ fixed points, a number that is attained by any hyperelliptic involution.*

Proof. Suppose X is a genus-g smooth curve with an order-m automorphism and suppose n is the number of fixed points of σ. Because the number of fixed points of σ is less than or equal to the number of fixed points of a power of σ, we may assume that the automorphism σ has prime order p. Let $Y = X/\langle\sigma\rangle$; then X is a branched covering space of Y. By the Riemann–Hurwitz theorem, we have $2g-2 \geq p(2g_Y - 2) + (p-1)n$ (with equality precisely in the tame case). Suppose that $n > 2g+2$; then we get $2g-2 \geq -2p + (p-1)(2g+3)$ (since $g_Y \geq 0$). So we have $g(4-2p) + 1 - p \geq 0$, which is a contradiction because $4 - 2p \leq 0$ and $1 - p < 0$ ($p \geq 2$). Therefore, $n \leq 2g+2$. If $n = 2g+2$ then we can take $g_Y = 0$ and $p = 2$; that is, there is a nontrivial order-2 automorphism β of a hyperelliptic curve of genus g such that β has $2g+2$ fixed points. Hence every nontrivial automorphism of a genus-g curve over K has at most $2g+2$ fixed points. □

REMARK 2.20. For $n > 2g+2$, if an automorphism α of a marked curve X fixes n points, then $\alpha = 1$. So if two automorphisms α and β of a marked curve X induce the same permutation, then $\alpha = \beta$. Also, for X corresponding to a point in the ordered moduli space $\mathfrak{M}_{g,n}$, the only automorphism α of X that fixes each marked point is the identity. (Note that the condition $n > 2g + 2$ is not really a restriction for $g = 0$ since $n_0(0) = 3$ and since we need $n \geq 4$ to get a nontrivial moduli space in genus 0.)

For unordered moduli spaces, however, there can be nonempty special loci even if n is large compared to g. This can be seen by taking a union of finitely many orbits of a finite-order automorphism of the underlying curve.

Therefore, if $n > 2g + 2$ then one can speak in terms of permutations rather than automorphisms, and these are easier to work with. This motivates the following proposition.

Let X be a marked curve with genus g in $\mathfrak{M}_{g,[n]}$. Let α be a finite-order automorphism of X that fixes the marked points as a set. We use $[\alpha]$ to denote the permutation induced by α for a point in the unordered moduli space.

PROPOSITION 2.21. *For $n > 2g + 2$, let $(X; x_1, \ldots, x_n)$ and $(X'; x'_1, \ldots, x'_n)$ be two unordered marked curves with genus g in $\mathfrak{M}_{g,[n]}$. Let α be a finite-order automorphism of X that fixes the marked points as a set. Then there exists a finite-order automorphism α' (which fixes the marked points as a set) of X' that is differentially equivalent to α if and only if there exists a finite-order automorphism α'' (which fixes the marked points as a set) of X' such that $[\alpha'']$ is conjugate to $[\alpha]$ in S_n.*

Proof. First suppose that there exists a finite-order automorphism α' (which fixes the marked points as a set) of X' that is differentially equivalent to α. Then there exists a diffeomorphism $\Psi\colon X \to X'$ that maps the set of marked points of X to the set of marked points of X' such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$

Then $\alpha' = \Psi\alpha\Psi^{-1}$. So α and α' induce conjugate permutations. Let $\alpha'' = \alpha'$; then $[\alpha]$ is conjugate to $[\alpha'']$.

Conversely, suppose that there exists a finite-order automorphism α'' (which fixes the marked points as a set) of X' such that $[\alpha'']$ is conjugate to $[\alpha]$. Choose parameterizations $\Phi\colon S \to X$ for X and $\Phi'\colon S \to X'$. Let $\psi\colon S \to S$ be the diffeomorphism induced by α and let $\psi'\colon S \to S$ be the diffeomorphism induced by α''. Because α and α'' induce conjugate permutations, we know that ψ and ψ' induce conjugate permutations.

Therefore, by the corollary to the isotopy lemma of [GP, p. 143], there exist a diffeomorphism $h\colon S \to S$ that is isotopic to identity and a diffeomorphism $\zeta\colon S \to S$ such that $\psi = \zeta^{-1}\psi'\zeta h$. In other words, the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\zeta h} & S \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi'} \\ S & \xrightarrow{\zeta} & S \end{array}$$

which is equivalent to the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\zeta} & S \\ {\scriptstyle h^{-1}}\downarrow & & \downarrow{\scriptstyle \psi'} \\ S & \xrightarrow{\zeta\psi} & S \end{array}$$

So we have the following commutative diagram:

$$\begin{array}{ccccc} S & \xrightarrow{\zeta} & S & \xrightarrow{\Phi'} & X' \\ {\scriptstyle h^{-1}}\downarrow & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \alpha''} \\ S & \xrightarrow{\zeta\psi} & S & \xrightarrow{\Phi'} & X' \end{array}$$

Hence the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi'\zeta} & X' \\ {\scriptstyle h^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha''} \\ S & \xrightarrow{\Phi'\zeta\psi} & X' \end{array}$$

Because $h^{-1}\colon S \to S$ is isotopic to the identity, we obtain that $(X', \Phi'\zeta)$ is fixed by the equivalence class φ of ψ in the full mapping class group. By Proposition 2.12(a), we know that (X, Φ) is fixed by φ. So by Proposition 2.12(b), there exists an automorphism α' of X' that is differentially equivalent to α; that is, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ X & \xrightarrow{\Psi} & X' \end{array}$$ □

REMARK 2.22. Note that, in the proof of Proposition 2.21, in the forward direction we can take $\alpha'' = \alpha'$. But in the converse direction, α' might have to be different from α'' in order for α' to be differentially equivalent to α.

COROLLARY 2.23. *For $n > 2g + 2$, let $(X; x_1, \ldots, x_n)$ be an unordered marked curve with genus g in $\mathfrak{M}_{g,[n]}$ and let α be a finite-order automorphism of X that fixes $\{x_1, \ldots, x_n\}$ as a set. The algebraic special locus of α is the set of points on the moduli space that have an automorphism whose induced permutation of the marked points is conjugate to $[\alpha]$ in S_n.*

The proof follows from Corollary 2.13, Theorem 2.18, and Proposition 2.21.

3. Examples of Special Loci in Low Genus

3.1. Special Loci in Genus 0 *over the Complex Numbers* $\mathbb{C}$

For the genus-0 case, a permutation τ of the ordered marked points can be realized as an automorphism of the marked Riemann surface [Sc1, Sec. 3.1.1]. Such points are not orbifold points on the ordered moduli space, but they are preimages of orbifold points on the unordered moduli space $\mathfrak{M}_{0,[n]}$ because the τ have less than $n!$ preimages under the action of S_n. The points corresponding to marked Riemann surfaces having special automorphism group determine where the special loci will lie in the unordered moduli space $\mathfrak{M}_{0,[n]}$.

Now we investigate the special loci in the genus-0 moduli spaces for arbitrary n. If S is a sphere with n marked points, then a finite-order element of the mapping class group $\Gamma_{g,[n]}$ is the class of a diffeomorphism that is simply a rotation around an axis. In fact, for $n \geq 5$, that all finite-order elements in $\Gamma_{0,[n]}$ are rotations follows from [MaH, Cor., p. 508] and [Sc2, Sec. 4.1]. For $n = 3$ we have $\Gamma_{0,[n]} = 1$. For $n = 4$ there are four conjugacy classes of finite-order elements, inducing different conjugate permutations [Sc2, Sec. 3, proof of Cor. 2]; we can see that each conjugate class comes from a rotation.

Let φ be a finite-order element of the mapping class group $\Gamma_{g,[n]}$. We may assume that φ is a rotation—say, around the axis through the north and south poles (corresponding to the points $\infty, 0$). The north and south poles of S may or may not be marked points, but they are always the only ramification points for φ. The permutation associated to a rotation φ is always of the form $c_1 \cdots c_k$, where the c_i are disjoint cycles of length j such that

$$jk = \begin{cases} n & \text{if the north and south poles are not marked,} \\ n-1 & \text{if one of the two poles is marked,} \\ n-2 & \text{if both poles are marked points.} \end{cases}$$

In [Sc1, Thm. 3.5.1] Schneps computed the special locus of φ, where φ is associated to a permutation $[\varphi]$ that is a product of k disjoint cycles of length j in the case $jk = n-2$ (i.e., when the two fixed points of φ are marked points). As noted in [Sc1, Sec. 3.5], the special locus of φ in the general case can then be deduced from this case. The reason is that the special locus in $\mathfrak{M}_{0,n}$ is just the image of the one we compute here in $\mathfrak{M}_{0,n+1}$ or $\mathfrak{M}_{0,n+2}$ under the morphism given by erasing the extra marked points.

3.2. Special Loci of genus 0 *in Characteristic* p

In characteristic p, we can also think about marked curves having special automorphism group in the ordered moduli space to determine the special loci in the unordered moduli space.

First we give several explicit examples of special loci in characteristic 5 and characteristic 3.

EXAMPLE 3.1. In characteristic 5 and in the case $\mathfrak{M}_{0,5}$, consider the permutation $\tau = (1,2,3,4,5)$ and a point $(\lambda, 0, 1, \infty, \mu)$ in $\mathfrak{M}_{0,5}$ in standard representation (with three marked points fixed at 0, 1, and ∞). The action of τ on the point takes it to $(\mu, \lambda, 0, 1, \infty)$, and then the transformation by the automorphism $x \mapsto \frac{x-\lambda}{\lambda x - \lambda}$ brings it back to $\left(\frac{\mu-\lambda}{\lambda\mu-\lambda}, 0, 1, \infty, \frac{1}{\lambda}\right)$. The fixed points of τ are given by (λ, μ) with

$$\lambda = \frac{\mu - \lambda}{\lambda\mu - \lambda} \quad \text{and} \quad \mu = \frac{1}{\lambda},$$

so λ is a root of $\lambda^3 - 2\lambda^2 + 1$. One root is $\lambda = 1$, but this is excluded in $\mathfrak{M}_{0,5}$. The remaining roots are $\lambda = \frac{1 \pm \sqrt{5}}{2} = \frac{1}{2} = 3$ (since this is in characteristic 5), so the only fixed point of τ is $(3, 0, 1, \infty, 2)$. In fact, the point $(3, 0, 1, \infty, 2)$ is equivalent to the point $(0, 1, 2, 3, 4)$ in the moduli space $\mathfrak{M}_{0,5}$ since $(3, 0, 1, \infty, 2)$ transforms to the point $(0, 1, 2, 3, 4)$ by the linear transformation $x \mapsto \frac{3x-9}{x-9}$. So $(0, 1, 2, 3, 4)$ is fixed by a translation τ.

REMARK 3.2. In Example 3.1, τ fixes only one point in characteristic 5 but fixes two points in characteristic 0 (cf. [Sc1, Thm. 3.5.1]).

EXAMPLE 3.3. In characteristic 5 and in the case $\mathfrak{M}_{0,10}$, consider the permutation $\tau = (1,2,3,4,5,6,7,8,9,10)$ and a point $(x_1, \ldots, x_7, 1, 0, \infty)$ in $\mathfrak{M}_{0,10}$ in standard representation (with three components fixed at 0, 1, and ∞); then the action of τ on the point takes it to $(\infty, x_1, \ldots, x_7, 1, 0)$. But in characteristic 5, there is no transformation that can bring it back to the original point $(x_1, \ldots, x_7, 1, 0, \infty)$, by a calculation similar to that used in Example 3.1. So there is no fixed point of τ in $\mathfrak{M}_{0,10}$.

REMARK 3.4. In Example 3.3, τ has no fixed point in characteristic 5 but fixes two disconnected 1-dimensional components in characteristic 0 (cf. [Sc1, Thm. 3.5.1]).

EXAMPLE 3.5. For $\mathfrak{M}_{0,4}$ in characteristic 3, consider the permutation $\tau = (1,2,3)$ and a point $(\lambda, 1, 0, \infty)$ in $\mathfrak{M}_{0,4}$ in standard representation (with three components fixed at 0, 1, and ∞). The action of τ on the point takes it to $(0, \lambda, 1, \infty)$. Then the transformation $y \mapsto y + 2$ brings the point $(0, 2, 1, \infty)$ back to the original point $(2, 1, 0, \infty)$. So the fixed point of τ in $\mathfrak{M}_{0,4}$ is $(2, 1, 0, \infty)$.

REMARK 3.6. In Example 3.5, τ fixes only one point in characteristic 3 but fixes two points in characteristic 0 (cf. [Sc1, Thm. 3.5.1]).

Now we give a result to describe the finite-order automorphism in genus-0 algebraic curves with marked points in characteristic p.

PROPOSITION 3.7. *Let K be an algebraically closed field. Every finite-order automorphism of $\mathbb{P}^1_K$ with marked points is the conjugacy class of a rotation around an axis* (*i.e., by multiplying roots of unity*) *or the conjugacy class of a translation* (*i.e., by adding an element in K*).

Proof. We already know that the group of automorphisms of $\mathbb{P}^1_K$ is isomorphic to $\mathrm{PGL}(2, K)$. Since K is an algebraically closed field it follows that, by Jordan canonical form, every element in $\mathrm{PGL}(2, K)$ is conjugate to either

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda, \lambda_1, \lambda_2$ are nonzero elements in K.

If an element is conjugate to A, then it has one fixed point ∞ and the corresponding fractional linear transformation is $z \mapsto z + \frac{1}{\lambda}$, which is just a translation. If the automorphism is of finite order, then the translation can happen only in characteristic p.

If an element is conjugate to B, then it has two fixed points 0 and ∞ and the corresponding fractional linear transformation is $z \mapsto \frac{\lambda_1}{\lambda_2} z$, which is a composition of a rotation and a dilation. If the automorphism is of finite order then it is just a rotation; this happens both in characteristic 0 and p. In characteristic 0, the rotation can have any order; in characteristic p, its order is prime to p because there are no primitive pth roots of unity. $\square$

The following result describes the special loci in characteristic p in the case $g = 0$ and $n \geq 3$.

Proposition 3.8. *Let $(X; x_1, \ldots, x_n)$ be an unordered marked curve of genus 0 over an algebraically closed field K, and let φ be a finite-order automorphism of X that fixes $\{x_1, \ldots, x_n\}$ as a set. Let $[\varphi]$ denote the permutation of marked points induced by φ. Let g' be the genus of X/φ and let n' be the number of marked points coming from the marked points of X. If $\mathfrak{M}_{0,[n]}(\varphi)$ is not empty then φ is of the form $c_1 \cdots c_k$, where the c_i are disjoint cycles of length j such that $jk = n$ or $n-1$ or $n-2$. Moreover, the following statements hold.*

(a) *If j is not a multiple of p, then $\mathfrak{M}_{0,[n]}(\varphi)$ has the same description in characteristic 0 and p.*
(b) *If $p|j$ and $j > p$, then $\mathfrak{M}_{0,[n]}(\varphi)$ is empty.*
(c) *If $p = j$ and $jk = n-2$, then $\mathfrak{M}_{0,[n]}(\varphi)$ is empty. If $p = j$ and $jk = n-1$ or $jk = n$, then $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to quotient of $\{\mathbb{P}^1 - \{0, 1, \infty\}\}^{k-2} - \Delta$ by S_k, where Δ denotes the multidiagonal of points with $x_i = x_j$ for some $i \neq j$.*

Proof. By Proposition 3.7, we know that if $\mathfrak{M}_{0,[n]}(\varphi)$ is not empty then φ is of the form $c_1 \cdots c_k$, where the c_i are disjoint cycles of length j such that $jk = n$ or $n-1$ or $n-2$.

(a) If j is not a multiple of p, then we have the jth roots of unity. Because the proof for characteristic 0 (cf. [Sc1, Thm. 3.5.1]) involves only the pure group theory, it also works for characteristic p in this case.

(b) If $p|j$ and $j > p$, then φ has no fixed point either as a rotation or as a translation.

(c) If $p = j$, then φ is a translation. If $jk = n-2$, then φ has no fixed point because a translation cannot fix two points pointwise.

If $jk = n-1$, then in the ordered moduli space $\mathfrak{M}_{0,n}$ we know that φ fixes $p-1$ disjoint connected components. Each such component is given by

$$C_i = (0, i, \ldots, (p-1)i, a_1, \ldots, a_1 + (p-1)i, \ldots, a_{k-1}, \ldots, a_{k-1} + (p-1)i, \infty),$$

where $i = 1, \ldots, p-1$ and $a_1, \ldots, a_{k-1}$ are any numbers in the field K such that all the marked points are distinct. In the unordered moduli space $\mathfrak{M}_{0,n}$, all the components C_i (as well as all those components corresponding to other rotations having the same cycle type as φ) become identified. So $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to one of C_i, say C_1, modulo its stabilizer in $S_n = S_{jk+1}$. We could determine its stabilizer by a procedure similar to that used in Schneps's proof of [Sc1, Thm. 3.5.1] (where the proof involved only the pure group theory). In fact, the stabilizer of C_1 is generated by two natural subgroups: the first, of order $k!$, corresponds to permuting the k disjoint cycles of $[\varphi]$; the second, of order j^k, is generated by the j cycles themselves. After computing the quotient of C_1 by its stabilizer, we obtain that $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to the quotient of $\{\mathbb{P}^1 - \{0, 1, \infty\}\}^{k-2} - \Delta$ by all permutations of marked points in X/φ that come from the marked points with the same ramification index in X (i.e., S_k), where Δ denotes the multidiagonal of points with $x_i = x_j$.

If $jk = n$, then a similar calculation as used in the case $jk = n-1$ yields that $\mathfrak{M}_{0,[n]}(\varphi)$ is isomorphic to the quotient of $\{\mathbb{P}^1 - \{0, 1, \infty\}\}^{k-2} - \Delta$ by S_k, where Δ denotes the multidiagonal of points with $x_i = x_j$. $\square$

Proposition 3.8 shows that there is no automorphism of order divisible by p in characteristic p unless the order is exactly p. Also, there is an automorphism of order p (namely, translation), but this automorphism behaves differently from an automorphism of the same order in characteristic 0.

3.3. *Special Loci in Genus 1*

There is also a generalization of Proposition 3.8 to higher genus. In particular, for $g = 1$, we give some results in the following proposition.

PROPOSITION 3.9. *Let X be a marked curve of genus* 1 *with n marked points $x_1, \ldots, x_n$ over an algebraically closed field K of characteristic $\neq 2, 3$ and let φ be a finite-order automorphism of X. Let $[\varphi] \in S_n$ be a permutation of marked points for $n \geq 5$, where $[\varphi]$ is the corresponding permutation of marked points of φ. Suppose that $\mathfrak{M}_{1,[n]}(\varphi)$ is not empty and write $[\varphi]$ as a product of disjoint cycles $c_1 \cdots c_k$. Then either the c_i are of the same length j such that $jk = n$ or each c_i has length* $2, 3, 4, 6$.

Proof. Let X be a marked curve of genus 1 with n marked points $x_1, \ldots, x_n$ over an algebraically closed field K of characteristic $\neq 2, 3$, and let φ be a finite-order automorphism of X. Let $[\varphi] \in S_n$ be a permutation of marked points for $n \geq 5$, where $[\varphi]$ is the corresponding permutation of marked points of φ. Let g_0 be the genus of $X/\langle\varphi\rangle$. Then the possible order m of φ and their branching data $(m_1, m_2, \ldots, m_r)$ are limited by the well-known Riemann–Hurwitz equation:

$$\frac{2g-2}{m} = (2g_0 - 2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right).$$

Here we consider the Riemann–Hurwitz formula only in the tame case, since if the characteristic of K is not 2 or 3 then all the branch coverings we considered are tame. Because we consider only the cyclic group $\langle\varphi\rangle$ of order m, let $M = \mathrm{lcm}(m_1, m_2, \ldots, m_r)$. Then the following conditions are satisfied [B, Cor. 9.4]:

(i) $\mathrm{lcm}(m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r) = M$ for all i;
(ii) M divides m, and if $g_0 = 0$ then $M = m$;
(iii) $r \neq 1$, and if $g_0 = 0$ then $r \geq 3$;
(iv) if M is even, then the number of m_i divisible by the maximum power of 2 dividing M is even.

By the classification of automorphisms of elliptic curves [Si, Thm. 10.1], we know the possible values of m_i are $2, 3, 4, 6$. Combining the previous conditions on the Riemann–Hurwitz equation (3.3), we get that the possible Galois coverings are:

(i) $m = 2, r = 4, m_i = 2$ for all i;
(ii) $m = 3, r = 3, m_i = 3$ for all i;
(iii) $m = 4, r = 3, (m_1, m_2, m_3) = (2, 4, 4)$;
(iv) $m = 6, r = 3, (m_1, m_2, m_3) = (2, 3, 6)$.

Since X has n marked points, by the possible Galois coverings just listed we can obtain the possible permutations $[\varphi]$ of marked points as in the proposition. □

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